

Article

Nonlinear ξ -Bi-Skew Lie Triple Derivations on $\ast$ -Algebras

Yimeng Yue ^{1,2}, Fenhong Li ^{1,*} , Liang Kong ¹ and Fugang Chao ¹

¹ School of Mathematics and Computer Application, Shangluo University, Shangluo 726000, China; 229022@slxy.edu.cn (Y.Y.); 229019@slxy.edu.cn (L.K.); 202105@slxy.edu.cn (F.C.)

² Key Laboratory of Intelligent Computing and Data Fusion, Shaanxi Provincial Institutions of Higher Education, Shangluo 726000, China

* Correspondence: 229011@slxy.edu.cn

Abstract

Let $\mathcal{A}$ be a unital associative $\ast$ -algebra over the complex field $\mathbb{C}$ containing a nontrivial projection and ξ be a nonzero scalar. In this paper, we give a characterization of ξ -bi-skew Lie triple derivation on $\mathcal{A}$. As applications, we apply the above result to prime $\ast$ -algebras, factor von Neumann algebras and standard operator algebras.

Keywords: ξ -bi-skew Lie triple derivation; $\ast$ -derivation; $\ast$ -algebra

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1. Introduction

Let $\mathcal{A}$ be a associative $\ast$ -algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, denote by $[A, B]_{\ast} = AB - BA^{\ast}$, $A \bullet B = AB + BA^{\ast}$, $[A, B]_{\diamond} = AB^{\ast} - BA^{\ast}$, $[A, B]_{\bullet} = AB^{\ast} + BA^{\ast}$ the skew Lie product, skew Jordan product, bi-skew Lie product, bi-skew Jordan product of A and B , respectively. Recently, many authors have studied various maps corresponding to these products on some operator algebras; see, for example, [1–15] and their references. Recall that a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is called an additive $\ast$ -derivation if it satisfies $\Phi(A + B) = \Phi(A) + \Phi(B)$, $\Phi(AB) = \Phi(A)B + A\Phi(B)$ and $\Phi(A^{\ast}) = \Phi(A)^{\ast}$ for all $A, B \in \mathcal{A}$. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear skew Lie triple derivation if $\Phi([[A, B]_{\ast}, C]_{\ast}) = [[\Phi(A), B]_{\ast}, C]_{\ast} + [[A, \Phi(B)]_{\ast}, C]_{\ast} + [[A, B]_{\ast}, \Phi(C)]_{\ast}$ for all $A, B, C \in \mathcal{A}$. Li et al. [10] proved that every nonlinear skew Lie triple derivation on factor von Neumann algebras is an additive $\ast$ -derivation. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear skew Jordan triple derivation if $\Phi(A \bullet B \bullet C) = \Phi(A) \bullet B \bullet C + A \bullet \Phi(B) \bullet C + A \bullet B \bullet \Phi(C)$ for all $A, B, C \in \mathcal{A}$. Zhao and Li [15] characterized nonlinear skew Jordan triple derivation on von Neumann algebras. In recent years, the theory of derivations has been intensively studied not only in associative $\ast$ -algebras but also in nonassociative algebras; see, for example, [16–20]. For instance, Brešar and Fošner [16] investigated rings with involution equipped with new products. Ferreira and Ferreira [17] studied the additivity of n -multiplicative maps on alternative rings. These works demonstrate that many results obtained in the associative setting can essentially be extended to alternative algebras or other nonassociative frameworks.

Furthermore, it is important to note that derivations are fundamentally linked to the Canonical Anticommutation Relations (CAR) algebra of fermions in mathematical physics. These maps often represent nonlinear transformations and interactions within the algebraic formulation of Quantum Field Theory (QED). Specifically, transformations such as Bogoliubov transformations or nonlinear dynamics in CAR algebras can be analyzed through the



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lens of generalized derivations. In cases where the derivation is self-adjoint, the map regulates the involution behavior within module algebras, providing a mathematical framework for describing dynamical symmetries in quantum mechanics.

Let ζ be a nonzero scalar. More generally, for $A, B \in \mathcal{A}$, denote by $A \diamond_{\zeta} B = AB + \zeta BA^*$ the ζ -skew Jordan product of A and B . A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear ζ -skew Jordan triple derivation if

$$\Phi(A \diamond_{\zeta} B \diamond_{\zeta} C) = \Phi(A) \diamond_{\zeta} B \diamond_{\zeta} C + A \diamond_{\zeta} \Phi(B) \diamond_{\zeta} C + A \diamond_{\zeta} B \diamond_{\zeta} \Phi(C)$$

for all $A, B, C \in \mathcal{A}$. Clearly, when $\zeta = -1$ ($\zeta = 1$), the nonlinear ζ -skew Jordan triple derivation is the nonlinear skew Lie (Jordan) triple derivation. Recently, Zhang [21] investigated nonlinear ζ -skew Jordan triple derivations on prime $*$ -algebras.

Inspired by the above works and based on the bi-skew Lie (Jordan) product, it is natural to introduce the nonlinear bi-skew Lie (Jordan) triple derivations. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear bi-skew Lie triple derivation if $\Phi([A, B]_{\diamond}, C)_{\diamond} = [[\Phi(A), B]_{\diamond}, C]_{\diamond} + [[A, \Phi(B)]_{\diamond}, C]_{\diamond} + [[A, B]_{\diamond}, \Phi(C)]_{\diamond}$ for all $A, B, C \in \mathcal{A}$. Khan [22] proved that every nonlinear bi-skew Lie triple derivation on factor von Neumann algebras is an additive $*$ -derivation. Zhang and Li [23] generalized the above result to the case of $*$ -algebras. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear bi-skew Jordan triple derivation if $\Phi([A, B]_{\bullet}, C)_{\bullet} = [[\Phi(A), B]_{\bullet}, C]_{\bullet} + [[A, \Phi(B)]_{\bullet}, C]_{\bullet} + [[A, B]_{\bullet}, \Phi(C)]_{\bullet}$ for all $A, B, C \in \mathcal{A}$. Ashraf et al. [4] proved that every nonlinear bi-skew Jordan triple derivation on factor von Neumann algebras is an additive $*$ -derivation. Later, Ashraf et al. [24] also generalized the above result to the case of $*$ -algebras. Let ζ be a nonzero scalar. For $A, B \in \mathcal{A}$, denote by $[A, B]_{\diamond}^{\zeta} = AB^* - \zeta BA^*$ the ζ -bi-skew Lie product of A and B . A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear ζ -bi-skew Lie derivation if

$$\Phi([A, B]_{\diamond}^{\zeta}) = [\Phi(A), B]_{\diamond}^{\zeta} + [A, \Phi(B)]_{\diamond}^{\zeta}$$

for all $A, B \in \mathcal{A}$. Very recently, the third author [8] obtained the concrete structure of nonlinear ζ -bi-skew Lie derivations on prime $*$ -algebras. Motivated by above-mentioned works, it is natural to introduce the nonlinear ζ -bi-skew Lie triple derivations. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ (without the additivity assumption) is called a nonlinear ζ -bi-skew Lie triple derivation if

$$\Phi([A, B]_{\diamond}^{\zeta}, C)_{\diamond}^{\zeta} = [[\Phi(A), B]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, \Phi(B)]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, B]_{\diamond}^{\zeta}, \Phi(C)]_{\diamond}^{\zeta}$$

for all $A, B, C \in \mathcal{A}$. In this paper, we will characterize the concrete structure of nonlinear ζ -bi-skew Lie triple derivation on $*$ -algebras.

2. The Main Result

The main result of this article is presented as follows:

Theorem 1. Let $\mathcal{A}$ be a unital associative $*$ -algebra with the unit I and ζ be a nonzero scalar. Assume that $\mathcal{A}$ contains a nontrivial projection P which satisfies

(♠) $XAP = 0$ implies $X = 0$,

(♣) $X\mathcal{A}(I - P) = 0$ implies $X = 0$.

A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies $\Phi([A, B]_{\diamond}^{\zeta}, C)_{\diamond}^{\zeta} = [[\Phi(A), B]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, \Phi(B)]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, B]_{\diamond}^{\zeta}, \Phi(C)]_{\diamond}^{\zeta}$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation and $\Phi(\zeta A) = \zeta \Phi(A)$ for all $A \in \mathcal{A}$.

Let $P_1 = P$, $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ ($i, j = 1, 2$). Then $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$, and for $A \in \mathcal{A}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{A}_{ij}$ ($i, j = 1, 2$). Let $\mathcal{M} = \{A \in \mathcal{A} : A^* = A\}$ and $\mathcal{N} = \{A \in \mathcal{A} : A^* = -A\}$.

Proof. Obviously, we need only prove the necessity. If $\xi = \pm 1$, then by the results of [23,24], Theorem 1 holds. In the following, we assume that $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. The Theorem 1 will be proved by checking several claims. $\square$

Claim 1. $\Phi(0) = 0$.

Proof. It is clear that

$$\Phi(0) = \Phi([0, 0]_{\diamond}^{\xi}, 0]_{\diamond}^{\xi}) = [[\Phi(0), 0]_{\diamond}^{\xi}, 0]_{\diamond}^{\xi} + [[0, \Phi(0)]_{\diamond}^{\xi}, 0]_{\diamond}^{\xi} + [[0, 0]_{\diamond}^{\xi}, \Phi(0)]_{\diamond}^{\xi} = 0.$$

$\square$

Claim 2. For every $A_{ii} \in \mathcal{A}_{ii}$, $B_{ij} \in \mathcal{A}_{ij}$ ($1 \leq i \neq j \leq 2$), we have

$$\Phi(A_{ii} + B_{ij}) = \Phi(A_{ii}) + \Phi(B_{ij}).$$

Proof. Let $T = \Phi(A_{11} + B_{12}) - \Phi(A_{11}) - \Phi(B_{12})$. Our target is to show that $T = 0$.

For any $X_{21} \in \mathcal{A}_{21}$, from $[[X_{21}, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$ and Claim 1, we have

$$\begin{aligned} & [[\Phi(X_{21}), A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{11} + B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[X_{21}, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ & = \Phi([X_{21}, A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = \Phi([X_{21}, A_{11}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([X_{21}, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = [[\Phi(X_{21}), A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{11}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[X_{21}, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi}. \end{aligned}$$

This implies that $[[X_{21}, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, and so

$$X_{21} T^* P_1 - \xi P_1 T X_{21}^* = 0. \quad (1)$$

Multiplying Equation (1) by P_1 from the left, we get $T_{11} = 0$ by ($\clubsuit$).

Since $[[P_2, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(P_2), A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_2, \Phi(A_{11} + B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[P_2, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([P_2, A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([P_2, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([P_2, B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(P_2), A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_2, \Phi(A_{11}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[P_2, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}, \end{aligned}$$

which implies that $[[P_2, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, that is

$$P_2 T^* P_2 - \xi T P_2 - \xi P_2 T P_2 + \xi \bar{\xi} P_2 T^* = 0. \quad (2)$$

Multiplying Equation (2) by P_1 from the left, we get $T_{12} = 0$.

Since $[[P_1, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(P_1), A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, \Phi(A_{11} + B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[P_1, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ & = \Phi([P_1, A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = \Phi([P_1, A_{11}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([P_1, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = [[\Phi(P_1), A_{11} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, \Phi(A_{11}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[P_1, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi}. \end{aligned}$$

It follows that $[[P_1, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, that is

$$P_1 T^* P_1 - \xi T P_1 - \xi P_1 T P_1 + \xi \bar{\xi} P_1 T^* = 0. \quad (3)$$

Multiplying Equation (3) by P_2 from the left, we obtain $T_{21} = 0$.

For any $X_{12} \in \mathcal{A}_{12}$, since $[[X_{12}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(X_{12}), A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{11} + B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{12}, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([X_{12}, A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([X_{12}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([X_{12}, B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(X_{12}), A_{11} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{11}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{12}, A_{11} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}. \end{aligned}$$

This implies that $[[X_{12}, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, and so

$$X_{12} T^* P_2 - \xi P_2 T X_{12}^* = 0. \quad (4)$$

Multiplying Equation (4) by P_2 from the left, we get $T_{22} = 0$ by $(\spadesuit)$. Hence, $T = 0$.

Similarly, we can show that $\Phi(A_{22} + B_{21}) = \Phi(A_{22}) + \Phi(B_{21})$. $\square$

Claim 3. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$$

and

$$\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Proof. Let $T = \Phi(A_{11} + B_{12} + C_{21}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21})$. Since $[[P_1, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = [[P_1, B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(P_1), A_{11} + B_{12} + C_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, \Phi(A_{11} + B_{12} + C_{21})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[P_1, A_{11} + B_{12} + C_{21}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([P_1, A_{11} + B_{12} + C_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([P_1, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([P_1, B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([P_1, C_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(P_1), A_{11} + B_{12} + C_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[P_1, A_{11} + B_{12} + C_{21}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}. \end{aligned}$$

It follows that $[[P_1, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = P_1 T^* P_2 - \xi P_2 T P_1 = 0$, and so $T_{21} = 0$.

From $[[A_{11}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = (\xi \bar{\xi} + \xi^2 \bar{\xi}^2) A_{11} - 2 \xi^2 \bar{\xi} A_{11}^*$, $[[B_{12}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = 0$, $[[C_{21}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = (\xi - \xi^2 \bar{\xi}) C_{21}^*$ and Claim 2, we have

$$\begin{aligned} & [[\Phi(A_{11} + B_{12} + C_{21}), P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(P_1)]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, P_1]_{\diamond}^{\xi}, \Phi(\xi \bar{\xi} P_1 - P_2)]_{\diamond}^{\xi} \\ & = \Phi([A_{11} + B_{12} + C_{21}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & = \Phi((\xi \bar{\xi} + \xi^2 \bar{\xi}^2) A_{11} - 2 \xi^2 \bar{\xi} A_{11}^* + (\xi - \xi^2 \bar{\xi}) C_{21}^*) \\ & = \Phi([A_{11}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) + \Phi([B_{12}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & + \Phi([C_{21}, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}), P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, \Phi(P_1)]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, P_1]_{\diamond}^{\xi}, \Phi(\xi \bar{\xi} P_1 - P_2)]_{\diamond}^{\xi}, \end{aligned}$$

which yields that

$$\begin{aligned} 0 = [[T, P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} & = \xi \bar{\xi} T P_1 - \xi^2 \bar{\xi} P_1 T^* P_1 + \xi P_1 T^* P_2 - \xi^2 \bar{\xi} P_1 T^* \\ & + \xi^2 \bar{\xi}^2 P_1 T P_1 - \xi \bar{\xi} P_2 T P_1. \end{aligned}$$

It follows from $T_{21} = 0$ that $(\xi \bar{\xi} + \xi^2 \bar{\xi}^2) P_1 T P_1 - 2 \xi^2 \bar{\xi} P_1 T^* P_1 = 0$. Then

$$(1 + \xi \bar{\xi}) P_1 T P_1 - 2 \xi P_1 T^* P_1 = 0. \quad (5)$$

From $[[A_{11}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = (-i \xi \bar{\xi} - i \xi^2 \bar{\xi}^2) A_{11} - 2 i \xi^2 \bar{\xi} A_{11}^*$, $[[B_{12}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = 0$, $[[C_{21}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} = (i \xi - i \xi^2 \bar{\xi}) C_{21}^*$ and Claim 2, we have

$$\begin{aligned} & [[\Phi(A_{11} + B_{12} + C_{21}), i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(i P_1)]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, i P_1]_{\diamond}^{\xi}, \Phi(\xi \bar{\xi} P_1 - P_2)]_{\diamond}^{\xi} \\ & = \Phi([A_{11} + B_{12} + C_{21}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & = \Phi((-i \xi \bar{\xi} - i \xi^2 \bar{\xi}^2) A_{11} - 2 i \xi^2 \bar{\xi} A_{11}^* + (i \xi - i \xi^2 \bar{\xi}) C_{21}^*) \\ & = \Phi([A_{11}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) + \Phi([B_{12}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & + \Phi([C_{21}, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}), i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, \Phi(i P_1)]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} \\ & + [[A_{11} + B_{12} + C_{21}, i P_1]_{\diamond}^{\xi}, \Phi(\xi \bar{\xi} P_1 - P_2)]_{\diamond}^{\xi}, \end{aligned}$$

which implies that

$$\begin{aligned} 0 = [[T, i P_1]_{\diamond}^{\xi}, \xi \bar{\xi} P_1 - P_2]_{\diamond}^{\xi} & = -i \xi \bar{\xi} T P_1 - i \xi^2 \bar{\xi} P_1 T^* P_1 + i \xi P_1 T^* P_2 - i \xi^2 \bar{\xi} P_1 T^* \\ & - i \xi^2 \bar{\xi}^2 P_1 T P_1 + i \xi \bar{\xi} P_2 T P_1. \end{aligned}$$

It follows from $T_{21} = 0$ that $(-i \xi \bar{\xi} - i \xi^2 \bar{\xi}^2) P_1 T P_1 - 2 i \xi^2 \bar{\xi} P_1 T^* P_1 = 0$, and so

$$(1 + \xi \bar{\xi}) P_1 T P_1 + 2 \xi P_1 T^* P_1 = 0. \quad (6)$$

By Equations (5) and (6), we can show that $T_{11} = 0$.

Since $[[A_{11}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = [[C_{21}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(A_{11} + B_{12} + C_{21}), P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(P_2)]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & \quad + [[A_{11} + B_{12} + C_{21}, P_2]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([A_{11} + B_{12} + C_{21}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([A_{11}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([B_{12}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([C_{21}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}), P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(P_2)]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & \quad + [[A_{11} + B_{12} + C_{21}, P_2]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}, \end{aligned}$$

which implies that $[[T, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$. It follows that

$$TP_2 - \xi P_2 T^* P_2 - \xi P_2 T^* + \xi \bar{\xi} P_2 T P_2 = 0. \quad (7)$$

Multiplying Equation (7) by P_1 from the left, we get $T_{12} = 0$.

For any $X_{12} \in \mathcal{A}_{12}$, since $[[A_{11}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = [[C_{21}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(A_{11} + B_{12} + C_{21}), X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(X_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & \quad + [[A_{11} + B_{12} + C_{21}, X_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([A_{11} + B_{12} + C_{21}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([A_{11}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([B_{12}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([C_{21}, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [\Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}), X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[A_{11} + B_{12} + C_{21}, \Phi(X_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & \quad + [[A_{11} + B_{12} + C_{21}, X_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}. \end{aligned}$$

It follows that $[[T, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, and so

$$-\xi X_{12} T^* P_2 + \xi \bar{\xi} P_2 T X_{12}^* = 0. \quad (8)$$

Multiplying Equation (8) by P_2 from the left, we get $T_{22} = 0$ by $(\spadesuit)$.

Similarly, we can obtain that $\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$. $\square$

Claim 4. For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Proof. Let

$$T = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22}).$$

For any $X_{21} \in \mathcal{A}_{21}$, since $[[X_{21}, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = [[X_{21}, C_{21}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = [[X_{21}, D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(X_{21}), A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & \quad + [[X_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ & = \Phi([X_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = \Phi([X_{21}, A_{11}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([X_{21}, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([X_{21}, C_{21}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & \quad + \Phi([X_{21}, D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \end{aligned}$$

$$\begin{aligned}
&= [[\Phi(X_{21}), A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[X_{21}, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[X_{21}, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi}.
\end{aligned}$$

It follows that $[[X_{21}, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, and so

$$X_{21}T^*P_1 - \xi P_1TX_{21}^* = 0. \quad (9)$$

Multiplying Equation (9) by P_1 from the left, we get $T_{11} = 0$ by (

Since $[[P_2, A_{11}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = [[P_2, C_{21}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = [[P_2, D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned}
&[[\Phi(P_2), A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_2, \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[P_2, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\
&= \Phi([[P_2, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\
&= \Phi([[P_2, A_{11}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([[P_2, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([[P_2, C_{21}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\
&\quad + \Phi([[P_2, D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\
&= [[\Phi(P_2), A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[P_2, \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[P_2, A_{11} + B_{12} + C_{21} + D_{22}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi},
\end{aligned}$$

which implies that $[[P_2, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, and so

$$P_2T^*P_1 - \xi P_1TP_2 = 0. \quad (10)$$

Multiplying Equation (10) by P_1 from the left, we obtain $T_{12} = 0$. Similarly, we can show that $T_{21} = T_{22} = 0$. $\square$

Claim 5. For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ ($1 \leq i \neq j \leq 2$), we have

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Proof. Let $T = \Phi(A_{12} + B_{12}) - \Phi(A_{12}) - \Phi(B_{12})$. Since $[[X_{21}, A_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$ for any $X_{21} \in \mathcal{A}_{21}$, we have

$$\begin{aligned}
&[[\Phi(X_{21}), A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{12} + B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[X_{21}, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\
&= \Phi([[X_{21}, A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\
&= \Phi([[X_{21}, A_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([[X_{21}, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\
&= [[\Phi(X_{21}), A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{12}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\
&\quad + [[X_{21}, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi},
\end{aligned}$$

which implies that $[[X_{21}, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, that is

$$X_{21}T^*P_1 - \xi P_1TX_{21}^* = 0. \quad (11)$$

Multiplying Equation (11) by P_1 from the left, we obtain $T_{11} = 0$ by (

Since $[[P_1, A_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} & [[\Phi(P_1), A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, \Phi(A_{12} + B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[P_1, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ & = \Phi([P_1, A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = \Phi([P_1, A_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) + \Phi([P_1, B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ & = [[\Phi(P_1), A_{12} + B_{12}]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, \Phi(A_{12}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} \\ & + [[P_1, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi}. \end{aligned}$$

This implies that

$$[[P_1, T]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = P_1 T^* P_1 - \xi T P_1 - \xi P_1 T P_1 + \xi \bar{\xi} P_1 T^* = 0. \quad (12)$$

Multiplying Equation (12) by P_2 from the left, we obtain $T_{21} = 0$.

Since $[[X_{12}, A_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$ for any $X_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} & [[\Phi(X_{12}), A_{12} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{12} + B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{12}, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([X_{12}, A_{12} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi([X_{12}, A_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([X_{12}, B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(X_{12}), A_{12} + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{12}) + \Phi(B_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{12}, A_{12} + B_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}. \end{aligned}$$

Which implies that $[[X_{12}, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, that is

$$X_{12} T^* P_2 - \xi P_2 T X_{12}^* = 0. \quad (13)$$

Multiplying Equation (13) by P_2 from the left, we obtain $T_{22} = 0$ by ().

Let $K_{A_{12}, B_{12}} = T_{12}$. Then $K_{A_{12}, B_{12}} \in \mathcal{A}_{12}$, and so

$$\Phi(A_{12} + B_{12}) = \Phi(A_{12}) + \Phi(B_{12}) + K_{A_{12}, B_{12}}. \quad (14)$$

Similarly, for $A_{21}, B_{21} \in \mathcal{A}_{21}$, there exist $G_{A_{21}, B_{21}} \in \mathcal{A}_{21}$ such that

$$\Phi(A_{21} + B_{21}) = \Phi(A_{21}) + \Phi(B_{21}) + G_{A_{21}, B_{21}}. \quad (15)$$

By Equation (15), there exist $G_{-\xi A_{12}^*, -\xi B_{12}^*} \in \mathcal{A}_{21}$ such that

$$\Phi(-\xi A_{12}^* - \xi B_{12}^*) = \Phi(-\xi A_{12}^*) + \Phi(-\xi B_{12}^*) + G_{-\xi A_{12}^*, -\xi B_{12}^*}. \quad (16)$$

From

$$[[P_1 + A_{12}, P_2 + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = A_{12} + B_{12} - \xi A_{12}^* - \xi B_{12}^*,$$

Claim 4 and Equation (16), we have

$$\begin{aligned} & \Phi(A_{12} + B_{12}) + \Phi(-\xi A_{12}^*) + \Phi(-\xi B_{12}^*) + G_{-\xi A_{12}^*, -\xi B_{12}^*} \\ & = \Phi(A_{12} + B_{12}) + \Phi(-\xi A_{12}^* - \xi B_{12}^*) \\ & = \Phi([P_1 + A_{12}, P_2 + B_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \end{aligned}$$

$$\begin{aligned}
&= [[\Phi(P_1) + \Phi(A_{12}), P_2 + B_{12}^*]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1 + A_{12}, \Phi(P_2) + \Phi(B_{12}^*)]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\
&\quad + [[P_1 + A_{12}, P_2 + B_{12}^*]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\
&= \Phi([P_1, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([P_1, B_{12}^*]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([A_{12}, P_2]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&\quad + \Phi([A_{12}, B_{12}^*]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&= \Phi(B_{12} - \xi B_{12}^*) + \Phi(A_{12} - \xi A_{12}^*) \\
&= \Phi(A_{12}) + \Phi(B_{12}) + \Phi(-\xi A_{12}^*) + \Phi(-\xi B_{12}^*).
\end{aligned}$$

It follows that

$$\Phi(A_{12} + B_{12}) = \Phi(A_{12}) + \Phi(B_{12}) - G_{-\xi A_{12}^*, -\xi B_{12}^*}. \quad (17)$$

Combining Equations (14) and (17), we have $K_{A_{12}, B_{12}} = -G_{-\xi A_{12}^*, -\xi B_{12}^*}$. This together with the fact that $K_{A_{12}, B_{12}} \in \mathcal{A}_{12}$ and $G_{-\xi A_{12}^*, -\xi B_{12}^*} \in \mathcal{A}_{21}$ yields that $K_{A_{12}, B_{12}} = 0$. It follows from Equation (14) that $\Phi(A_{12} + B_{12}) = \Phi(A_{12}) + \Phi(B_{12})$.

Similarly, we can show that $\Phi(A_{21} + B_{21}) = \Phi(A_{21}) + \Phi(B_{21})$. $\square$

Claim 6. For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ ($i = 1, 2$), we have

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Proof. Let

$$T = \Phi(A_{11} + B_{11}) - \Phi(A_{11}) - \Phi(B_{11}).$$

Since $[[P_2, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned}
&[[\Phi(P_2), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_2, \Phi(A_{11} + B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\
&\quad + [[P_2, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\
&= \Phi([P_2, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&= \Phi([P_2, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([P_2, B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&= [[\Phi(P_2), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_2, \Phi(A_{11}) + \Phi(B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\
&\quad + [[P_2, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}.
\end{aligned}$$

It follows that $[[P_2, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, and so

$$P_2 T^* P_2 - \xi T P_2 - \xi P_2 T P_2 + \xi \bar{\xi} P_2 T^* = 0. \quad (18)$$

Multiplying Equation (18) by P_1 from the left, we get $T_{12} = 0$.

Since $[[X_{12}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned}
&[[\Phi(X_{12}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{11} + B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\
&\quad + [[X_{12}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\
&= \Phi([X_{12}, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&= \Phi([X_{12}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([X_{12}, B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\
&= [[\Phi(X_{12}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{12}, \Phi(A_{11}) + \Phi(B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\
&\quad + [[X_{12}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}.
\end{aligned}$$

Which implies that $[[X_{12}, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, and then

$$X_{12}T^*P_2 - \xi P_2TX_{12}^* = 0. \quad (19)$$

Multiplying Equation (19) by P_2 from the left, we get $T_{22} = 0$ by ($\spadesuit$).
Since

$$[[X_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = -\xi A_{11}X_{21}^* - \xi B_{11}X_{21}^* + \xi\bar{\xi}X_{21}A_{11}^* + \xi\bar{\xi}X_{21}B_{11}^*,$$

we have from Claims 4 and 5 that

$$\begin{aligned} & [[\Phi(X_{21}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{11} + B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([X_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi(-\xi A_{11}X_{21}^* + \xi\bar{\xi}X_{21}A_{11}^*) + \Phi(-\xi B_{11}X_{21}^* + \xi\bar{\xi}X_{21}B_{11}^*) \\ & = \Phi([X_{21}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([X_{21}, B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(X_{21}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[X_{21}, \Phi(A_{11}) + \Phi(B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[X_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}, \end{aligned}$$

which implies that

$$[[X_{21}, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = X_{21}T^*P_2 - \xi TX_{21}^* - \xi P_2TX_{21}^* + \xi\bar{\xi}X_{21}T^* = 0. \quad (20)$$

Multiplying Equation (20) by P_1 from the left, we obtain $T_{11} = 0$ by ($\clubsuit$).
Since

$$[[iX_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = i\xi A_{11}X_{21}^* + i\xi B_{11}X_{21}^* + i\xi\bar{\xi}X_{21}A_{11}^* + i\xi\bar{\xi}X_{21}B_{11}^*,$$

we have from Claims 4 and 5 that

$$\begin{aligned} & [[\Phi(iX_{21}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[iX_{21}, \Phi(A_{11} + B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[iX_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ & = \Phi([iX_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = \Phi(i\xi A_{11}X_{21}^* + i\xi\bar{\xi}X_{21}A_{11}^*) + \Phi(i\xi B_{11}X_{21}^* + i\xi\bar{\xi}X_{21}B_{11}^*) \\ & = \Phi([iX_{21}, A_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) + \Phi([iX_{21}, B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ & = [[\Phi(iX_{21}), A_{11} + B_{11}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[iX_{21}, \Phi(A_{11}) + \Phi(B_{11})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} \\ & + [[iX_{21}, A_{11} + B_{11}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi}. \end{aligned}$$

Then $[[iX_{21}, T]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$, that is

$$X_{21}T^*P_2 + \xi TX_{21}^* + \xi P_2TX_{21}^* + \xi\bar{\xi}X_{21}T^* = 0. \quad (21)$$

Using Equations (20) and (21), we can obtain that $T_{21} = 0$.

Similarly, we can show that $\Phi(A_{22} + B_{22}) = \Phi(A_{22}) + \Phi(B_{22})$. $\square$

Claim 7. Φ is additive on $\mathcal{A}$.

Proof. We have $A = \sum_{i,j=1}^2 A_{ij}$ and $B = \sum_{i,j=1}^2 B_{ij}$ for every $A, B \in \mathcal{A}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. From Claims 4–6, we obtain

$$\Phi(A + B) = \sum_{i,j=1}^2 \Phi(A_{ij} + B_{ij}) = \sum_{i,j=1}^2 \Phi(A_{ij}) + \sum_{i,j=1}^2 \Phi(B_{ij}) = \Phi(A) + \Phi(B).$$

Therefore, Φ is additive. $\square$

Claim 8. $\Phi(I) = 0$.

Proof. Since $[[P_1, P_2]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} 0 &= \Phi([P_1, P_2]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ &= [[\Phi(P_1), P_2]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, \Phi(P_2)]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_1, P_2]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ &= -\xi P_2 \Phi(P_1)^* P_1 + \xi \bar{\xi} P_1 \Phi(P_1) P_2 + P_1 \Phi(P_2)^* P_1 - \xi \Phi(P_2) P_1 - \xi P_1 \Phi(P_2) P_1 \\ &\quad + \xi \bar{\xi} P_1 \Phi(P_2)^*. \end{aligned} \quad (22)$$

Multiplying Equation (22) by P_1 from both sides, we get

$$(1 + \xi \bar{\xi}) P_1 \Phi(P_2)^* P_1 - 2\xi P_1 \Phi(P_2) P_1 = 0. \quad (23)$$

Since $[[P_2, P_1]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} = 0$, we have

$$\begin{aligned} 0 &= \Phi([P_2, P_1]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi}) \\ &= [[\Phi(P_2), P_1]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_2, \Phi(P_1)]_{\diamond}^{\xi}, P_1]_{\diamond}^{\xi} + [[P_2, P_1]_{\diamond}^{\xi}, \Phi(P_1)]_{\diamond}^{\xi} \\ &= \Phi(P_2) P_1 - \xi P_1 \Phi(P_2)^* P_1 - \xi P_1 \Phi(P_2)^* + \xi \bar{\xi} P_1 \Phi(P_2) P_1 + P_2 \Phi(P_1)^* P_1 \\ &\quad - \xi P_1 \Phi(P_1) P_2. \end{aligned} \quad (24)$$

Multiplying Equation (24) by P_1 from both sides, we have

$$(1 + \xi \bar{\xi}) P_1 \Phi(P_2) P_1 - 2\xi P_1 \Phi(P_2)^* P_1 = 0. \quad (25)$$

Combining Equations (23) and (25), we get $((1 + \xi \bar{\xi})^2 - 4\xi^2) P_1 \Phi(P_2) P_1 = 0$. Since $\xi \neq \pm 1$, we have $(1 + \xi \bar{\xi})^2 - 4\xi^2 \neq 0$, and so $P_1 \Phi(P_2) P_1 = 0$. Similarly, we can show that $P_2 \Phi(P_1) P_2 = 0$.

Since $[[P_1, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = 0$ for any $X_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} 0 &= \Phi([P_1, X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ &= [[\Phi(P_1), X_{12}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, \Phi(X_{12})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, X_{12}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ &= -\xi X_{12} \Phi(P_1)^* P_2 + \xi \bar{\xi} P_2 \Phi(P_1) X_{12}^* + P_1 \Phi(X_{12})^* P_2 - \xi P_2 \Phi(X_{12}) P_1. \end{aligned} \quad (26)$$

Multiplying Equation (26) by P_2 from the left and by P_1 from the right, we get $\xi \bar{\xi} P_2 \Phi(P_1) X_{12}^* - \xi P_2 \Phi(X_{12}) P_1 = 0$. This together with the fact that $P_2 \Phi(P_1) P_2 = 0$ yields that $P_2 \Phi(X_{12}) P_1 = 0$.

It follows from $[[P_1, X_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} = X_{21}^* - \xi X_{21}$ and Claim 7 that

$$\begin{aligned} &\Phi(X_{21}^*) + \Phi(-\xi X_{21}) \\ &= \Phi([P_1, X_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi}) \\ &= [[\Phi(P_1), X_{21}]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, \Phi(X_{21})]_{\diamond}^{\xi}, P_2]_{\diamond}^{\xi} + [[P_1, X_{21}]_{\diamond}^{\xi}, \Phi(P_2)]_{\diamond}^{\xi} \\ &= \Phi(P_1) X_{21}^* - \xi X_{21} \Phi(P_1)^* P_2 - \xi X_{21} \Phi(P_1)^* + \xi \bar{\xi} P_2 \Phi(P_1) X_{21}^* + P_1 \Phi(X_{21})^* P_2 \end{aligned}$$

$$-\zeta P_2 \Phi(X_{21})P_1 + X_{21}^* \Phi(P_2)^* - \zeta X_{21} \Phi(P_2)^* - \zeta \Phi(P_2)X_{21} + \zeta \bar{\zeta} \Phi(P_2)X_{21}^*. \quad (27)$$

Multiplying Equation (27) by P_2 from the left and by P_1 from the right, and using the fact that $P_1 \Phi(P_2)P_1 = 0$ and $P_2 \Phi(X_{12})P_1 = 0$, we get

$$P_2 \Phi(-\zeta X_{21})P_1 = -\zeta X_{21} \Phi(P_1)^* P_1 - \zeta P_2 \Phi(X_{21})P_1 - \zeta P_2 \Phi(P_2)X_{21}. \quad (28)$$

From $[[P_2, X_{12}]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} = X_{12}^* - \zeta X_{12}$ and Claim 7, we have

$$\begin{aligned} & \Phi(X_{12}^*) + \Phi(-\zeta X_{12}) \\ &= \Phi([[P_2, X_{12}]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta}) \\ &= [[\Phi(P_2), X_{12}]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} + [[P_2, \Phi(X_{12})]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} + [[P_2, X_{12}]_{\diamond}^{\zeta}, \Phi(P_1)]_{\diamond}^{\zeta} \\ &= \Phi(P_2)X_{12}^* - \zeta X_{12} \Phi(P_2)^* P_1 - \zeta X_{12} \Phi(P_2)^* + \zeta \bar{\zeta} P_1 \Phi(P_2)X_{12}^* + P_2 \Phi(X_{12})^* P_1 \\ & \quad - \zeta P_1 \Phi(X_{12})P_2 + X_{12}^* \Phi(P_1)^* - \zeta X_{12} \Phi(P_1)^* - \zeta \Phi(P_1)X_{12} + \zeta \bar{\zeta} \Phi(P_1)X_{12}^*. \end{aligned} \quad (29)$$

Multiplying Equation (29) by P_2 from the left and by P_1 from the right, and using the fact that $P_2 \Phi(P_1)P_2 = 0$ and $P_2 \Phi(X_{12})P_1 = 0$, we have

$$P_2 \Phi(X_{12}^*)P_1 = P_2 \Phi(P_2)X_{12}^* + P_2 \Phi(X_{12})^* P_1 + X_{12}^* \Phi(P_1)^* P_1. \quad (30)$$

It follows from $[[X_{12}, P_2]_{\diamond}^{\zeta}, P_2]_{\diamond}^{\zeta} = X_{12} - \zeta X_{12}^*$ and Claim 7 that

$$\begin{aligned} & \Phi(X_{12}) + \Phi(-\zeta X_{12}^*) \\ &= \Phi([[X_{12}, P_2]_{\diamond}^{\zeta}, P_2]_{\diamond}^{\zeta}) \\ &= [[\Phi(X_{12}), P_2]_{\diamond}^{\zeta}, P_2]_{\diamond}^{\zeta} + [[X_{12}, \Phi(P_2)]_{\diamond}^{\zeta}, P_2]_{\diamond}^{\zeta} + [[X_{12}, P_2]_{\diamond}^{\zeta}, \Phi(P_2)]_{\diamond}^{\zeta} \\ &= \Phi(X_{12})P_2 - \zeta P_2 \Phi(X_{12})^* P_2 - \zeta P_2 \Phi(X_{12})^* + \zeta \bar{\zeta} P_2 \Phi(X_{12})P_2 + X_{12} \Phi(P_2)^* P_2 \\ & \quad - \zeta P_2 \Phi(P_2)X_{12}^* + X_{12} \Phi(P_2)^* - \zeta X_{12}^* \Phi(P_2)^* - \zeta \Phi(P_2)X_{12}^* + \zeta \bar{\zeta} \Phi(P_2)X_{12}. \end{aligned} \quad (31)$$

Multiplying Equation (31) by P_2 from the left and by P_1 from the right, and using the fact that $P_2 \Phi(X_{12})P_1 = 0$ and $P_1 \Phi(P_2)P_1 = 0$, we have

$$P_2 \Phi(-\zeta X_{12}^*)P_1 = -\zeta P_2 \Phi(X_{12})^* P_1 - 2\zeta P_2 \Phi(P_2)X_{12}^*.$$

This together with Equation (28) gives

$$\begin{aligned} & -\zeta X_{12}^* \Phi(P_1)^* P_1 - \zeta P_2 \Phi(X_{12}^*)P_1 - \zeta P_2 \Phi(P_2)X_{12}^* \\ &= P_2 \Phi(-\zeta X_{12}^*)P_1 \\ &= -\zeta P_2 \Phi(X_{12})^* P_1 - 2\zeta P_2 \Phi(P_2)X_{12}^*. \end{aligned}$$

It follows that

$$P_2 \Phi(X_{12}^*)P_1 = P_2 \Phi(P_2)X_{12}^* + P_2 \Phi(X_{12})^* P_1 - X_{12}^* \Phi(P_1)^* P_1. \quad (32)$$

We can see that $X_{12}^* \Phi(P_1)^* P_1 = 0$ by comparing Equations (30) and (32). Then by ($\clubsuit$), we get $P_1 \Phi(P_1)P_1 = 0$. Similarly, we can prove that $P_2 \Phi(P_2)P_2 = 0$.

Since $[[P_1, P_1]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} = (1 - 2\zeta + \zeta \bar{\zeta})P_1$ and the fact that $P_1 \Phi(P_1)P_1 = 0$, we have

$$\begin{aligned} & \Phi((1 - 2\zeta + \zeta \bar{\zeta})P_1) \\ &= \Phi([[P_1, P_1]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta}) \\ &= [[\Phi(P_1), P_1]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} + [[P_1, \Phi(P_1)]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} + [[P_1, P_1]_{\diamond}^{\zeta}, \Phi(P_1)]_{\diamond}^{\zeta} \end{aligned}$$

$$= (1 - 2\zeta + \zeta\bar{\zeta})\Phi(P_1)P_1 + (1 - 2\zeta + \zeta\bar{\zeta})P_1\Phi(P_1)^*. \quad (33)$$

It follows from Equation (33) and $[(1 - 2\zeta + \zeta\bar{\zeta})P_1, P_2]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} = 0$ that

$$\begin{aligned} 0 &= \Phi([(1 - 2\zeta + \zeta\bar{\zeta})P_1, P_2]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta}) \\ &= [\Phi((1 - 2\zeta + \zeta\bar{\zeta})P_1), P_2]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} + [(1 - 2\zeta + \zeta\bar{\zeta})P_1, \Phi(P_2)]_{\diamond}^{\zeta}, P_1]_{\diamond}^{\zeta} \\ &= (-\zeta + 2\zeta\bar{\zeta} - \zeta^2\bar{\zeta})P_2\Phi(P_1)P_1 + (\zeta\bar{\zeta} - 2\zeta^2\bar{\zeta} + \zeta^2\bar{\zeta}^2)P_1\Phi(P_1)^*P_2 \\ &\quad + (-\zeta + 2\zeta\bar{\zeta} - \zeta^2\bar{\zeta})\Phi(P_2)P_1 + (\zeta\bar{\zeta} - 2\zeta^2\bar{\zeta} + \zeta^2\bar{\zeta}^2)P_1\Phi(P_1)^*P_2. \end{aligned} \quad (34)$$

Multiplying Equation (34) by P_2 from the left and by P_1 from the right, we have

$$(-\zeta + 2\zeta\bar{\zeta} - \zeta^2\bar{\zeta})P_2\Phi(P_1)P_1 + (-\zeta + 2\zeta\bar{\zeta} - \zeta^2\bar{\zeta})P_2\Phi(P_2)P_1 = 0.$$

Since $\zeta \neq \pm 1$, we have $-\zeta + 2\zeta\bar{\zeta} - \zeta^2\bar{\zeta} \neq 0$, and so

$$P_2\Phi(P_1)P_1 + P_2\Phi(P_2)P_1 = 0. \quad (35)$$

Similarly, we can obtain that

$$P_1\Phi(P_2)P_2 + P_1\Phi(P_1)P_2 = 0. \quad (36)$$

Combining Equations (35) and (36) and the fact that $P_2\Phi(P_1)P_2 = P_1\Phi(P_2)P_1 = P_1\Phi(P_1)P_1 = P_2\Phi(P_2)P_2 = 0$, we have

$$\begin{aligned} \Phi(I) &= \Phi(P_1) + \Phi(P_2) \\ &= P_1\Phi(P_1)P_2 + P_2\Phi(P_1)P_1 + P_1\Phi(P_2)P_2 + P_2\Phi(P_2)P_1 \\ &= 0. \end{aligned}$$

□

Claim 9. $\Phi(M)^* = \Phi(M)$ for all $M \in \mathcal{M}$.

Proof. Let $M \in \mathcal{M}$. From $[[I, M]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} = (1 - 2\zeta + \zeta\bar{\zeta})M$ and Claim 8, we have

$$\begin{aligned} \Phi((1 - 2\zeta + \zeta\bar{\zeta})M) &= \Phi([I, M]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta}) \\ &= [[\Phi(I), M]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} + [[I, \Phi(M)]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} + [[I, M]_{\diamond}^{\zeta}, \Phi(I)]_{\diamond}^{\zeta} \\ &= (1 + \zeta\bar{\zeta})\Phi(M)^* - 2\zeta\Phi(M). \end{aligned} \quad (37)$$

From $[[M, I]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} = (1 - 2\zeta + \zeta\bar{\zeta})M$ and Claim 8, we have

$$\begin{aligned} \Phi((1 - 2\zeta + \zeta\bar{\zeta})M) &= \Phi([M, I]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta}) \\ &= [[\Phi(M), I]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} + [[M, \Phi(I)]_{\diamond}^{\zeta}, I]_{\diamond}^{\zeta} + [[M, I]_{\diamond}^{\zeta}, \Phi(I)]_{\diamond}^{\zeta} \\ &= (1 + \zeta\bar{\zeta})\Phi(M) - 2\zeta\Phi(M)^*. \end{aligned} \quad (38)$$

Comparing Equations (37) and (38), we can see that $(1 + 2\zeta + \zeta\bar{\zeta})\Phi(M)^* = (1 + 2\zeta + \zeta\bar{\zeta})\Phi(M)$. Since $\zeta \neq \pm 1$, we have $1 + 2\zeta + \zeta\bar{\zeta} \neq 0$. Hence $\Phi(M)^* = \Phi(M)$. □

Claim 10. $\Phi(N)^* = -\Phi(N)$ for all $N \in \mathcal{N}$.

Proof. Let $N \in \mathcal{N}$. Then $[[N, I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = (1 + 2\xi + \xi\bar{\xi})N$. By Claim 8, we have

$$\begin{aligned} & \Phi((1 + 2\xi + \xi\bar{\xi})N) \\ &= \Phi([N, I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(N), I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[N, \Phi(I)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[N, I]_{\diamond}^{\xi}, \Phi(I)]_{\diamond}^{\xi} \\ &= (1 + \xi\bar{\xi})\Phi(N) - 2\xi\Phi(N)^*. \end{aligned} \quad (39)$$

It follows from $[[I, N]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = -(1 + 2\xi + \xi\bar{\xi})N$ and Claim 8 that

$$\begin{aligned} & -\Phi((1 + 2\xi + \xi\bar{\xi})N) \\ &= \Phi([I, N]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(I), N]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[I, \Phi(N)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[I, N]_{\diamond}^{\xi}, \Phi(I)]_{\diamond}^{\xi} \\ &= (1 + \xi\bar{\xi})\Phi(N)^* - 2\xi\Phi(N). \end{aligned} \quad (40)$$

Comparing Equations (39) and (40), we obtain that $(1 - 2\xi + \xi\bar{\xi})(\Phi(N) + \Phi(N)^*) = 0$. Since $\xi \neq \pm 1$, we have $1 - 2\xi + \xi\bar{\xi} \neq 0$. Hence $\Phi(N)^* = -\Phi(N)$. $\square$

Claim 11. $\Phi(iI) = 0$.

Proof. Since $iI \in \mathcal{N}$, we have from Claims 8 and 10 that

$$\begin{aligned} & \Phi((1 - 2\xi + \xi\bar{\xi})I) = \Phi([iI, iI]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(iI), iI]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[iI, \Phi(iI)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[iI, iI]_{\diamond}^{\xi}, \Phi(iI)]_{\diamond}^{\xi} \\ &= -2i(1 - 2\xi + \xi\bar{\xi})\Phi(iI). \end{aligned} \quad (41)$$

Since $[[I, I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = (1 - 2\xi + \xi\bar{\xi})I$, we have from Claim 8 that

$$\begin{aligned} & \Phi((1 - 2\xi + \xi\bar{\xi})I) = \Phi([I, I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(I), I]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[I, \Phi(I)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[I, I]_{\diamond}^{\xi}, \Phi(I)]_{\diamond}^{\xi} \\ &= 0. \end{aligned} \quad (42)$$

Comparing Equations (41) and (42) yields $\Phi(iI) = 0$. $\square$

Claim 12. $\Phi(iA) = i\Phi(A)$ for all $A \in \mathcal{A}$.

Proof. Since $[[iI, iM]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = (1 - 2\xi + \xi\bar{\xi})M$ for any $M \in \mathcal{M}$, we have from Claims 8, 10 and 11 that

$$\Phi((1 - 2\xi + \xi\bar{\xi})M) = \Phi([iI, iM]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) = [[iI, \Phi(iM)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = -i(1 - 2\xi + \xi\bar{\xi})\Phi(iM). \quad (43)$$

It follows from Equation (37) and Claim 9 that

$$\Phi((1 - 2\xi + \xi\bar{\xi})M) = (1 - 2\xi + \xi\bar{\xi})\Phi(M). \quad (44)$$

Comparing Equations (43) and (44), we obtain that

$$\Phi(iM) = i\Phi(M) \quad (45)$$

for all $M \in \mathcal{M}$.

Let $A \in \mathcal{A}$. Then $A = Q + iR$, where $Q, R \in \mathcal{M}$. It follows from Equation (45) that

$$\Phi(iA) = \Phi(iQ) - \Phi(R) = i\Phi(Q) + i\Phi(iR) = i\Phi(Q + iR) = i\Phi(A).$$

□

Claim 13. $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$.

Proof. Let $A \in \mathcal{A}$. Then $A = Q + iR$, where $Q, R \in \mathcal{M}$. It follows from Claims 9 and 12 that

$$\Phi(A^*) = \Phi(Q) - i\Phi(R) = (\Phi(Q) + i\Phi(R))^* = \Phi(Q + iR)^* = \Phi(A)^*.$$

□

Claim 14. $\Phi(\xi A) = \xi\Phi(A)$ for all $A \in \mathcal{A}$.

Proof. For any $A \in \mathcal{A}$, since $[[I, A]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = (1 + \xi\bar{\xi})A^* - 2\xi A$, we have from Claims 7 and 8 that

$$\begin{aligned}\Phi(A^*) + \Phi(\xi\bar{\xi}A^*) + \Phi(-2\xi A) &= \Phi([I, A]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[I, \Phi(A)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} \\ &= \Phi(A)^* + \xi\bar{\xi}\Phi(A)^* - 2\xi\Phi(A).\end{aligned}$$

By Claim 13, we have

$$\Phi(\xi\bar{\xi}A^*) - 2\Phi(\xi A) = \xi\bar{\xi}\Phi(A)^* - 2\xi\Phi(A). \quad (46)$$

Replacing A by iA in Equation (46) yields that

$$\Phi(\xi\bar{\xi}(iA)^*) - 2\Phi(\xi iA) = \xi\bar{\xi}\Phi(iA)^* - 2\xi\Phi(iA). \quad (47)$$

From Claim 12 and Equation (47), we get

$$\Phi(\xi\bar{\xi}A^*) + 2\Phi(\xi A) = \xi\bar{\xi}\Phi(A)^* + 2\xi\Phi(A). \quad (48)$$

By Equations (46) and (48), we can show that $\Phi(\xi A) = \xi\Phi(A)$. □

Claim 15. Φ is an additive $*$ -derivation on $\mathcal{A}$.

Proof. Let $Q, R \in \mathcal{M}$. It follows from $[[Q, R]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = (1 + \xi\bar{\xi})QR - 2\xi RQ$, Claims 7–9 and 14 that

$$\begin{aligned}\Phi((1 + \xi\bar{\xi})QR) - 2\xi\Phi(RQ) &= \Phi([Q, R]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(Q), R]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[Q, \Phi(R)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[Q, R]_{\diamond}^{\xi}, \Phi(I)]_{\diamond}^{\xi} \\ &= (1 + \xi\bar{\xi})\Phi(Q)R - 2\xi R\Phi(Q) + (1 + \xi\bar{\xi})Q\Phi(R) - 2\xi\Phi(R)Q.\end{aligned} \quad (49)$$

From $[[Q, iR]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} = -i(1 + \xi\bar{\xi})QR - 2i\xi RQ$, Claims 7, 8, 10, 12 and 14, we have

$$\begin{aligned}-i\Phi((1 + \xi\bar{\xi})QR) - 2i\xi\Phi(RQ) &= \Phi([Q, iR]_{\diamond}^{\xi}, I]_{\diamond}^{\xi}) \\ &= [[\Phi(Q), iR]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[Q, \Phi(iR)]_{\diamond}^{\xi}, I]_{\diamond}^{\xi} + [[Q, iR]_{\diamond}^{\xi}, \Phi(I)]_{\diamond}^{\xi} \\ &= -i(1 + \xi\bar{\xi})\Phi(Q)R - 2i\xi R\Phi(Q) - i(1 + \xi\bar{\xi})Q\Phi(R) - 2i\xi\Phi(R)Q.\end{aligned}$$

It follows that

$$\Phi((1 + \xi\bar{\xi})QR) + 2\xi\Phi(RQ) = (1 + \xi\bar{\xi})\Phi(Q)R + 2\xi R\Phi(Q) + (1 + \xi\bar{\xi})Q\Phi(R) + 2\xi\Phi(R)Q. \quad (50)$$

Combining Equations (49) and (50), we have

$$\Phi(RQ) = \Phi(R)Q + R\Phi(Q) \quad (51)$$

for all $Q, R \in \mathcal{M}$.

For any $A, B \in \mathcal{A}$, we have $A = A_1 + iA_2, B = B_1 + iB_2$, where $A_i, B_i \in \mathcal{M}$ ($i = 1, 2$). By Equation (51), Claims 7 and 12, we have

$$\begin{aligned} \Phi(AB) &= \Phi(A_1B_1 + iA_1B_2 + iA_2B_1 - A_2B_2) \\ &= \Phi(A_1B_1) + i\Phi(A_1B_2) + i\Phi(A_2B_1) - \Phi(A_2B_2) \\ &= \Phi(A_1)B_1 + A_1\Phi(B_1) + i\Phi(A_1)B_2 + iA_1\Phi(B_2) + i\Phi(A_2)B_1 + iA_2\Phi(B_1) \\ &\quad - \Phi(A_2)B_2 - A_2\Phi(B_2) \\ &= (\Phi(A_1) + i\Phi(A_2))(B_1 + iB_2) + (A_1 + iA_2)(\Phi(B_1) + i\Phi(B_2)) \\ &= \Phi(A)B + A\Phi(B). \end{aligned}$$

By Claims 7 and 13, we get Φ is an additive $*$ -derivation on $\mathcal{A}$. $\square$

As applications of Theorem 1, we have the following corollaries.

Corollary 1. Let $\mathcal{A}$ be a unital prime $*$ -algebra containing a nontrivial projection. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies $\Phi([A, B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi}) = [[\Phi(A), B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, \Phi(B)]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, B]_{\diamond}^{\xi}, \Phi(C)]_{\diamond}^{\xi}$ for any $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation and $\Phi(\xi A) = \xi\Phi(A)$ for all $A \in \mathcal{A}$.

Corollary 2. Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with $\dim(\mathcal{A}) > 1$. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies $\Phi([A, B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi}) = [[\Phi(A), B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, \Phi(B)]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, B]_{\diamond}^{\xi}, \Phi(C)]_{\diamond}^{\xi}$ for any $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation and $\Phi(\xi A) = \xi\Phi(A)$ for all $A \in \mathcal{A}$.

Corollary 3. Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathcal{H}$, which is closed under the adjoint operation and contain a nontrivial projection. A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies $\Phi([A, B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi}) = [[\Phi(A), B]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, \Phi(B)]_{\diamond}^{\xi}, C]_{\diamond}^{\xi} + [[A, B]_{\diamond}^{\xi}, \Phi(C)]_{\diamond}^{\xi}$ for any $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$ -derivation and $\Phi(\xi A) = \xi\Phi(A)$ for all $A \in \mathcal{A}$.

3. Open Questions

Building upon the foundational framework of nonlinear ξ -bi-skew Lie derivations and nonlinear ξ -bi-skew Lie triple derivations, we proceed to develop them in a natural way. Fix a positive integer $n \geq 2$, and consider the following sequence of polynomials:

$$\begin{aligned} p_1(X_1) &= X_1, \\ p_2(X_1, X_2) &= [p_1(X_1), X_2]_{\diamond}^{\xi} = [X_1, X_2]_{\diamond}^{\xi}, \\ p_3(X_1, X_2, X_3) &= [p_2(X_1, X_2), X_3]_{\diamond}^{\xi} = [[X_1, X_2]_{\diamond}^{\xi}, X_3]_{\diamond}^{\xi}, \\ p_4(X_1, X_2, X_3, X_4) &= [p_3(X_1, X_2, X_3), X_4]_{\diamond}^{\xi} = [[[X_1, X_2]_{\diamond}^{\xi}, X_3]_{\diamond}^{\xi}, X_4]_{\diamond}^{\xi}, \\ &\vdots \\ p_n(X_1, X_2, \dots, X_n) &= [p_{n-1}(X_1, X_2, \dots, X_{n-1}), X_n]_{\diamond}^{\xi} \\ &= [\dots [[X_1, X_2]_{\diamond}^{\xi}, X_3]_{\diamond}^{\xi}, \dots, X_{n-1}]_{\diamond}^{\xi}, X_n]_{\diamond}^{\xi}. \end{aligned}$$

In this context, a nonlinear ζ -bi-skew Lie n -derivation is defined as a map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ satisfying the condition

$$\Phi(p_n(X_1, X_2, \dots, X_n)) = \sum_{k=1}^n p_n(X_1, \dots, X_{k-1}, \Phi(X_k), X_{k+1}, \dots, X_n)$$

for all $X_1, X_2, \dots, X_n \in \mathcal{A}$. Clearly, every nonlinear ζ -bi-skew Lie derivation is a nonlinear ζ -bi-skew Lie 2-derivation and every nonlinear ζ -bi-skew Lie triple derivation is a nonlinear ζ -bi-skew Lie 3-derivation. It is easily checked that every nonlinear ζ -bi-skew Lie derivation on $\ast$ -algebra is a nonlinear ζ -bi-skew Lie triple derivation. But we do not know whether the converse statement is correct. Nonlinear ζ -bi-skew Lie 2-derivations, nonlinear ζ -bi-skew Lie 3-derivations and nonlinear ζ -bi-skew Lie n -derivations are collectively referred to as nonlinear ζ -bi-skew Lie-type derivations.

This observation naturally gives rise to the following open problem, which is in fact more interesting.

Question 1. Let $\mathcal{A}$ be a unital associative $\ast$ -algebra with the unit I , containing a nontrivial projection P which satisfying $XAP = 0$ implies $X = 0$, $X\mathcal{A}(I - P) = 0$ implies $X = 0$ and ζ be a nonzero scalar. If a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ be a nonlinear ζ -bi-skew Lie n -derivation, then what is the structure of Φ ?

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{D} = \{\Phi_m\}_{m \in \mathbb{N}}$ be a family of nonlinear mappings $\Phi_m : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Phi_0 = \text{id}_{\mathcal{A}}$, the identity mapping on $\mathcal{A}$. Then $\mathfrak{D}$ is called a nonlinear ζ -bi-skew Lie n -higher derivation if $\mathfrak{D}$ satisfies the condition

$$\Phi_m(p_n(X_1, X_2, \dots, X_n)) = \sum_{i_1+i_2+\dots+i_n=m} p_n(\Phi_{i_1}(X_1), \Phi_{i_2}(X_2), \dots, \Phi_{i_n}(X_n))$$

for all $X_1, X_2, \dots, X_n \in \mathcal{A}$. In the cases of $n = 2$ and $n = 3$, Φ_m is respectively called a nonlinear ζ -bi-skew Lie higher derivation and a nonlinear ζ -bi-skew Lie triple higher derivation. Inspired by the results of prior work, we give the subsequent open problem:

Question 2. Let $\mathcal{A}$ be a unital associative $\ast$ -algebra with the unit I , containing a nontrivial projection P which satisfying $XAP = 0$ implies $X = 0$, $X\mathcal{A}(I - P) = 0$ implies $X = 0$ and ζ be a nonzero scalar. If a map $\Phi_m : \mathcal{A} \rightarrow \mathcal{A}$ be a nonlinear ζ -bi-skew Lie n -higher derivation, then what is the structure of Φ_m ?

Question 3. (Conjecture) Let $\mathcal{A}$ be an alternative unital $\ast$ -algebra with the unit I , and ζ be a nonzero scalar. Suppose $\mathcal{A}$ contains a nontrivial projection P satisfying:

- (♠) $XAP = 0$ implies $X = 0$,
- (♣) $X\mathcal{A}(I - P) = 0$ implies $X = 0$.

A map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies

$$\Phi([A, B]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta}) = [[\Phi(A), B]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, \Phi(B)]_{\diamond}^{\zeta}, C]_{\diamond}^{\zeta} + [[A, B]_{\diamond}^{\zeta}, \Phi(C)]_{\diamond}^{\zeta}$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $\ast$ -derivation and satisfies $\Phi(\zeta A) = \zeta \Phi(A), \forall A \in \mathcal{A}$.

Question 4. Let $\mathcal{A}$ be an alternative unital $\ast$ -algebra with the unit I , containing a nontrivial projection P which satisfying $XAP = 0$ implies $X = 0$, $X\mathcal{A}(I - P) = 0$ implies $X = 0$ and ζ be a nonzero scalar. If a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ be a nonlinear ζ -bi-skew Lie-type derivation, then what is the structure of Φ ?

Question 5. Let $\mathcal{A}$ be an alternative unital $\ast$ -algebra with the unit I , containing a nontrivial projection P which satisfying $XAP = 0$ implies $X = 0$, $X\mathcal{A}(I - P) = 0$ implies $X = 0$ and ζ be a nonzero scalar. If a map $\Phi_m : \mathcal{A} \rightarrow \mathcal{A}$ be a nonlinear ζ -bi-skew Lie n -higher derivation, then what is the structure of Φ_m ?

Question 6. Can the characterization of structure of nonlinear ξ -bi-skew Lie triple derivations be further expanded with concrete applications of Quantum Electrodynamics (QED) in quantum field theory?

4. Conclusions

This study characterizes the structure of nonlinear ξ -bi-skew Lie triple derivations on unital associative $*$ -algebras, proving they are equivalent to additive $*$ -derivations satisfying $\Phi(\xi A) = \xi \Phi(A)$. The above result is applied to prime $*$ -algebras, factor von Neumann algebras, and standard operator algebras. Future research could extend these findings to alternative $*$ -algebras and further explore the applications in Quantum Electrodynamics (QED).

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