

Article

A Super-Multiplicative Inequality for the Number of Finite Unlabeled and T_0 Topologies

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Abstract

Let n be a nonnegative integer, and let $f(n)$ denote the number of unlabeled finite topologies on n points. We show that the inequality $f(n+m) \geq f(n)f(m)$ holds both for both labeled and unlabeled finite topologies. In addition, we establish a similar inequality for labeled and unlabeled T_0 topologies.

Keywords: finite topology; T_0 topology; number of unlabeled topologies; inequalities; growth rate

MSC: 05A20; 05A16; 54B10

1. Introduction

Finite topological spaces raise interesting combinatorial questions, most notably the number $T(n)$ of distinct topologies on n points. Exhaustive enumeration has established $T(n)$ for $n \leq 18$ [1,2]; however, a general formula remains out of reach.

The enumeration can be further refined by counting $T(n, k)$, the number of topologies on n points with k open sets. This remains a long-standing open problem, though some known cases exist. Key contributions come from Ern  and Stege, who computed $T(n, k)$ for $n \leq 11$ and arbitrary k in [3], including related counts for T_0 , connected topologies, and homeomorphism classes. They provided a complete description of all $T(n, k)$ for $k \leq 23$ [3]. It has been confirmed and extended in subsequent work [4].

Asymptotic behavior is complex. Finding a simple closed-form expression or a straightforward asymptotic formula for $T(n)$ has proved to be highly challenging. However, there are some known bounds and asymptotic estimates. For instance, $T(n)$ grows on the order of $T(n) = 2^{\Theta(n^2)}$, which is of the same exponential scale as $2^{\binom{n}{2}}$ [3]. However, very little is known about the growth behavior of $f(n)$, which denotes the number of unlabeled topologies on n points.

Super-multiplicative inequalities involving $f(n+m)$, $f(n)$, and $f(m)$, for any non-negative integers n and m , are also highly relevant for this question. In another field of combinatorics, matroid theory, a famous conjecture is that $f(n+m) \geq f(n)f(m)$, where $f(n)$ is the number of non-isomorphic matroids on n -element set [5] (see [6], p. 594). It remained unresolved for nearly three decades [7,8]. If we establish a similar inequality for the number of unlabeled finite topologies on n points, it would provide interesting structural properties, describing how topologies on disjoint sets combine to form topologies on their union. We can then get lower bounds on the growth of $f(n)$. For example, since



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$f(2) = 3$ [9], $f(n) = f(2 + (n - 2)) \geq f(2)f(n - 2) = 3f(n - 2)$, suggesting at least exponential growth with a base related to $\sqrt{3}$. Such inequalities could be a stepping stone towards a more precise understanding of the asymptotic growth rate. In addition to the analogy with matroid theory, this kind of inequality provides recursive approximation via maximum product, thereby enabling recursive lower bounds.

$$f(n) \geq \max_{1 \leq k \leq n-1} \{f(k) \cdot f(n - k)\}.$$

This formulation is useful for estimating $f(n)$ when exact enumeration is difficult, and it reflects the idea that the most prolific decompositions dominate the growth.

Let τ_i , $i = 1, 2$, be two topologies defined on finite and disjoint sets E_i . The direct sum $\tau = \tau_1 \oplus \tau_2$ is the collection $\{O_1 \cup O_2 \text{ with } (O_1, O_2) \in \tau_1 \times \tau_2\}$. A straightforward consequence is that the direct sum defines a topology on the disjoint union $E_1 \cup E_2$. For the labeled case, it is not difficult to see that, by using the direct sum, we achieve our super-multiplicative inequality. However, in the unlabeled case, we cannot prove it by using the direct sum. Indeed, if τ_i , $i = 1, 2$, are two non-homeomorphic finite topologies, then $\tau_1 \oplus \tau_2 = \tau_2 \oplus \tau_1$. So, a more sophisticated (non-commutative) operation should be used for this purpose. It is what we call the w -sum. This operation will be introduced in a later section.

Recent work has explored generalizations of finite topological structures, including finite fuzzy topologies on Boolean algebras [10], as well as metrization problems in fuzzy and $[0, 1]$ -valued topologies [11]. These developments highlight the continued interest in structural and combinatorial properties of finite and generalized topologies. In particular, we have an analogous inequality for $T(n, B_t)$, the number of (labeled) fuzzy topologies on Boolean algebras $B_t = \{0, 1\}^t$ on an n -set,

$$T(n + m, B_t) \geq T(n, B_t) T(m, B_t).$$

The number of unlabeled finite topologies on n points, $f(n)$, serves as a fundamental measure of structural variety in discrete systems. A defining characteristic of these spaces is their super-multiplicative growth as expressed by the inequality

$$f(n + m) \geq f(n) \cdot f(m)$$

This inequality is not merely a combinatorial curiosity; it represents a guarantee of compositional complexity, ensuring that when two independent systems of “size” n and m are integrated, the resulting configuration space possesses a topological richness that exceeds the simple product of its components. This principle underpins the scalability and modeling capacity of finite topologies across several real-life applications, including but not limited to a wide range of industrial sectors such as petrochemicals, electronics, automation, geospatial technology, and data science.

Chemistry: Topologies support the analysis of molecular graphs and indices used to predict chemical properties [12]. The super-multiplicative bound helps explain the vast diversity of isomers possible during molecular synthesis.

Image Analysis and GIS: These spaces provide a rigorous framework for modeling pixel connectivity and digital surfaces [13]. In Geographic Information Systems (GIS), they model spatial relationships whose adjacency complexity increases as regions are merged [14].

Robotics and Automation: Finite topologies inform configuration space analysis and motion planning [15]. In automata theory, they aid in the classification of state-transition systems, where the growth of $f(n)$ relates to the increasing capacity of control systems [16].

Data Science: In topological data analysis (TDA), they contribute to the study of shape and connectivity in high-dimensional data, facilitating the discovery of non-trivial features in large data sets [17].

The rest of the paper is organized as follows: in Section 2, we introduce the w -sum operation and prove some of its properties; in Section 3, we establish the main result; the final section summarizes our findings and outlines potential directions for future work.

2. The w -Sum Operation

Definition 1. Let n be a nonnegative integer, E a finite set of cardinality n , and τ a collection of subsets of E . We say that τ is a topology on E , or on n points, if

(w1) $\emptyset, E \in \tau$;

(w2) τ is closed under union; that is, if $A, B \in \tau$, then $A \cup B \in \tau$;

(w3) τ is closed under intersection; that is, if $A, B \in \tau$, then $A \cap B \in \tau$.

Moreover, E is called the ground set of τ , and the members of τ are called open sets.

Let us recall what we mean by a topologies homeomorphism.

Definition 2. Let τ_i , $i = 1, 2$, be two topologies defined on finite sets E_i , respectively, and φ a mapping from E_1 to E_2 . We say that φ is an homeomorphism from τ_1 to τ_2 if

(1) φ is bijective;

(2) $Y = \varphi(X) \in \tau_2$, if and only if $X \in \tau_1$.

In this case, we say that X and Y are homeomorphic, and τ_1 and τ_2 are homeomorphic.

In general, two topologies are homeomorphic if such homeomorphism exists.

The class of unlabeled topologies is the class of non-homeomorphic topologies.

A direct consequence of the above definition is as follows.

Proposition 1. Two homeomorphic open sets have the same cardinality. Moreover, homeomorphisms keep inclusion (order), intersection, union, and complement.

We denote by $T(n)$ and $f(n)$ the number of labeled and unlabeled topologies on n points, respectively, while the corresponding class for unlabeled topologies is denoted by $\mathcal{T}(n)$.

Definition 3. Let τ be a finite topology defined on E , $X \in \tau$, and let Y be a finite set that is disjoint from E .

(1) The intersection topology $\tau(X) = \tau \cap X$ is the collection $\{O \cap X \text{ for all } O \in \tau\}$.

(2) The shift topology $\tau - X$ is the collection $\{O \setminus X \text{ for all } O \in \tau\}$.

(3) The inverse shift topology $\tau + Y$ is the collection $\{O \cup Y \text{ for all } O \in \tau\}$.

We recall the notion of co-topology.

Definition 4. Let τ be a finite topology defined on E . Its co-topology denoted τ^C is the collection $\{E \setminus X \text{ for all } X \in \tau\}$.

It is evident that a co-topology is again a topology defined on the same ground set. From the above definitions, we deduce the following properties.

Lemma 1. Let τ be a finite topology defined on E , $X \in \tau$, and let Y be a finite set that is disjoint from E . Then,

(i) $\tau \cap X$ is a finite topology on X .

(ii) $\tau - X = [\tau^C \cap (E \setminus X)]^C$.

- (iii) $\tau - X$ is a finite topology on $E \setminus X$.
 (iv) $\tau + Y$ is closed under union and intersection.

Proof.

- (i) Let $A, B \in \tau \cap X$. It follows that $A, B \subseteq X$. This yields $A \cup B, A \cap B \subseteq X$; that is, $A \cup B, A \cap B \in \tau \cap X$. Since $\emptyset, X \in \tau \cap X \subseteq \tau$, $\tau \cap X$ is a topology on X . This completes the proof.
 (ii) Let $Y \in \tau - X$; that is, $Y = O \setminus X \subseteq E \setminus X$, for some $O \in \tau$. Thus, $(E \setminus X) \setminus Y = (E \setminus X) \cap (E \setminus O) \in (E \setminus X) \cap \tau^C = \tau^C \cap (E \setminus X)$. In other words, $Y \in [\tau^C \cap (E \setminus X)]^C$, and vice versa.
 (iii) Since a co-topology is also a topology on the same ground set and according to the above (i)–(ii), $\tau - X$ is a topology on $E \setminus X$.
 (iv) Similarly to (i), we can prove that $\tau + Y$ is closed under union and intersection. Its smallest member is Y , while its larger one is $E \cup Y$. $\square$

Definition 5. We say that a topology τ defined on E is a T_0 one, if, in addition, it satisfies one of the following equivalent conditions:

- (t01) For any two distinct points $x, y \in E$, there exists an open set $A \in \tau$, such that either $x \in A$ and $y \notin A$, or $y \in A$ and $x \notin A$.
 (t02) For any two distinct points $x, y \in E$, there exists an open set $A \in \tau$, such that $|\{x, y\} \cap A| = 1$.

The numbers of unlabeled and labeled T_0 -topologies on n points are denoted by $f_0(n)$, and $T_0(n)$, respectively.

We have analogous results to Lemma 1.

Lemma 2. Let τ be a T_0 topology defined on E , $X \in \tau$, and let Y be a finite set that is disjoint from E . Then,

- (i) $\tau \cap X$ is a T_0 topology.
 (ii) τ^C is a T_0 topology.
 (iii) $\tau - X$ is a T_0 topology.

Proof.

- (i) Let $x, y \in X \subseteq E$ be two distinct points. It follows that there exists an open set $A \in \tau$, such that $|\{x, y\} \cap A| = 1$. This yields $A \cap X \in \tau \cap X$. Furthermore, $|\{x, y\} \cap (A \cap X)| = |(\{x, y\} \cap X) \cap A| = |\{x, y\} \cap A| = 1$. This is what we request.
 (ii) Let $x, y \in E$ be two distinct points. It follows that there exists an open set $A \in \tau$, such that $|\{x, y\} \cap A| = 1$. This means that $|\{x, y\} \cap (E \setminus A)| = 1$, because $|\{x, y\} \cap E| = 2$. Since $E \setminus A \in \tau^C$, we are done.
 (iii) Combining (i)–(ii) of the current lemma with (ii) of Lemma 1 implies the result. $\square$

Now, we define the w -sum of two finite topologies.

Definition 6. Let E_i , $i = 1, 2$, be two disjoint finite sets, and let τ_i be two topologies defined on E_i , with $|E_i| = n_i$, respectively. The w -sum of τ_1 and τ_2 is

$$\tau_1 \oplus_w \tau_2 = \tau_1 \cup (\tau_2 + E_1).$$

Example 1. $n = 2$ and $m = 1$. (See Figure 1)

$$\begin{aligned} E_1 &= \{a, b\}, E_2 = \{c\}, \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \text{ and } \tau_2 = \{\emptyset, \{c\}\}. \\ \tau_2 + E_1 &= \{\emptyset \cup E_1 = \{a, b\}, \{c\} \cup E_1 = \{a, b, c\}\} = \{\{a, b\}, \{a, b, c\}\}. \\ \tau_{12} &= \tau_1 \oplus_w \tau_2 = \tau_1 \cup (\tau_2 + E_1) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}. \\ \tau_1 + E_2 &= \{\emptyset \cup E_2 = \{c\}, \{a\} \cup E_2 = \{a, c\}, \{b\} \cup E_2 = \{b, c\}, \{a, b\} \cup E_2 \end{aligned}$$

$$= \{a, b, c\} = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

$$\tau_{21} = \tau_2 \oplus_w \tau_1 = \tau_2 \cup (\tau_1 + E_2) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

It is clear that τ_{12} and τ_{21} are two topologies on $2 + 1 = 3$ points. Moreover, they are non-homeomorphic, because the first one has two singletons, while the second one has only one singleton.

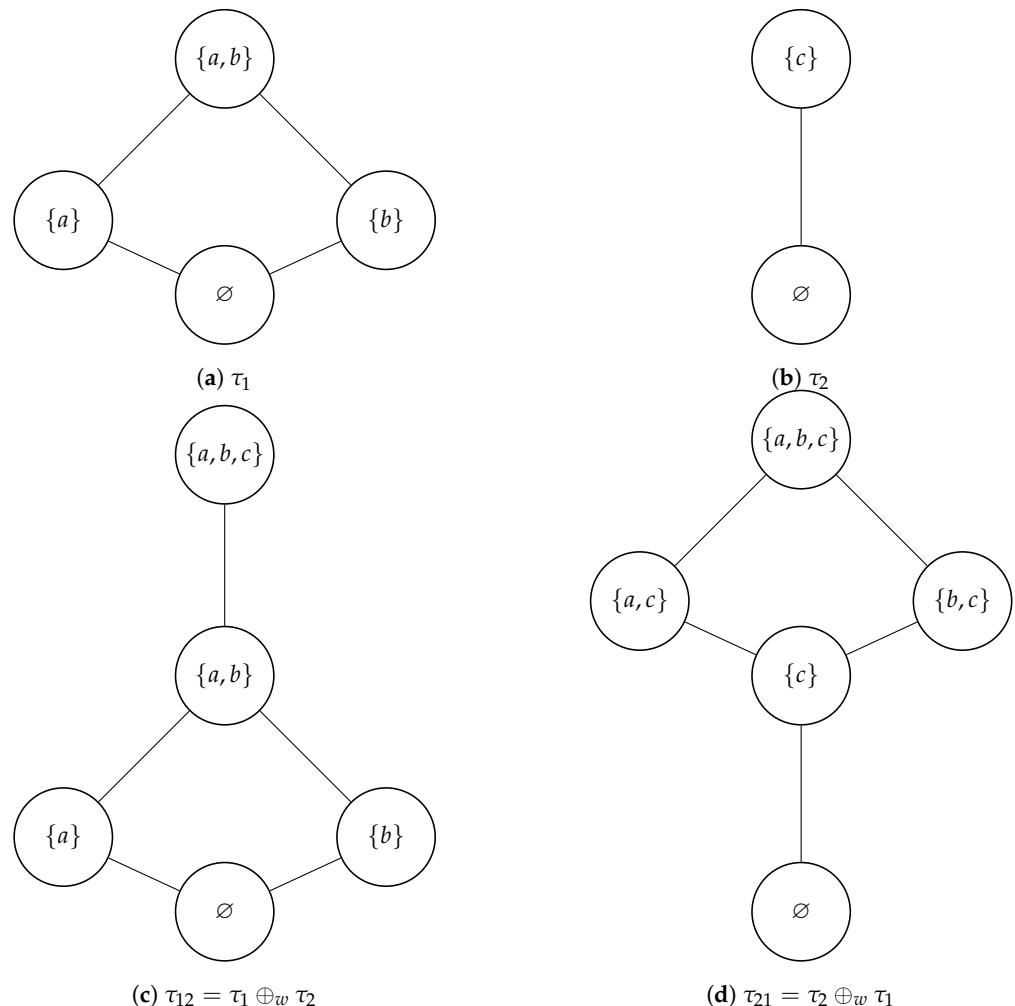


Figure 1. Illustration of Example 1 via Hasse diagrams.

Through Example 1, we can see that, roughly speaking, the Hasse diagram of the w -sum is obtained by gluing the Hasse diagram of the second topology on the ground set of the Hasse diagram of the first one. This implies that, unless both topologies are homeomorphic or they are T_0 chain topologies, this operation cannot be commutative. As explained in the Introduction, we have to take care only in the case $n = m$. Indeed, if $n = m$, then the w -sum is commutative when both involved topologies are homeomorphic. This guarantees the injectivity of the mapping introduced in Theorem 2, in contrast to the direct sum which can be commutative even if the involved topologies are non homeomorphic as we can see in the following example.

Example 2. $n = m = 2$. (See Figures 2 and 3)

$E_1 = \{a, b\}$, $E_2 = \{c, d\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, E_1\}$, and $\tau_2 = \{\emptyset, \{c\}, E_2\}$. It is clear that τ_1 and τ_2 are non-homeomorphic, since they have different cardinalities, namely, $|\tau_1| = 4 \neq 3 = |\tau_2|$.

$$\tau_{12} = \tau_1 \oplus \tau_2 = \{\emptyset, \{a\}, \{b\}, E_1\} \cup \{\{c\}, \{c, a\}, \{c, b\}, \{c, a, b\}\} \cup \{E_2, \{c, d, a\}, \{c, d, b\}, E_1 \cup E_2\}.$$

$$\tau_{21} = \tau_2 \oplus \tau_1 = \{\emptyset, \{c\}, E_2\} \cup \{\{a\}, \{a, c\}, \{a, c, d\}\} \cup \{\{b\}, \{b, c\}, \{b, c, d\}\} \cup \{E_1, \{a, b, c\},$$

$E_1 \cup E_2\}$.

It is clear that τ_{12} and τ_{21} are two homeomorphic topologies on $2 + 2 = 4$ points. Therefore, the direct sum is commutative even if the involved topologies are non-homeomorphic.

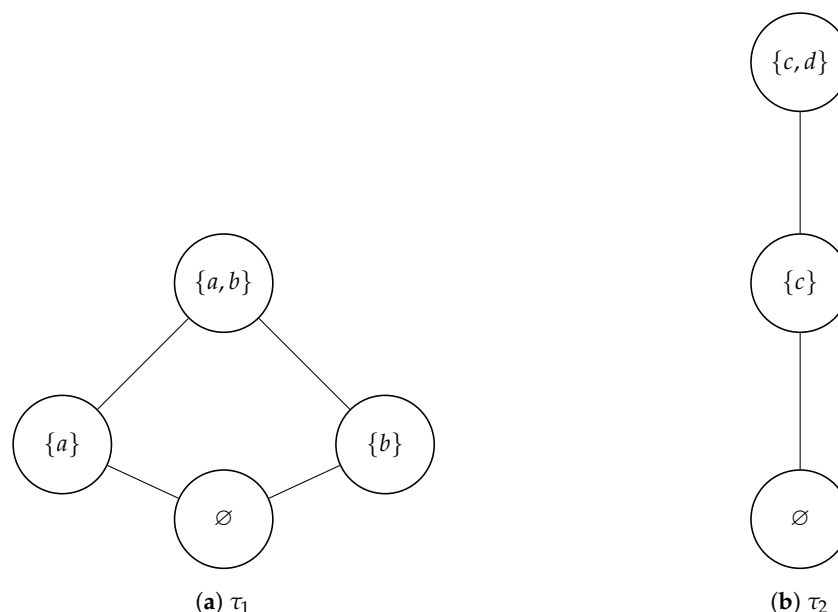


Figure 2. Hasse diagrams of the component topologies in Example 2.

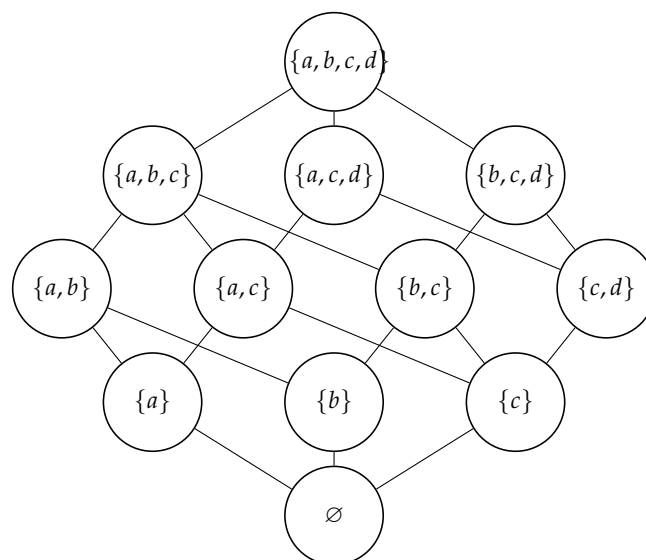


Figure 3. Hasse diagram of the direct sum $\tau_1 \oplus \tau_2$ in Example 2.

Furthermore, we have the following.

Proposition 2.

If $\tau = \tau_1 \oplus_w \tau_2$ is a topology, then $\tau_1 = \tau \cap E_1$, and $\tau_2 = \tau - E_1$.

Proof.

Let $A \in \tau_1 \subseteq \tau$. This means that $A \subseteq E_1$, i.e., $A \in \tau \cap E_1$, and vice versa. For $B \in \tau_2$, we have $B \cup E_1 \in \tau_2 + E_1 \subseteq \tau$, i.e., $B = (B \cup E_1) \setminus E_1 \in \tau - E_1$, and vice versa. $\square$

Lemma 3. Let τ_i , $i = 1, 2$, be two finite topologies defined on two disjoint sets, and let $\tau = \tau_1 \oplus_w \tau_2$ be a topology. Then, the following assertions are equivalent.

- (i) τ is a T_0 topology.
- (ii) τ_1 and τ_2 are T_0 topologies.

Proof.

(i) $\Rightarrow$ (ii): Proposition 2 implies that $\tau_1 = \tau \cap E_1$ and $\tau_2 = \tau - E_1$. They are two T_0 topologies according to (i) and (iii) of Lemma 2 when τ is a T_0 topology.

(ii) $\Rightarrow$ (i): Since $\tau = \tau_1 \cup [E_1 + \tau_2]$, its largest open set is $E = E_1 \cup E_2$ and its smallest one is the empty set. Let $x, y \in E$ be two distinct points. If both belong to E_1 (E_2 , respectively), then there exists an appropriate open set $X \in \tau_1$ (or $Y \in \tau_2$, i.e., $Y \cup E_1 \in \tau_2 + E_1$, respectively) that distinguish them. Now, if $x \in E_1$ and $y \in E_2$, then $y \notin E_1$, and $E_1 \in \tau_1 \subseteq \tau$ is the right open set since $E_1 \cap E_2 = \emptyset$. $\square$

The main result of this section is stated as follows.

Theorem 1.

(1) $\tau = \tau_1 \oplus_w \tau_2$ is a topology defined on $E_1 \cup E_2$, that is, a topology on $|E_1| + |E_2|$ points.

(2) If τ_1 and τ_2 are T_0 topologies, then so is τ .

Proof.

(1) Let $A, B \in \tau$. Without loss of generality, we can suppose that $A \in \tau_1$, and $B \in \tau_2 + E_1$, because the latter is closed under union and intersection according to (iv) of Lemma 1. In this case,

$$B = O_2 \cup E_1 \supseteq A, \quad (1)$$

for some $O_2 \in \tau_2$. Hence, $A \cup B = B \in \tau$, and $A \cap B = A \in \tau$. Therefore, τ is a topology on $E_1 \cup E_2 \in E_1 + \tau_2$, since $\emptyset \in \tau_1 \subseteq \tau$. So, it is a topology on $|E_1 \cup E_2| = |E_1| + |E_2|$ points, and we are done.

(2) Statement (2) follows directly from Lemma 3. $\square$

3. Main Result

First, we prove that intersection topologies and shift topologies keep homeomorphism.

Lemma 4. Let τ_i be two homeomorphic topologies, and let $X_i \in \tau_i$ and $i = 1, 2$, be two homeomorphic open sets. Then,

(i) $\tau_1 \cap X_1$ and $\tau_2 \cap X_2$ are homeomorphic.

(ii) τ_1^C and τ_2^C are homeomorphic.

(iii) $\tau_1 - X_1$ and $\tau_2 - X_2$ are homeomorphic.

Proof.

Let φ be a homeomorphism from τ_1 to τ_2 . This means that φ is bijective from E_1 to E_2 and $\varphi(Y_1) \in \tau_2$, if and only if $Y_1 \in \tau_1$.

(i) Now let φ_{X_1} be the restriction of φ to $X_1 \subseteq E_1$. Since X_1 and X_2 are homeomorphic, i.e., they have the same cardinality, φ_{X_1} is bijective from X_1 to X_2 . Moreover, inclusions are kept by φ ; hence, $\tau_1 \cap X_1$ and $\tau_2 \cap X_2$ are homeomorphic.

(ii) For the co-topologies, consider the same bijective mapping φ from E_1 to E_2 . Since an homeomorphism keeps intersection, union, and complement, $\varphi(E_1 \setminus O_1) = \varphi(E_1) \setminus \varphi(O_1) = E_2 \setminus O_2 \in \tau_2^C$, if and only if $E_1 \setminus O_1 \in \tau_1^C$.

(iii) (i)–(ii) of the current lemma and (ii) of Lemma 1 allows us to conclude. $\square$

We introduce the following mapping φ from $\mathcal{T}(n) \times \mathcal{T}(m)$ to $\mathcal{T}(n + m)$ as follows. For any $(\tau_1, \tau_2) \in \mathcal{T}(n) \times \mathcal{T}(m)$, $\varphi(\tau_1, \tau_2) = \tau_1 \oplus_w \tau_2$.

Theorem 2. φ is an injective mapping.

Proof.

First, it is clear that φ is well defined according to assertion (i) of Theorem 1. To prove injectivity, let $\tau_{ij} \in \mathcal{T}(i)$, $i = n, m$, and $\tau_j = \tau_{nj} \oplus_w \tau_{mj}$, $j = 1, 2$. Suppose that τ_1 and τ_2 are homeomorphic and that the ground sets of τ_{n1} and τ_{m1} are E_{n1} and E_{m1} , respectively, while those of τ_{n2} and τ_{m2} are F_{n2} and F_{m2} , respectively. Then, Theorem 1 and (i)–(iii) of Lemma 4 imply that

(i) $\tau_{n1} = \tau_1 \cap E_{n1}$ and $\tau_{n2} = \tau_2 \cap F_{n1}$ are homeomorphic.

(ii) $\tau_{m1} = \tau_1 - E_{n1}$ and $\tau_{m2} = \tau_2 - F_{n1}$ are homeomorphic.

Thus, φ is injective. $\square$

Corollary 1.

(i) $f(n+m) \geq f(n)f(m)$.

(ii) $f(n+m) \geq \max\{f(i)f(m+n-i) \text{ for all } 1 \leq i \leq n+m-1\}$.

Corollary 2.

(i) $T(n+m) \geq T(n)T(m)$.

(ii) $T(n+m) \geq \max\{T(i)T(m+n-i) \text{ for all } 1 \leq i \leq n+m-1\}$.

Now, we give the corresponding inequalities for T_0 topologies.

Corollary 3.

(i) $f_0(n+m) \geq f_0(n)f_0(m)$.

(ii) $f_0(n+m) \geq \max\{f_0(i)f_0(m+n-i) \text{ for all } 1 \leq i \leq n+m-1\}$.

Corollary 4.

(i) $T_0(n+m) \geq T_0(n)T_0(m)$.

(ii) $T_0(n+m) \geq \max\{T_0(i)T_0(m+n-i) \text{ for all } 1 \leq i \leq n+m-1\}$.

4. Conclusions

While the lower bounds of our super-multiplicative inequality may not be sharp, due to the use of simple formulas, our goal was to prove a similar non-trivial inequality to that established for matroids. Furthermore, the w -sum introduced to achieve the main result may be used in future research related to finite topologies.

The new w -sum operation and the newly introduced notions—such as the intersection, shift, and inverse shift—involve a high degree of abstraction, and they constitute a rigorous new framework. The strength of our main result is underpinned by the development of these novel theoretical tools, and we believe these tools represent a significant independent contribution to the field.

On the other hand, improvements of our inequality must use a sum of products as a lower bound. To illustrate this, consider the following inequality:

$$f(n+m) \geq [f(n)f(m)]^\alpha, \quad (2)$$

where α is a positive real number. Hence,

$$\log_2 f(n+m) \geq \alpha \log_2 [f(n)f(m)] = \alpha [\log_2 f(n) + \log_2 f(m)].$$

Since the logarithm (base 2) of the number of topologies grows according to the Kleitman–Rothschild formula [18],

$$\log_2 T(n) \sim \log_2 f(n) \sim \frac{n^2}{4},$$

for sufficiently large values of n and m , we obtain

$$\frac{(n+m)^2}{4} \geq \alpha \left[\frac{n^2}{4} + \frac{m^2}{4} \right];$$

that is,

$$(n+m)^2 \geq \alpha(n^2 + m^2),$$

or equivalently,

$$2nm \geq (\alpha - 1)(n^2 + m^2),$$

Specifically, for sufficiently large n and m , the left-hand side is eventually overtaken by the right-hand side unless $\alpha \leq 1$. Consequently, the best constant power that holds for all unlabeled finite topologies is the super-multiplicative identity that we have proven, namely $\alpha = 1$.

For example, if $\alpha = 1.5$, then inequality (2) fails when $n = 3$ and $m = 2$. Indeed, according to [9], $f(3+2) = f(5) = 139$, $f(3) = 9$, and $f(2) = 3$. So, $f(3)f(2) = 27$, and $[f(3)f(2)]^{1.5} > 140 > f(3+2)$. While for $\alpha = 1.15$, $f(13+1) = f(14) = 1,856,792,610,995$, $f(13) = 49,872,339,897$, and $[f(13)f(1)]^{1.15} = [f(13)]^{1.15} > 2,006,959,952,507 > f(13+1)$; similar examples can be obtained analogously.

Inserting some multiplicative factors to the left hand-side of the inequality fails again. For example, the inequality

$$f(n+m) \geq (n+m)f(n)f(m)$$

is not true ($f(1+3) = f(4) = 33 < 36 = (1+3) \times 1 \times 9 = (1+3)f(1)f(3)$).

Another type of multiplicative factor is

$$f(n+m) \geq nmf(n)f(m),$$

which also fails ($f(2+2) = f(4) = 33 < 36 = 2 \times 2 \times 3 \times 3 = 2 \times 2 \times f(2)f(2)$).

Finally, trying to generalize the w -sum by replacing the ground set of the first topology by any open set also fails. To illustrate this limitation for the reader, we provide here a counter-example. Let $E_1 = \{a, b\}$ and $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ be the discrete topology on E_1 , and $\tau_2 = \{\emptyset, \{c\}, \{c, d\}\}$. If $Y = \{b\}$, the collection $\tau = \tau_1 \cup (Y + \tau_2) = \tau_1 \cup \{\{b\}, \{b, c\}, \{b, c, d\}\}$ is not closed under union (specifically, while $\{a\}, \{b, c\} \in \tau$, $\{a\} \cup \{b, c\} = \{a, b, c\} \notin \tau$). We note that this result holds regardless of whether Y is an open set or not.

We think that improvements may be of the following form:

$$f(n+m) \geq \sum_{i,j} f(i)f(j).$$

We plan to investigate this kind of approach in the future.

Moreover, since T_0 topologies are in one-to-one correspondence with posets, we have a similar inequality for the number of non-isomorphic posets.

As a final remark, we observe that all the inequalities established above remain valid for a larger class of combinatorial structures than finite topologies, namely, those referred to as w -topologies. A w -topology is any finite collection of finite sets that is closed under union and intersection; unlike genuine topologies, it need not contain either the empty set or the ground set. Consequently, every finite topology is a w -topology, although the converse does not hold. For example, the inverse shift topology $Y + \tau$ is a w -topology that is not a topology according to assertion (iv) of Lemma 1, since it is closed under union and intersection but does not contain the empty set or the ground set. By analyzing its proof,

we deduce that this property holds when τ is a w -topology and not necessarily when it is a classical topology. Indeed, any w -topology is an inverse shift of a classical topology. To illustrate this, let $\tau = \{\{a\}, \{a, b\}, \{a, b, c\}\} = \{a\} + \{\emptyset, \{b\}, \{b, c\}\} = Y + \tau_0$. Y here is the smallest member of τ , that is, the intersection of all members of τ . So, $\tau_0 = \tau - Y$. This holds for any w -topology. Just think that your w -topology has a unique corresponding classical topology for which a particular inverse shift will provide it. All w -topologies having the same corresponding classical topology τ_0 form an equivalence class with τ_0 as its representative.

Thus, the notions of T_0 w -topology, w -sum, homeomorphic w -topologies, and so on are defined in direct analogy with their counterparts for finite topologies. This extension is natural, since the proofs rely essentially on closure under union and intersection.

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