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Finite-Series Solutions of Hybrid PDE Systems for Conditional Moments of Regime-Switching Extended CEV Processes with Applications in Finance

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Abstract

This paper develops finite-series solutions of a hybrid system of interconnected partial differential equations for computing the conditional moments of regime-switching extended constant elasticity of variance processes with generalized drift and diffusion coefficients. The regime-switching mechanism is modeled by a continuous-time, finite-state, irreducible Markov chain with m regimes, for any integer $m \geq 1$. For any real $\gamma > 0$, we identify a tractable class of processes where the γ^{th} conditional moment admits an explicit finite power series representation in the initial state, arising from the polynomial structure. The analytical framework is derived via a Feynman–Kac representation adapted for regime-switching diffusions and validated for accuracy and efficiency using Monte Carlo simulations. In addition, we investigate the asymptotic behavior of the first conditional moment for a two-state regime-switching constant elasticity of variance process with nonlinear drift, emphasizing the effects of symmetry in the Markov intensity matrix and comparisons with the corresponding linear-drift case. Applications in futures pricing demonstrate the framework's relevance for derivative pricing and risk management.

Keywords: regime-switching extended constant elasticity of variance process; conditional moments; hybrid system of PDEs; finite series solutions

MSC: 60J28; 91G20

Check for updates

Academic Editors: Ioannis S. Stamatiou and Christos P. Kitsos

Received: 24 January 2026
Revised: 19 February 2026
Accepted: 26 February 2026
Published: 28 February 2026

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1. Introduction

Stochastic differential equations (SDEs) with regime switching have become an important modeling framework for systems exhibiting abrupt structural changes over time. Such changes are frequently observed in financial markets, where asset dynamics are influenced by shifts in economic conditions, market sentiment, and policy environments. Classical diffusion models with constant parameters are often inadequate to capture these features. By allowing model parameters to switch across a finite number of states governed by a continuous-time Markov chain, regime-switching models provide a flexible and realistic representation of evolving market dynamics [1–5].

In financial modeling, regime-switching diffusions have been widely applied to asset prices, interest rates, volatility indices, and derivative securities. Empirical evidence suggests that markets alternate between distinct regimes, such as high- and low-volatility

periods or expansionary and contractionary phases. Incorporating regime dependence into stochastic models has been shown to significantly improve descriptive power and pricing performance compared with single-regime models [4–9].

A central object in the analysis of stochastic processes is the study of conditional moments, which characterize the distributional properties of the underlying dynamics. Conditional moments play a fundamental role in risk management, statistical inference, and derivative pricing. In particular, the valuation of futures contracts often relies on the conditional expectation of the future spot price given current market information, while higher-order conditional moments provide insight into risk exposure and uncertainty surrounding futures returns. Despite their importance, analytical expressions for conditional moments are rarely available for nonlinear stochastic models, especially in the presence of regime switching.

Among nonlinear diffusion models, the constant elasticity of variance (CEV) process occupies a prominent position due to its ability to capture state-dependent volatility and leverage effects. Since its introduction by Cox [10,11], the CEV model has been extensively studied and applied in option pricing and asset modeling [12–15]. Several generalizations have been proposed to enhance its flexibility, including extensions with nonlinear drift, time-dependent parameters, jumps, and stochastic volatility [16–20].

Recent studies have focused on deriving closed-form expressions for conditional moments of generalized CEV-type processes. Rujivan [21] obtained analytical formulas for conditional moments of the extended Cox–Ingersoll–Ross (ECIR) process. Chumpong et al. [22] derived simple closed-form moment formulas for inhomogeneous nonlinear drift CEV processes, while Sutthimat et al. [23] further generalized these results to nonlinear drift CEV models with time-dependent coefficients. These contributions demonstrate that, under suitable structural conditions, explicit moment representations can be obtained even for nonlinear diffusions.

More recently, attention has shifted toward incorporating regime-switching mechanisms into these nonlinear frameworks. Chumpong et al. [24] derived a closed-form formula for the conditional moment-generating function of a regime-switching nonlinear drift CEV process and demonstrated its applicability to option pricing problems. Their results provide important insight into how regime-dependent parameters affect moment dynamics and derivative values. Complementing this line of research, Rujivan et al. [25] developed an analytical framework for computing conditional moments in the ECIR process with regime switching (RS-ECIR) by solving a hybrid system of interconnected partial differential equations (PDEs). Their work represents a significant step forward in bridging nonlinear diffusion theory with regime-switching dynamics and financial applications.

Despite extensive research on CEV-type models and regime-switching diffusions, analytical results for conditional moments of regime-switching extended CEV (RS-ECEV) processes remain scarce. Most existing studies rely on numerical methods or Monte Carlo (MC) simulations, which can be computationally expensive and offer limited analytical insight. Moreover, closed-form transition densities for such hybrid processes are generally unavailable, making direct computation of conditional moments infeasible.

This paper addresses this gap by developing an analytically tractable framework for computing conditional moments of RS-ECEV processes with generalized drift and diffusion coefficients that allow time-dependent parameters. The underlying regime-switching mechanism is modeled by a continuous-time, finite-state, irreducible Markov chain with an arbitrary number of regimes. Within this framework, we identify a class of models for which conditional moments of arbitrary positive order admit explicit representations in finite form.

The analytical approach is based on the Feynman–Kac representation adapted to regime-switching diffusions, proposed by Baran et al. [26], which leads to a system of interconnected PDEs characterizing the evolution of conditional moments. By exploiting the polynomial structure induced by the extended CEV (ECEV) specification, we derive finite series representations for the conditional moments, yielding exact and computationally efficient solutions. Unlike numerical PDE schemes or simulation-based methods, the proposed approach avoids discretization errors and significantly reduces computational cost.

Focusing on the m-state RS-ECEV process governed by an underlying finite-state Markovian mechanism, this paper presents a rigorous mathematical approach for moment evaluation. By utilizing a generalized Feynman–Kac representation tailored for switching diffusions, we facilitate the computation of conditional moments for an arbitrary positive real order γ and any number of states m. Such a formulation extends the analytical capabilities for a wide spectrum of $\gamma > 0$ and $m \geq 1$.

The key contributions of this paper are as follows:

1. By applying the Feynman–Kac formula adapted to regime-switching diffusions, we identify a tractable class of such processes for which the γ^{th} conditional moment, for any positive γ, admits an explicit representation as a finite power series in the initial state, which, to the best of our knowledge, has not been achieved in the existing literature. This result offers several important advantages. Moreover, we validate our analytical results using MC simulations to assess the accuracy and computational efficiency of the proposed finite series solutions. The finite series approach developed here significantly reduces computational cost while maintaining a high degree of accuracy.
2. We illustrate the practical relevance of the proposed methodology through applications in financial modeling, with particular emphasis on the pricing of futures in regime-switching environments. The finite series representations of conditional moments derived in this paper enable explicit pricing formulas for rapid numerical evaluation of derivative prices under regime-switching dynamics that can avoid costly numerical integration or MC simulation when computing moments, providing a valuable tool for practitioners in quantitative finance and risk management.
3. For the two-state regime-switching CEV (RS-CEV) process with nonlinear drift, we analyze the asymptotic properties of the first conditional moment, emphasizing the effects of symmetry in the Markov chain's intensity matrix, comparisons with the linear-drift case, and the impact of alternative parameter configurations.

The novelty of this work lies in providing a unified, analytically tractable framework that generalizes existing approaches to m-state RS-ECEV processes. Unlike the single-state models in [22,23] or the time-invariant cases in [24], our methodology extends to any finite number of regimes ($m \geq 1$) and accommodates time-dependent parameters. This approach offers a broader theoretical scope, valid for all positive γ and $\beta \in [0, 2]$. Furthermore, our framework ensures mathematical consistency with the RS-ECIR model of [25] while significantly enhancing the model's flexibility to capture complex market dynamics.

The remainder of the paper is organized as follows. Section 2 introduces the m-state RS-ECEV process and establishes conditions ensuring the finiteness of the γ^{th} conditional moments. It also presents a hybrid system of PDEs, derived via the Feynman–Kac representation for regime-switching diffusions, for computing these moments. Section 3 derives finite power-series solutions to this system and identifies a tractable class of processes for which the γ^{th} conditional moments ($\gamma > 0$) admit an explicit representation in the initial state. Section 4 provides explicit formulas for the first conditional moments of the two-state RS-CEV process and illustrates their application to futures pricing. Section 5 reports numerical experiments demonstrating accuracy and computational efficiency, examines the

asymptotic behavior of the first conditional moment, and discusses the implications of the results. Finally, Section 6 concludes the paper.

2. Preliminary

2.1. The m-State RS-ECEV Process and Its Conditional Moments

Let $(m, \beta) \in \mathbb{N} \times \mathbb{R}_0^+$. Consider the m-state RS-ECEV process $X_{m,\beta} := \left(X_{m,\beta,t}\right)_{t\geq 0}$ governed by the SDE:

$$dX_{m,\beta,t} = \kappa\left(t, \xi_t^{(m)}\right)\left(\theta\left(t, \xi_t^{(m)}\right) X_{m,\beta,t}^{-(1-\beta)} - X_{m,\beta,t}\right)dt + \sigma\left(t, \xi_t^{(m)}\right) X_{m,\beta,t}^{\beta/2} dW_t, \tag{1}$$

defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0})$ with initial condition $X_{m,\beta,0} = x_0 > 0$.

Here, m denotes the number of regimes in the underlying Markov chain governing the switching dynamics, while β is the elasticity parameter that determines the degree of state dependence in the diffusion term and characterizes the model class (e.g., CIR-type when $\beta = 1$ and Geometric Brownian Motion (GBM)-type when $\beta = 2$). These parameters jointly control the structural complexity of the process and the behavior of its conditional moments.

Remark 1. *The drift term involving $X_{m,\beta,t}^{-(1-\beta)}$ in Equation (1) generalizes several classical diffusion specifications. In particular, when $\beta = 1$, the model reduces to a CIR-type mean-reverting structure, while other values of β recover ECEV dynamics with state-dependent drift and volatility. This formulation therefore provides a unified framework encompassing both CIR and CEV-type processes under regime switching.*

The process $\xi^{(m)} := \left(\xi_t^{(m)}\right)_{t\geq 0}$ is a continuous-time irreducible m-state Markov chain with state space $\mathcal{M}_m := \{1, 2, \ldots, m\}$ and intensity matrix $\mathbf{Q}_m := \left[q_{ij}^{(m)}\right]_{m\times m}$. The process $W := (W_t)_{t\geq 0}$ is a standard Brownian motion, and it is assumed that W and $\xi^{(m)}$ are independent. Furthermore, $(\mathcal{F}_t)_{t\geq 0}$ represents the $\mathbb{P}$-complete filtration capturing the historical paths of both W and $\xi^{(m)}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Let $\kappa(t,i)$, $\theta(t,i)$, and $\sigma(t,i)$ denote the mean-reversion rate, long-term mean, and volatility coefficient, respectively, formulated as real-valued functions mapping from the domain $\mathcal{D}_X := [0, T] \times \mathcal{M}_m$. To streamline the notation, we introduce the diagonal matrices $\mathbf{K}[t, m] := \operatorname{diag}[\kappa(t,1), \ldots, \kappa(t,m)]$, $\mathbf{\Theta}[t, m] := \operatorname{diag}[\theta(t,1), \ldots, \theta(t,m)]$, and $\mathbf{\Sigma}[t, m] := \operatorname{diag}[\sigma^2(t,1), \ldots, \sigma^2(t,m)]$. The transpose operator for any arbitrary matrix $\mathbf{A}$ is indicated by $\mathbf{A}^\top$. Furthermore, for any $p \in \mathbb{N}$, the matrix exponential of a $p \times p$ matrix $\mathbf{A}$ is defined via the infinite series $\mathbf{e}^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$, where $\mathbf{A}^0$ corresponds to the p-dimensional identity matrix $\mathbf{I}_p$. For a time-varying matrix function $\mathbf{A}(t) = [a_{ij}(t)]_{p\times q}$, its integral over a bounded interval $0 \leq u \leq v < \infty$ is evaluated element-wise, yielding $\int_u^v \mathbf{A}(\xi)\, d\xi := \left[\int_u^v a_{ij}(\xi)\, d\xi\right]_{p\times q}$. Whenever X is governed by an m-state RS-CEV process—indicating that the parameters κ, θ, and σ exhibit no explicit time dependence—we compactly express the corresponding constant diagonal matrices as $\mathbf{K}_m$, $\mathbf{\Theta}_m$, and $\mathbf{\Sigma}_m$. Ultimately, by restricting the state space to $m = 1$, the dynamics of X naturally collapse to the classical mean-reverting CEV framework.

For the Markov chain $\xi^{(m)}$, its associated intensity matrix, denoted as $\mathbf{Q}_m = \left[q_{ij}^{(m)}\right]_{m\times m}$, must fulfill the following requirements [27]:

$$q_{ij}^{(m)} \geq 0 \quad \text{for all } i, j \in \mathcal{M}_m,\ i \neq j, \qquad \sum_{j\in\mathcal{M}_m} q_{ij}^{(m)} = 0 \quad \text{for all } i \in \mathcal{M}_m. \tag{2}$$

Let $\gamma \in \mathbb{R}^+$ and $T \in \mathbb{R}^+$. The real-valued function $U^{(\gamma,m,\beta)} : \mathbb{R}^+ \times [0,T] \times \mathcal{M}_m \to \mathbb{R}^+$ is defined as the γ^{th} conditional moment of $X_{m,\beta}$, governed by the SDE (1):

$$U^{(\gamma,m,\beta)}(x,\tau,i) := \mathbb{E}^{\mathbb{P}}[X^{\gamma}_{m,\beta,T} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[X^{\gamma}_{m,\beta,T} \mid X_{m,\beta,t} = x, \xi^{(m)}_t = i], \tag{3}$$

where $\tau = T - t$ for any $t \in [0,T]$, provided the conditional expectation on the right-hand side is well-defined. The second equality follows from the Markov property. Although we can compute $U^{(\gamma,m,\beta)}(x,\tau,i)$ using numerical approaches such as MC simulations [28], due to the high computational cost of MC methods, this paper focuses on developing an analytical approach to derive a closed-form formula for $U^{(\gamma,m,\beta)}(x,\tau,i)$, significantly reducing computational effort.

Before proceeding, it is necessary to impose specific conditions that ensure the γ^{th} conditional moments of the m-state RS-ECEV process $X_{m,\beta}$ are well-defined and finite. The conditions build on those established in [23] for the single-state ECEV process and are further extended to accommodate the regime-switching structure introduced above. In particular, the switching-related components follow standard formulations in the regime-switching literature (cf. [2]), which together lead to the sufficient conditions presented below.

Assumption 1. *For each state $i \in \mathcal{M}_m$, we require the parameters $\kappa(t,i)$, $\theta(t,i)$, and $\sigma(t,i)$ to be strictly positive, continuously differentiable functions of time on $[0,T]$, with the supplementary condition that the quotient $\kappa(t,i)/\sigma^2(t,i)$ remains locally bounded over this interval.*

Assumption 2. *For each state $i \in \mathcal{M}_m$, the parameters are constrained by the standard Feller condition, satisfying $2\kappa(t,i)\theta(t,i) \geq \sigma^2(t,i)$ throughout the interval $t \in [0,T]$.*

Assumptions 1 and 2 are imposed to ensure the well-posedness of the regime-switching diffusion and the finiteness of the conditional moments considered in this study. These conditions are primarily technical in nature and are standard in the analysis of CIR- and CEV-type processes. In particular, the regularity and positivity requirements guarantee the existence and uniqueness of a strong solution, while the Feller-type condition prevents boundary degeneracy and ensures non-negativity of the process. Although these assumptions may not be minimal in a strict mathematical sense, they provide a convenient and widely adopted framework under which the hybrid PDE system and the associated finite-series representations can be rigorously derived. Relaxing these conditions would require additional technical arguments and is left for future research.

2.2. Application of the Feynman–Kac Formula

To analytically evaluate the conditional expectation (3), we employ the Feynman–Kac formula adapted for regime-switching diffusions in Theorem 6 of [26]. Consequently, this requires determining the solution to a hybrid coupled PDE system defined via the infinitesimal generator of $X_{m,\beta}$, which takes the form:

$$\begin{aligned}\mathcal{L}_E[f(x,t,i)] := {} & \frac{1}{2}\sigma^2(t,i)x^{\beta}\frac{\partial^2 f(x,t,i)}{\partial x^2} + \kappa(t,i)\Big(\theta(t,i)x^{-(1-\beta)} - x\Big)\frac{\partial f(x,t,i)}{\partial x} \\ & + \sum_{\substack{l \in \mathcal{M}_m \\ l \neq i}} q^{(m)}_{il}\big(f(x,t,l) - f(x,t,i)\big),\end{aligned} \tag{4}$$

applicable to any function $f(\cdot,t,i) \in C^2(\mathbb{R}^+;\mathbb{R})$ that is twice continuously differentiable.

To justify the application of the Feynman–Kac formula in the following theorem, we assume that the function $U^{(\gamma,m,\beta)}(x,\tau,i)$ satisfies the standard regularity conditions for regime-switching diffusions. In particular, for each regime $i \in \mathcal{M}_m$, the function

$U^{(\gamma,m,\beta)}(\cdot,\cdot,i)$ is assumed to belong to the class of $C^2(\mathbb{R}^+;\mathbb{R})$ and the parameter functions are assumed to satisfy appropriate growth and integrability conditions ensuring that the infinitesimal generator is well-defined and the associated expectations are finite. These assumptions are consistent with the regime-switching Feynman–Kac framework and align with the conditions stated in Theorem 6 of [26].

Theorem 1. *Let $(\gamma, m, \beta) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}_0^+$ be fixed. The functions $U^{(\gamma,m,\beta)}(x,\tau,i)$, $i \in \mathcal{M}_m$, satisfy the hybrid system of interconnected PDEs:*

$$-\frac{\partial}{\partial\tau}U^{(\gamma,m,\beta)}(x,\tau,i) + \mathcal{L}_E\left[U^{(\gamma,m,\beta)}(x,\tau,i)\right] = 0, \tag{5}$$

for all $(x,\tau) \in \mathbb{R}^+ \times (0,T]$, subject to the initial conditions:

$$U^{(\gamma,m,\beta)}(x,0,i) = x^\gamma, \tag{6}$$

for $x \in \mathbb{R}^+$ and $i \in \mathcal{M}_m$.

Proof. See Appendix A.1. □

3. Derivation of the Finite Power Series Solution

Inspired by the method proposed by Rujivan et al. [25], we aim to represent $U^{(\gamma,m,\beta)}(x,\tau,i)$ as a finite power series with respect to the initial value x. This section focuses on deriving the finite power series solution for the interconnected PDEs (5) and (6), established in Theorem 1. Before deriving an analytical formula for $U^{(\gamma,m,\beta)}(x,\tau,i)$, we note that finding the n^{th} conditional moment associated with the m-state RS-ECIR process, as proposed by [25], is a special case of our problem when $\gamma \in \mathbb{N}$, $m \in \mathbb{N}$, and $\beta = 1$. Our problem generalizes the framework set by [25].

For a given $T \in \mathbb{R}^+$ and $(\gamma, m, n) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$, we seek an m-state RS-ECEV process $X_{m,\beta}$ for some $\beta \in \mathbb{R}_0^+$, such that $U^{(\gamma,m,\beta)}(x,\tau,i)$ can be expressed as a finite power series: $\sum_{k=0}^{n} a_{i,k}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}$, where $a_{i,k}(\tau;n,m,\beta)$, $k = 0,1,\ldots,n$, are differentiable functions of $\tau = T - t$, defined on $[0,\infty)$.

3.1. The Main Theorem

Our methodology proceeds by defining a matrix $\mathbf{B}_j(\tau;n,m,\beta)$, applicable for all $\tau \geq 0$ and $j = 0,\ldots,n$, expressed as

$$\mathbf{B}_j(\tau;n,m,\beta) := \begin{bmatrix} b_{1,j}(\tau;n,m,\beta) \\ \vdots \\ b_{m,j}(\tau;n,m,\beta) \end{bmatrix}_{m\times 1}, \tag{7}$$

where $b_{i,j}(\tau;n,m,\beta)$ is a differentiable function of $\tau = T - t$, defined on $[0,\infty)$ for each $i \in \mathcal{M}_m$.

The matrices $\mathbf{B}_j(\tau;n,m,\beta)$, for $j = 0,\ldots,n$, satisfy the following m-dimensional linear ODE system

$$\frac{d}{d\tau}\mathbf{B}_j(\tau;n,m,\beta) = \Psi_j^{(1)}(\tau;n,m,\beta)\mathbf{B}_j(\tau;n,m,\beta) + \Psi_j^{(-1)}(\tau;n,m,\beta)\mathbf{B}_{j-1}(\tau;n,m,\beta), \tag{8}$$

for $\tau > 0$, with initial conditions

$$\mathbf{B}_j(0;n,m,\beta) = \begin{cases} \mathbf{1}_m^\top & \text{if } j=0, \\ \mathbf{0}_m^\top & \text{if } j=1,\ldots,n, \end{cases} \tag{9}$$

where $\mathbf{1}_m$ and $\mathbf{0}_m$ denote the m-dimensional row vectors consisting entirely of ones and zeros, respectively.

The recursive ODE system in (8) reflects how the polynomial coefficients evolve over time under the combined effects of regime switching and state-dependent dynamics. The matrix $\Psi_j^{(1)}(\tau;n,m,\beta)$ governs the within-order evolution of the coefficient vector $\mathbf{B}_j(\tau;n,m,\beta)$, capturing contributions from the Markov switching mechanism and the drift component of the process. In contrast, $\Psi_j^{(-1)}(\tau;n,m,\beta)$ links adjacent polynomial orders by transferring information from $\mathbf{B}_{j-1}(\tau;n,m,\beta)$ to $\mathbf{B}_j(\tau;n,m,\beta)$, primarily reflecting the influence of the diffusion term and nonlinear state dependence. Together, these matrices encode the hierarchical structure of the polynomial expansion and determine how coefficients at different orders interact over time.

Within the system (8), the associated time-dependent coefficient matrices are formulated as follows:

$$\Psi_j^{(1)}(\tau;n,m,\beta) := \mathbf{Q}_m - (2-\beta)(n-j)\mathbf{K}[T-\tau,m], \tag{10}$$

$$\Psi_j^{(-1)}(\tau;n,m,\beta) := (2-\beta)(n-j+1)\left(\frac{1}{2}((2-\beta)(n-j+1)-1)\mathbf{\Sigma}[T-\tau,m] + \mathbf{K}[T-\tau,m]\mathbf{\Theta}[T-\tau,m]\right), \tag{11}$$

Furthermore, to ensure the recursive system is fully specified, we adopt the convention $\mathbf{B}_{-1}(\tau;n,m,\beta) = \mathbf{0}_m^\top$ uniformly across all $\tau \geq 0$.

Leveraging the aforementioned matrices $\mathbf{B}_j(\tau;n,m,\beta)$ for $j=0,\ldots,n$, we deduce the explicit finite power series solution to the interconnected PDEs (5) and (6) established in Theorem 1. This finite power-series representation in (12) is motivated by the polynomial-preserving structure of the infinitesimal generator associated with the RS-ECEV process. In particular, when the generator acts on functions that are polynomial in the initial value x, the resulting expressions remain within the same polynomial class, with coefficients evolving according to a coupled system determined by the model parameters and the regime-switching mechanism. This structural property suggests seeking solutions to the hybrid PDE system in the form of finite power series in x, as such representations are closed under the action of the generator and therefore compatible with the Feynman–Kac framework. This observation provides the analytical basis for the construction of the representation in (12).

Theorem 2. *Let $T>0$ and $(\gamma,m,n) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$ be fixed with $\frac{\gamma}{n} \leq 2$. Define a real-valued function $U^{(\gamma,m,\beta)}(x,\tau,i)$ as*

$$U^{(\gamma,m,\beta)}(x,\tau,i) := \sum_{k=0}^{n} a_{i,k}(\tau;n,m,\beta)x^{(2-\beta)k}, \tag{12}$$

for all $(x,\tau,i) \in \mathbb{R}^+ \times [0,T] \times \mathcal{M}_m$ and $\beta = 2 - \frac{\gamma}{n} > 0$, where the coefficients $a_{i,k}(\tau;n,m,\beta)$, $i \in \mathcal{M}_m$, are determined by solving the recursive matrix ODEs (8) under the initial conditions (9) as described above through the matrix relations:

$$\begin{bmatrix} \mathbf{A}_1(\tau;n,m,\beta) \\ \vdots \\ \mathbf{A}_m(\tau;n,m,\beta) \end{bmatrix} = \begin{bmatrix} \mathbf{B}_0(\tau;n,m,\beta) & \cdots & \mathbf{B}_n(\tau;n,m,\beta) \end{bmatrix}, \tag{13}$$

where

$$\mathbf{A}_i(\tau; n, m, \beta) := \begin{bmatrix} a_{i,n}(\tau; n, m, \beta) & \cdots & a_{i,0}(\tau; n, m, \beta) \end{bmatrix}_{1\times(n+1)}.$$

Then $U^{(\gamma,m,\beta)}(x, \tau, i), i \in \mathcal{M}_m$, *solve the interconnected PDEs* (5) *and* (6).

Proof. See Appendix A.2. □

The restriction $\frac{\gamma}{n} \leq 2$, in Theorem 2, arises from the requirement that the polynomial ansatz remains consistent under the action of the hybrid PDE system. When substituting the finite power-series form into the PDEs, the drift and diffusion terms generate contributions involving different powers of x. To ensure closure of the resulting system—so that no higher-order terms beyond those included in the assumed expansion appear—the exponents must balance across the differential operators. This balance leads to the condition $\frac{\gamma}{n} \leq 2$, which guarantees that the diffusion term does not produce powers exceeding those represented in the finite series and thus allows the recursive system to remain finite-dimensional.

Theorem 2 characterizes the values of β for which the solution $U^{(\gamma,m,\beta)}(x, \tau, i)$ of the interconnected PDEs (5) and (6) can be represented as a finite power series in the initial state x. However, in many practical applications, β is fixed. Consequently, the following corollary identifies the specific values of γ, for a given β, that ensure the solution admits such a finite series representation.

Corollary 1. *Let* $T > 0$, $(m, n) \in \mathbb{N} \times \mathbb{N}$ *and* $\beta \in [0, 2)$ *be fixed. Set* $\gamma = n(2 - \beta)$. *Then, the* γ^{th} *conditional moment of* $X_{m,\beta}$ *defined in* (3), *satisfies* (12).

Corollary 1 is particularly useful in practical applications where the elasticity parameter $\beta \in [0, 2)$ is specified a priori. In this case, n can be chosen as any positive integer satisfying $\gamma = n(2 - \beta)$. In practice, one typically selects the positive integer n that yields the desired conditional moment order γ.

It should be remarked that Theorem 2 and Corollary 1 are valid only for $\beta \in [0, 2)$. By considering a special case where $\beta = 2$, we obtain the following theorem, which illustrates this particular case.

Theorem 3. *Let* $\beta = 2, T > 0$ *and* $(\gamma, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ *with* $\gamma = n$ *be fixed. Define a real-valued function* $U^{(n,m,2)}(x, \tau, i)$ *as*

$$U^{(n,m,2)}(x, \tau, i) := a_{i,n}(\tau; n, m, \beta)x^n, \tag{14}$$

for all $(x, \tau, i) \in \mathbb{R}^+ \times [0, T] \times \mathcal{M}_m$, *where the coefficients* $a_{i,n}(\tau; n, m, 2)$, $i \in \mathcal{M}_m$, *are determined by solving the matrix differential equations*

$$\frac{d}{d\tau}\mathbf{B}_j(\tau; n, m, 2) = \Psi_j(\tau; n, m, 2)\mathbf{B}_j(\tau; n, m, 2), \tag{15}$$

for $\tau > 0$, $j = 0, \ldots, n$, *where*

$$\Psi_j(\tau; n, m, 2) := \mathbf{Q}_m + (n - j)\left(\frac{1}{2}(n - j - 1)\mathbf{\Sigma}[T - \tau, m] + \mathbf{K}[T - \tau, m]\Big(\mathbf{\Theta}[T - \tau, m] - \mathbf{I}_m\Big)\right), \tag{16}$$

with initial conditions

$$\mathbf{B}_j(0; n, m, 2) = \begin{cases} \mathbf{1}_m^\top & \text{if } j = 0, \\ \mathbf{0}_m^\top & \text{if } j = 1, \ldots, n, \end{cases} \tag{17}$$

through the matrix relations (13)*, then* $U^{(n,m,2)}(x,\tau,i)$, $i \in \mathcal{M}_m$*, solve the interconnected PDEs* (5) *and* (6)*.*

Proof. See Appendix A.3. □

3.2. Comparison with Other Approaches

The recent literature has witnessed notable advancements regarding the exact analytical derivation of conditional moments of an m-state RS-ECEV process. In the single-state case ($m = 1$), Sutthimat et al. [23] studied the ECEV process, or so-called NLD-CEV, in their work, for $\beta \in [0,2) \cup (2,\infty)$, while Chumpong et al. [22] considered $\beta \in [0,2)$. Both works established a comprehensive analytical framework and showed that it outperforms standard approaches for computing conditional fractional moments, including MC simulations.

For the multi-state case ($m > 1$) with the specific constant elasticity of variance parameter $\beta = 1$, Rujivan et al. [25] developed a general framework that incorporates regime switching with the extended CIR process. Their method enabled the computation of positive integer conditional moments and was applied to the pricing of VIX futures and VIX options, where the accuracy of the results was confirmed through comparisons with MC simulations. Additionally, they studied the long-run behavior of the first conditional moments for the two-state RS-CIR process, specifically examining how symmetry in the transition intensity matrix influences these dynamics. Chumpong et al. [24] deduced a closed-form conditional moment-generating function for a regime-switching nonlinear drift CEV process. Operating under a time-independent two-state switching mechanism, their work demonstrated practical utility in derivative pricing.

The methodology in the present paper clearly extends the earlier approaches of [22,23] to finite-state cases ($m \geq 1$) for $\beta \in [0,2)$. We also provide a unified framework valid for all positive values of γ and for any number of regimes $m \geq 1$. In the special case $m = 1$ of our work, the intensity matrix $\mathbf{Q}_1$ reduces to a one-dimensional null matrix. Within this reduced framework, our results agree with Theorem 2 in [23] and Corollary 3 of [22]. Moreover, our framework can be viewed as a generalization of [24], as it allows the process parameters to be time-dependent.

Relating our findings to the m-state RS-ECIR framework studied by Rujivan et al. [25] (representing the special case $\beta = 1$ in our model), their derived ODE system and associated parameters—specifically their Equations (4.2) through (4.5)—are in complete agreement with the structure of (8) and (9) derived in the present study when restricted to $\beta = 1$. Additionally, we analyze the long-term dynamics of the first conditional moment governed by a two-state RS-CEV model with $\beta = \frac{3}{2}$. As detailed in Section 5.2, our investigation emphasizes the influence of a symmetric intensity matrix, ultimately yielding markedly different asymptotic outcomes.

4. Financial Applications

In the prior section, exact finite power series representations in terms of the starting value x were constructed for the hybrid PDE framework (5) and (6), as established in Theorem 1. These solutions, obtained for the m-state RS-ECEV process (1) for some $\beta \in [0,2)$ and for $\beta = 2$, were rigorously characterized in Theorems 2 and 3 respectively. In particular, for each $\gamma \in \mathbb{R}^+$, we define the set

$$\begin{aligned} \mathcal{B}_\gamma :&= \left\{ \beta \in [0,2) : U^{(\gamma,m,\beta)}(x,t,i) \text{ admits a finite power series representation in } x \right\} \\ &= \left\{ \beta = 2 - \frac{\gamma}{n} : \text{for all positive integer } n \geq \frac{\gamma}{2} \right\}, \end{aligned} \tag{18}$$

which characterizes the admissible elasticity parameters β leading to tractable solutions for the γ-th conditional moment.

Drawing upon these formulations, Theorems 2 and 3 enable the exact evaluation of the first conditional moment for the m-state RS-ECEV process (1). This expectation, denoted by $U^{(\gamma,m,\beta)}(x,\tau,i)$ at $\gamma = 1$ as formulated in (3), plays a crucial role in establishing fair strike prices for futures contracts within derivative pricing [9,29,30], particularly when the underlying asset is governed by m-state RS-ECEV dynamics.

This section is dedicated to formulating explicit representations of $U^{(\gamma,m,\beta)}(x,\tau,i)$, with primary emphasis on the case $\gamma = 1$. These analytical results provide market practitioners with vital tools for accurately pricing financial derivatives under the m-state RS-ECEV framework. Moreover, our proposed analytical approach offers a substantial computational advantage over the traditional MC simulations frequently relied upon for derivative valuation. To comprehensively cover the dynamics of the m-state RS-ECEV process (1), we classify and present these explicit formulas across three parametric cases: $\beta \in \mathcal{B}_1$, $\beta \notin \mathcal{B}_1$, and $\beta = 2$.

4.1. The First Conditional Moment When $\beta \in \mathcal{B}_1$

This subsection details the explicit derivation of $U^{(\gamma,m,\beta)}(x,\tau,i)$ specifically tailored to the scenario where $\gamma = 1$ and for $\beta \in \mathcal{B}_1 = \left\{2 - \frac{1}{n} : \text{for positive integer } n\right\}$.

Assuming $\beta \in \mathcal{B}_1$ and operating under the assumptions of Theorem 2, the exact first conditional moment for the m-regime RS-ECEV model takes the form:

$$U^{(1,m,\beta)}(x,\tau,i) = \sum_{k=0}^{\frac{1}{2-\beta}} a_{i,k}(\tau; \tfrac{1}{2-\beta}, m, \beta) x^{(2-\beta)k}, \tag{19}$$

for $x > 0, \tau > 0$, and $i \in \mathcal{M}_m$, where the coefficients $a_{i,k}(\tau; \frac{1}{2-\beta}, m, \beta)$, $i \in \mathcal{M}_m$ are determined by solving the recursive matrix ODEs (8), governed by the initial constraints (9) established previously.

4.2. The First Conditional Moment When $\beta \in [0,2)\backslash\mathcal{B}_1$

Apart from the values of β considered in the previous subsection, we consider the m-state RS-ECEV process (1) in the case where $\beta \in [0,2)\backslash\mathcal{B}_1$, that is, when the solutions cannot be expressed as finite power series in the initial value x. Consequently, a different approach is required to obtain the first conditional moment. We begin by defining a new random variable

$$Y := X_{m,\beta,t}^{(2-\beta)}, \qquad Y > 0,$$

that is,

$$X_{m,\beta,t} := Y^{\frac{1}{(2-\beta)}},$$

Set $q := \frac{1}{(2-\beta)} > 0$, and let p be an integer such that $p < q < p+1$. The regularized integral representation of the Gamma function for a negative number is (see [31,32]):

$$\Gamma(-q) = \int_0^\infty t^{-q-1}\left(\mathbf{e}^{-t} - \sum_{k=0}^{p} \frac{(-1)^k t^k}{k!}\right) dt \tag{20}$$

Here, the summation $\sum_{k=0}^{p} \frac{(-1)^k t^k}{k!}$ removes the singularities of order $t^{-q-1}, t^{-q}, \ldots$ at $t \to 0$, ensuring the integral converges.

Since Y is a positive random variable ($Y > 0$), we employ the variable substitution $t = sY$, and $dt = Y\,ds$. Substituting this into Equation (20):

$$\begin{aligned}\Gamma(-q) &= \int_0^\infty (sY)^{-q-1}\left(\mathbf{e}^{-sY} - \sum_{k=0}^{p}\frac{(-1)^k(sY)^k}{k!}\right)Y\,ds \\ &= \int_0^\infty s^{-q-1}Y^{-q-1}\left(\mathbf{e}^{-sY} - \sum_{k=0}^{p}\frac{(-1)^k s^k Y^k}{k!}\right)Y\,ds.\end{aligned}$$

We can factor out the terms involving Y that are constant with respect to the summation index k (note that the Y^k inside the sum remains):

$$\Gamma(-q) = Y^{-q-1}\cdot Y\int_0^\infty s^{-q-1}\left(\mathbf{e}^{-sY} - \sum_{k=0}^{p}\frac{(-1)^k Y^k s^k}{k!}\right)ds.$$

Rearranging the equation to solve for Y^q:

$$Y^q = \frac{1}{\Gamma(-q)}\int_0^\infty s^{-q-1}\left(\mathbf{e}^{-sY} - \sum_{k=0}^{p}\frac{(-1)^k Y^k s^k}{k!}\right)ds. \tag{21}$$

We now rename the variable s back to t and take the conditional expectation $\mathbb{E}^{\mathbb{P}}[\cdot \mid \mathcal{F}_t]$ on both sides of (21). Assuming the conditions for Fubini's theorem are met (allowing the interchange of expectation and integration), we get:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[Y^q \mid \mathcal{F}_t] &= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\left(\mathbf{e}^{-tY} - \sum_{k=0}^{p}\frac{(-1)^k Y^k t^k}{k!}\right)dt \mid \mathcal{F}_t\right] \\ &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\left(\mathbb{E}^{\mathbb{P}}[\mathbf{e}^{-tY} \mid \mathcal{F}_t] - \mathbb{E}^{\mathbb{P}}\left[\sum_{k=0}^{p}\frac{(-1)^k Y^k t^k}{k!} \mid \mathcal{F}_t\right]\right)dt.\end{aligned}$$

By the linearity of expectation, we can move $\mathbb{E}^{\mathbb{P}}[\cdot \mid \mathcal{F}_t]$ inside the summation to obtain

$$\mathbb{E}^{\mathbb{P}}[Y^q \mid \mathcal{F}_t] = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\left(\mathbb{E}^{\mathbb{P}}[\mathbf{e}^{-tY} \mid \mathcal{F}_t] - \sum_{k=0}^{p}\frac{(-1)^k\mathbb{E}^{\mathbb{P}}[Y^k \mid \mathcal{F}_t]}{k!}t^k\right)dt. \tag{22}$$

where $\mathbb{E}^{\mathbb{P}}[\mathbf{e}^{-tY} \mid \mathcal{F}_t]$ is the Laplace transform of Y. Since the Laplace transform can be expanded in terms of the integer moments

$$\mathbb{E}^{\mathbb{P}}[\mathbf{e}^{-tY} \mid \mathcal{F}_t] = \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\mathbb{E}^{\mathbb{P}}[Y^k \mid \mathcal{F}_t],$$

this gives

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[Y^q \mid \mathcal{F}_t] &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\Big(\sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\mathbb{E}^{\mathbb{P}}[Y^k \mid \mathcal{F}_t] - \sum_{k=0}^{p}\frac{(-t)^k}{k!}\mathbb{E}^{\mathbb{P}}[Y^k \mid \mathcal{F}_t]\Big)dt, \\ &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\Big(\sum_{k=p+1}^{\infty}\frac{(-t)^k}{k!}\mathbb{E}^{\mathbb{P}}[Y^k \mid \mathcal{F}_t]\Big)dt,\end{aligned} \tag{23}$$

It is noteworthy that the n^{th} conditional moments of Y are

$$\mathbb{E}^{\mathbb{P}}[Y^n \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[X_{m,\beta,T}^{n(2-\beta)} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[X_{m,\beta,T}^{\gamma} \mid \mathcal{F}_t],$$

for $n \in \mathrm{N}$. According to Corollary 1, for a given β, the conditional γ^{th}-moments with $\gamma = n(2-\beta)$ can be expressed in finite power series of the initial value x for $n \in \mathrm{N}$, and therefore, $\mathbb{E}^{\mathbb{P}}[Y^n \mid \mathcal{F}_t]$ is known to us for $n \in \mathrm{N}_0$. Finally, we can compute $\mathbb{E}^{\mathbb{P}}[X_{m,\beta,T} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[Y^{\frac{1}{(2-\beta)}} \mid \mathcal{F}_t]$, where $q = \frac{1}{(2-\beta)}$, as follows:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[X_{m,\beta,T} \mid \mathcal{F}_t] &= \mathbb{E}^{\mathbb{P}}[Y^{\frac{1}{(2-\beta)}} \mid \mathcal{F}_t] \\ &= \frac{1}{\Gamma(-\frac{1}{(2-\beta)})} \int_0^\infty t^{-\frac{1}{(2-\beta)}-1} \Big(\sum_{k=p+1}^{\infty} \frac{(-t)^k}{k!} \mathbb{E}^{\mathbb{P}}[X^{k(2-\beta)} \mid \mathcal{F}_t] \Big) dt.\end{aligned}$$

4.3. The First Conditional Moment When $\beta = 2$

For the case where $\beta = 2$, the underlying m-state RS-ECEV model (1) directly degenerates into an m-state regime-switching extended GBM. Consequently, relying on Theorem 2, the exact first conditional moment tailored for the m-state RS-ECEV process (1) with $\beta = 2$ takes the form:

$$U^{(1,m,2)}(x,\tau,i) = a_{i,1}(\tau;1,m,2)x$$

defined over $x > 0$, $\tau > 0$, and $i \in \mathcal{M}_m$, where the coefficients $a_{i,1}(\tau;1,m,2)$, $i \in \mathcal{M}_m$ are determined by solving the matrix ODEs (15) in conjunction with their corresponding initial states (17).

4.4. Application to Pricing Futures

While Elliott et al. [33,34] employ the regime-switching Esscher transform to identify an equivalent martingale pricing measure, our study bypasses this derivation. Instead, we proceed under the assumption that a risk-neutral probability space, denoted by $(\Omega, \mathcal{F}, \mathbb{Q})$, is already well-defined. Consequently, under this risk-neutral measure $\mathbb{Q}$, the dynamics of the underlying asset price are governed by

$$dX_{m,\beta,t} = \hat{\kappa}\Big(t,\xi_t^{(m)}\Big)\left(\hat{\theta}\Big(t,\xi_t^{(m)}\Big)X_{m,\beta,t}^{-(1-\beta)} - X_{m,\beta,t}\right)dt + \sigma\Big(t,\xi_t^{(m)}\Big)X_{m,\beta,t}^{\beta/2}d\hat{W}_t, \quad 0 \le t \le T,$$

in which $\big(\hat{W}_t\big)_{t\ge 0}$ represents the $\mathbb{Q}$-Brownian motion given by

$$d\hat{W}_t = dW_t + \frac{\kappa\Big(t,\xi_t^{(m)}\Big)\left(\theta\Big(t,\xi_t^{(m)}\Big)X_{m,\beta,t}^{-(1-\beta)} - X_{m,\beta,t}\right) - \hat{\kappa}\Big(t,\xi_t^{(m)}\Big)\left(\hat{\theta}\Big(t,\xi_t^{(m)}\Big)X_{m,\beta,t}^{-(1-\beta)} - X_{m,\beta,t}\right)}{\sigma\Big(t,\xi_t^{(m)}\Big)X_{m,\beta,t}^{\beta/2}}dt.$$

Furthermore, we require the risk-neutral parameters $\hat{\kappa}(t,i)$ and $\hat{\theta}(t,i)$ to strictly adhere to Assumptions 1 and 2. This condition guarantees that the fundamental structure of the m-state RS-ECEV model (1) is invariant under the measure transformation, differing only in its specific parameterization.

Considering a futures contract written on the underlying asset with a finite expiration T, its arbitrage-free value at any intermediate time $t \in [0,T]$ can be explicitly formulated. Specifically, conditional on the current state $X_{m,\beta,t} = x > 0$ and the active regime $\xi_t^{(m)} = i \in \mathcal{M}_m$, the fair delivery price, denoted by $F_T^{(m)}(x,t,i)$, is evaluated via the risk-neutral expectation:

$$F_T^{(m)}(x,t,i) = \mathbb{E}^{\mathbb{Q}}[X_{m,\beta,T} \mid X_{m,\beta,t} = x, \xi_t^{(m)} = i].$$

By evaluating (12) at $\gamma = 1$ and replacing the drift parameters $\kappa(t,i)$ and $\theta(t,i)$ in Theorem 2 with their risk-neutral counterparts $\hat{\kappa}(t,i)$ and $\hat{\theta}(t,i)$, we establish the exact equivalence

$F_T^{(m)}(x,t,i) = U^{(1,m,\beta)}(x,T-t,i)$. This identification effectively resolves the hybrid PDE system (5) and (6).

Building on the derivations established in Section 4.1, whenever $\beta \in \mathcal{B}_1$, we can explicitly express the fair valuation of the futures contract as

$$F_T^{(m)}(x,t,i) = \sum_{k=0}^{\frac{1}{2-\beta}} a_{i,k}(T-t;\tfrac{1}{2-\beta},m,\beta)x^{(2-\beta)k},$$

defined over $t \in [0,T]$, $x > 0$, and $i \in \mathcal{M}_m$. The associated time-dependent coefficients $a_{i,k}(T-t;\frac{1}{2-\beta},m,\beta)$ for $k = 0,\ldots,\frac{1}{2-\beta}$ are systematically extracted via the matrix relationships defined in (13).

Turning to the scenario where $\beta \in [0,2) \setminus \mathcal{B}_1$, as analyzed in Section 4.2, the arbitrage-free futures price is formulated via the integral representation:

$$F_T^{(m)}(x,t,i) = \frac{1}{\Gamma(-\frac{1}{(2-\beta)})}\int_0^\infty \Big(\sum_{k=p+1}^{\infty}\frac{(-t)^k}{k!}M_k\Big)t^{-\frac{1}{(2-\beta)}-1}dt,$$

where the conditional moment factor is defined as $M_k = \mathbb{E}^{\mathbb{P}}[X_{m,\beta,T}^{k(2-\beta)} \mid X_{m,\beta,t} = x, \xi_t^{(m)} = i]$.

Lastly, following Section 4.3 for the specific case $\beta = 2$, the fair delivery price collapses to

$$F_T^{(m)}(x,t,i) = a_{i,1}(T-t;1,m,2)x,$$

valid for $t \in [0,T], x > 0$, and $i \in \mathcal{M}_m$. The respective coefficient $a_{i,1}(T-t;1,m,2)$ is derived by evaluating the matrix differential system (15) under the initial states (17), combined with the relational matrices in (13).

5. Numerical Results and Discussions

The analytical frameworks constructed in Section 3 via Theorems 2 and 3 serve as the primary mathematical engine for evaluating the γ^{th} conditional moments of the m-state RS-ECEV dynamics. Notably, this unified framework accommodates any positive real γ and any finite number of regimes $m \geq 1$, these explicit results capture the behaviors governed by both $\beta \in [0,2)$ and $\beta = 2$. Securing these explicit moment representations is central to fulfilling the core objectives of this research. Building upon this mathematical foundation, Section 4 illustrated the practical utility of our approach in financial mathematics, specifically leveraging the finite series solutions for the first conditional moment to explicitly determine the fair prices of futures contracts.

This section systematically investigates the numerical properties of the explicit formulas established in Section 4.1. In Section 5.1, we conduct a robust validation of the $m = 2$ case using MC simulations, while simultaneously assessing the execution time efficiency of our analytical solutions relative to MC estimates for select parameters $\beta \in \{1, \frac{3}{2}, \frac{5}{3}\} \subset \mathcal{B}_1$. Furthermore, Section 5.2 provides an asymptotic analysis of the first conditional moment for the two-state RS-CEV process at $\beta = \frac{3}{2}$. These long-term dynamics are subsequently compared to the $\beta = 1$ subcase—representing the two-state RS-CIR process—to contextualize our results against the prior work of Rujivan et al. [25].

To operationalize this numerical framework, we specify the transition intensity matrix for the two-state RS-ECEV process as

$$\mathbf{Q}_2 = \begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix}, \qquad q_1, q_2 > 0.$$

Subsequently, the foundational matrices governing the time-dependent parameters of the process are constructed as follows.

First, the mean-reversion rate matrix $\mathbf{K}[t,2]$ is specified as

$$\mathbf{K}[t,2] = \begin{bmatrix} \kappa(t,1) & 0 \\ 0 & \kappa(t,2) \end{bmatrix},$$

for $t \geq 0$, and for each $i \in \{1,2\}$,

$$\kappa(t,i) = \kappa_{i0} + \kappa_{i1} e^{-\kappa_{i2} t}, \tag{24}$$

where κ_{i0}, κ_{i1} and κ_{i2} are constants with $\kappa_{i0} > 0$.

Next, the long-run mean matrix $\boldsymbol{\Theta}[t,2]$ is defined as

$$\boldsymbol{\Theta}[t,2] = \begin{bmatrix} \theta(t,1) & 0 \\ 0 & \theta(t,2) \end{bmatrix},$$

for $t \geq 0$, and for each $i \in \{1,2\}$,

$$\theta(t,i) = \theta_{i0} + \theta_{i1} t + \theta_{i2} e^{\sin(\theta_{i3} t)}, \tag{25}$$

where θ_{i0}, θ_{i1}, θ_{i2}, and θ_{i3} are constants, specifically requiring $\theta_{i0} > 0$ to maintain the integrity of the model.

Finally, the volatility rate matrix $\boldsymbol{\Sigma}[t,2]$ is characterized by

$$\boldsymbol{\Sigma}[t,2] = \begin{bmatrix} \sigma^2(t,1) & 0 \\ 0 & \sigma^2(t,2) \end{bmatrix},$$

for $t \geq 0$, and for each $i \in \{1,2\}$,

$$\sigma(t,i) = \sigma_{i0} + \sigma_{i1} e^{-\sigma_{i2} t}, \tag{26}$$

where σ_{i0}, σ_{i1}, and σ_{i2} are deterministic constants, inherently requiring $\sigma_{i0} > 0$ to ensure valid diffusion.

We emphasize that the parameterizations chosen for our numerical investigation are rigorously informed by both theoretical considerations and the prevailing empirical literature [8,35–37]. These specific parameters were selected for their proven ability to effectively model the temporal trends and seasonal patterns within the underlying dynamics of X.

Importantly, the selected parameters admit a natural financial interpretation. The mean-reversion rates $\kappa(t,i)$ dictate the speed at which the asset price returns to a regime-dependent long-run level, capturing market stabilization forces under varying economic conditions. The functions $\theta(t,i)$ represent regime-specific equilibrium levels, corresponding to distinct market phases such as expansionary versus contractionary periods. Finally, the volatility parameters $\sigma(t,i)$ quantify the magnitude of stochastic fluctuations, characterizing the regime-dependent risk intensity.

To simulate the sample paths required for the MC framework, we implement the Euler–Maruyama discretization introduced by Mao et al. [38]. This specific scheme is chosen as it theoretically ensures the convergence of the numerical approximations to the true trajectories of the Markovian-switching SDE (1). Furthermore, all numerical evaluations and visual renderings were performed in MATHEMATICA, running on a desktop environment equipped with an Intel Core i7-6700 CPU (3.40 GHz) and 32 GB of RAM, operating on a 64-bit Windows 10 platform.

5.1. The Accuracy and Efficiency of Our Analytical Formulas for First Conditional Moments for the Two-State RS-ECEV Process

Example 1. *This numerical example serves to demonstrate both the exactness and the computational superiority of the analytical solutions derived in Theorem 2. Specifically, we evaluate the first conditional moment of the two-state RS-ECEV process, explicitly formulated in (19), across a selected subset of elasticity parameters* $\beta \in \{1, \frac{3}{2}, \frac{5}{3}\} \subset \mathcal{B}_1$. *The assessment is conducted through a detailed comparison with MC simulation results.*

To specify the model configuration, we consider the intensity matrix $\mathbf{Q}_2$ *with transition rates* $q_1 = 0.3$ *and* $q_2 = 0.5$. *The mean-reversion coefficients for the function* $\mathbf{K}[t, 2]$ *defined in (24) are chosen as*

$$\kappa_{10} = 0.01, \qquad \kappa_{11} = \kappa_{12} = 1,$$

$$\kappa_{20} = 0.02, \qquad \kappa_{21} = \kappa_{22} = 1.$$

The long-run mean functions $\mathbf{\Theta}[t, 2]$ *defined in (25) are parametrized by*

$$\theta_{10} = 0.5,\ \theta_{11} = 0,\ \theta_{12} = 2,\ \theta_{13} = 10,$$

$$\theta_{20} = 2,\ \theta_{21} = 0,\ \theta_{22} = 3,\ \theta_{23} = 20,$$

while the volatility functions $\mathbf{\Sigma}[t, 2]$ *defined in (26) are set to*

$$\sigma_{10} = 0.1,\ \sigma_{11} = 0.02,\ \sigma_{12} = 2,$$

$$\sigma_{20} = 0.2,\ \sigma_{21} = 0.03,\ \sigma_{22} = 3.$$

This parameter selection satisfies all conditions stated in Assumptions 1 and 2.

By incorporating these defined parameters into the analytical expression (19) and fixing the time horizon at $T = 1$, *we graphically present the time-dependent coefficient functions* $a_{i,k}(\tau; n, 2, \beta)$. *These visualizations specifically detail the configurations where the variance elasticity* $\beta \in \{1, \frac{3}{2}, \frac{5}{3}\}$, *evaluated across both regimes* $i \in \{1, 2\}$ *for the specific order* $k = n$. *The resulting plots reveal that the coefficient functions* $a_{i,n}(\tau; n, 2, \beta)$ *coincide for all* $\tau \in [0, 1]$ *and for all considered values of* β, *as illustrated in Figure 1a. This observation follows from the fact that for* $\beta \in \mathcal{B}_1$ *we have* $n = 1/(2 - \beta)$, *and the solution of the system (8) and (9) for*

$$b_{i,0}(\tau; n, 2, \beta) = a_{i,n}(\tau; n, 2, \beta)$$

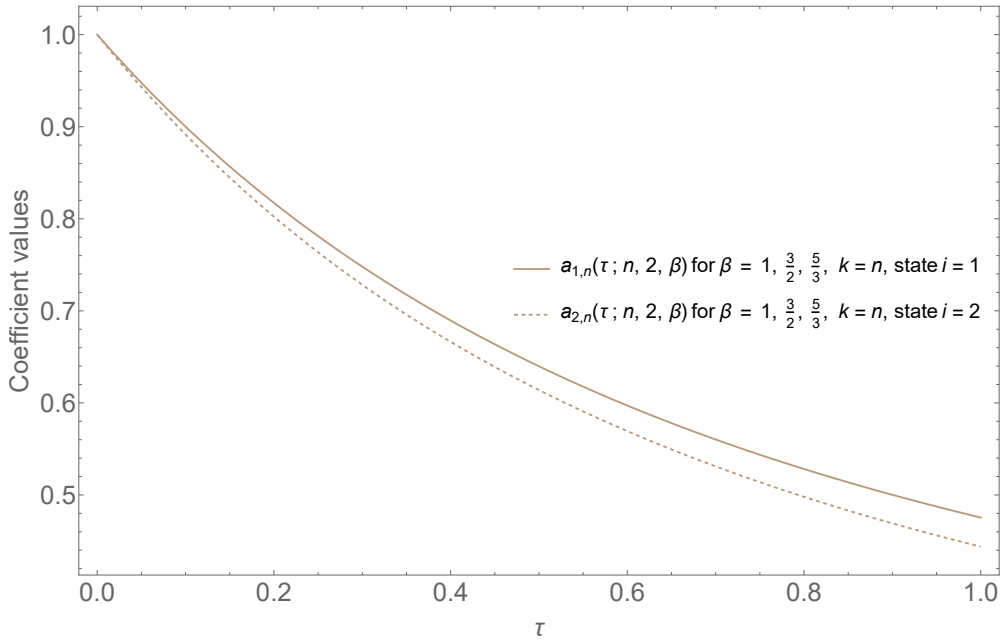
depends only on the matrices $\mathbf{Q}_2$ *and* $\mathbf{K}[t, 2]$, *and not on* β. *Moreover, the coefficient functions* $a_{i,k}(\tau; n, 2, \beta)$ *for* $k = n - 1, n - 2, \ldots, 0$ *exhibit similar oscillatory behaviors over* $\tau \in [0, 1]$, *as shown in Figure 1b,c. These graphical results further confirm the theoretical foundation of the proposed analytical approach.*

For additional validation, the exact solutions for the first conditional moments $U^{(1,2,\beta)}(x, \tau, i)$ *are evaluated against MC simulations employing* 10^6 *generated paths with* $X_0 = 1$. *This assessment covers the time domain* $\tau \in [0, 1]$, *operational states* $i \in \{1, 2\}$, *and elasticity values* $\beta \in \{1, \frac{3}{2}, \frac{5}{3}\}$. *The graphical evidence presented in Figure 2 demonstrates a tight consensus between the analytical and simulated results, firmly establishing the exactness of our formulas.*

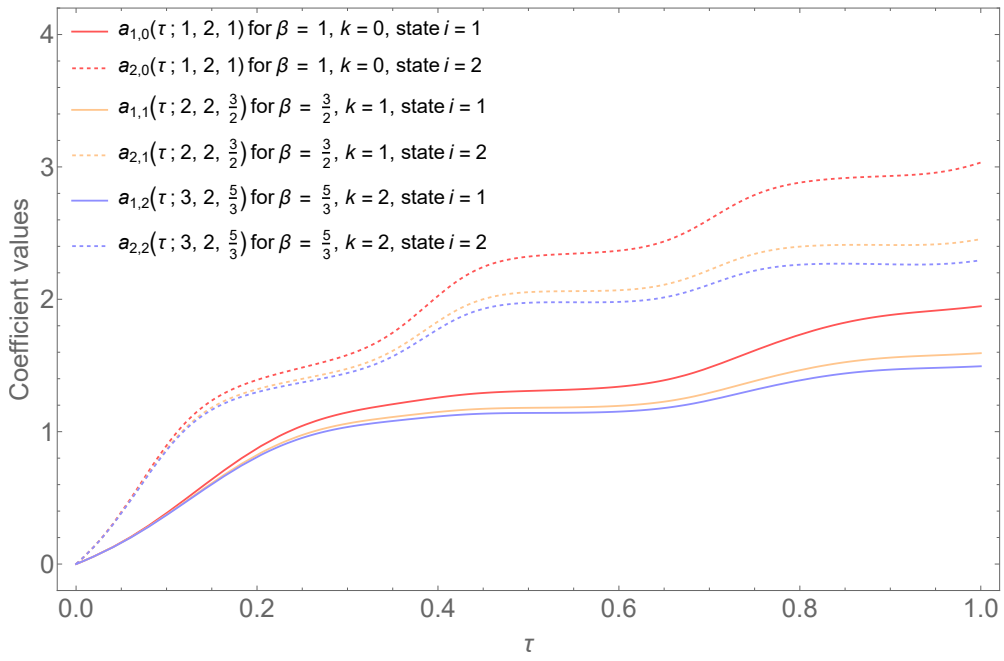
To assess computational efficiency, we benchmark the execution time of the analytical method (T_A) *against MC simulations* (T_{MC}) *employing* $N_p \in \{1000, 10{,}000, 100{,}000, 1{,}000{,}000\}$ *sample paths. Evaluating the explicit formulas requires an average execution time of merely* 0.00298, 0.00369, *and* 0.00569 *s for* $\beta = 1, \frac{3}{2}$, *and* $\frac{5}{3}$, *respectively. Naturally,* T_A *scales upward because the required number of terms in the finite series expansion grows as* β *increases. In contrast, the execution time for MC simulations is strictly dependent on the number of sample paths and remains invariant across all* β *values, consuming approximately* 23, 230, 2300, *and* 23,000 *s for the respective* N_p *values. As detailed in Table 1a–c, the analytical approach demonstrates a substantial*

speed advantage. Specifically, to maintain an average relative error (ϵ) below 0.5% for the $\beta = \frac{5}{3}$ case, the MC method necessitates at least 1,000,000 trajectories. This translates to our analytical method achieving an efficiency gain of approximately 4,000,000 times per evaluation point relative to the MC simulation.

In conclusion, our numerical investigations definitively establish the exactness and the immense computational superiority of the proposed analytical framework for the two-state RS-ECEV process. The seamless alignment with MC benchmarks, coupled with a drastic reduction in execution time, firmly substantiates the theoretical robustness and practical viability of our methodology for derivative pricing.

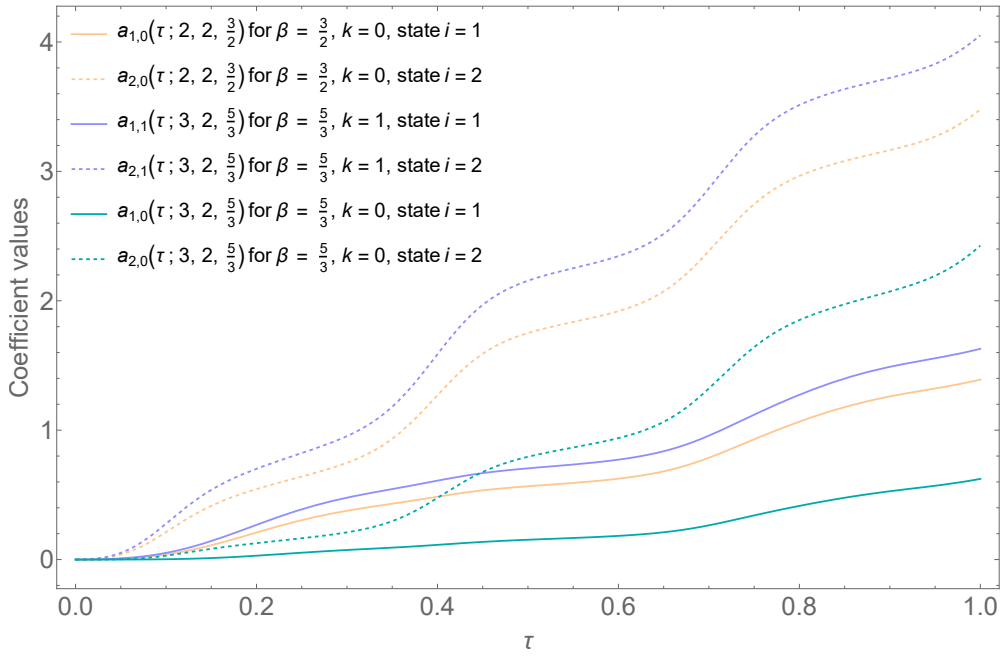


(**a**)



(**b**)

Figure 1. *Cont.*



(**c**)

Figure 1. Time evolution of the coefficient functions $a_{i,k}(\tau; n, 2, \beta)$ associated with the term x^k, requisite for computing the first conditional moment $U^{(1,2,\beta)}(x, \tau, i)$ via (19). The plots illustrate the dynamics across regimes $i \in \{1, 2\}$ for orders $k \in \{n, n-1, n-2, n-3\}$, based on the setup in Example 1. (**a**) $k = n$; (**b**) $k = n-1$; (**c**) $k = n-2, n-3$.

Table 1. Comparative analysis of average execution times (in seconds) between MC simulations (T_{MC}) and the proposed analytical approach (T_A) derived in (19). The tables also report the average percentage relative absolute error (ϵ) and the computational speedup factor at the evaluation point $(x, \tau) = (1, 1)$. These metrics highlight the substantial efficiency gains and precision of the exact formulas detailed in Example 1.

(a) Average execution times for computing $U^{(1,2,1)}(x, \tau, 1)$ and $U^{(1,2,1)}(x, \tau, 2)$				
N_p	ϵ (%)	T_{MC} (seconds)	T_A (seconds)	Efficiency Gain (times)
1×10^3	0.26	23.6637	0.00298	7941
1×10^4	0.16	231.1925	0.00298	77,581
1×10^5	0.11	2367.8342	0.00298	794,575
1×10^6	0.06	23,105.6504	0.00298	7,967,466
(b) Average execution times for computing $U^{(1,2,\frac{3}{2})}(x, \tau, 1)$ and $U^{(1,2,\frac{3}{2})}(x, \tau, 2)$				
N_p	ϵ (%)	T_{MC} (seconds)	T_A (seconds)	Efficiency Gain (times)
1×10^3	0.47	22.7584	0.00369	6168
1×10^4	0.32	230.5955	0.00369	62,492
1×10^5	0.28	2350.8762	0.00369	637,094
1×10^6	0.22	23,303.6218	0.00369	6,615,345
(c) Average execution times for computing $U^{(1,2,\frac{5}{3})}(x, \tau, 1)$ and $U^{(1,2,\frac{5}{3})}(x, \tau, 2)$				
N_p	ϵ (%)	T_{MC} (seconds)	T_A (seconds)	Efficiency Gain (times)
1×10^3	1.44	22.6257	0.00569	3976
1×10^4	0.61	232.7335	0.00569	40,902
1×10^5	0.52	2342.3501	0.00569	411,661
1×10^6	0.43	22,891.9674	0.00569	4,023,193

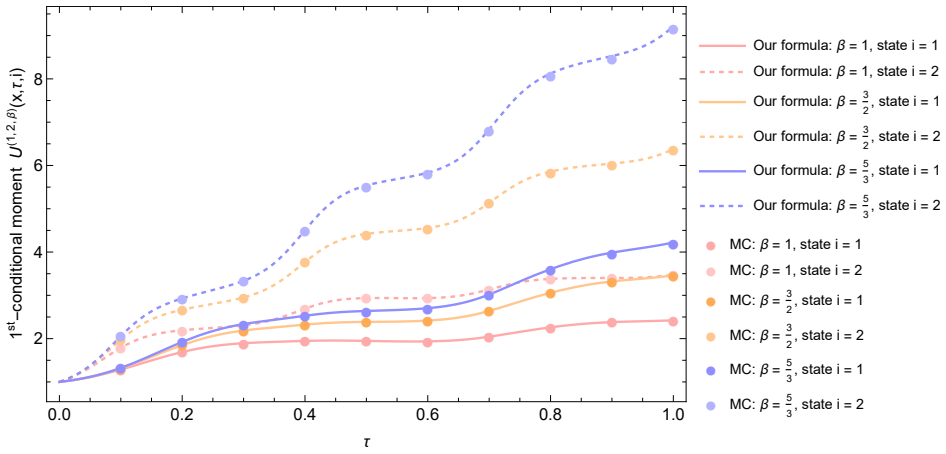


Figure 2. Validation of the analytical solutions (19) against MC simulations for the first conditional moments $U^{(1,2,\beta)}(x,\tau,i)$ under a two-state RS-ECEV framework. The figure highlights the exactness of the derived formulas and demonstrates the state-driven variability of the moments, as detailed in Example 1.

5.2. The Asymptotic Characteristic of the First Conditional Moments Within the Two-State RS-ECEV Process

Example 2. *This example investigates the asymptotic dynamics of the first conditional moment for the two-state RS-CEV process with $\beta = \frac{3}{2}$, providing a comparative study against the two-state RS-CIR process ($\beta = 1$). The latter serves as a benchmark, having been analyzed extensively in Example 6.2 of Rujivan et al. [25].*

Our analysis is bifurcated into two distinct scenarios based on the structural properties of the transition intensity matrix. In the first scenario, we utilize the non-symmetric matrix $\mathbf{Q}_2$ as specified in Example 1. In the second scenario, we examine the impact of a symmetric intensity matrix, defined by

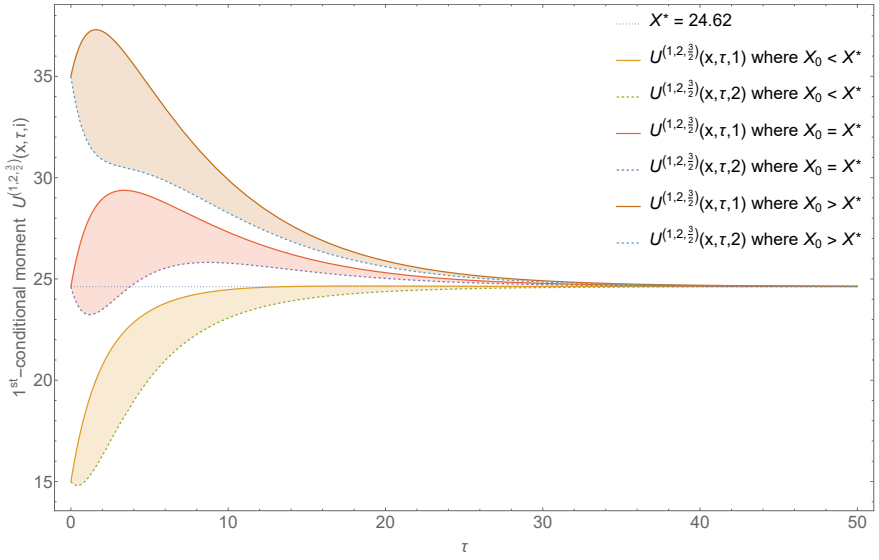
$$\widehat{\mathbf{Q}}_2 = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}.$$

In both cases, the parameters of the mean-reversion matrix $\mathbf{K}(t,2) \equiv \mathbf{K}_2$ are set to $\kappa_{10} = \kappa_{20} = 0.3$ and $\kappa_{11} = \kappa_{12} = \kappa_{21} = \kappa_{22} = 0$. For the long-term mean matrix $\mathbf{\Theta}(t,2)$, we choose $\theta_{10} = 8$ and $\theta_{20} = 3$, with all remaining coefficients equal to zero. The volatility parameters are specified as $\sigma_{10} = 2$ and $\sigma_{20} = 0.5$, with all remaining coefficients set to zero. These parameter choices are carefully selected to satisfy all conditions imposed in Assumptions 1 and 2.

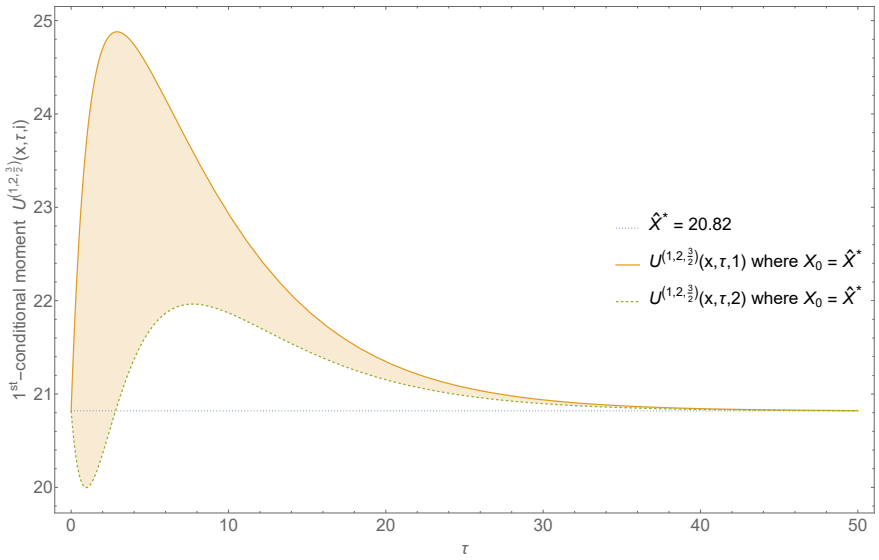
The asymptotic limits of the first conditional moment for $\beta = \frac{3}{2}$, denoted by X^* and $\widehat{X}^*$, are evaluated as 24.62 and 20.82 for the intensity matrices $\mathbf{Q}_2$ and $\widehat{\mathbf{Q}}_2$, respectively. These results are consistently derived from the stationary mean expression (A36) provided in Appendix B. Figure 3a illustrates the convergence of the first conditional moment to X^* as $\tau \to \infty$ for all conditions $X_0 < X^*$, $X_0 > X^*$, and $X_0 = X^*$. A key observation is that the limiting value X^* differs from both θ_{10} and θ_{20}, and this discrepancy is attributable to the influence of the underlying Markov chain. This phenomenon also arises in the two-state RS-CIR process ($\beta = 1$), as shown in Figure 5c of [25]. According to Corollary 5.5 in [25] ($\beta = 1$), when $X_0 = X^*$, the first conditional moment deviates from X^* for both states $i = 1, 2$ in opposite directions as τ increases from zero, before converging back to X^* as $\tau \to \infty$. Moreover, under a symmetric intensity matrix and with $X_0 = \widehat{X}^*$, these deviations are symmetric across the two states. However, this symmetry property does not generally hold for the two-state RS-CEV process with $\beta = \frac{3}{2}$. In this case, when $X_0 = X^*$, the first conditional moment may deviate from X^* for both states either in the same direction or in opposite directions, and may even cross the limiting value X^* as τ increases from zero, before converging back to X^* as $\tau \to \infty$, see Figure 3a. Furthermore, even under a symmetric intensity matrix and with $X_0 = \widehat{X}^*$, symmetric deviations of the first conditional moment across regimes are no longer guaranteed,

as illustrated in Figure 3b. The observed behavior arises because the long-term mean of the two-state RS-CEV process is sensitive to the full parameter set $\{q_i, \kappa_i, \theta_i, \sigma_i\}$*, according to (A36) in Appendix B. Conversely, for the linear-drift RS-CIR case* ($\beta = 1$)*, the stationary expectation is determined solely by the transition intensities* q_i *and long-run targets* θ_i *(see Corollary 5.4 of [25]), highlighting the more complex dependency structure introduced by the nonlinear elasticity in the CEV framework.*

In conclusion, contrary to the symmetric transient deviations observed in the two-state RS-CIR process ($\beta = 1$) *under a symmetric intensity matrix, the nonlinear drift of the two-state RS-CEV process with* $\beta = \frac{3}{2}$ *induces significantly different dynamics. In particular, the transient dynamics of the first conditional moment may be asymmetric and regime-dependent, even under symmetric intensity matrices, prior to eventual convergence in the long run.*



(**a**)



(**b**)

Figure 3. Long-term evolutionary dynamics of the first conditional moments for the two-state RS-CEV process under different intensity matrix configurations, as detailed in Example 2. The plots illustrate the convergence of the moments toward their respective stationary limits as $\tau \to \infty$. (**a**) The first conditional moments calculated via the non-symmetric intensity matrix $\mathbf{Q}_2$, with initial condition $X_0 > X^*$, $X_0 = X^*$ and $X_0 < X^*$. (**b**) The first conditional moments calculated via the symmetric intensity matrix $\widehat{\mathbf{Q}}_2$, with initial condition $X_0 = \widehat{X}^*$.

6. Conclusions

In this study, we have developed exact finite-series solutions for a hybrid system of interconnected PDEs to evaluate the conditional moments of RS-ECEV processes characterized by generalized drift. The underlying regime-switching mechanism is governed by a continuous-time, finite-state, and irreducible Markov chain across m regimes, for any integer $m \geq 1$. By solving the hybrid PDE system derived from a modified Feynman–Kac formula, we have identified a tractable class of regime-switching diffusions. For these processes, the γ^{th} conditional moment, for any $\gamma > 0$, admits an explicit representation as a finite power series in the initial state. The robustness, precision, and superior computational efficiency of this analytical framework have been rigorously validated through extensive MC simulations.

The MC simulations conducted in this work have served two main purposes. First, they have provided rigorous numerical validation of the analytical moment formulas derived in Section 4. Across a wide range of parameter configurations, the MC estimates of the first conditional moment have been in excellent agreement with the analytical results, thereby confirming the correctness and robustness of the proposed methodology. Second, a comparison of computational efficiency has demonstrated a substantial reduction in computation time when the analytical explicit expressions have been employed. This efficiency gain has become increasingly pronounced as the number of simulated paths or the time horizon increases.

In addition, the asymptotic and transient analysis of the two-state RS-CEV process with nonlinear drift has highlighted important qualitative differences relative to the two-state RS-CIR case. While both models exhibit convergence of the first conditional moment to a regime-influenced long-run limit, the two-state RS-CEV process displays richer and potentially asymmetric transient dynamics, even under symmetric transition intensities. In particular, the first conditional moment may deviate asymmetrically across regimes or cross its limiting value before convergence, reflecting its dependence not only on the Markov transition structure but also on regime-specific drift and volatility parameters. These findings underscore the impact of nonlinear drift on conditional moment behavior and further illustrate the flexibility and descriptive power of the proposed RS-ECEV framework.

From a financial perspective, the application to futures pricing has illustrated the practical relevance of the proposed approach. Since the fair price of a futures contract depends directly on the conditional expectation of the underlying asset price, the availability of accurate and computationally efficient moment formulas has provided a clear advantage over purely simulation-based methods. Moreover, the regime-switching feature has enabled the model to incorporate macroeconomic or market-state effects in a parsimonious yet mathematically tractable manner.

The results of this study have suggested several promising directions for future research. These include extensions to higher-order conditional moments and joint moments, the incorporation of time-dependent or state-dependent transition intensities, and applications to the pricing and hedging of more complex derivative products, such as volatility swaps or options under regime-switching volatility dynamics. Overall, the proposed framework has provided a robust and flexible foundation for both theoretical investigation and practical implementation in stochastic modeling and mathematical finance.

Author Contributions: Conceptualization, P.J., S.R. and B.D.; methodology, P.J. and S.R.; software, P.J.; validation, P.J., S.R. and B.D.; formal analysis, P.J., S.R. and B.D.; investigation, P.J., S.R. and B.D.; writing—original draft, P.J., S.R. and B.D.; writing—review and editing, P.J.; visualization, P.J.; supervision, S.R. and B.D.; funding acquisition, P.J., S.R. and B.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Walailak University Graduate Scholarship (Contract No. 11/2022) and the Verg Foundation in Sweden.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors appreciate the financial support from the Walailak University Graduate Scholarship and the Verg Foundation in Sweden. They also sincerely thank the anonymous reviewers for their valuable comments and constructive suggestions, which have significantly contributed to improving the quality of this manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CEV	Constant Elasticity of Variance
CIR	Cox–Ingersoll–Ross
ECEV	Extended Constant Elasticity of Variance
ECIR	Extended Cox–Ingersoll–Ross
GBM	Geometric Brownian Motion
MC	Monte Carlo
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
RS-CEV	Regime-Switching Constant Elasticity of Variance
RS-CIR	Regime-Switching Cox–Ingersoll–Ross
RS-ECEV	Regime-Switching Extended Constant Elasticity of Variance
RS-ECIR	Regime-Switching Extended Cox–Ingersoll–Ross
SDE	Stochastic Differential Equation

Appendix A. Omitted Proofs

Appendix A.1. Proof of Theorem 1

Proof. Let $(\gamma, m, \beta) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}_0^+$ be fixed. The process $X_{m,\beta}$ in (1) is a switching diffusion of the type explored by Baran et al. [26] with generator $\mathcal{L}_E$ defined in (4). Utilizing the temporal shift $\tau = T - t$ in conjunction with Theorem 6 of Baran et al. [26], we apply Dynkin's formula to $\mathcal{L}_E[U^{(\gamma,m,\beta)}(x,\tau,i)]$. This confirms that $U^{(\gamma,m,\beta)}(x,\tau,i)$ is the solution to the hybrid system of coupled PDEs (5) with initial state requirements (6). □

Appendix A.2. Proof of Theorem 2

Proof. Let $T > 0$ and $(\gamma, m, n) \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N}$ be fixed with $\frac{\gamma}{n} \leq 2$, and set $\beta = 2 - \frac{\gamma}{n} \geq 0$. By Theorem 1, upon introducing the time-to-maturity variable $\tau = T - t$, the functions $U^{(\gamma,m,\beta)}(x,\tau,i)$, for all $i \in \mathcal{M}_m$, satisfy the hybrid system of interconnected PDEs (5) and (6). Next, we assume that the solution $U^{(\gamma,m,\beta)}(x,\tau,i)$ admits a representation in the form of a finite power series as follows:

$$U^{(\gamma,m,\beta)}(x,\tau,i) = \sum_{k=0}^{n} a_{i,k}(\tau;n,m,\beta)x^{(2-\beta)k} = \sum_{k=0}^{n} a_{i,k}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}, \tag{A1}$$

for all $i \in \mathcal{M}_m$. The coefficients $a_{i,k}(\tau;n,m,\beta)$ for each $k = 0,\ldots,n$ are derived by utilizing the hybrid system of coupled PDEs (5) in conjunction with the conditions specified in (6). In the case of $\beta = 2 - \frac{\gamma}{n}$, they become

$$\begin{aligned} 0 = -\frac{\partial}{\partial \tau} U^{(\gamma,m,\beta)}(x,\tau,i) &+ \frac{1}{2}\sigma^2(T-\tau,i)x^{(2-\frac{\gamma}{n})}\frac{\partial^2}{\partial x^2}U^{(\gamma,m,\beta)}(x,\tau,i) \\ &+ \kappa(T-\tau,i)\Big(\theta(T-\tau,i)x^{(1-\frac{\gamma}{n})} - x\Big)\frac{\partial}{\partial x}U^{(\gamma,m,\beta)}(x,\tau,i) \\ &+ \sum_{\substack{l\in\mathcal{M}_m \\ l\neq i}} q_{il}^{(m)}\Big(U^{(\gamma,m,\beta)}(x,\tau,l) - U^{(\gamma,m,\beta)}(x,\tau,i)\Big) \end{aligned} \tag{A2}$$

defined on $(x,\tau) \in \mathbb{R}^+ \times (0,T]$, with the initial conditions:

$$U^{(\gamma,m,\beta)}(x,0,i) = x^\gamma \tag{A3}$$

for $x > 0$ and $i \in \mathcal{M}_m$. We now proceed to simplify the LHS of (A2) specifically for the components involving $\mathbf{Q}_m$. Substituting the solution form (A1) yields

$$\sum_{\substack{l\in\mathcal{M}_m \\ l\neq i}} q_{il}^{(m)}\Big(U^{(\gamma,m,\beta)}(x,\tau,l) - U^{(\gamma,m,\beta)}(x,\tau,i)\Big) = \sum_{k=0}^{n}\left(\sum_{\substack{l\in\mathcal{M}_m \\ l\neq i}} q_{il}^{(m)}a_{l,k}(\tau;n,m,\beta) - a_{i,k}(\tau;n,m,\beta)\sum_{\substack{l\in\mathcal{M}_m \\ l\neq i}} q_{il}^{(m)}\right)x^{\frac{\gamma k}{n}}. \tag{A4}$$

In light of (2), the equality $\sum_{l\in\mathcal{M}_m, l\neq i} q_{il}^{(m)} = -q_{ii}^{(m)}$ allows further simplification of (A4) to

$$\sum_{\substack{l\in\mathcal{M}_m \\ l\neq i}} q_{il}^{(m)}\Big(U^{(\gamma,m,\beta)}(x,\tau,l) - U^{(\gamma,m,\beta)}(x,\tau,i)\Big) = \sum_{k=0}^{n}\sum_{l\in\mathcal{M}_m} q_{il}^{(m)}a_{l,k}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}. \tag{A5}$$

Utilizing the series expansion (A1), the required partial derivatives are computed as follows:

$$\begin{aligned} \frac{\partial}{\partial \tau}U^{(\gamma,m,\beta)}(x,\tau,i) &= \sum_{k=0}^{n}\frac{d}{d\tau}a_{i,k}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}, \\ x\frac{\partial}{\partial x}\tilde{U}^{(\gamma,m,\beta)}(x,\tau,i) &= \sum_{k=0}^{n}(\frac{\gamma k}{n})a_{i,k}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}, \\ x^{(1-\frac{\gamma}{n})}\frac{\partial}{\partial x}U^{(\gamma,m,\beta)}(x,\tau,i) &= \sum_{k=0}^{n-1}(\frac{\gamma(k+1)}{n})a_{i,k+1}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}, \\ x^{(2-\frac{\gamma}{n})}\frac{\partial^2}{\partial x^2}U^{(\gamma,m,\beta)}(x,\tau,i) &= \sum_{k=0}^{n-1}(\frac{\gamma(k+1)}{n})(\frac{\gamma(k+1)}{n}-1)a_{i,k+1}(\tau;n,m,\beta)x^{\frac{\gamma k}{n}}. \end{aligned} \tag{A6}$$

Subsequently, by substituting the expressions (A5) and (A6) into the hybrid system of PDEs (A2), we arrive at the following equation:

$$\begin{aligned} 0 = &\left(\frac{d}{d\tau}a_{i,n}(\tau;n,m,\beta) + \gamma\kappa(T-\tau,i)a_{i,n}(\tau;n,m,\beta) - \sum_{l\in\mathcal{M}_m} q_{il}^{(m)}a_{l,n}(\tau;n,m,\beta)\right)x^\gamma \\ &+ \sum_{k=0}^{n-1}\left(\frac{d}{d\tau}a_{i,k}(\tau;n,m,\beta) + (\frac{\gamma k}{n})\kappa(T-\tau,i)a_{i,k}(\tau;n,m,\beta) - \sum_{l\in\mathcal{M}_m} q_{il}^{(m)}a_{l,k}(\tau;n,m,\beta)\right. \\ &\left. - (\frac{\gamma(k+1)}{n})\left(\frac{1}{2}(\frac{\gamma(k+1)}{n}-1)\sigma^2(T-\tau,i) + \kappa(T-\tau,i)\theta(T-\tau,i)\right)a_{i,k+1}(\tau;n,m,\beta)\right)x^{\frac{\gamma k}{n}}. \end{aligned} \tag{A7}$$

By equating the coefficients of $x^{\frac{\gamma k}{n}}$ in (A7) to zero for each $k \in \{n,\ldots,0\}$, we derive the following systems of ODEs. Specifically, the governing equation for the leading order $k = n$

is given in (A8), while the recursive relations for the subsequent orders $k \in \{n-1, \ldots, 0\}$ are detailed in (A9):

$$\frac{d}{d\tau} a_{i,n}(\tau; n, m, \beta) = \sum_{l \in \mathcal{M}_m} q_{il}^{(m)} a_{l,n}(\tau; n, m, \beta) - \gamma\kappa(T-\tau, i) a_{i,n}(\tau; n, m, \beta), \tag{A8}$$

$$\begin{aligned} \frac{d}{d\tau} a_{i,k}(\tau; n, m, \beta) = & \sum_{l \in \mathcal{M}_m} q_{il}^{(m)} a_{l,k}(\tau; n, m, \beta) - \left(\frac{\gamma k}{n}\right) \kappa(T-\tau, i) a_{i,k}(\tau; n, m, \beta) \\ & + \left(\frac{\gamma(k+1)}{n}\right) \left(\frac{1}{2} \left(\frac{\gamma(k+1)}{n} - 1 \right) \sigma^2(T-\tau, i) + \kappa(T-\tau, i) \theta(T-\tau, i) \right) a_{i,k+1}(\tau; n, m, \beta). \end{aligned} \tag{A9}$$

To establish the initial conditions for the system of ODEs in (A8) and (A9), we substitute the series solution (A1) into the terminal condition (A3) of the hybrid PDE system (A2) at $\tau = 0$. By matching the coefficients of the resulting series with the power $x^{\frac{\gamma k}{n}}$, we derive the initial values for the ODE system as follows:

$$a_{i,k}(0; n, m, \beta) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n \end{cases} \tag{A10}$$

for each $i \in \mathcal{M}_m$.

Furthermore, we demonstrate that the system of ODEs in (A8) and (A9) can be decoupled into $n+1$ systems, each consisting of m first-order linear ODEs. This decomposition is facilitated by the specific structure of the initial conditions (A10), as detailed in the following derivation:

Since the ODEs (A8) and (A9) hold for each $i \in \mathcal{M}_m$, Equation (A8) for $k = n$ characterizes a homogeneous system of m first-order linear ODEs, hereafter referred to as System I. In parallel, the equations in (A9) for $k = n-1, \ldots, 0$ give rise to n distinct non-homogeneous systems of m first-order linear ODEs, designated as System II.

We partition the index set $\mathcal{I} = \{0, \ldots, n\}$ into two disjoint subsets: $\mathcal{I}_1 = \{0, \ldots, n-1\}$ and $\mathcal{I}_2 = \{n\}$, such that $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$. For $k \in \mathcal{I}_2$, System I is solved by imposing the initial conditions (A10), where $a_{i,n}(0; n, m, \beta) = 1$ for all $i \in \mathcal{M}_m$, thereby facilitating the determination of the coefficient functions $a_{i,n}(\tau; n, m, \beta)$.

For $k \in \mathcal{I}_1$, System II is solved under the initial conditions $a_{i,k}(0; n, m, \beta) = 0$ for all $i \in \mathcal{M}_m$. It is important to note that the preceding results establish $a_{i,n}(\tau; n, m, \beta) \neq 0$. Consequently, the term $a_{i,k+1}(\tau; n, m, \beta)$ in (A9) may remain non-vanishing for certain $i \in \mathcal{M}_m$, thereby acting as a non-homogeneous source term in the recursive system.

For a more systematic investigation of the solution properties, the system (A8)–(A10) is transformed into a recurrent matrix-valued differential equation. This is achieved by implementing a change in index notation, as specified by the relationship:

$$a_{i,j_1}(\tau; n, m, \beta) = b_{i,n-j_1}(\tau; n, m, \beta), \tag{A11}$$

for $j_1 = n, n-1, \ldots, 0$. The resulting equivalence in matrix format is subsequently detailed in (13).

By reintroducing a change in variable $j = n - j_1$ to the index in (A11), we can reformulate the original system. Incorporating this relationship into Equations (A8) and (A9) and the initial conditions (A10) yields a sequence of $n+1$ systems, each comprising m first-order linear ODEs:

$$\frac{d}{d\tau} b_{i,0}(\tau; n, m, \beta) = \sum_{l \in \mathcal{M}_m} q_{il}^{(m)} b_{l,0}(\tau; n, m, \beta) - \gamma\kappa(T-\tau, i) b_{i,0}(\tau; n, m, \beta), \tag{A12}$$

and

$$\begin{aligned}\frac{d}{d\tau}b_{i,j}(\tau;n,m,\beta) = &\sum_{l\in\mathcal{M}_m} q_{il}^{(m)} b_{l,j}(\tau;n,m,\beta) - \left(\frac{\gamma(n-j)}{n}\right)\kappa(T-\tau,i)b_{i,j}(\tau;n,m,\beta)\\ &+\left(\frac{\gamma(n-j+1)}{n}\right)\left(\frac{1}{2}\left(\frac{\gamma(n-j+1)}{n}-1\right)\sigma^2(T-\tau,i)+\kappa(T-\tau,i)\theta(T-\tau,i)\right)b_{i,j-1}(\tau;n,m,\beta),\end{aligned} \tag{A13}$$

accompanied by the initial conditions:

$$b_{i,j}(0;n,m,\beta) = \begin{cases} 1 & \text{if } j=0,\\ 0 & \text{if } j\neq 0\,,\end{cases} \tag{A14}$$

for $j = 1,\ldots,n$ and for all states $i \in \mathcal{M}_m$.

By substituting $2-\beta$ for the ratio $\frac{\gamma}{n}$, it is straightforward to demonstrate that the collection of first-order linear ODEs (A12) and (A13) and their initial states (A14) can be succinctly formulated into the recurrent matrix-valued differential equations presented in (8). Under this parameterization, the system satisfies the boundary conditions in (9), thereby concluding the proof. □

Appendix A.3. Proof of Theorem 3

Proof. The derivation proceeds by analogy with Theorem 2, adjusted for the case $\beta = 2$ and $T > 0$ under the constraint $\gamma = n$ for fixed $(\gamma, m, n) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. In a departure from the assumptions in Theorem 2, we postulate that the solution is representable as a finite polynomial expansion of the form:

$$U^{(n,m,2)}(x,\tau,i) = \sum_{k=0}^{n} a_{i,k}(\tau;n,m,2)x^k. \tag{A15}$$

The determination of the coefficient functions $a_{i,k}(\tau;n,m,2)$ for $k = 0,\ldots,n$ relies on the hybrid system of coupled PDEs (5)–(6). Specifically, when $\beta = 2$, the governing equations for the conditional moments $U^{(n,m,2)}(x,\tau,i)$ are reformulated as:

$$\begin{aligned}0 = -\frac{\partial}{\partial\tau}U^{(n,m,2)}(x,\tau,i) &+ \frac{1}{2}\sigma^2(T-\tau,i)x^2\frac{\partial^2}{\partial x^2}U^{(n,m,2)}(x,\tau,i)\\ &+\kappa(T-\tau,i)(\theta(T-\tau,i)x-x)\frac{\partial}{\partial x}U^{(n,m,2)}(x,\tau,i)\\ &+\sum_{\substack{l\in\mathcal{M}_m\\ l\neq i}} q_{il}^{(m)}\left(U^{(n,m,2)}(x,\tau,l)-U^{(n,m,2)}(x,\tau,i)\right)\end{aligned} \tag{A16}$$

for $\tau \in (0,T]$ and $x > 0$, with initial conditions:

$$U^{(n,m,2)}(x,0,i) = x^n \tag{A17}$$

for $x > 0$ and $i \in \mathcal{M}_m$.

Next, we employ a similar argument to that of Theorem 2. By inserting the polynomial form (A15) into the governing hybrid system of PDEs (A16) and (A17) and equating the coefficients of x^k to zero for each $k = n,\ldots,0$, we obtain the following decoupled set of linear ODEs:

$$\begin{aligned}\frac{d}{d\tau}a_{i,k}(\tau;n,m,2) = &\sum_{l\in\mathcal{M}_m} q_{il}^{(m)} a_{l,k}(\tau;n,m,2)\\ &+k\left(\frac{1}{2}(k-1)\sigma^2(T-\tau,i)+\kappa(T-\tau,i)\left(\theta(T-\tau,i)-1\right)\right)a_{i,k}(\tau;n,m,2).\end{aligned} \tag{A18}$$

The initial values for the ODE system (A18) are derived by evaluating the series expansion (A15) at $\tau = 0$ and substituting it into the PDE initial boundary (A17). By matching the resulting polynomial with the power x^k, the following initial conditions are established for each coefficient function:

$$a_{i,k}(0; n, m, 2) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n \end{cases}$$

for each $i \in \mathcal{M}_m$.

The subsequent steps of the derivation are consistent with the methodology detailed in the proof of Theorem 2. Since the underlying reasoning remains valid for this case, the proof is hereby complete. □

Appendix B. Stationary First Moment of the Two-State RS-CEV Process with $\beta = \frac{3}{2}$

In this section, for $m = 2$ and $\beta = \frac{3}{2}$, utilizing Theorem 2, we establish an explicit closed-form expression for the stationary first moment of the two-state RS-CEV process in the case $\beta = \frac{3}{2}$, defined as $X^* := \lim_{\tau\to\infty} U^{(1,2,\frac{3}{2})}(x, \tau, i)$.

To simplify the analysis, we assume the coefficients κ_i, θ_i, and σ_i are time-invariant for each state $i \in \{1, 2\}$, such that $\mathbf{K}[t, 2] = \mathbf{K}_2$, $\mathbf{\Theta}[t, 2] = \mathbf{\Theta}_2$, and $\mathbf{\Sigma}[t, 2] = \mathbf{\Sigma}_2$. Under this time-independent framework, the two-state RS-ECEV model (1) simplifies to the two-state RS-CEV process. By applying the results from Theorem 2 and further assuming identical mean-reversion rates ($\kappa_1 = \kappa_2 = \kappa$), the first conditional moment is representable as a polynomial of x (19) as follows:

$$U^{(1,2,\frac{3}{2})}(x, \tau, i) = a_{i,0}(\tau; 2, 2, \tfrac{3}{2}) + a_{i,1}(\tau; 2, 2, \tfrac{3}{2})x^{\frac{1}{2}} + a_{i,2}(\tau; 2, 2, \tfrac{3}{2})x. \tag{A19}$$

To keep the notation simple, we leave out the indices $(2, 2, \frac{3}{2})$ for the rest of this section. Next, applying the relation (A11) to replace the coefficients a with b, Equation (A19) becomes

$$U^{(1,2,\frac{3}{2})}(x, \tau, i) = b_{i,2}(\tau) + b_{i,1}(\tau)x^{\frac{1}{2}} + b_{i,0}(\tau)x, \tag{A20}$$

for $x > 0, \tau > 0$ and $i = 1, 2$. Then, the coefficients in (A20) can be expressed in the matrix form $\mathbf{B}_j(\tau)$ for $j = 0, 1, 2$, as defined in (7):

$$\mathbf{B}_0(\tau) := \begin{bmatrix} b_{1,0}(\tau) \\ b_{2,0}(\tau) \end{bmatrix}, \quad \mathbf{B}_1(\tau) := \begin{bmatrix} b_{1,1}(\tau) \\ b_{2,1}(\tau) \end{bmatrix}, \quad \mathbf{B}_2(\tau) := \begin{bmatrix} b_{1,2}(\tau) \\ b_{2,2}(\tau) \end{bmatrix},$$

satisfying the system of two linear ODEs in (8):

$$\frac{d}{d\tau}\mathbf{B}_0(\tau) = \Psi_0^{(1)}\mathbf{B}_0(\tau), \tag{A21}$$

$$\frac{d}{d\tau}\mathbf{B}_1(\tau) = \Psi_1^{(1)}\mathbf{B}_1(\tau) + \Psi_1^{(-1)}\mathbf{B}_0(\tau), \tag{A22}$$

$$\frac{d}{d\tau}\mathbf{B}_2(\tau) = \Psi_2^{(1)}\mathbf{B}_2(\tau) + \Psi_2^{(-1)}\mathbf{B}_1(\tau), \tag{A23}$$

subject to the initial conditions (9):

$$\mathbf{B}_j(0) = \begin{cases} \mathbf{1}_2^\top & \text{if } j = 0, \\ \mathbf{0}_2^\top & \text{if } j = 1, 2, \end{cases} \tag{A24}$$

where $\mathbf{1}_2$ and $\mathbf{0}_2$ are the 1×2 matrices of ones and zeros, respectively. The coefficient matrices, as specified in (10) and (11), become constants:

$$\Psi_0^{(1)} := \mathbf{Q}_2 - \mathbf{K}_2, \quad \Psi_1^{(1)} := \mathbf{Q}_2 - \tfrac{1}{2}\mathbf{K}_2, \quad \Psi_2^{(1)} := \mathbf{Q}_2,$$

$$\Psi_1^{(-1)} := \mathbf{K}_2\boldsymbol{\Theta}_2, \quad \Psi_2^{(1)} := -\tfrac{1}{8}\boldsymbol{\Sigma}_2 + \tfrac{1}{2}\mathbf{K}_2\boldsymbol{\Theta}_2.$$

We proceed by solving the ODEs (A21)–(A23) with initial conditions (A24) to obtain the solutions:

$$\mathbf{B}_0(\tau) = \mathbf{e}^{\tau\Psi_0^{(1)}}\mathbf{1}_2^\top, \tag{A25}$$

$$\mathbf{B}_1(\tau) = \int_0^\tau \mathbf{e}^{(\tau-\eta)\Psi_1^{(1)}}\Psi_1^{(-1)}\mathbf{B}_0(\eta)d\eta, \tag{A26}$$

$$\mathbf{B}_2(\tau) = \int_0^\tau \mathbf{e}^{(\tau-\eta)\Psi_2^{(1)}}\Psi_2^{(-1)}\mathbf{B}_1(\eta)d\eta, \tag{A27}$$

respectively.

By the Lagrange–Sylvester Interpolation formula, (see Chapter 7 of [39]), we can express an analytic function f of a $p \times p$ matrix $\mathbf{G} = \tau\mathbf{A}$ as a polynomial in $\mathbf{G}$ of degree at most $p - 1$:

$$f(\mathbf{G}) = \sum_{k=1}^{p} f(\tau\lambda_i)\mathbf{G}_i,$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are eigenvalues of the matrix $\mathbf{A}$, then $\tau\lambda_1, \tau\lambda_2, ..., \tau\lambda_p$ are eigenvalues of the matrix $\mathbf{G}$, and $\mathbf{G}_i$, the i^{th} spectral projectors of $\mathbf{G}$, is given by

$$\mathbf{G}_i = \prod_{\substack{k=1 \\ k\neq i}}^{p} \frac{\mathbf{G} - \tau\lambda_k\mathbf{I}_p}{\tau\lambda_i - \tau\lambda_k} = \prod_{\substack{k=1 \\ k\neq i}}^{p} \frac{\mathbf{A} - \lambda_k\mathbf{I}_p}{\lambda_i - \lambda_k} = \mathbf{A}_i.$$

In order to apply this approach to the matrix exponential $f(G) = \mathbf{e}^G$, which in our case is $\mathbf{e}^{\tau\Psi_0^{(1)}}$, $\mathbf{e}^{(\tau-\eta)\Psi_1^{(1)}}$, and $\mathbf{e}^{(\tau-\eta)\Psi_2^{(1)}}$, for $i = 1, 2$ and $j = 0, 1, 2$, we then find and denote the eigenvalues of $\Psi_j^{(1)}$ by $\lambda_{j,i}$, where

$$\begin{aligned} \lambda_{0,1} &:= -(q_1 + q_2 + \kappa) < 0, & \lambda_{0,2} &:= -\kappa < 0, \\ \lambda_{1,1} &:= -\frac{1}{2}(2q_1 + 2q_2 + \kappa) < 0, & \lambda_{1,2} &:= -\frac{1}{2}\kappa < 0, \\ \lambda_{2,1} &:= -(q_1 + q_2) < 0, & \lambda_{2,2} &:= 0, \end{aligned}$$

Moreover, the i^{th} spectral projectors of $\Psi_j^{(1)}$ denoted by $\Psi_{j,i}^{(1)} := \prod_{\substack{k=1 \\ k\neq i}}^{2} \frac{\boldsymbol{\Psi}_\mathbf{j}^{(\mathbf{1})} - \lambda_k\mathbf{I}_2}{\lambda_i - \lambda_k}$, are

$$\Psi_{0,1}^{(1)} = \Psi_{1,1}^{(1)} = \Psi_{2,1}^{(1)} = \begin{bmatrix} \frac{q_1}{q_1+q_2} & \frac{-q_1}{q_1+q_2} \\ \frac{-q_2}{q_1+q_2} & \frac{q_2}{q_1+q_2} \end{bmatrix},$$

$$\Psi_{0,2}^{(1)} = \Psi_{1,2}^{(1)} = \Psi_{2,2}^{(1)} = \begin{bmatrix} \frac{q_2}{q_1+q_2} & \frac{q_1}{q_1+q_2} \\ \frac{q_2}{q_1+q_2} & \frac{q_1}{q_1+q_2} \end{bmatrix}.$$

Subsequently, (A25) can be expressed as

$$\mathbf{B}_0(\tau) = \sum_{i=1}^{2} \mathbf{e}^{\tau\lambda_{0,i}} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top, \tag{A28}$$

Substituting $\mathbf{B}_0(\tau)$ into (A26), we obtain

$$\mathbf{B}_1(\tau) = \sum_{k=1}^{2}\sum_{i=1}^{2} \frac{\mathbf{e}^{\tau\lambda_{0,i}} - \mathbf{e}^{\tau\lambda_{1,k}}}{\lambda_{0,i} - \lambda_{1,k}} \Psi_{1,k}^{(1)} \Psi_1^{(-1)} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top, \tag{A29}$$

Substituting $\mathbf{B}_1(\tau)$ into (A27), we obtain

$$\mathbf{B}_2(\tau) = \sum_{l=1}^{2}\sum_{k=1}^{2}\sum_{i=1}^{2} \frac{1}{\lambda_{0,i} - \lambda_{1,k}} \left(\frac{\mathbf{e}^{\tau\lambda_{0,i}} - \mathbf{e}^{\tau\lambda_{2,l}}}{\lambda_{0,i} - \lambda_{2,l}} - \frac{\mathbf{e}^{\tau\lambda_{1,k}} - \mathbf{e}^{\tau\lambda_{2,l}}}{\lambda_{1,k} - \lambda_{2,l}} \right) \Psi_{2,l}^{(1)} \Psi_2^{(-1)} \Psi_{1,k}^{(1)} \Psi_1^{(-1)} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top, \tag{A30}$$

Since $\lambda_{2,2} = 0$, $\mathbf{B}_2(\tau)$ in (A30) can be expressed as

$$\begin{aligned} \mathbf{B}_2(\tau) = & \sum_{k=1}^{2}\sum_{i=1}^{2} \frac{1}{\lambda_{0,i} - \lambda_{1,k}} \left(\frac{\mathbf{e}^{\tau\lambda_{0,i}} - \mathbf{e}^{\tau\lambda_{2,1}}}{\lambda_{0,i} - \lambda_{2,1}} - \frac{\mathbf{e}^{\tau\lambda_{1,k}} - \mathbf{e}^{\tau\lambda_{2,1}}}{\lambda_{1,k} - \lambda_{2,1}} \right) \Psi_{2,1}^{(1)} \Psi_2^{(-1)} \Psi_{1,k}^{(1)} \Psi_1^{(-1)} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top \\ & + \sum_{k=1}^{2}\sum_{i=1}^{2} \frac{1}{\lambda_{0,i} - \lambda_{1,k}} \left(\frac{\mathbf{e}^{\tau\lambda_{0,i}} - 1}{\lambda_{0,i}} - \frac{\mathbf{e}^{\tau\lambda_{1,k}} - 1}{\lambda_{1,k}} \right) \Psi_{2,2}^{(1)} \Psi_2^{(-1)} \Psi_{1,k}^{(1)} \Psi_1^{(-1)} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top, \end{aligned} \tag{A31}$$

It is noteworthy that the rest of $\lambda_{j,i}$ are all negative. Taking the limit of $\mathbf{B}_j(\tau)$ in (A28), (A29), and (A31) as $\tau \to \infty$, we obtain

$$\lim_{\tau\to\infty} \mathbf{B}_0(\tau) = \mathbf{0}_2^\top, \tag{A32}$$

$$\lim_{\tau\to\infty} \mathbf{B}_1(\tau) = \mathbf{0}_2^\top, \tag{A33}$$

and

$$\lim_{\tau\to\infty} \mathbf{B}_2(\tau) = \sum_{k=1}^{2}\sum_{i=1}^{2} \frac{1}{\lambda_{0,i} - \lambda_{1,k}} \left(\frac{-1}{\lambda_{0,i}} - \frac{-1}{\lambda_{1,k}} \right) \Psi_{2,2}^{(1)} \Psi_2^{(-1)} \Psi_{1,k}^{(1)} \Psi_1^{(-1)} \Psi_{0,i}^{(1)} \mathbf{1}_2^\top, \tag{A34}$$

Using basic matrix algebra to compute the RHS of (A34), we arrive at

$$\lim_{\tau\to\infty} \mathbf{B}_2(\tau) = \frac{4q_2\kappa(2q_2+\kappa)\theta_1^2 - q_2\theta_1\big(-16q_1\kappa\theta_2 + (2q_2+\kappa)\sigma_1^2 + 2q_1\sigma_2^2\big) + q_1\theta_2\big(4\kappa(2q_1+\kappa)\theta_2 - 2q_2\sigma_1^2 - (2q_1+\kappa)\sigma_2^2\big)}{4(q_1+q_2)\kappa(2q_1+2q_2+\kappa)}. \tag{A35}$$

Taking the limit of $U^{(1,2,\frac{3}{2})}(x,\tau,i)$ in (A19) and (A20) as $\tau \to \infty$ and injecting the results in (A32), (A33), and (A35), we are then able to obtain an explicit closed-form expression for the stationary first moment of the two-state RS-CEV process in the case $\beta = \frac{3}{2}$ and $\kappa_1 = \kappa_2 = \kappa$ as follows:

$$\begin{aligned} X^* :&= \lim_{\tau\to\infty} U^{(1,2,\frac{3}{2})}(x,\tau,i) \\ &= \lim_{\tau\to\infty} a_{i,0}(\tau) + \lim_{\tau\to\infty} a_{i,1}(\tau) x^{\frac{1}{2}} + \lim_{\tau\to\infty} a_{i,2}(\tau) x \\ &= \lim_{\tau\to\infty} b_{i,2}(\tau) + \lim_{\tau\to\infty} b_{i,1}(\tau) x^{\frac{1}{2}} + \lim_{\tau\to\infty} b_{i,0}(\tau) x \\ &= \frac{4q_2\kappa(2q_2+\kappa)\theta_1^2 - q_2\theta_1\big(-16q_1\kappa\theta_2 + (2q_2+\kappa)\sigma_1^2 + 2q_1\sigma_2^2\big) + q_1\theta_2\big(4\kappa(2q_1+\kappa)\theta_2 - 2q_2\sigma_1^2 - (2q_1+\kappa)\sigma_2^2\big)}{4(q_1+q_2)\kappa(2q_1+2q_2+\kappa)}. \end{aligned} \tag{A36}$$

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