

## Article

# Geodesic Boundary of Parabolic Surfaces and Existence of H-Fillable Curves in $\mathbb{H}^2 \times \mathbb{R}$

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## Abstract

This article provides a geometric characterization of the geodesic boundary for surfaces invariant under parabolic isometries in  $\mathbb{H}^2 \times \mathbb{R}$ . We present an alternative, constructive proof for the existence of minimal surfaces with rectangular asymptotic boundaries by utilizing a specific family of invariant surfaces. Furthermore, we generalize these existence results to surfaces with constant mean curvature  $H \in (0, 1/2)$ . By analyzing the variation in the relative asymptotic height, we establish the existence of properly embedded H-surfaces whose geodesic boundary is a rectangle of arbitrary height, provided it exceeds a specific lower bound determined by the parabolic solution.

**Keywords:** geodesic boundary; minimally fillable; parabolic surface; asymptotic height; H-fillable curves

**MSC:** 53A10; 53C42

## 1. Introduction

The theory of asymptotic boundaries in Hadamard manifolds, introduced by Eberlein and O'Neill [1], established a robust geometric framework for analyzing the behavior of geodesics at infinity. This compactification has allowed for classical problems in differential geometry to be extended to broader settings, most notably the asymptotic Plateau problem: given a Jordan curve in the asymptotic boundary, find a surface of constant mean curvature that spans it.

In the specific context of the product space  $\mathbb{H}^2 \times \mathbb{R}$ , the existence of minimal surfaces with prescribed asymptotic boundaries has been a central topic of research. Foundational to this field is the work of Jenkins and Serrin [2], who established necessary and sufficient conditions for the existence of minimal graphs over polygonal domains with infinite boundary values. This analytical program was significantly extended to  $\mathbb{H}^2 \times \mathbb{R}$  by Nelli and Rosenberg [3], proving existence results for general continuous curves. Subsequently, the theory was expanded to constant mean curvature (CMC) surfaces, with key results on vertical ends [4], uniqueness properties [5], and investigations into the general CMC asymptotic Plateau problem [6].

In recent years, the study of CMC surfaces in product spaces has remained a highly active area of research, as evidenced by recent investigations into entire CMC graphs [7]



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and the development of slab and half-space theorems in  $\mathbb{H}^2 \times \mathbb{R}$  [8]. Furthermore, the characterization of invariant surfaces continues to be refined; notably, Del Prete and Gimeno i Garcia (2024) [9] characterized the parabolicity of invariant surfaces in ambient spaces admitting Killing vector fields, while recent work by Del Prete (2024) [10] addresses Dirichlet problems for CMC graphs with prescribed asymptotic values. These developments highlight the ongoing relevance of understanding boundary behavior in product manifolds.

While the aforementioned classical and recent results primarily rely on analytical methods—such as partial differential equations and maximum principles—our approach in this paper is fundamentally *constructive*. By utilizing rotationally invariant surfaces (parabolic and hyperbolic), we explicitly generate geometric barriers. This method offers significant adaptability; unlike conjugate surface methods, which are technically involved for  $H \neq 0$  due to the lack of a simple Weierstrass representation, our approach focuses on the *relative asymptotic height* (following the framework in [11]). This provides a clear geometric intuition of how boundary height is controlled by the curvature parameter  $H$ , allowing us to generalize existence results for  $H \in (0, 1/2)$  without exclusive reliance on complex PDE existence theories.

This work complements the classification of screw motion surfaces by Sa Earp and Toubiana [12–14] and the recent characterizations of parabolicity in product spaces. We distinguish between the product boundary and the geodesic boundary [15], working with the latter as defined by equivalence classes of rays. Finally, we provide an alternative proof for the existence of minimal surfaces with rectangular boundaries and generalize this to  $H \in (0, 1/2)$  for rectangles of arbitrary height.

## 2. Geodesic and Product Compactification in $\mathbb{H}^2 \times \mathbb{R}$

Let  $M$  be a Hadamard manifold. A geodesic ray in  $M$  is a geodesic  $\gamma : [0, +\infty) \rightarrow M$  with  $|\gamma'(t)| = 1$ . Moreover, we say that two geodesic rays  $\gamma_1, \gamma_2$  are asymptotic if there exists  $C \geq 0$  such that

$$d(\gamma_1, \gamma_2) \leq C, \forall t \in [0, +\infty).$$

The asymptotic relation  $\sim$  is an equivalence relation. In this sense, the *asymptotic boundary* of  $M$  is defined as

$$\partial_g M := \{\text{geodesic rays}\} / \sim$$

referred to as the *geodesic boundary* throughout the text (initially defined in [1]). The equivalence classes will be denoted as  $[\gamma]$ . The following Proposition 1 implies that the asymptotic boundary is well-defined, and moreover, fixing  $p \in M$ , there exists a bijective correspondence between the unit sphere  $S_p$  in  $T_p M$  and  $\partial_g M$ , namely,  $v \in S_p \mapsto [\gamma_v] \in \partial_g M$ , where  $\gamma_v(t) = \exp_p(tv)$ , Theorem 2.10 of [1].

**Proposition 1** (Theorem 1.2 of [1]). *Given  $\alpha : [0, \infty) \rightarrow M$  geodesic ray and  $p \in M$ , there exists a unique  $\gamma : [0, \infty) \rightarrow M$  geodesic ray with  $\gamma(0) = p$  and  $\gamma \sim \alpha$ .*

The above is a construction aimed at defining a topological compactification on Hadamard manifolds. For this purpose, it is necessary to define a topology on the asymptotic boundary; in this case, it will be the cone topology (Definition 2.2 of [1]). A cone with vertex at  $p \in M$ , aperture  $\theta \in (0, \pi)$ , and axis  $v \in T_p M$  has the form

$$C_p(v, \theta) = \{\exp_p(tv) \mid \angle(w, v) < \theta, \text{ with } t > 0\}$$

and additionally, a truncated cone is defined as  $T_p(v, \theta, R) := C_p(v, \theta) \setminus \overline{B(p, R)}$  where  $\overline{B(p, R)}$  is the closed ball centered at  $p$  with radius  $R > 0$ . Finally, the open balls of  $M$  and

the truncated cones are a topological basis for the set  $\overline{M}^g := M \cup \partial_g M$ , and  $\partial_g M$  has the subspace topology.

In [1], it is shown that  $\overline{B(p, 1)}$  and  $\overline{M}^g$  are homeomorphic; consequently,  $\overline{M}^g$  is compact. This topological compactification will be referred to as the geodesic compactification, and more details can be found in [1].

In this text, we focus on the Hadamard manifold  $\mathbb{H}^2 \times \mathbb{R}$ , where  $\mathbb{H}^2$  is the hyperbolic space of dimension two, i.e., the unique complete, simply connected, two-dimensional manifold with constant sectional curvature equal to  $-1$ .

Two well-known models of  $\mathbb{H}^2$  are the upper half-plane model

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; y > 0\},$$

equipped with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

The other is the disk model

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\},$$

with the metric

$$ds^2 = \left( \frac{2}{1 - (x^2 + y^2)} \right)^2 (dx^2 + dy^2).$$

Throughout the text, illustrations and drawings refer to the disk model. In this context, we present the product metric in  $\mathbb{H}^2 \times \mathbb{R}$ :

$$d\tilde{s}^2 = \left( \frac{2}{1 - (x^2 + y^2)} \right)^2 (dx^2 + dy^2) + dt^2.$$

We will describe the geodesic compactification

$$\overline{\mathbb{H}^2 \times \mathbb{R}}^g = (\mathbb{H}^2 \times \mathbb{R}) \cup \partial_g(\mathbb{H}^2 \times \mathbb{R}).$$

Each geodesic ray  $\gamma$  in  $\mathbb{H}^2 \times \mathbb{R}$  takes the form  $(\gamma_1, \gamma_2)$ , where  $\gamma_1, \gamma_2$  are geodesic rays in  $\mathbb{H}^2$  and  $\mathbb{R}$ , respectively. For example, fixing a point in  $a \in \mathbb{H}^2$ , we have the geodesic rays  $\{a\} \times \mathbb{R}^+, \{a\} \times \mathbb{R}^-$  which will be called poles and denoted as  $\{p^+, p^-\}$ . Another example of a geodesic ray is the product  $\alpha \times \{C\}$  where  $\alpha$  is a geodesic in  $\mathbb{H}^2$  and  $C \in \mathbb{R}$  is a constant. Also, observe that  $\alpha \times \{C\} \sim \alpha \times \{0\}$ , which means that such geodesic rays form the boundary of the slice  $\mathbb{H}^2 \times \{0\}$  denoted by  $\partial_g \mathbb{H}^2$  and called the “equator”, as shown in Figure 1.

An important observation regarding notation is that poles can be denoted as  $p^+ = [(p, +\infty)]$ ,  $p^- = [(p, -\infty)]$ . Moreover, geodesic rays in  $\mathbb{R}$  are of the form  $\beta = a + mt$ , so equivalence classes can be identified with the slope of the geodesic ray, i.e.,  $[\beta] \equiv m$ . Thus, any element of  $\partial_g(\mathbb{H}^2 \times \mathbb{R}) \setminus \{p^+, p^-\}$  can be denoted as  $(\gamma, m)$ , where  $\gamma$  is a geodesic in  $\mathbb{H}^2$  and  $m$  is the slope of a geodesic in  $\mathbb{R}$ .

The “vertical lines” in  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  connecting  $p^+$  with  $p^-$  passing through  $p \in \partial_g \mathbb{H}^2$  are called Weyl chambers (introduced in [16]) and will be denoted as  $W(p)$ , i.e., given  $p \in \partial_g \mathbb{H}^2$ ,

$$W(p) = \{[(\gamma_1, \gamma_2)] | [\gamma_1] = p \in \partial_g \mathbb{H}^2\} \cup \{p^-\} \cup \{p^+\}.$$

Similarly, we define

$$W^+(p) = \{[(\gamma_1, \gamma_2)] | [\gamma_1] = p \in \partial_g \mathbb{H}^2, \gamma_2 = mt \text{ y } 0 \leq m \leq +\infty\},$$

$W^-(p)$  is defined analogously but with slopes going towards  $-\infty$ .

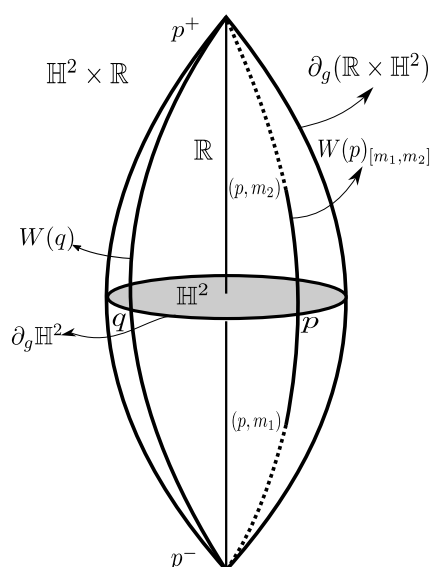


**Figure 1.** Geodesic boundary.

Note that while the manifold  $\mathbb{H}^2 \times \mathbb{R}$  is non-compact, its product compactification  $\overline{\mathbb{H}^2 \times \mathbb{R}}^\times := \overline{\mathbb{H}^2} \times \overline{\mathbb{R}}$  is compact under the product topology. Consequently, the product boundary  $\partial_\times(\mathbb{H}^2 \times \mathbb{R}) := (\partial_g \mathbb{H}^2) \times \overline{\mathbb{R}}$  is well-defined. Although the geodesic boundary  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  and the product boundary are topologically distinct, for the vertical graphs considered in this work, there is a natural identification. The “equator” corresponds to geodesic rays lying in the slice  $\mathbb{H}^2 \times \{0\}$ , i.e., points of the form  $([\gamma], 0)$ , see Figure 2.

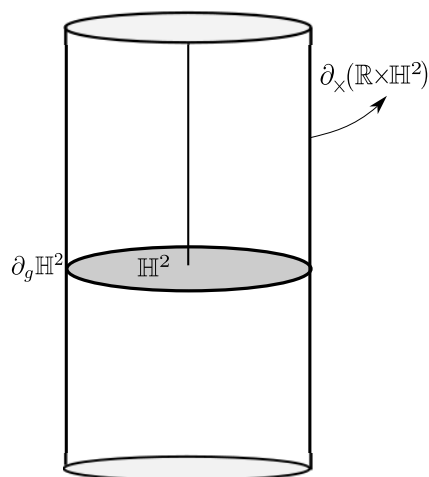


**Figure 2.** Product boundary on  $\mathbb{H}^2 \times \mathbb{R}$ .

### 3. Invariant Surfaces

In a Riemannian manifold  $M$  of dimension  $n$ , the mean curvature  $H$  of an oriented hypersurface  $S \subset M$  with normal vector  $\eta$  at each point  $p \in S$  is defined as

$$H(p) = \frac{1}{n-1} \text{tr}(S_\eta(p))$$

$S_\eta$  is the shape operator, defined at  $p \in M$  as

$$S_\eta(p) : \begin{array}{ccc} T_p M & \rightarrow & T_p M \\ v & \mapsto & -(\nabla_v S)^T \end{array} \quad (1)$$

where the term  $(\cdot)^T$  denotes the tangential component of the vector to the hypersurface,  $S$  is a local extension of  $\eta$  to the manifold, and  $\bar{\nabla}$  is the connection of the manifold.

There is the particular case of a manifold of the form  $M \times \mathbb{R}$  and for a hypersurface  $S$  which is the graph of a function  $u : \Omega \rightarrow \mathbb{R}$ , where  $\Omega \subset M$ . Defining

$$\begin{aligned} F : \Omega \times \mathbb{R} &\rightarrow \mathbb{R} \\ (p, x) &\mapsto u(p) - x \end{aligned} \quad (2)$$

orienting the graph upwards, i.e., taking the normal vector as  $\eta = -\frac{\nabla F}{|\nabla F|}$ , and making the necessary calculations, we obtain that the mean curvature equation

$$\operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) - 2H = 0 \quad (3)$$

where  $\operatorname{div}$  and  $\nabla$  are the divergence and gradient operators in  $M$ .

In the manifold  $\mathbb{H}^2 \times \mathbb{R}$ , the mean curvature equation formula obtained with a function  $\phi \circ s$  will be used, where  $s$  is defined on a domain of  $\mathbb{H}^2$  and  $\phi$  is a real function defined on an interval  $I \subset \mathbb{R}$ . Thus, the divergence (3) becomes

$$\begin{aligned} \operatorname{div} \left( \frac{\phi'(s) \nabla s}{\sqrt{1 + |\phi'(s) \nabla s|^2}} \right) &= \frac{d}{ds} \left( \frac{\phi'(s)}{\sqrt{1 + |\phi'(s) \nabla s|^2}} \right) \langle \nabla s, \nabla s \rangle \\ &+ \left( \frac{\phi'(s)}{\sqrt{1 + |\phi'(s) \nabla s|^2}} \right) \Delta s. \end{aligned} \quad (4)$$

In  $\mathbb{H}^2$ , there are three types of families of 1-parameter isometries: the elliptic ones, which are rotations around a common point in  $\mathbb{H}^2$ ; the parabolic isometries, which can be considered as rotations around a point  $p \in \partial_g \mathbb{H}^2$ ; and the hyperbolic isometries. In the work of P. Klaser, R. Soares, and M. Telichevesky [17], we can find constant mean curvature (CMC) surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  that are invariant under each type of 1-parameter isometry family in  $\mathbb{H}^2$  and extended to  $\mathbb{H}^2 \times \mathbb{R}$ .

### 3.1. Unduloid

Revolution surfaces are the first surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  invariant under a 1-parameter isometry family constructed in work [17], in this case, elliptic isometries. These surfaces are graphs of functions defined in domains like  $\mathbb{H}^2 \setminus B_r(o)$  (the exterior of a ball in  $\mathbb{H}^2$  centered at  $o \in \mathbb{H}^2$  with radius  $r \geq 0$ ) and symmetric with respect to the axis  $o \times \mathbb{R}$ , i.e., graphs of functions of the form  $u = \phi \circ s : \mathbb{H}^2 \setminus B_r(o) \rightarrow \mathbb{R}$ , where  $s : \mathbb{H}^2 \setminus B_r(o) \rightarrow [0, +\infty)$  is the distance function to the circle  $\partial B_r(o)$ , and  $\phi : [0, T] \rightarrow \mathbb{R}$  is a real function.

As was mentioned, the mean curvature equation for the graph of  $u$  oriented with the normal vector  $\eta = -\frac{\nabla F}{|\nabla F|}$  is

$$Q_H(u) := \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) - 2H = 0, \quad (5)$$

but the graph has the form  $Gr(u) = \{(p, \phi(s(p))) \in \mathbb{H}^2 \times \mathbb{R} \mid p \in \mathbb{H}^2 \setminus B_r(o)\}$ . Therefore, we can rewrite the mean curvature equation using Equation (4) and the fact that  $|\nabla s| = 1$  and  $\Delta s = \coth(s + r)$ , obtaining that the equation reduces to  $\phi$  satisfying the following ODE:

$$\left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right)' + \left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right) \Delta s(s + r) - 2H = 0. \quad (6)$$

Thus, fixing  $r \geq 0$ , constant  $H$ , and solving the ODE (6) with the initial condition  $\phi'(0) = +\infty$ , we obtain the unduloid  ${}_H\text{-und}_r$ , which is a rotational constant mean curvature surface (CMC  $-H$ ) tangent to the cylinder  $\partial B_r(o) \times \mathbb{R}$ . Its corresponding general expression is

$${}_H\text{-und}_r(s) = \int_0^s \frac{2H(\cosh(r+t) - \cosh(r)) + \sinh(r)}{\sqrt{\sinh^2(r+t) - (2H(\cosh(r+t) - \cosh(r)) + \sinh(r))^2}} dt, \quad (7)$$

defined on  $[0, +\infty]$ , where  $0 \leq H \leq 1/2$ , and the complete unduloid is obtained by joining  ${}_H\text{-und}_r$  with  $-{}_H\text{-und}_r$  on the sphere  $\partial B_r(o)$ .

### 3.2. Horonduloids

The  $H$ -horonduloids are graphs of functions defined in the exterior of a horodisc, whose level curves are the horocycles passing through point  $a \in \partial_\infty \mathbb{H}^2$  on the boundary of  $\mathbb{H}^2$ . Although the function  ${}_H\text{-und}_\infty$  that provides the parabolic surfaces can be obtained by letting  $r \rightarrow +\infty$  in the expression  ${}_H\text{-und}_r$ , it is also possible to find it by solving (5) but in this case assuming that the solution depends on the distance  $s$  to a fixed horocycle  $C_\infty$  containing  $a$ . Thus, the distance function  $s$  satisfies again  $\Delta s = 1$ , and we obtain the equation

$$\left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right)' + \left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right) - 2H = 0. \quad (8)$$

Again, fixing the constant  $H$  and solving the ordinary differential Equation (8) with the initial condition  $\phi'(0) = +\infty$ , we obtain the following explicit expression:

$$\begin{aligned} {}_H\text{-und}_\infty(s) &= \int_0^s \frac{2H + (1 - 2H)e^{-t}}{\sqrt{1 - (2H + (1 - 2H)e^{-t})^2}} dt \\ &= \theta + \frac{4H}{\sqrt{1 - 4H^2}} \tanh^{-1} \left( \sqrt{\frac{1 + 2H}{1 - 2H}} \tan \left( \frac{\theta}{2} \right) \right), \text{ if } H < 1/2, \end{aligned} \quad (9)$$

where  $0 \leq \theta \leq \pi$  is such that  $\cos \theta = (1 - 2H)e^{-s} + 2H$ .

In the same way as with the unduloids, the  ${}_H$ -horosuperficie of a fixed  $C_\infty$  is the surface obtained by joining the graph of  ${}_H\text{-und}_\infty$  with the graph of  $-{}_H\text{-und}_\infty$ , the horocycles passing through point  $a$ , as illustrated in Figure 3. More details in [17].

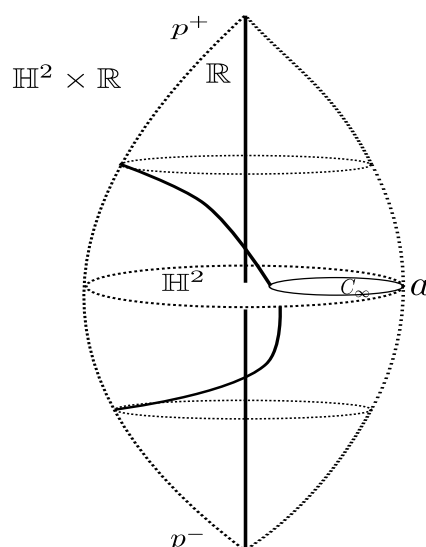


Figure 3. Horonduloid passing through  $a$ .

#### 4. Geodesic Boundary of Invariant Surfaces

This work will focus on the study of curves on the geodesic boundary of  $\mathbb{H}^2 \times \mathbb{R}$  that are boundaries of properly embedded surfaces with constant mean curvature. This problem can be interpreted as an “asymptotic H-Plateau problem”. The most relevant reference regarding curves that are boundaries of CMC surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  comes from the work of B. Nelli and H. Rosenberg [3], specifically Theorem 4. See Figure 4.

**Theorem 1** (Theorem 4 of [3]). *Let  $\Gamma$  be a continuous Jordan curve in  $\partial_g(\mathbb{H}^2) \times \mathbb{R}$ , which is a vertical graph over  $\partial_g(\mathbb{H}^2)$ . Then, there exists a minimal vertical graph  $\Sigma$  over  $\mathbb{H}^2$  such that  $\Gamma$  is the asymptotic boundary of  $\Sigma$ . The graph is unique.*

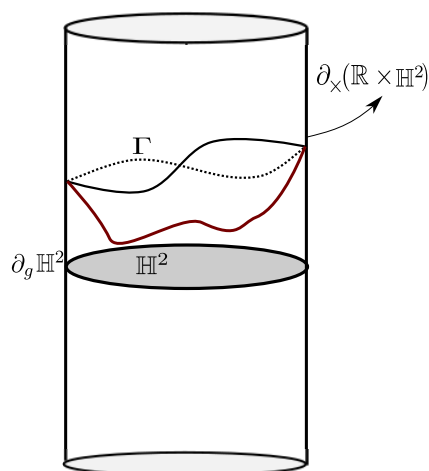


Figure 4. Asymptotic boundary of  $\Sigma$ .

It is important to note that, in this case, the Plateau problem occurs on the boundary product of  $\mathbb{H}^2 \times \mathbb{R}$ , and the solution is a minimal surface with a continuous curve as boundary, which is a graph over the geodesic boundary of  $\mathbb{H}^2$ . Another important reference is the work of R. Sa Earp and E. Toubiana [13], where curves are presented on the boundary product of  $\mathbb{H}^2 \times \mathbb{R}$  that are boundaries of minimal surfaces, see Figure 5.

**Proposition 2** (Proposition 2.1 of [13]). *Let  $a, b \in \mathbb{R}$  be and  $\widehat{q_1, q_2}$ , an arc in  $\partial_g \mathbb{H}^2$  with  $q_1 \neq q_2$  and  $b - a \geq \pi$ , and there exists a properly embedded minimal surface  $\Sigma$  such that its asymptotic boundary is the rectangle  $\widehat{q_1, q_2} \times [a, b]$ .*

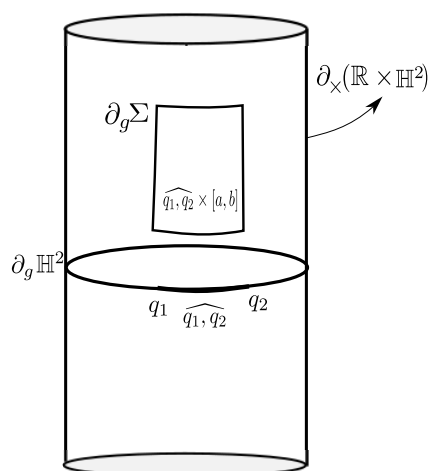


Figure 5. Asymptotic boundary is the rectangle  $\widehat{q_1, q_2} \times [a, b]$ .

The curves presented in the previous works are the geodesic boundary of minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ . In this sense, B. Kloeckner and R. Mazzeo in Section 4 of work [15] give a definition for these types of curves.

**Definition 1** (Minimally Fillable). *A curve is minimally fillable if it is the asymptotic boundary of a properly embedded minimal surface in  $\mathbb{H}^2 \times \mathbb{R}$ , where “curve” means a Jordan curve or a disjoint union of Jordan curves.*

Kloeckner and Mazzeo initially construct curves on the boundary product of  $\mathbb{H}^2 \times \mathbb{R}$  to generalize Theorem 4 by B. Nelli and H. Rosenberg [3], as shown in the following proposition. See Figure 6.

**Proposition 3** (Proposition 4.1 of [15]). *Let  $\sigma \in \partial_\times(\mathbb{H}^2 \times \mathbb{R})$  be a curve, and suppose that each  $p \in \partial_\times(\mathbb{H}^2 \times \mathbb{R})/\sigma$  can be separated from  $\sigma$  by the boundary  $\partial_\times \Sigma_p$  of a properly embedded minimal surface  $\Sigma_p$ . Then,  $\sigma$  is minimally fillable.*

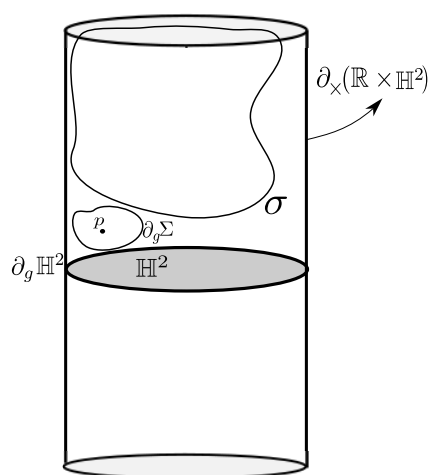


Figure 6. Boundary  $\partial_\times \Sigma_p$ .

However, their main goal is to classify all minimally fillable curves on the geodesic boundary  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  in Theorem 5.1 and Corollary 5.2 of the same work [15]. Following this idea, in work [11] they define curves in  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  that are the geodesic boundaries of surfaces with constant mean curvature  $H$  for  $H \in [0, 1/2]$ , generalizing the minimally fillable curves.

**Definition 2** (Definition 3.1 of [11]). *A curve is  $H$ -fillable if it is the geodesic boundary of a properly embedded surface in  $\mathbb{H}^2 \times \mathbb{R}$  with constant mean curvature  $H$ , where  $0 \leq H \leq 1/2$ , and “curve” means a Jordan curve or a disjoint union of Jordan curves.*

It is important to note that the orientation considered in this case is the one pointing “inwards” in  $\mathbb{H}^2 \times \mathbb{R}$  when it reaches the curve at infinity. We observe that the only situation where “inwards” does not make sense is when the mean curvature vector is parallel to the  $\mathbb{R}$  factor of  $\mathbb{H}^2 \times \mathbb{R}$  on the prescribed geodesic boundary. However, in this case, we would have a surface reaching the boundary  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  parallel to a section  $\mathbb{H}^2 \times c$ , where  $c$  is a real constant.

#### 4.1. Geodesic Boundary of Parabolic Surfaces

The following result characterizes the curves on the geodesic boundary  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  that are geodesic boundaries of surfaces invariant under parabolic isometries (parabolic surfaces) in  $\mathbb{H}^2 \times \mathbb{R}$ . The following proposition shows the geodesic boundary of the

horocylinder, which turns out not to be an example of an  $H$ -fillable curve. To simplify writing, the subsets of  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$  given by all geodesic rays of  $\partial_g \mathbb{H}^2$  with a fixed slope  $m$  in the  $\mathbb{R}$  factor will be called circles and denoted as  $\partial_g \mathbb{H}^2 \times \{m\}$ . An important constant that was defined by [18] and used throughout the text is

$$l_H := \frac{2H}{\sqrt{1-4H^2}}, \text{ where } l_{1/2} := +\infty. \quad (10)$$

**Proposition 4.** Let  $\sigma = W_{[-l_H, l_H]}(p) \cup (\partial_g \mathbb{H}^2 \times \{-l_H, l_H\})$ , where  $p \in \partial_g \mathbb{H}^2$  and  $l_H = \frac{2H}{\sqrt{1-4H^2}}$  is the constant defined in Equation (10). Then, there exists a properly embedded surface  $\Sigma$  with constant mean curvature  $H$  ( $0 \leq H \leq 1/2$ ) such that its geodesic boundary is  $\partial_g \Sigma = \sigma$ .

**Proof.** Let  $p \in \partial_g \mathbb{H}^2$ . Consider the parabolic surface  $\Sigma$  defined as the graph of  $H\text{-und}_\infty$  over the exterior of a horodisc  $C$  centered at  $p$ . Recall from Equation (11) that

$$H\text{-und}_\infty(s) = \int_0^s \frac{2H + (1-2H)e^{-t}}{\sqrt{1 - (2H + (1-2H)e^{-t})^2}} dt. \quad (11)$$

To determine the boundary of  $\Sigma$ , we analyze the asymptotic behavior of the graph along geodesic rays  $\gamma(t)$  in  $\mathbb{H}^2$  starting from  $C$ . It is a known result (see Proposition 3.1 in [11]) that if a graph  $u(\gamma(t))$  over a geodesic ray satisfies  $\lim_{t \rightarrow \infty} u'(\gamma(t)) = m$ , then the ray  $(\gamma(t), u(\gamma(t)))$  in  $\mathbb{H}^2 \times \mathbb{R}$  converges to the point  $([\gamma], m)$  in the geodesic boundary  $\partial_g(\mathbb{H}^2 \times \mathbb{R})$ .

Differentiating the integral expression with respect to the distance  $s$ ,

$$\lim_{s \rightarrow \infty} \frac{d}{ds}(H\text{-und}_\infty(s)) = \lim_{t \rightarrow \infty} \frac{2H + (1-2H)e^{-t}}{\sqrt{1 - (2H + (1-2H)e^{-t})^2}} = \frac{2H}{\sqrt{1-4H^2}} = l_H. \quad (12)$$

Thus, the graph approaches the boundary with a vertical slope exactly equal to  $l_H$ . This implies that the asymptotic boundary of the graph contains the set  $\partial_g \mathbb{H}^2 \times \{l_H\}$ .

Furthermore, since the construction is invariant under parabolic isometries fixing  $p$ , and considering the convexity of the domain, the vertical segments over  $p$  are filled up to the maximum slope. Specifically, using the same argument as in Proposition 3.11 of [11], the surface fills the Weyl chamber  $W_{[0, l_H]}(p)$ . By symmetry (Alexandrov reflection), the surface also fills the lower part. Therefore, the geodesic boundary is exactly the union of these sets:

$$\partial_g \Sigma = W_{[-l_H, l_H]}(p) \cup (\partial_g \mathbb{H}^2 \times \{-l_H, l_H\}).$$

This completes the proof, see Figure 7.  $\square$

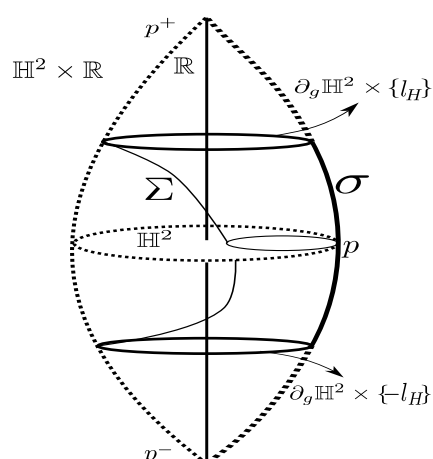


Figure 7. Proposition 4.

#### 4.2. Geodesic Boundary of Tall Rectangles

The study of tall rectangles on the product boundary  $\partial_\times(\mathbb{H}^2 \times \mathbb{R})$  was initiated by Sa Earp and Toubiana [13], who proved that rectangles with height at least  $\pi$  are minimally fillable curves. In this section, we provide an alternative proof of this classical result (Proposition 2.1 of [13]) by employing the geometric framework of Klaser, Soares, and Telichevesky [17]. This constructive approach not only recovers the minimal case but also serves as the necessary foundation for our main result: the generalization to constant mean curvature surfaces and the analysis of their relative asymptotic height.

Hyperbolic surfaces can be constructed by modifying the domain using the level curves of the distance function. Geometrically, this implies considering the distance to an equidistant curve (a hypercycle) instead of a geodesic. The mathematical advantage of using equidistant curves as the domain boundary is that the distance parameter  $s$  extends to  $[0, +\infty)$ . In contrast, for elliptic surfaces (unduloids), the domain is bounded between spheres, and for surfaces based on geodesics, the geometry restricts the asymptotic behavior. The non-compact domain provided by the hypercycle is essential for our construction because it allows for the invariant family to “reach” the asymptotic boundary at infinity, which is necessary to define the parabolic and hyperbolic invariant families that solve the asymptotic Plateau problem.

Choosing  $r > R := \tanh^{-1}(2H)$  and  $\Delta s = \tanh(r + s)$ , the ODE (6) is rewritten:

$$\left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right)' + \left( \frac{\phi'(s)}{\sqrt{1 + \phi'(s)^2}} \right) \tanh(r + s) - 2H = 0. \quad (13)$$

Taking  $\phi'(0) = +\infty$  as an initial condition, the solution to the ODE (13) is

$$\text{h-hnod}_r(s) = \int_0^s \frac{\cosh r - 2H \sinh r + 2H \sinh(r + t)}{\sqrt{\cosh^2(r + t) - (\cosh r - 2H \sinh r + 2H \sinh(r + t))^2}} dt. \quad (14)$$

The theorem refers only to minimal surfaces, so the surfaces considered will all have zero mean curvature ( $H = 0$ ), i.e.,  $\text{0-nod}_\infty(s)$  and  $\text{0-hiper}_r(s)$ .

**Theorem 2** (Proposition 2.1 de [13]). *For  $a, b \in \mathbb{R}$  and  $\widehat{q_1, q_2}$  given, an arc in  $\partial_g \mathbb{H}^2$  with  $q_1 \neq q_2$  and  $b - a \geq \pi$ , there exists an embedded minimal surface  $\Sigma$  with its asymptotic boundary at the boundary of the rectangle  $\widehat{q_1, q_2} \times [a, b]$ .*

**Proof.** The idea of the proof is to calculate the lower and upper limits of the heights of  $\partial_\times \text{0-hiper}_r$  on the boundary of the product  $\partial_\times(\mathbb{H}^2 \times \mathbb{R})$  as  $r$  varies, where  $\text{0-hiper}_r$  is the minimal graph generated around the geodesic connecting  $q_1$  with  $q_2$ . The geodesic boundary of  $\text{0-hiper}_r$  is an equatorial arc, so the boundary of the product  $\partial_\times \text{0-hiper}_r$  will describe half of a rectangle in the boundary of the product  $\partial_\times(\mathbb{H}^2 \times \mathbb{R})$  (see Figure 8), and the limitations of these halves will be calculated. First, observe that  $\lim_{r \rightarrow \infty} \tanh(r + s) = 1$ , so  $\text{0-hiper}_\infty(s) = \text{0-nod}_\infty(s)$ ; also note that from Equation (9),

$$\text{0-nod}_\infty(s) = \theta$$

where  $0 \leq \theta \leq \pi$  is such that  $\cos \theta = (1 - 2H)e^{-s} + 2H$ , i.e., if  $H = 0$ , then  $\cos \theta = e^{-s}$ , so  $\lim_{s \rightarrow +\infty} \text{0-nod}_\infty(s) = \pi/2$ .

Next, we show that the family of minimal surfaces  $0\text{-}hiper_r$  is strictly decreasing with respect to the parameter  $r \in (0, \infty)$ . Differentiating the integrand with respect to  $r$ ,

$$\frac{\partial}{\partial r} \left( \frac{\cosh r}{\cosh(r+t)} \right) = \frac{\sinh r \cosh(r+t) - \cosh r \sinh(r+t)}{\cosh^2(r+t)} \quad (15)$$

$$= \frac{\sinh(r - (r+t))}{\cosh^2(r+t)} = \frac{-\sinh(t)}{\cosh^2(r+t)}. \quad (16)$$

Since  $t > 0$ , this derivative is strictly negative. Thus,  $\frac{\partial}{\partial r} 0\text{-}hiper_r(s) < 0$  for all  $s$ , proving the monotonicity.

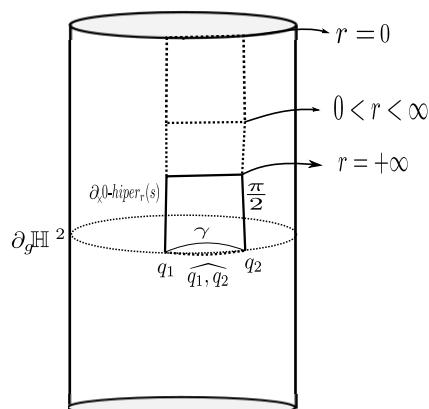


Figure 8. Theorem 3.

Combining this with the fact that  $0\text{-}nod_\infty(s) = \lim_{r \rightarrow \infty} 0\text{-}hiper_r(s)$ , we confirm the lower bound  $\pi/2$ .

On the other hand, we analyze the behavior as  $r \rightarrow 0^+$ . In the minimal case ( $H = 0$ ), the parameter  $r$  can approach 0. The integral becomes

$$\lim_{r \rightarrow 0^+} 0\text{-}hiper_r(s) = \int_0^s \frac{1}{\sinh t} dt = [\ln \tanh(t/2)]_0^s = +\infty. \quad (17)$$

This confirms that the family sweeps all heights from  $\pi/2$  (at  $r \rightarrow \infty$ ) to  $+\infty$  (at  $r \rightarrow 0$ ). Combining these limits with the Aleksandrov reflection, we obtain the desired result, which will describe half of a rectangle in the boundary (see Figure 8).  $\square$

#### 4.3. Existence of $H$ -Fillable Curves with Arbitrary Height

In this section, we generalize the previous existence result for surfaces with constant mean curvature  $H \in (0, 1/2)$ . Unlike the minimal case, the asymptotic slope  $l_H$  is strictly positive. Following the definition introduced in [11], we measure the *asymptotic height* of a surface relative to a fixed reference.

Let  $u_{cap}$  be the graph of a rotationally symmetric  $H$ -cap centered at the origin. Since all hyperbolic invariant surfaces  $H\text{-}hnod_r$  (for  $r > R$ ) share the same asymptotic slope  $l_H$  as the  $H$ -cap, the difference between their graphs at infinity is finite. We define the relative asymptotic height  $h(r)$  as follows:

$$h(r) := \lim_{s \rightarrow +\infty} (H\text{-}hnod_r(s) - u_{cap}(s)). \quad (18)$$

Let  $\mathcal{H}_{par}$  be the relative asymptotic height of the parabolic surface  $H\text{-}und_\infty$ , defined as  $\mathcal{H}_{par} = \lim_{s \rightarrow \infty} (H\text{-}und_\infty(s) - u_{cap}(s))$ .

**Remark 1** (Continuity of the Height Function). *The relative asymptotic height function  $h(r)$  is defined in Equation (18) as an integral depending on the parameter  $r$ . Its continuity with respect to  $r \in (R, +\infty)$  is a direct consequence of the theorem on continuous dependence of solutions to ordinary differential equations (ODEs) with respect to initial conditions and parameters. Specifically, the function  $\phi'(s)$  is the solution to the ODE Equation (13), which has smooth coefficients for  $r > R$ . Since the integrand in the definition of  $h(r)$  is a composition of continuous functions of  $r$  and  $\phi'(s)$ , the resulting integral varies continuously with  $r$ . This ensures that the image of  $h(r)$  is connected, allowing for the use of the Intermediate Value Theorem.*

**Theorem 3.** *Let  $H \in (0, 1/2)$ . For any value  $h \in (\mathcal{H}_{par}, +\infty)$ , there exists a properly embedded surface invariant under hyperbolic isometries with mean curvature  $H$  and relative asymptotic height  $h$ .*

**Proof.** The proof relies on the continuity of the height function  $h(r)$  and an analysis of its behavior at the boundaries of the domain  $(R, +\infty)$ , as well as its monotonic variation with respect to the parameter  $r$ . We first aim to show that the relative height  $h(r)$  is a strictly decreasing function. Since the reference term  $u_{cap}$  does not depend on  $r$ , it suffices to study the derivative of the hyperbolic surface integral  $H\text{-}h\text{nod}_r(s)$  given in Equation (13). Let us denote the integrand by  $\mathcal{I}(r, t)$ . Its structure is of the form  $\mathcal{I} = \frac{N}{\sqrt{C^2 - N^2}}$ , where  $N(r, t)$  and  $C(r, t)$  are the numerator and the hyperbolic cosine term in the denominator, respectively. Since this function is increasing with respect to the ratio  $N/C$ , the sign of  $\frac{\partial}{\partial r} H\text{-}h\text{nod}_r(s)$  is determined by the sign of the derivative of this ratio:

$$\frac{\partial}{\partial r} \left( \frac{N(r, t)}{C(r, t)} \right) = \frac{C \cdot \partial_r N - N \cdot \partial_r C}{C^2}.$$

Substituting the expressions  $N = \cosh r - 2H \sinh r + 2H \sinh(r + t)$  and  $C = \cosh(r + t)$ , and computing the partial derivatives with respect to  $r$ , the numerator of the derivative becomes  $\mathcal{X}(t) = -\sinh(t) + 2H(1 - \cosh t)$ . For any  $t > 0$ , we have  $\sinh(t) > 0$ , making the first term negative. Similarly, since  $\cosh t > 1$  and  $H > 0$ , the second term  $2H(1 - \cosh t)$  is strictly negative. Consequently,  $\frac{\partial}{\partial r} H\text{-}h\text{nod}_r(s) < 0$  for all  $s > 0$ , which implies that the asymptotic height  $h(r)$  decreases strictly as  $r$  increases.

Having established monotonicity, we now determine the range of the function by evaluating its limits. Geometrically, as the parameter  $r$  grows indefinitely ( $r \rightarrow +\infty$ ), the neck of the hyperbolic surface widens and recedes. In this limit, the functions defining the hyperbolic family  $H\text{-}h\text{nod}_r$  converge uniformly on compact sets to the generating function of the parabolic surface  $H\text{-}und_\infty$ . Since the relative height is defined by the difference at infinity, we obtain that  $\lim_{r \rightarrow +\infty} h(r) = \mathcal{H}_{par}$ . This confirms that the interval of attainable heights is bounded below by the height of the parabolic solution.

On the other hand, we analyze the behavior of the integral defining the surface height near the critical value  $R = \tanh^{-1}(2H)$ . The convergence or divergence of the height depends on the behavior of the integrand near  $t = 0$ . Let  $\Delta(r, t) = \cosh^2(r + t) - N(r, t)^2$  be the term inside the square root in the denominator. Performing a Taylor expansion of  $\Delta(r, t)$  with respect to  $t$  around  $t = 0$ , we observe that the critical behavior is governed by the linear coefficient, which is given by  $2 \cosh^2 r (\tanh r - 2H)$ . For any  $r > R$ , since  $\tanh r > 2H$ , this linear coefficient is positive, ensuring the integral converges. However, as  $r$  approaches the critical value  $R^+$ , we have  $\tanh r \rightarrow 2H$ , causing the linear term to vanish. In this critical limit, the dominant term of the expansion becomes quadratic ( $O(t^2)$ ). Consequently, the denominator behaves like  $\sqrt{t^2} = t$ , leading to a logarithmic integral of the form  $\int \frac{1}{t} dt$ , which diverges to  $+\infty$ . Thus, we have proven that  $\lim_{r \rightarrow R^+} h(r) = +\infty$ .

Since  $h(r)$  is a continuous function on  $(R, +\infty)$  that is strictly decreasing, and its image covers the range  $(\mathcal{H}_{par}, +\infty)$ , by the Intermediate Value Theorem, for any prescribed height  $\mathfrak{h} > \mathcal{H}_{par}$ , there exists a unique  $r$  such that the surface  $\Sigma_r$  has exactly that asymptotic height. Applying the Alexandrov reflection principle, we obtain the complete surface with total asymptotic height  $2\mathfrak{h}$ .  $\square$

## 5. Conclusions

In this work, we have provided a complete geometric characterization of the geodesic boundary for surfaces invariant under parabolic isometries in the product space  $\mathbb{H}^2 \times \mathbb{R}$ . By utilizing a constructive approach, we established the existence of properly embedded surfaces with constant mean curvature  $H \in [0, 1/2)$  whose boundaries at infinity are rectangles of arbitrary height, provided they exceed a specific lower bound determined by the parabolic solution. The primary advantage of this constructive proof, as opposed to traditional analytical methods based on the existence theory of partial differential equations, lies in its directness and geometric intuition. Our method explicitly generates geometric barriers that allow for a clear visualization of how the boundary height is controlled by the curvature parameter  $H$ , which is particularly relevant for  $H \neq 0$  where Weierstrass-type representations are technically complex.

The characterization and existence theorems presented here open several avenues for further mathematical applications. These constructive families can serve as essential barriers to solve more general Plateau problems for non-invariant curves by providing necessary height estimates. Furthermore, the analysis of the relative asymptotic height offers a potential path toward a more comprehensive classification of vertical ends for CMC surfaces, complementing the uniqueness results currently available in the literature. It is also important to note that this construction can be naturally extended to higher-dimensional manifolds  $\mathbb{H}^n \times \mathbb{R}$  by considering hyperspheres instead of geodesic circles. However, the analysis of the geodesic boundary in higher dimensions becomes more involved as  $\partial_g \mathbb{H}^n$  is a sphere  $S^{n-1}$ , leaving the precise characterization of  $H$ -fillable boundaries as a fertile ground for future research.

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