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Novel Operational Properties and Inductive Relationships of TL-Fuzzy Filters and TL-Fuzzy Congruences in Residuated Lattices

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Abstract

Filters and congruences are fundamental concepts in residuated lattices for characterizing their structure. In this paper, using a complete lattice L as the truth-value set and based on a triangular norm T and its induced operator ν_T, we investigate new operational properties of TL-fuzzy filters and TL-fuzzy congruences. We first define the $\otimes_T$ and ν_T-operations on L-fuzzy sets and study their effects on TL-fuzzy filters. Next, we examine the congruence-preserving properties and operational rules of TL-fuzzy congruences under $*_T$ and $\diamond_{\nu_T}$-compositions. Finally, leveraging the correspondence between TL-fuzzy filters and TL-fuzzy congruences, we explore the interplay and internal relationships among their respective operations.

Keywords: lattice; TL-fuzzy filter; TL-fuzzy congruence; operation; composition

MSC: 03G25; 03B52; 08A72

1. Introduction

The genesis of residuated lattices is deeply intertwined with the foundational inquiries into logic and algebra in the early 20th century. They emerged not as an isolated construct, but as a natural algebraic counterpart to non-classical logical systems, particularly substructural logics, which lack some of the structural rules of classical logic, such as contraction or weakening. The primary motivation was to provide a robust algebraic semantics for the implication connective in these logics. In this study, the key aim was to abstract the essential properties of the interaction between conjunction and implication. Specifically, the fundamental residuation property, $a \otimes b \leqslant c$ if and only if $a \leqslant b \to c$, formally captures the adjoint relationship between a monoidal multiplication (modeling logical conjunction) and a residuum (modeling logical implication). This principle was first systematically investigated in the context of ideal theory in rings by M. Ward and R.P. Dilworth [1] in the 1930s. Since then, residuated lattices have evolved into a central unifying framework, generalizing a spectrum of well-known algebraic structures including Boolean algebras [2], MV-algebras [3], Heyting algebras [4], and BL-algebras [5] providing the cornerstone for the algebraic study of fuzzy logics, linear logic, and many other non-classical reasoning paradigms.

As we all know, filters and congruences play fundamental and interconnected roles in the algebraic study of residuated lattices [6,7]. Filters, specifically the lattice filters

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that are closed under the monoidal operation, serve as the primary algebraic tool for capturing logical truth and consequence. They are pivotal for defining a deductive system and for constructing quotient algebras. Congruences, on the other hand, provide the structural framework for factorization by identifying elements that are to be considered equivalent. The profound connection lies in the fact that in a residuated lattice, there is a natural one-to-one correspondence between certain filters (e.g., implicative filters) and congruences. This duality means that every such filter uniquely determines a congruence relation, and vice versa, allowing the formation of a quotient structure that itself remains a residuated lattice. Consequently, this interplay is indispensable for algebraic representation theorems, completeness proofs for fuzzy logic and many-valued logic, and the systematic analysis of the structure of residuated lattices.

As fuzzy mathematics evolves, the study of fuzzy filters and fuzzy congruences in residuated lattices and related logical algebras has attracted considerable attention from scholars. For example, the concepts of fuzzy filters, fuzzy prime filters, and cosets of a fuzzy filter in BL-algebras were first introduced by Liu and Li [8], who also explored several corresponding properties. Y.B. Jun et al. [9] provided some equivalent characterizations of fuzzy filters in MTL-algebras and established the induced relationship between these fuzzy filters and fuzzy congruences. Zhu and Xu [10] extended some particular types of filters and fuzzy filters in BL-algebras and MTL-algebras naturally to general residuated lattices, and provided characterizations of certain special residuated lattices based on filters and fuzzy filters. Gao et al. [11] proposed the notion of fuzzy extended filters and investigated their related properties. They proved that all fuzzy extended filters on a residuated lattice form a complete Heyting algebra under specific operations. Previous research in this area has largely been valued in the unit interval [0, 1]. However, driven by the extensive use of L-fuzzy sets [12], researchers have progressively extended the theories of fuzzy filters and fuzzy congruences to more universal frameworks. This has resulted in the introduction of TL-fuzzy filters and TL-congruences, where L denotes an arbitrary complete lattice and T is a triangular norm defined on it. It is noteworthy that Wang [13] was the first to examine the fundamental properties of TL-fuzzy filters within residuated lattices and provided a formulation for generating TL-fuzzy filters from L-fuzzy sets. Meanwhile, R.A. Borzooei and M. Bakhshi [14] established some equivalent characterizations of TL-fuzzy filters in BL-algebras. Furthermore, they established that there is a bijective correspondence between TL-fuzzy filters and TL-fuzzy congruences in BL-algebras under the condition that the triangular norm T is idempotent. Building on the work of [14], Liu et al. [15] established the quotient structure and homomorphism theorems of residuated lattices via TL-fuzzy congruences. A.A. Ramadan [16] investigated the profound connections between L-fuzzy filters, L-fuzzy topological spaces, and L-fuzzy pre-proximity spaces in complete residuated lattices from a categorical perspective. Recently, An and Pang [17] investigated problems related to L-fuzzy filters in effect algebras by employing the idea of degree, and proved that the degree of fuzzy filters in effect algebras can induce an L-fuzzy convex structure. Based on the one-to-one correspondence between fuzzy filters and fuzzy congruences, Zhou and Dong [18] studied the relevant properties of fuzzy rough sets based on fuzzy filters within the framework of residuated lattices, thereby providing a new approach for research on residuated lattices, fuzzy rough sets, and fuzzy convex structures. Related work on fuzzy filters and fuzzy congruences in other logical algebras can be found in references [19–26].

As indicated by the above introduction, although research on fuzzy filters and fuzzy congruences in residuated lattices and other logical algebras has made progress in equivalent characterizations, mutual generation, and induced quotient structures, there are significant gaps: the existing studies focus on "static properties" and rarely explore their "dynamic operational properties" based on the intrinsic operations of residuated lattices

(e.g., intersection, union, adjoint operation, etc.), which is crucial for revealing the essence and improving the theory. There are two key unresolved issues: first, whether the composition of different congruence relations can preserve congruence (directly affecting the analysis of quotient structures); and second, the deep operational correlation between such composition and fuzzy substructures (e.g., filters, ideals, etc.), whose clarification is the core of establishing a unified operational framework. These gaps limit theoretical expansion and fail to meet application needs, thus possessing research value. Based on this, this paper breaks through the limitation of "static characterization" to systematically study the operational properties of TL-fuzzy filters and the composition of TL-fuzzy congruences in residuated lattices, exploring their correlations and rules under two types of key operations. This research can fill the gaps, clarify congruence preservation and deep correlations, improve the operational theory of residuated lattice fuzzy algebras, and provide key operational basis for the application in complex logical reasoning.

The basic structure of this paper can be summarized as follows. Basic concepts and properties related to residuated lattices, triangular norms, fuzzy filters, and fuzzy binary relations are laid out in Section 2. In Section 3, we define the $\otimes_T$-operation and $\rightarrow_{\nu_T}$-operation of L-fuzzy sets and discuss the relevant properties of TL-fuzzy filters in residuated lattices. In Section 4, based on $*_T$-composition and $\diamond_T$-composition operations, we emphasize the study of the congruence-preserving properties and operational rules of TL-fuzzy congruences. In Section 5, we further investigate some new operational association properties of TL-fuzzy filters and TL-fuzzy congruences in residuated lattices.

2. Preliminaries

In this section, we will recall some essential concepts and results on residuated lattices, triangular norms, L-fuzzy sets, L-fuzzy relations, and L-fuzzy filters that are foundational to this work. For concepts not defined in this paper, the reader can refer to [6,7,10,12].

A residuated lattice, which is endowed with an order structure, a variety of algebraic operations, and an adjoint implication operation, often constitutes the truth-value lattice for multivalued logical reasoning within fuzzy mathematics and associated fields. As a fundamental logical algebra, the internal structure and properties of residuated lattices have themselves been the subject of extensive study. Although the noncommutative version was introduced by Ward and Dilworth [1], our discussion will be confined to the commutative setting.

Definition 1 ([7]). *An algebraic structure $(X; \wedge, \vee, \otimes, \rightarrow)$ of type (2,2,2,2) is defined to be a residuated lattice provided that it meets the following requirements:*

(R1) *$(X; \wedge, \vee)$ is a bounded lattice equipped with the top element 1_X and the bottom element 0_X;*
(R2) *$(X, \otimes, 1_X)$ is a commutative monoid with identity 1_X;*
(R3) *$\gamma \otimes \zeta \leqslant \sigma$ if and only if $\zeta \leqslant \gamma \rightarrow \sigma$ for any $\gamma, \zeta, \sigma \in X$.*

Formally, a residuated lattice is termed a complete residuated lattice if its underlying lattice X is complete. In a complete residuated lattice X, Condition (R3) is equivalent to (R3′) $\gamma \otimes (\bigvee_{j\in J} \zeta_j) = \bigvee_{j\in J} \gamma \otimes \zeta_j$, for all $\gamma \in X$ and $\{\zeta\}_{j\in J} \subseteq X$.

Next, we will recall the relevant definitions of triangular norms on complete lattices, which plays a crucial role in our subsequent discussion of the properties of TL-fuzzy filters in residuated lattices.

Definition 2 ([6]). *Let L be a complete lattice. A triangular norm on L refers to a mapping $T : L \times L \rightarrow L$ that fulfills the subsequent conditions: for each $\gamma, \zeta, \sigma \in L$*

(T1) $T\big(T(\gamma, \zeta), \sigma\big) = T\big(\gamma, T(\zeta, \sigma)\big)$;

(T2) $T(\gamma,\zeta) = T(\zeta,\gamma)$;
(T3) $\zeta \leqslant \sigma$ *implies* $T(\gamma,\zeta) \leqslant T(\gamma,\sigma)$;
(T4) $T(\gamma,1_L) = \gamma$.

If $T(\gamma,\bigvee_{j\in J}\lambda_j) = \bigvee_{j\in J}T(\gamma,\lambda_j)$ for all $\gamma \in X$ and $\{\lambda_j\}_{j\in J} \subseteq X$, then T is called a left continuous triangular norm. For any left continuous triangular norm T, consider the subsequent binary operation over L defined as follows:

$$\nu_T(\gamma,\zeta) = \bigvee\{\lambda \in L : T(\gamma,\lambda) \leqslant \zeta\}.$$

This operation shall be termed the implication operator induced by T.

Remark 1. *For a triangular norm T on a complete lattice L, it follows from Conditions* (T2) *and* (T4) *in Definition 2 that the triple $(L,T,1_L)$ is a commutative monoid with identity 1_L. Furthermore, as a direct consequence of Condition* (R3′)*, the left continuity of T implies that $(L;\wedge,\vee,T,\nu_T)$ is a complete residuated lattice.*

There are many canonical examples of complete residuated lattices in lattice theory and logic, for example the complete Heyting algebra and the unit interval equipped with a left continuous triangular norm. Here, we give two special examples which will be used in the subsequent discussion.

Example 1. *Let $X = \{0_X,\alpha,\beta,1_X\}$ be a lattice whose Hasse diagram is below (see Figure 1).*

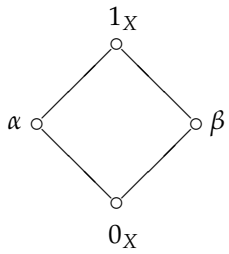

Figure 1. The Hasse diagram of X.

For each $x,y \in X$, we define two binary operations $\otimes$ and $\to$ on X as follows:

$$x \otimes y = x \wedge y, \qquad x \to y = \begin{cases} 1_X, & x \leqslant y; \\ \beta, & x = \alpha, y \in \{0_X,\beta\}; \\ \alpha, & x = \beta, y \in \{0_X,\alpha\}. \end{cases}$$

It is easy to verify that $(X;\wedge,\vee,\otimes,\to,0_X,1_X)$ is a complete residuated lattice. More precisely, it is a complete Heyting algebra.

Example 2. *Let X be the real unit interval $[0,1]$ with lattice operations $\wedge = min$ and $\vee = max$.*

(1) *If the operations on X are defined as follows:*

$$x \otimes y = min\{x,y\}, \qquad x \to y = \begin{cases} 1, & x \leqslant y; \\ y, & otherwise, \end{cases}$$

where $x,y \in X$. Then, the algebraic structure $([0,1];\wedge,\vee,\otimes,\to,0,1)$ is called the Gödel residuated lattice on the real unit interval.

(2) *If the operations on X are defined as follows:*

$$x \otimes y = max\{x+y-1,0\}, \qquad x \to y = min\{1-x+y,1\},$$

for all $x, y \in X$. Then, the algebraic structure $([0,1]; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called the Łukasiewicz residuated lattice on the real unit interval.

(3) *If the operations on X are defined as follows:*

$$x \otimes y = x \cdot y, \qquad x \rightarrow y = \begin{cases} 1, & x \leqslant y; \\ \frac{y}{x}, & \text{otherwise}, \end{cases}$$

where $x, y \in X$. Then, the algebraic structure $([0,1]; \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is called the Goguen residuated lattice on the real unit interval.

The subsequent properties are satisfied by complete residuated lattices.

Proposition 1 ([6,7]). *Let X be a complete residuated lattice. For each $\gamma, \zeta, \sigma, \omega \in X$ and each of the two families $(\delta_j)_{j\in J}$ and $(\lambda_j)_{j\in J}$ of X, the following statements hold:*

(1) $\gamma \rightarrow \zeta = 1_X$ *if and only* $\gamma \leqslant \zeta$;
(2) $\gamma \rightarrow 1_X = 1_X$ *and* $1_X \rightarrow \gamma = \gamma$;
(3) $\gamma \leqslant \zeta$ *implies* $\sigma \rightarrow \gamma \leqslant \sigma \rightarrow \zeta$ *and* $\zeta \rightarrow \sigma \leqslant \gamma \rightarrow \sigma$;
(4) $(\gamma \otimes \zeta) \rightarrow \sigma = \gamma \rightarrow (\zeta \rightarrow \sigma) = \zeta \rightarrow (\gamma \rightarrow \sigma)$;
(5) $(\gamma \rightarrow \zeta) \otimes (\zeta \rightarrow \sigma) \leqslant \gamma \rightarrow \sigma$;
(6) $(\gamma \rightarrow \zeta) \otimes (\sigma \rightarrow \omega) \leqslant (\gamma \otimes \sigma) \rightarrow (\zeta \otimes \omega)$;
(7) $\gamma \otimes (\bigvee_{j\in I} \delta_j) = \bigvee_{j\in J} (\gamma \otimes \delta_j)$;
(8) $(\bigvee_{j\in J} \lambda_j) \rightarrow \zeta = \bigwedge_{j\in J} (\lambda_j \rightarrow \zeta)$;
(9) $\zeta \rightarrow (\bigwedge_{j\in J} \delta_j) = \bigwedge_{j\in J} (\zeta \rightarrow \delta_j)$.

From Remark 1, it follows that $(L; \wedge, \vee, T, \nu_T)$ is a complete residuated lattice when T is a left continuous triangular norm on the complete lattice L. This indicates that $(L; \wedge, \vee, T, \nu_T)$ also satisfies the properties of Proposition 1.

For the sake of convenience in writing, in this paper, we always use $(X; \wedge, \vee, \otimes, \rightarrow)$ to denote a complete residuated lattice, abbreviated as X. It is particularly important to note that the object of study in this paper is a complete residuated lattice, and the lattice-valued setting considered is a left continuous triangular norm on a complete lattice L. According to Remark 1, the lattice-valued setting itself is also a complete residuated lattice. These are two distinct residuated lattices, and this distinction should be kept in mind.

Generalizing Zadeh's fuzzy sets [27], Goguen [12] first introduced the concept of L-fuzzy sets. An L-fuzzy set A on X is defined as a mapping $A : X \rightarrow L$. We denote the collection of all L-fuzzy sets over X by L^X, and the partial order on L can be extended to L^X in a pointwise manner. That is, for each $A, B \in L^X$, $A \leqslant B \Leftrightarrow A(x) \leqslant B(x)$ for all $x \in X$.

A mapping $R : X \times X \rightarrow L$ is referred to as an L-fuzzy relation on X, i.e., $R \in L^{X\times X}$. In what follows, we elaborate on two composition operations for L-fuzzy relations.

Definition 3 ([6]). *Let R and S be L-fuzzy relations on X and T be a left continuous triangular of L. The $*_T$-composition and $\diamond_{\nu_T}$-composition of R and S are defined as follows:*

$$(R *_T S)(\gamma, \zeta) = \bigvee_{\delta\in X} T\big(R(\gamma, \delta), S(\delta, \zeta)\big)$$

and

$$(R \diamond_{\nu_T} S)(\gamma, \zeta) = \bigwedge_{\delta\in X} \nu_T\big(R(\gamma, \delta), S(\delta, \zeta)\big),$$

for all $\gamma, \zeta \in X$.

Definition 4 ([13]). *An L-fuzzy set A of X is termed a TL-fuzzy filter if it fulfills the subsequent conditions: for each $\gamma, \zeta \in X$*

(1) $A(1_X) = 1_L$;
(2) $\gamma \leqslant \zeta$ *implies* $A(\gamma) \leqslant A(\zeta)$;
(3) $A(\gamma \otimes \zeta) \geqslant T\big(A(\gamma), A(\zeta)\big)$.

Remark 2. (1) *The second statement in the above definition essentially indicates that A is an order-preserving mapping. In fact, an L-fuzzy set that satisfies the order-preserving property is also termed an L-upper set.*

(2) *Although many researchers call the above definition an L-fuzzy filter, we adopt Wang's [13] nomenclature of a TL-fuzzy filter to emphasize the role of the triangular norm T.*

(3) *According to the results in [18], Conditions (2) and (3) in Definition 4 are equivalent to the condition "$\lambda \otimes \gamma \leqslant \zeta, A(\zeta) \geqslant T\big(A(\lambda), A(\gamma)\big)$", and thus can be replaced by it.*

In the discussion that follows, we present some examples of a TL-fuzzy filter in the general case of a residuated lattice.

Example 3. *Let $X = \{0_X, \gamma, \zeta, 1_X\}$ be a linearly ordered set such that $0_X < \gamma < \zeta < 1_X$. Define the lattice operations $\wedge$ and $\vee$ on X by*

$$\gamma \wedge \zeta = \min\{\gamma, \zeta\}, \quad \gamma \vee \zeta = \max\{\gamma, \zeta\}.$$

Furthermore, define the operations $\otimes$ and $\to$ by

$$\gamma \otimes \zeta = \gamma \wedge \zeta, \quad \gamma \to \zeta = \begin{cases} 1_X, & \gamma \leqslant \zeta; \\ \zeta, & \text{otherwise.} \end{cases}$$

Then $(X; \wedge, \vee, \otimes, \to, 0_X, 1_X)$ is a linearly ordered Heyting algebra and, in particular, a complete residuated lattice.

(1) *Suppose that $L = 2^{\{m,n,p,q\}}$, the power set of $\{m, n, p, q\}$), ordered by set inclusion. Then $(L, \cap, \cup)$ is a complete Boolean algebra. Define the operations T and ν_T on L by*

$$T(H, K) = H \cap K, \quad \nu_T(H, K) = H^c \cup K, \quad \text{for all } H, K \in L,$$

where H^c denotes the complement of H in $\{m, n, p, q\}$. Then $(L; \cap, \cup, T, \nu_T, \emptyset, \{m, n, p, q\})$ is a complete residuated lattice. Define the L-fuzzy set A on X by

$$A(0_X) = A(\gamma) = \{n, p\}, \ \ A(\zeta) = \{n, p, q\}, \ \ A(1_X) = \{m, n, p, q\}.$$

It is straightforward to verify that A is a TL-fuzzy filter on X.

(2) *Let $L = ([0, 1]; \wedge, \vee, T, \nu_T, 0, 1)$ be the Łukasiewicz residuated lattice in Example 2(2). Define the L-fuzzy set B on X by*

$$B(0_X) = 0.2, \ \ B(\gamma) = 0.5, \ \ B(\zeta) = 0.8, \ \ B(1_X) = 1.$$

It is easy to verify that B is a TL-fuzzy filter on X.

(3) *Let L be the complete residuated lattice in (1). Define the L-fuzzy set C on X by*

$$C(0_X) = \{m\}, \ \ C(\gamma) = \{n\}, \ \ C(\zeta) = \{m, n\}, \ \ C(1_X) = \{m, n, p, q\}.$$

It is easy to verify that C satisfies Conditions (1) and (3) in Definition 4, but fails to satisfy Condition (2). Hence, it is not a TL-fuzzy filter on X.

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3. Operations of $\otimes_T$ and $\rightarrow_{\nu_T}$ on TL-Fuzzy Filters

In this section, building on the triangular norm T and its associated induced implication operator ν_T, we formally define two novel binary operations on L-fuzzy sets within the framework of residuated lattices. These operations are designated as the $\otimes_T$-operation and the $\rightarrow_{\nu_T}$-operation, respectively. With these two fundamental operations in place, we proceed to investigate and discuss the relevant structural and operational properties of TL-fuzzy filters under their action. To lay a solid foundation for the subsequent analysis, we first present the following key definition.

Definition 5. *Let $A, B \in L^X$. Define $A \otimes_T B$ and $A \otimes_{\nu_T} B \in L^X$ as follows:*

$$(A \otimes_T B)(\lambda) = \bigvee_{\lambda \geqslant \gamma \otimes \zeta} T\big(A(\gamma), B(\zeta)\big)$$

and

$$(A \rightarrow_{\nu_T} B)(\lambda) = \bigwedge_{\lambda \leqslant \gamma \rightarrow \zeta} \nu_T\big(A(\gamma), B(\zeta)\big),$$

for all $\lambda \in X$.

In [15], the authors provided some different equivalent characterizations of TL-fuzzy filters in residuated lattices. Based on the $\otimes_T$ operation, we present a new characterization of TL-fuzzy filters.

Theorem 1. *Let X be a residuated lattice and L be a complete lattice. Then, A is a TL-fuzzy filter if and only if $A(1_X) = 1_L$ and $A \otimes_T A = A$.*

Proof. "⇒". Let A be a TL-fuzzy filter of X. Then, we obtain $A(1_X) = 1_L$ and

$$\begin{aligned}(A \otimes_T A)(\lambda) &= \bigvee_{\lambda \geqslant \gamma \otimes \zeta} T\big(A(\gamma), A(\zeta)\big) \\ &\leqslant \bigvee_{\lambda \geqslant \gamma \otimes \zeta} A(\gamma \otimes \zeta) \\ &\leqslant A(\lambda),\end{aligned}$$

for all $\lambda \in X$. This means that $A \otimes_T A \leqslant A$.

Furthermore, by $\lambda \geqslant 1_X \otimes \lambda$, we obtain

$$\begin{aligned}(A \otimes_T A)(\lambda) &= \bigvee_{\lambda \geqslant \gamma \otimes \zeta} T\big(A(\gamma), A(\zeta)\big) \\ &\geqslant T\big(A(1_X), A(\lambda)\big) \\ &= T\big(1_L, A(\lambda)\big) \\ &= A(\lambda),\end{aligned}$$

for all $\lambda \in X$. This implies that $A \otimes_T A \geqslant A$. Hence, $A \otimes_T A = A$.

"⇐". If $A(1_X) = 1_L$ and $A \otimes_T A = A$, let $\gamma \leqslant \zeta$, then we obtain

$$\begin{aligned}A(\zeta) &= (A \otimes_T A)(\zeta) \\ &= \bigvee_{\zeta \geqslant \gamma \otimes \sigma} T\big(A(\gamma), A(\sigma)\big) \\ &\geqslant T\big(A(1_X), A(\gamma)\big) \\ &= T\big(1_L, A(\gamma)\big) \\ &= A(\gamma).\end{aligned}$$

Moreover, since $\gamma \otimes \zeta \geqslant \gamma \otimes \zeta$, we obtain

$$A(\gamma \otimes \zeta) = (A \otimes_T A)(\gamma \otimes \zeta) = \bigvee_{\gamma\otimes\zeta \geqslant \sigma\otimes\omega} T\big(A(\sigma), A(\omega)\big) \geqslant T\big(A(\lambda), A(\zeta)\big).$$

This thus verifies that A constitutes a TL-fuzzy filter of X. □

Next, we will consider the properties related to the $\otimes_T$ and $\rightarrow_{\nu_T}$ operations of TL-fuzzy filters. With this goal in mind, we first formulate the subsequent lemma.

Lemma 1. *Let A, B and $C \in L^X$. If A and C are L-upper sets, then $A \otimes_T B \leqslant C$ if and only if $A \leqslant B \rightarrow_{\nu_T} C$.*

Proof. "⇒". Let $A \otimes_T B \leqslant C$. Then, we obtain $(A \otimes_T B)(\lambda) \leqslant C(\lambda)$, for all $\lambda \in X$. We proceed to demonstrate that

$$A(\lambda) \leqslant (B \rightarrow_{\nu_T} C)(\lambda) = \bigwedge_{\lambda\leqslant m\rightarrow n} \nu_T\big(B(m), C(n)\big).$$

In fact, for each $\lambda \leqslant m \rightarrow n$, by virtue of the fact that A is an L-upper set, we deduce that $A(\lambda) \leqslant A(m \rightarrow n)$. It is worth noting that $(m \rightarrow n) \otimes m \leqslant n$, so we can infer that

$$\begin{aligned} T\big(A(m \rightarrow n), B(m)\big) &\leqslant \bigvee_{n\geqslant\gamma\otimes\zeta} T\big(A(\gamma), B(\zeta)\big)\big) \\ &= (A \otimes_T B)(n) \\ &\leqslant C(n). \end{aligned}$$

This implies that $A(m \rightarrow n) \leqslant \nu_T\big(B(m), C(n)\big)$. Thus, we obtain that

$$A(\lambda) \leqslant \bigwedge_{\lambda\leqslant m\rightarrow n} \nu_T\big(B(m), C(n)\big) = (B \rightarrow_{\nu_T} C)(\lambda),$$

for all $\lambda \in X$. Hence, $A \leqslant B \rightarrow_{\nu_T} C$.

"⇐". Let $A \leqslant B \rightarrow_{\nu_T} C$. Then we obtain $A(\lambda) \leqslant (B \rightarrow_{\nu_T} C)(\lambda)$, for all $\lambda \in X$. We need to prove that

$$(A \otimes_T B)(\lambda) = \bigvee_{\lambda\geqslant m\otimes n} T\big(A(m), B(n)\big) \leqslant C(\lambda).$$

In fact, for each $\lambda \geqslant m \otimes n$, by virtue of the fact that C is an L-upper set, we deduce that $C(m \otimes n) \leqslant C(\lambda)$.

Notice that the inequality $m \leqslant n \rightarrow (m \otimes n)$ holds, whence we obtain that

$$A(m) \leqslant (B \rightarrow_{\nu_T} C)(m) = \bigwedge_{m\leqslant\gamma\rightarrow\zeta} \nu_T\big(B(\gamma), C(\zeta)\big) \leqslant \nu_T\big(B(n), C(m \otimes n)\big).$$

We thus obtain the inequality $T\big(A(m), B(n)\big) \leqslant C(m \otimes n)$, for all $m, n \in X$. Thus, we can deduce that

$$\begin{aligned} (A \otimes_T B)(\lambda) &= \bigvee_{\lambda\geqslant m\otimes n} T\big(A(m), B(n)\big) \\ &\leqslant \bigvee_{\lambda\geqslant m\otimes n} C(m \otimes n) \\ &\leqslant C(\lambda), \end{aligned}$$

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for all $\lambda \in X$. Therefore, $A \otimes_T B \leqslant C$. □

Proposition 2. *If A and B are TL-fuzzy filters of X, then $C = A \otimes_T B$ is the smallest TL-fuzzy filter of X such that $A \to_{\nu_T} C \geqslant B$.*

Proof. Let A and B be two TL-fuzzy filters of X. Then, we obtain

$$\begin{aligned}(A \otimes_T B)(1_X) &= \bigvee_{1_X \geqslant a \otimes b} T\big(A(a), B(b)\big)\\ &\geqslant T\big(A(1_X), B(1_X)\big)\\ &= T(1_L, 1_L)\\ &= 1_L.\end{aligned}$$

This implies that $(A \otimes_T B)(1_X) = 1_L$.

For each $\lambda, \gamma, \zeta \in X$ with $\lambda \otimes \gamma \leqslant \zeta$, we obtain

$$\begin{aligned}T\Big((A \otimes_T B)(\lambda), (A \otimes_T B)(\gamma)\Big) &= T\Big(\bigvee_{\lambda \geqslant a \otimes b} T\big(A(a), B(b)\big), \bigvee_{\gamma \geqslant c \otimes d} T\big(A(c), B(d)\big)\Big)\\ &= \bigvee_{\lambda \geqslant a \otimes b} \bigvee_{\gamma \geqslant c \otimes d} T\Big(T\big(A(a), B(b)\big), T\big(A(c), B(d)\big)\Big)\\ &= \bigvee_{\lambda \geqslant a \otimes b} \bigvee_{\gamma \geqslant c \otimes d} T\Big(A(c), T\big(A(a), T(B(b), B(d))\big)\Big)\\ &\leqslant \bigvee_{\lambda \geqslant a \otimes b} \bigvee_{\gamma \geqslant c \otimes d} T\Big(A(c), T\big(A(a), B(b \otimes d)\big)\Big)\\ &= \bigvee_{\lambda \geqslant a \otimes b} \bigvee_{\gamma \geqslant c \otimes d} T\Big(B(b \otimes d), T\big(A(a), A(c)\big)\Big)\\ &\leqslant \bigvee_{\lambda \otimes \gamma \geqslant a \otimes c \otimes b \otimes d} T\big(A(a \otimes c), B(b \otimes d)\big) \quad \text{(by Definition 4(3))}\\ &\leqslant \bigvee_{\zeta \geqslant m \otimes n} T\big(A(m), B(n)\big)\\ &= (A \otimes_T B)(\zeta).\end{aligned}$$

Consequently, based on the assertions in Remark 2(3), we can establish that $A \otimes_T B$ is a TL-fuzzy filter of X.

Furthermore, since $\gamma \otimes (\gamma \to \zeta) \leqslant \zeta$, we obtain

$$\begin{aligned}\big(A \to_{\nu_T} (A \otimes_T B)\big)(\lambda) &= \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\big(A(\gamma), (A \otimes_T B)(\zeta)\big)\\ &= \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\Big(A(\gamma), \bigvee_{\zeta \geqslant c \otimes d} T\big(A(c), B(d)\big)\Big)\\ &\geqslant \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\Big(A(\gamma), T\big(A(\gamma), B(\gamma \to \zeta)\big)\Big)\\ &\geqslant \bigwedge_{\lambda \leqslant \gamma \to \zeta} B(\gamma \to \zeta)\\ &\geqslant B(\lambda).\end{aligned}$$

This implies that $A \to_{\nu_T} C \geqslant B$.

In what follows, we establish that C is the smallest TL-fuzzy filter fulfilling the condition $A \to_{\nu_T} C \geqslant B$. In fact, if C^* is a TL-fuzzy filter with $A \to_{\nu_T} C^* \geqslant B$, then we thus deduce from Lemma 1 that $C = A \otimes_T B \leqslant C^*$. The proof is completed. □

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Lemma 2. *Let $A, B \in L^X$. If B is a TL-fuzzy filter with $A \leqslant B$, then $B \leqslant A \to_{\nu_T} B$.*

Proof. Let B be a TL-fuzzy filter. In view of the fact that $A \leqslant B$, we deduce from Definition 5 that

$$(A \to_{\nu_T} B)(\lambda) = \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\big(A(\gamma), B(\zeta)\big) \geqslant \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\big(B(\gamma), B(\zeta)\big).$$

Notice that B is a TL-fuzzy filter and $\gamma \otimes (\gamma \to \zeta) \leqslant \zeta$, so we obtain

$$T\big(B(\gamma), B(\gamma \to \zeta)\big) \leqslant B\big(\gamma \otimes (\gamma \to \zeta)\big) \leqslant B(\zeta).$$

This implies that $\nu_T\big(B(\gamma), B(\zeta)\big) \geqslant B(\gamma \to \zeta)$. It follows that

$$(A \to_{\nu_T} B)(\lambda) \geqslant \bigwedge_{\lambda \leqslant \gamma \to \zeta} \nu_T\big(B(\gamma), B(\zeta)\big) \geqslant \bigwedge_{\lambda \leqslant \gamma \to \zeta} B(\gamma \to \zeta) \geqslant B(\lambda).$$

Hence, $B \leqslant A \to_{\nu_T} B$. □

Proposition 3. *Let A and B be two TL-fuzzy filters of X. If $A \leqslant B$, then $D = A \to_{\nu_T} B$ is the largest TL-fuzzy filter of X such that $A \otimes_T D \leqslant B$.*

Proof. Let A and B be TL-fuzzy filters with $A \leqslant B$. On the basis of Lemma 2, it can be concluded that $A \to_{\nu_T} B \geqslant B$. Then we obtain $(A \to_{\nu_T} B)(1_X) \geqslant B(1_X) = 1_L$. This means that $(A \to_{\nu_T} B)(1_X) = 1_L$.

Let $\lambda \leqslant \gamma$. For each $\gamma \leqslant \zeta \to \sigma$, we obtain $\lambda \leqslant \zeta \to \sigma$. From Definition 5, it holds that

$$\begin{aligned}(A \to_{\nu_T} B)(\lambda) &= \bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big) \\ &\leqslant \bigwedge_{\gamma \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big) \\ &= (A \to_{\nu_T} B)(\gamma).\end{aligned}$$

Since B is a TL-fuzzy filter of X, we obtain that $B(\lambda \otimes \gamma) \leqslant B(m \to n)$ holds, for all $\lambda \otimes \gamma \leqslant m \to n$. In view of the fact that $A \leqslant B$, we deduce that

$$\begin{aligned}&\nu_T\Big(T\big((A \to_{\nu_T} B)(\lambda), (A \to_{\nu_T} B)(\gamma)\big), (A \to_{\nu_T} B)(\lambda \otimes \gamma)\Big) \\ =&\nu_T\Big((A \to_{\nu_T} B)(\lambda), \nu_T\big((A \to_{\nu_T} B)(\gamma), (A \to_{\nu_T} B)(\lambda \otimes \gamma)\big)\Big) \\ =&\nu_T\Big(\bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big), \nu_T\big(\bigwedge_{\gamma \leqslant c \to d} \nu_T\big(A(c), B(d)\big), \bigwedge_{\lambda \otimes \gamma \leqslant m \to n} \nu_T\big(A(m), B(n)\big)\big)\Big) \\ \geqslant&\nu_T\Big(\bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big), \nu_T\big(B(\gamma), \bigwedge_{\lambda \otimes \gamma \leqslant m \to n} \nu_T\big(A(m), B(n)\big)\big)\Big) \ \text{(by Proposition 1(3))} \\ \geqslant&\nu_T\Big(\bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big), \nu_T\big(B(\gamma), \bigwedge_{\lambda \otimes \gamma \leqslant m \to n} \nu_T\big(B(m), B(n)\big)\big)\Big) \ \text{(since } A \leqslant B\text{)} \\ \geqslant&\nu_T\Big(\bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big), \nu_T\big(B(\gamma), \bigwedge_{\lambda \otimes \gamma \leqslant m \to n} B(m \to n)\big)\Big) \ \text{(by Lemma 2)} \\ \geqslant&\nu_T\Big(\bigwedge_{\lambda \leqslant \zeta \to \sigma} \nu_T\big(A(\zeta), B(\sigma)\big), \nu_T\big(B(\gamma), B(\lambda \otimes \gamma)\big)\Big) \\ \geqslant&\nu_T\Big(B(\lambda), \nu_T\big(B(\gamma), B(\lambda \otimes \gamma)\big)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \nu_T\Big(T\big(B(\lambda), B(\gamma)\big), B(\lambda \otimes \gamma)\Big) \\
&\geqslant \nu_T\Big(B(\lambda \otimes \gamma), B(\lambda \otimes \gamma)\Big) \\
&= 1_L.
\end{aligned}
$$

We obtain $\nu_T\Big(T\big((A \to_{\nu_T} B)(\lambda), (A \to_{\nu_T} B)(\gamma)\big), (A \to_{\nu_T} B)(\lambda \otimes \gamma)\Big) = 1_L$. This enables us to infer that

$$(A \to_{\nu_T} B)(\lambda \otimes \gamma) \geqslant T\big((A \to_{\nu_T} B)(\lambda), (A \to_{\nu_T} B)(\gamma)\big).$$

Hence, $A \to_{\nu_T} B$ is a TL-fuzzy filter of X.

Furthermore, we demonstrate that D is the largest TL-fuzzy filter satisfying the condition $A \otimes_T D \leqslant B$. In fact, if D^* is a TL-fuzzy filter with $A \otimes_T D^* \leqslant B$, then by virtue of Lemma 1, we deduce that $D^* \leqslant A \to_{\nu_T} B = D$. The proof is completed. □

Remark 3. *The two propositions above indicate that TL-fuzzy filters are closed under the $\otimes_T$-operation and the ν_T-operation (the latter requires two TL-fuzzy filters to satisfy an order relation). These properties will play a significant role in the research of Section 5.*

4. The $*_T$-Composition and $\diamond_{\nu_T}$-Composition of TL-Fuzzy Congruences

In this section, we focus on the composition properties of two specific types of TL-fuzzy congruence relations on residuated lattices, termed $*_T$-composition and $\diamond_{\nu_T}$-composition, respectively. Based on these two composition operations, we will systematically investigate their congruence-preserving properties and corresponding operational laws. A crucial part of our analysis involves examining how these compositions maintain or alter the defining characteristics of TL-fuzzy congruences, such as reflexivity, symmetry, transitivity, and compatibility with the algebraic operations of the residuated lattice. To establish a clear foundation for this study, we first introduce the formal definition and essential properties of TL-fuzzy congruences in the context of residuated lattices.

Definition 6. *An L-fuzzy relation R over X is termed a TL-fuzzy congruence if it fulfills the subsequent conditions: for each $\lambda, \gamma, \zeta \in X$*

(FC1) $R(\lambda, \lambda) = 1_L$;
(FC2) $R(\lambda, \gamma) = R(\gamma, \lambda)$;
(FC3) $T\big(R(\lambda, \zeta), R(\zeta, \gamma)\big) \leqslant R(\lambda, \gamma)$;
(FC4) $R(\lambda \otimes \zeta, \gamma \otimes \zeta) \geqslant R(\lambda, \gamma)$;
(FC5) $T\big(R(\lambda \to \zeta, \gamma \to \zeta), R(\zeta \to \lambda, \zeta \to \gamma)\big) \geqslant R(\lambda, \gamma)$.

Remark 4. (1) *An L-fuzzy relation R is termed fuzzy reflexive, fuzzy symmetric, and T-fuzzy transitive if it fulfills conditions* (FC1), (FC2), *and* (FC3) *in turn. Additionally, if R simultaneously satisfies* (FC1), (FC2), *and* (FC3), *it is referred to as TL-fuzzy equivalence.*

(2) *The condition* (FC3) *in Definition 6 can be replaced by $R *_T R \leqslant R$. That is, R is T-fuzzy transitive if and only if $R *_T R \leqslant R$ holds. Moreover, if R is a fuzzy reflexive and T-fuzzy transitive relation of X, then $R *_T R = R$.*

(3) *The above definition of TL-fuzzy congruences differs from the notion of T-congruence presented in [13]. Specifically, Condition* (FC5) *in Definition 6 is formulated based on the triangular norm T, whereas the corresponding condition in [13] is independent of any triangular norm. In particular, when T is taken as the Gödel triangular norm, the two definitions become equivalent. Therefore, the definition of TL-fuzzy congruences on residuated lattices proposed in this paper is novel.*

Example 4. *Let* $X = \{0_X, \alpha, \beta, 1_X\}$ *be the residuated lattice as in Example 1 and* $L = [0,1]$. *The triangular norm* T *on* L *is defined by*

$$T(\lambda, \mu) = max\{\lambda + \mu - 1, 0\}, \text{ for all } \lambda, \mu \in [0,1].$$

(1) *The L-fuzzy relation* R_1 *on* X *is given in Table 1.*

Table 1. The definition of R_1, where $x, y \in X$.

$R_1(x,y)$	0_X	α	β	1_X
0_X	1	0.6	0.6	0.3
α	0.6	1	0.6	0.6
β	0.6	0.6	1	0.6
1_X	0.3	0.6	0.6	1

It is easy to verify that R_1 *satisfies Conditions* (FC1)–(FC4) *in Definition 6, but it does not satisfy Condition* (FC5). *In fact, if* $\lambda = 0_X, \gamma = 1_X, \zeta = \alpha$, *then*

$$0_X \to \alpha = 1_X, \ \ 1_X \to \alpha = \alpha, \ \ \alpha \to 0_X = \beta \ \text{ and } \ \alpha \to 1_X = 1_X.$$

Thus, we obtain

$$T\big(R_1(1_X, \alpha), R_1(\beta, 1_X)\big) = T(0.6, 0.6) = 0.2 < 0.3 = R_1(0_X, 1_X).$$

This shows that R_1 *is a TL-fuzzy equivalence relation, but not a TL-fuzzy congruence.*

(2) *The L-fuzzy relation* R_2 *on* X *is given in Table 2.*

Table 2. The definition of R_2, where $x, y \in X$.

$R_2(x,y)$	0_X	α	β	1_X
0_X	1	0.2	0.2	0.3
α	0.2	1	0.7	0.9
β	0.2	0.7	1	0.9
1_X	0.3	0.9	0.9	1

It is easy to verify that R_2 *satisfies Conditions* (FC1)–(FC3) *and* (FC5) *in Definition 6, but it does not satisfy Condition* (FC4). *In fact, if* $\lambda = \alpha, \gamma = \beta, \zeta = \alpha$, *then*

$$R_2(\alpha \otimes \alpha, \beta \otimes \alpha) = R_2(\alpha, 0_X) = 0.2 < 0.7 = R_2(\alpha, \beta).$$

This indicates that R_2 *is a TL-fuzzy equivalence relation, rather than a TL-fuzzy congruence.*

(3) *The L-fuzzy relation* R_3 *on* X *is given by Table 3.*

Table 3. The definition of R_3, where $x, y \in X$.

$R_3(x,y)$	0_X	α	β	1_X
0_X	1	0.8	0.8	0.6
α	0.8	1	0.6	0.8
β	0.8	0.6	1	0.8
1_X	0.6	0.8	0.8	1

One can easily verify that R_3 *satisfies Conditions* (FC1)–(FC5) *in Definition 6. This means that* R_3 *is a TL-fuzzy congruence on* X.

(4) *The L-fuzzy set* A *of* X *is defined as follows:*

$$A(0_X) = 0.3, \ \ A(\alpha) = A(\beta) = 0.6, \ \ A(1_X) = 1.$$

Obviously, A is a TL-fuzzy filter of X. If we define the L-fuzzy relation R_A as follows:

$$R_A(x,y) = T\big(A(x \to y), A(y \to x)\big),$$

then we can represent R_A using Table 4.

Table 4. The definition of R_A, where $x, y \in X$.

$R_A(x,y)$	0_X	α	β	1_X
0_X	1	0.6	0.6	0.3
α	0.6	1	0.2	0.6
β	0.6	0.2	1	0.6
1_X	0.3	0.6	0.6	1

After verification, the L-fuzzy relation R_A is a TL-fuzzy congruence. We call R_A the TL-fuzzy congruence induced by the TL-fuzzy filter A.

The following proposition investigate the properties related to T-fuzzy transitivity and TL-fuzzy equivalence from the definitions of $\diamond_{\nu_T}$-composition, respectively.

Proposition 4. *Let R be an L-fuzzy relation on X.*

(1) *If R is fuzzy symmetric, then R is T-fuzzy transitive if and only if $R \leqslant R \diamond_{\nu_T} R$;*
(2) *If R is TL-fuzzy equivalence, then $R = R \diamond_{\nu_T} R$.*

Proof. (1) Let R be an L-fuzzy symmetric relation. Since R is T-fuzzy transitive, we obtain $T\big(R(\zeta,\lambda), R(\lambda,\gamma)\big) \leqslant R(\zeta,\gamma)$ for all $\lambda, \gamma, \zeta \in X$. Thus,

$$\begin{aligned}
\nu_T\Big(R(\lambda,\gamma), (R \diamond_{\nu_T} R)(\lambda,\gamma)\Big) &= \nu_T\Big(R(\lambda,\gamma), \bigwedge_{\zeta\in X} \nu_T\big(R(\lambda,\zeta), R(\zeta,\gamma)\big)\Big) \\
&= \bigwedge_{\zeta\in X} \nu_T\Big(R(\lambda,\gamma), \nu_T\big(R(\lambda,\zeta), R(\zeta,\gamma)\big)\Big) \quad \text{(by Proposition 1(9))} \\
&= \bigwedge_{\zeta\in X} \nu_T\Big(T\big(R(\lambda,\gamma), R(\lambda,\zeta)\big), R(\zeta,\gamma)\Big) \quad \text{(by Proposition 1(4))} \\
&= \bigwedge_{\zeta\in X} \nu_T\Big(T\big(R(\lambda,\gamma), R(\zeta,\lambda)\big), R(\zeta,\gamma)\Big) \quad \text{(by Proposition 1(3))} \\
&\geqslant \bigwedge_{\zeta\in X} \nu_T\big(R(\zeta,\gamma), R(\zeta,\gamma)\big) \\
&= 1_L.
\end{aligned}$$

Relying on Proposition 1(1), we obtain the result that $R(\lambda,\gamma) \leqslant (R \diamond_{\nu_T} R)(\lambda,\gamma)$. Hence, $R \leqslant R \diamond_{\nu_T} R$.

For the converse direction, since $R \leqslant R \diamond_{\nu_T} R$, we obtain

$$\forall \zeta \in X, \nu_T\big(R(\lambda,\zeta), R(\zeta,\gamma)\big) \geqslant R(\lambda,\gamma).$$

This implies that $T\big(R(\lambda,\gamma), R(\lambda,\zeta)\big) \leqslant R(\zeta,\gamma)$.

Notice that R is fuzzy symmetric, so we obtain

$$T\big(R(\zeta,\lambda), R(\lambda,\gamma)\big) \leqslant R(\zeta,\gamma),$$

for all $\lambda, \gamma, \zeta \in X$. This means that R is T-fuzzy transitive.

(2) Let R be a TL-fuzzy equivalence relation. Then, it follows from (1) that $R \leqslant R \diamond_{\nu_T} R$. Notice that R is fuzzy reflexive, so we obtain

$$\begin{aligned}(R \diamond_{\nu_T} R)(\lambda, \gamma) &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), R(\zeta, \gamma)\big) \\ &\leqslant \nu_T\big(R(\lambda, \lambda), R(\lambda, \gamma)\big) \\ &= \nu_T\big(1_L, R(\lambda, \gamma)\big) \\ &= R(\lambda, \gamma).\end{aligned}$$

This implies that $R \leqslant R \diamond_{\nu_T} R$. Therefore, $R = R \diamond_{\nu_T} R$. □

In the following, we will examine the operational properties of the $*_T$-composition and $\diamond_{\nu_T}$-composition of fuzzy congruences in residuated lattices. For the convenience of subsequent discussions, the following lemma is provided.

Lemma 3. *Let R and S be two TL-fuzzy equivalence relations on X. Then, we obtain the following statements:*

(1) *If $R *_T S = S *_T R$, then $R *_T S$ is a TL-fuzzy equivalence relation on X;*
(2) *If $R \leqslant S$, then $R \diamond_{\nu_T} S$ is a TL-fuzzy equivalence relation on X.*

Proof. (1) Since R and S are fuzzy reflexive, we obtain

$$\begin{aligned}(R *_T S)(\lambda, \lambda) &= \bigvee_{\zeta \in X} T\big(R(\lambda, \zeta), S(\zeta, \lambda)\big) \\ &\geqslant T\big(R(\lambda, \lambda), S(\lambda, \lambda)\big) \\ &= T(1_L, 1_L) \\ &= 1_L.\end{aligned}$$

This implies that $R *_T S$ is fuzzy reflexive.

Notice that the composition of two TL-fuzzy equivalence relations is not guaranteed to be a TL-fuzzy equivalence relation in general. This is because the symmetry of the composition relies critically on the commutativity condition.

Since R and S are fuzzy symmetric and $R *_T S = S *_T R$, it follows that

$$\begin{aligned}(R *_T S)(\lambda, \gamma) &= \bigvee_{\zeta \in X} T\big(R(\lambda, \zeta), S(\zeta, \gamma)\big) \\ &= \bigvee_{\zeta \in X} T\big(S(\gamma, \zeta), R(\zeta, \lambda)\big) \\ &= (S *_T R)(\gamma, \lambda) \\ &= (R *_T S)(\gamma, \lambda).\end{aligned}$$

This shows that $R *_T S$ is fuzzy symmetric.

Moreover, since R and S are T-fuzzy transitive, it follows from Remark 2(2) that

$$\begin{aligned}(R *_T S) *_T (R *_T S) &= \big((R *_T S) *_T R\big) *_T S \\ &= \big(R *_T (S *_T R)\big) *_T S \\ &= \big(R *_T (R *_T S)\big) *_T S \\ &= \big((R *_T R) *_T S\big) *_T S\end{aligned}$$

$$\begin{aligned} &= (R *_T R) *_T (S *_T S) \\ &\leqslant R *_T S. \end{aligned}$$

This means that $R *_T S$ is T-fuzzy transitive. Hence, $R *_T S$ is a TL-fuzzy equivalence relation on X.

(2) Since S is fuzzy symmetric and $R \leqslant S$, we obtain

$$\begin{aligned} (R \diamond_{\nu_T} S)(\lambda, \lambda) &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), S(\zeta, \lambda)\big) \\ &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), S(\lambda, \zeta)\big) \\ &= 1_L, \end{aligned}$$

for all $\lambda \in X$. This shows that $R \diamond_{\nu_T} S$ is fuzzy reflexive.

Let $m \leqslant \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), S(\zeta, \gamma)\big)$. Then,

$$m \leqslant \nu_T\big(R(\lambda, \lambda), S(\lambda, \gamma)\big) = \nu_T\big(1_L, S(\lambda, \gamma)\big) = S(\lambda, \gamma) = S(\gamma, \lambda).$$

Notice that S is T-fuzzy transitive and $R \leqslant S$, so we obtain

$$S(\zeta, \lambda) \geqslant T\big(S(\zeta, \gamma), S(\gamma, \lambda)\big) \geqslant T\big(R(\zeta, \gamma), S(\gamma, \lambda)\big).$$

This implies $T\big(m, R(\gamma, \zeta)\big) \leqslant S(\zeta, \lambda)$. Thus, $m \leqslant \nu_T\big(R(\gamma, \zeta), S(\zeta, \lambda)\big)$, for all $\zeta \in X$. In view of the arbitrariness of m, we deduce that

$$\begin{aligned} (R \diamond_{\nu_T} S)(\lambda, \gamma) &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), S(\zeta, \gamma)\big) \\ &\leqslant \bigwedge_{\zeta \in X} \nu_T\big(R(\gamma, \zeta), S(\zeta, \lambda)\big) \\ &= (R \diamond_{\nu_T} S)(\gamma, \lambda). \end{aligned}$$

Analogously, we can demonstrate that $(R \diamond_{\nu_T} S)(\gamma, \lambda) \leqslant (R \diamond_{\nu_T} S)(\lambda, \gamma)$. It follows that $(R \diamond_{\nu_T} S)(\lambda, \gamma) = (R \diamond_{\nu_T} S)(\gamma, \lambda)$. This shows that $R \diamond_{\nu_T} S$ is fuzzy symmetric.

Furthermore, by the fuzzy reflexivity of R, we deduce that

$$\begin{aligned} \bigwedge_{n \in X} \nu_T\big(R(\gamma, n), S(n, \zeta) &\leqslant \nu_T\big(R(\gamma, y), S(\gamma, \zeta)\big) \\ &= \nu_T\big(1_L, S(\gamma, \zeta)\big) \\ &= S(\gamma, \zeta). \end{aligned}$$

Notice that S is T-fuzzy transitive, so we obtain

$$T\Big(S(m, \gamma), \textstyle\bigwedge_{n \in X} \nu_T\big(R(\gamma, n), S(n, \zeta)\big)\Big) \leqslant T\big(S(m, \gamma), S(\gamma, \zeta)\big) \leqslant S(m, \zeta).$$

Thus,

$$S(m, \gamma) \leqslant \nu_T\Big(\textstyle\bigwedge_{n \in X} \nu_T\big(R(\gamma, n), S(n, \zeta)\big), S(m, \zeta)\Big).$$

Consequently, we deduce that

$$\begin{aligned} \bigwedge_{m \in X} \nu_T\big(R(\lambda, m), S(m, \gamma)\big) &\leqslant \bigwedge_{m \in X} \nu_T\Big(R(\lambda, m), \nu_T\big(\bigwedge_{n \in X} \nu_T\big(R(\gamma, n), S(n, \zeta)\big), S(m, \zeta)\big)\Big) \\ &= \bigwedge_{r \in X} \nu_T\Big(R(\lambda, r), \nu_T\big(\bigwedge_{n \in X} \nu_T\big(R(\gamma, n), S(n, \zeta)\big), S(r, \zeta)\big)\Big) \end{aligned}$$

$$= \bigwedge_{r\in X} \nu_T\Big(\bigwedge_{n\in X} \nu_T\big(R(\gamma,n),S(n,\zeta)\big), \nu_T\big(R(\lambda,r),S(r,\zeta)\big)\Big)$$
$$= \nu_T\Big(\bigwedge_{n\in X} \nu_T\big(R(\gamma,n),S(n,\zeta)\big), \bigwedge_{r\in X} \nu_T\big(R(\lambda,r),S(r,\zeta)\big)\Big).$$

This verifies the fact that

$$T\Big(\textstyle\bigwedge_{m\in X} \nu_T\big(R(\lambda,m),S(m,\gamma)\big), \bigwedge_{n\in X} \nu_T\big(R(\gamma,n),S(n,\zeta)\big)\Big) \leqslant \bigwedge_{r\in X} \nu_T\big(R(\lambda,r),S(r,\zeta)\big).$$

Thus, we obtain

$$T\big((R\diamond_{\nu_T} S)(\lambda,\gamma),(R\diamond_{\nu_T} S)(\gamma,\zeta)\big) \leqslant (R\diamond_{\nu_T} S)(\lambda,\zeta).$$

This indicates that $R\diamond_{\nu_T} S$ is T-fuzzy transitive. The proof is thereby completed. □

Remark 5. *In general, the $*_T$-composition and $\diamond_{\nu_T}$-composition of two L-fuzzy equivalence relations are not closed, because symmetry is not preserved under these composition operations. Therefore, the conditions $R *_T S = S *_T R$ and $R \leqslant S$ in the above lemma are indispensable.*

Next, we will discuss the relevant properties of the $*_T$-composition and $\diamond_{\nu_T}$-composition operations for TL-fuzzy congruences.

Proposition 5. *Let R and S be two TL-fuzzy congruences on X. If $R *_T S = S *_T R$, then $R *_T S$ is a TL-fuzzy congruence on X.*

Proof. Since R and S are TL-fuzzy congruences on X and satisfy the commutative property $R *_T S = S *_T R$, it can be concluded from Lemma 3(1) that $R *_T S$ is a TL-fuzzy equivalence relation on X.

We need to verify that $R *_T S$ satisfies Conditions (FC4) and (FC5) in Definition 6. In reality, for every $m,\lambda,\gamma,\zeta \in X$, notice that R and S are TL-fuzzy congruences, so we deduce that

$$\begin{aligned}(R *_T S)(\lambda\otimes\zeta,\gamma\otimes\zeta) &= \bigvee_{m\in X} T\big(R(\lambda\otimes\zeta,m),S(m,\gamma\otimes\zeta)\big)\\ &\geqslant T\big(R(\lambda\otimes\zeta,\lambda\otimes\zeta),S(\lambda\otimes\zeta,\gamma\otimes\zeta)\big)\\ &\geqslant T\big(R(\lambda,m),S(m,\gamma)\big).\end{aligned}$$

Taking the arbitrariness of m into account, we have that

$$\begin{aligned}(R *_T S)(\lambda\otimes\zeta,\gamma\otimes\zeta) &\geqslant \bigvee_{m\in X} T\big(R(\lambda,m),S(m,\gamma)\big)\\ &= (R *_T S)(\lambda,\gamma).\end{aligned}$$

Furthermore, we obtain that

$$\begin{aligned}&T\big((R *_T S)(\lambda\to\zeta,\gamma\to\zeta),(R *_T S)(\zeta\to\lambda,\zeta\to\gamma)\big)\\ =&T\Big(\bigvee_{m\in X} T\big(R(\lambda\to\zeta,m),S(m,\gamma\to\zeta)\big), \bigvee_{n\in X} T\big(R(\zeta\to\lambda,n),S(n,\zeta\to\gamma)\big)\Big)\\ \geqslant&T\Big(T\big(R(\lambda\to\zeta,p\to\zeta),S(p\to\zeta,\gamma\to\zeta)\big),T\big(R(\zeta\to\lambda,\zeta\to p),S(\zeta\to p,\zeta\to\gamma)\big)\Big)\\ =&T\Big(R(\zeta\to\lambda,\zeta\to p),T\big(R(\lambda\to\zeta,p\to\zeta),T(S(p\to\zeta,\gamma\to\zeta),S(\zeta\to p,\zeta\to\gamma))\big)\Big)\\ =&T\Big(T\big(R(\lambda\to\zeta,p\to\zeta),R(\zeta\to\lambda,\zeta\to p)\big),T\big(S(p\to\zeta,\gamma\to\zeta),S(\zeta\to p,\zeta\to\gamma)\big)\Big)\\ \geqslant&T\big(R(\lambda,p),S(p,\gamma)\big)\end{aligned}$$

for every $\lambda, \gamma, \zeta, p \in X$.

By the arbitrariness of p, we obtain

$$T\big((R *_T S)(\lambda \to \zeta, \gamma \to \zeta), (R *_T S)(\zeta \to \lambda, \zeta \to \gamma)\big) \geqslant \bigvee_{p \in X} T\big(R(\lambda, p), S(p, \gamma)\big).$$

This leads to the conclusion that

$$T\big((R *_T S)(\lambda \to \zeta, \gamma \to \zeta), (R *_T S)(\zeta \to \lambda, \zeta \to \gamma)\big) \geqslant (R *_T S)(\lambda, \gamma).$$

Hence, $R *_T S$ is a TL-fuzzy congruence on X. □

Proposition 6. *Let R and S be two TL-fuzzy congruences on X. If $R \leqslant S$, then $R \diamond_{\nu_T} S$ is a TL-fuzzy congruence on X.*

Proof. In view of the fact that R and S are TL-fuzzy congruences on X and $R \leqslant S$ holds, we conclude by Lemma 3(2) that $R \diamond_{\nu_T} S$ is a TL-fuzzy equivalence relation on X.

We need to verify that $R \diamond_{\nu_T} S$ satisfies Conditions (FC4) and (FC5) in Definition 6. Actually, for each $\lambda, \gamma, \zeta \in X$, notice that S is T-fuzzy transitive, so we obtain

$$T\big(S(\lambda \otimes \zeta, m), S(\lambda \otimes \zeta, \gamma \otimes \zeta)\big) \leqslant S(m, \gamma \otimes \zeta),$$

which means $S(\lambda \otimes \zeta, m) \leqslant \nu_T\big(S(\lambda \otimes \zeta, \gamma \otimes \zeta), S(m, \gamma \otimes \zeta)\big)$. It follows that

$$\begin{aligned}
&\nu_T\Big((R \diamond_{\nu_T} S)(\lambda, \gamma), (R \diamond_{\nu_T} S)(\lambda \otimes \zeta, \gamma \otimes \zeta)\Big)\\
=&\nu_T\Big(\bigwedge_{n \in X} \nu_T\big(R(\lambda, n), S(n, \gamma)\big), \bigwedge_{m \in X} \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big)\\
=&\bigwedge_{m \in X} \nu_T\Big(\bigwedge_{n \in X} \nu_T\big(R(\lambda, n), S(n, \gamma)\big), \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big) \quad \text{(by Proposition 1(9))}\\
\geqslant&\bigwedge_{m \in X} \nu_T\Big(\nu_T\big(R(\lambda, \lambda), S(\lambda, \gamma)\big), \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big) \quad \text{(by Proposition 1(3))}\\
=&\bigwedge_{m \in X} \nu_T\Big(\nu_T\big(1_L, S(\lambda, \gamma)\big), \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big)\\
=&\bigwedge_{m \in X} \nu_T\Big(S(\lambda, \gamma), \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big) \quad \text{(by Proposition 1(4))}\\
\geqslant&\bigwedge_{m \in X} \nu_T\Big(S(\lambda \otimes \zeta, \gamma \otimes \zeta), \nu_T\big(R(\lambda \otimes \zeta, m), S(m, \gamma \otimes \zeta)\big)\Big) \quad \text{(by Definition 6(FC4))}\\
=&\bigwedge_{m \in X} \nu_T\Big(R(\lambda \otimes \zeta, m), \nu_T\big(S(\lambda \otimes \zeta, \gamma \otimes \zeta), S(m, \gamma \otimes \zeta)\big)\Big) \quad \text{(by Proposition 1(4))}\\
\geqslant&\bigwedge_{m \in X} \nu_T\big(R(\lambda \otimes \zeta, m), S(\lambda \otimes \zeta, m)\big)\\
=&1_L.
\end{aligned}$$

Thus, $\nu_T\Big((R \diamond_{\nu_T} S)(\lambda, \gamma), (R \diamond_{\nu_T} S)(\lambda \otimes \zeta, \gamma \otimes \zeta)\Big) = 1_L$. This implies that $(R \diamond_{\nu_T} S)(\lambda \otimes \zeta, \gamma \otimes \zeta) \geqslant (R \diamond_{\nu_T} S)(\lambda, \gamma)$.

Moreover, by Proposition 4(2), we obtain that $S = S \diamond_{\nu_T} S$. It can be deduced that

$$\begin{aligned}
&\nu_T\Big((R \diamond_{\nu_T} S)(\lambda, \gamma), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big)\\
=&\nu_T\Big(\bigwedge_{c \in X} \nu_T\big(R(\lambda, c), S(c, \gamma)\big), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big)\\
\geqslant&\nu_T\Big(\nu_T\big(R(\lambda, \lambda), S(\lambda, \gamma)\big), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \nu_T\Big(\nu_T\big(1_L, S(\lambda,\gamma)\big), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big) \\
&= \nu_T\Big(S(\lambda,\gamma), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big) \ \text{(by Proposition 1(2))} \\
&\geqslant \nu_T\Big(S(\lambda,\gamma), T\big((S \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (S \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big) \ \text{(since } R \leqslant S\text{)} \\
&= \nu_T\Big(S(\lambda,\gamma), T\big(S(\lambda \to \zeta, \gamma \to \zeta), S(\zeta \to \lambda, \zeta \to \gamma)\big)\Big) \\
&\geqslant \nu_T\big(S(\lambda,\gamma), S(\lambda,\gamma)\big) \\
&= 1_L.
\end{aligned}
$$

Thus,

$$\nu_T\Big((R \diamond_{\nu_T} S)(\lambda,\gamma), T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big)\Big) = 1_L.$$

This means $T\big((R \diamond_{\nu_T} S)(\lambda \to \zeta, \gamma \to \zeta), (R \diamond_{\nu_T} S)(\zeta \to \lambda, \zeta \to \gamma)\big) \geqslant (R \diamond_{\nu_T} S)(\lambda,\gamma)$. Hence, $R \diamond_{\nu_T} S$ is a TL-fuzzy congruence on X. □

Remark 6. *The above two propositions, respectively, demonstrate that TL-fuzzy congruences are closed under the $*_T$-operation and the $\diamond_{\nu_T}$-operation. The former requires the L-fuzzy relation to satisfy commutativity with respect to the $*_T$-operation, while the latter requires two fuzzy congruences to satisfy an order relation. These properties will play a significant role in further studying the interrelations between TL-fuzzy filters and TL-fuzzy congruences.*

Proposition 7. *Let R and S be two TL-fuzzy congruences on X. If $R *_T S = S *_T R$, then $P = R *_T S$ is the smallest TL-fuzzy congruence on X such that $R \diamond_{\nu_T} P \geqslant S$.*

Proof. Since R, S are TL-fuzzy congruences and the commutativity $R *_T S = S *_T R$ holds, we deduce from Proposition 5 that $P = R *_T S$ is a TL-fuzzy congruence on X. For each $\lambda, \gamma, \zeta \in X$, we obtain

$$T\big(S(\lambda,\gamma), R(\lambda,\zeta)\big) \leqslant \bigvee_{m \in X} T\big(R(\zeta,m), S(m,\gamma)\big).$$

This implies that

$$S(\lambda,\gamma) \leqslant \nu_T\Big(R(\lambda,\zeta), \bigvee_{m \in X} T\big(R(\zeta,m), S(m,\gamma)\big)\Big).$$

Thus, we obtain

$$
\begin{aligned}
\big(R \diamond_{\nu_T} (R *_T S)\big)(\lambda,\gamma) &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda,\zeta), (R *_T S)(\zeta,\gamma)\big) \\
&= \bigwedge_{\zeta \in X} \nu_T\Big(R(\lambda,\zeta), \bigvee_{m \in X} T\big(R(\zeta,m), S(m,\gamma)\big)\Big) \\
&\geqslant S(\lambda,\gamma).
\end{aligned}
$$

This shows that $R \diamond_{\nu_T} P \geqslant S$.

Let P^* be a TL-fuzzy congruence on X with $R \diamond_{\nu_T} P^* \geqslant S$. We aim to prove that $P = R *_T S \leqslant P^*$. In fact, since

$$\forall \lambda, \gamma, \zeta \in X, T\Big(R(\lambda,\zeta), \nu_T\big(R(\lambda,\zeta), P^*(\lambda,\gamma)\big)\Big) \leqslant P^*(\lambda,\gamma),$$

it follows that

$$R(\lambda,\zeta) \leqslant \nu_T\Big(\bigwedge_{m \in X} \nu_T\big(R(\zeta,m), P^*(m,\gamma)\big), P^*(\lambda,\gamma)\Big),$$

for all $\zeta \in X$. Thus, we obtain

$$\bigvee_{\zeta \in X} T\Big(R(\lambda,\zeta), \bigwedge_{m \in X} \nu_T\big(R(\zeta,m), P^*(m,\gamma)\big)\Big) \leqslant P^*(\lambda,\gamma).$$

This implies that

$$\begin{aligned}(R *_T S)(\lambda,\gamma) &= \bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,\gamma)\big)\\ &\leqslant \bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), (R \diamond_{\nu_T} P^*)(\zeta,\gamma)\big) \ \ \big(\text{since } S \leqslant R \diamond_{\nu_T} P^*\big)\\ &= \bigvee_{\zeta\in X} T\Big(R(\lambda,\zeta), \bigwedge_{m\in X} \nu_T\big(R(\zeta,m), P^*(m,\gamma)\big)\Big)\\ &\leqslant P^*(\lambda,\gamma).\end{aligned}$$

This establishes the fact that $P = R *_T S \leqslant P^*$. □

Proposition 8. *Let R and S be two TL-fuzzy congruences on X. If $R \leqslant S$, then $Q = R \diamond_{\nu_T} S$ is the largest TL-fuzzy congruence on X such that $R *_T Q \leqslant S$.*

Proof. Since R and S are TL-fuzzy congruences and $R \leqslant S$ holds, we deduce from Proposition 6 that $Q = R \diamond_{\nu_T} S$ is a TL-fuzzy congruence on X. For every $\lambda,\gamma,\zeta \in X$, we establish $T\Big(R(\zeta,\lambda), \nu_T\big(R(\zeta,\lambda), S(\lambda,\gamma)\big)\Big) \leqslant S(\lambda,\gamma)$, which implies

$$T\Big(R(\lambda,\zeta), \textstyle\bigwedge_{m\in X} \nu_T\big(R(\zeta,m), S(m,\gamma)\big)\Big) \leqslant S(\lambda,\gamma),$$

for all $m,\zeta \in X$. In view of the arbitrariness of ζ, we deduce that

$$\begin{aligned}\big(R *_T (R \diamond_{\nu_T} S)\big)(\lambda,\gamma) &= \bigvee_{\zeta\in X} T\Big(R(\lambda,\zeta), (R \diamond_{\nu_T} S)(\zeta,\gamma)\Big)\\ &= \bigvee_{\zeta\in X} T\Big(R(\lambda,\zeta), \bigwedge_{m\in X} \nu_T\big(R(\zeta,m), S(m,\gamma)\big)\Big)\\ &\leqslant S(\lambda,\gamma).\end{aligned}$$

This shows that $R *_T Q \leqslant S$.

Suppose that Q^* is a TL-fuzzy congruence on X such that $R *_T Q^* \leqslant S$. We shall demonstrate that $Q = R *_T S \geqslant Q^*$. In fact, since

$$T\big(Q^*(\lambda,\gamma), R(\lambda,\zeta)\big) \leqslant \textstyle\bigvee_{m\in X} T\big(R(\zeta,m), Q^*(m,\gamma)\big),$$

we obtain

$$Q^*(\lambda,\gamma) \leqslant \nu_T\Big(R(\lambda,\zeta), \textstyle\bigvee_{m\in X} T\big(R(\zeta,m), Q^*(m,\gamma)\big)\Big),$$

for all $\zeta \in X$. As a result, we conclude that

$$\begin{aligned}(R \diamond_{\nu_T} S)(\lambda,\gamma) &= \bigwedge_{\zeta\in X} \nu_T\big(R(\lambda,\zeta), S(\zeta,\gamma)\big)\\ &\geqslant \bigwedge_{\zeta\in X} \nu_T\big(R(\lambda,\zeta), (R *_T Q^*)(\zeta,\gamma)\big)\\ &= \bigwedge_{\zeta\in X} \nu_T\Big(R(\lambda,\zeta), \bigvee_{m\in X} T\big(R(\zeta,m), Q^*(m,\gamma)\big)\Big)\\ &\geqslant Q^*(\lambda,\gamma).\end{aligned}$$

This shows that $Q = R \diamond_{\nu_T} S \geqslant Q^*$. □

5. The Operational Correlation Between *TL*-Fuzzy Filters and *TL*-Fuzzy Congruences

Filters and congruences are two fundamental and important concepts in residuated lattices, and they can induce each other. This intrinsic connection has been extended to the fuzzy setting, where the interplay between TL-fuzzy filters and TL-fuzzy congruences forms a rich area of study. In [13], Wang systematically investigated this relationship within residuated lattices, establishing foundational correspondences. Building upon this work, the present section will further explore two specific types of operational properties for these structures. Our primary focus is to deepen the investigation into the operational correlations between TL-fuzzy filters and TL-fuzzy congruences, analyzing how operations defined on one structure translate or relate to operations on the other. As a necessary prelude to this analysis, we first introduce the two key propositions stated below, which serve as the starting point for our discussion.

Proposition 9. *Assume that A is a TL-fuzzy filter of X. Then the L-fuzzy relation R_A which is defined by the formula*

$$\forall \lambda, \gamma \in X,\ R_A(\lambda, \gamma) = T\big(A(\lambda \to \gamma), A(\gamma \to \lambda)\big)$$

is a TL-fuzzy congruence on X.

Proof. The proof is analogous to that of Theorem 6.2 in [13], and thus is omitted here. □

Proposition 10. *Let R be a TL-fuzzy congruence on X. If the L-fuzzy set A_R satisfies $A_R(\lambda) = R(\lambda, 1_X)$, for all $\lambda \in X$, then A_R forms a TL-fuzzy filter of X.*

Proof. The proof is analogous to that of Theorem 6.1 in [13], and thus is omitted here. □

Remark 7. *It should be noted that Wang [13] merely presented the method of mutual induction between TL-fuzzy filters and TL-fuzzy congruences on a residuated lattice, without proving that there is a one-to-one correspondence between them. In [14], the authors established that a bijective correspondence holds between TL-fuzzy filters and TL-fuzzy congruences in BL-algebras under the condition that the triangular norm T is idempotent. In fact, this result also holds in residuated lattices when T satisfies the idempotent property.*

Based on Propositions 9 and 10, we are able to derive the subsequent properties.

Proposition 11. *Suppose that A and B are two TL-fuzzy filters of X. Then, we obtain*

(1) $R_{A \otimes_T B} = R_A *_T R_B$;
(2) $A \leqslant B$ *implies* $R_{A \to_{\nu_T} B} = R_A \diamond_{\nu_T} R_B$.

Proof. (1) Let A and B be TL-fuzzy filters of X. Then, relying on Proposition 2, it can be established that $A \otimes_T B$ is a TL-fuzzy filter. Since $\lambda \geqslant \lambda \otimes 1_X$, we obtain

$$\begin{aligned}(A \otimes_T B)(\lambda) &= \bigvee_{\lambda \geqslant m \otimes n} T\big(A(m), B(n)\big) \\ &\geqslant T\big(A(\lambda), B(1_X)\big) \\ &= T\big(A(\lambda), 1_L\big) \\ &= A(\lambda).\end{aligned}$$

This implies that $A \otimes_T B \geqslant A$.

For each $\lambda, \gamma \in X$, it is readily derived from Proposition 9 that

$$\begin{aligned} R_{A\otimes_T B}(\lambda,\gamma) &= T\big((A\otimes_T B)(\lambda\to\gamma),(A\otimes_T B)(\gamma\to\lambda)\big) \\ &\geqslant T\big(A(\lambda\to\gamma),A(\gamma\to\lambda)\big) \ \ \big(\text{since } A\otimes_T B\geqslant A\big) \\ &= R_A(\lambda,\gamma). \end{aligned}$$

Analogously, we have $R_{A\otimes_T B}(\lambda,\gamma)\geqslant R_B(\lambda,\gamma)$.
This means that $R_{A\otimes_T B}\geqslant R_A$ and $R_{A\otimes_T B}\geqslant R_B$. It follows that

$$R_{A\otimes_T B} *_T R_{A\otimes_T B}\geqslant R_A *_T R_B.$$

Notice that $R_{A\otimes_T B}$ is a TL-fuzzy congruence of X, Proposition 4 allows us to derive

$$R_{A\otimes_T B} *_T R_{A\otimes_T B} = R_{A\otimes_T B}.$$

Therefore, $R_{A\otimes_T B}\geqslant R_A *_T R_B$.
Next, we intend to demonstrate that $R_{A\otimes_T B}\leqslant R_A *_T R_B$. For each $a, b, m, n \in X$ with

$$\lambda\to\gamma\geqslant a\otimes b \text{ and } \ \gamma\to\lambda\geqslant m\otimes n,$$

we obtain that

$$\lambda\otimes a\otimes b\leqslant\gamma \text{ and } \ \gamma\otimes m\otimes n\leqslant\lambda.$$

Let $p=(\lambda\otimes a)\vee(\gamma\otimes n)$. Then, $\lambda\otimes a\leqslant p$. This thereby entails that $a\leqslant\lambda\to p$. Similarly, we deduce that $n\leqslant\gamma\to p$. Notice that

$$\begin{aligned} p\otimes m &= \big((\lambda\otimes a)\vee(\gamma\otimes n)\big)\otimes m \\ &= (\lambda\otimes a\otimes m)\vee(\gamma\otimes n\otimes m) \\ &\leqslant \lambda\vee\lambda \\ &= \lambda, \end{aligned}$$

so we obtain $m\leqslant p\to\lambda$. Similarly, it holds that $b\leqslant p\to\gamma$. On the grounds that A and B are TL-fuzzy filters, we are able to conclude that

$$\begin{aligned} (R_A *_T R_B)(\lambda,\gamma) &= \bigvee_{\zeta\in X} T\big(R_A(\lambda,\zeta),R_B(\zeta,\gamma)\big) \\ &= \bigvee_{\zeta\in X} T\Big(T\big(A(\lambda\to\zeta),A(\zeta\to\lambda)\big),T\big(B(\zeta\to\gamma),B(\gamma\to\zeta)\big)\Big) \\ &\geqslant T\Big(T\big(A(\lambda\to p),A(p\to\lambda)\big),T\big(B(p\to\gamma),B(\gamma\to p)\big)\Big) \\ &\geqslant \bigvee_{\lambda\to\gamma\geqslant a\otimes b,\gamma\to\lambda\geqslant m\otimes n} T\Big(T\big(A(a),A(m)\big),T\big(B(b),B(n)\big)\Big) \\ &= T\Big(\bigvee_{\lambda\to\gamma\geqslant a\otimes b} T\big(A(a),B(b)\big),\bigvee_{\gamma\to\lambda\geqslant m\otimes n} T\big(A(m),B(n)\big)\Big) \\ &= T\big((A\otimes_T B)(\lambda\to\gamma),(A\otimes_T B)(\gamma\to\lambda)\big) \\ &= R_{A\otimes_T B}(\lambda,\gamma). \end{aligned}$$

This means that $R_A *_T R_B\geqslant R_{A\otimes_T B}$. Hence, $R_{A\otimes_T B}=R_A *_T R_B$.
(2) Let A and B be TL-fuzzy filters of X. Consequently, R_A and R_B are TL-fuzzy congruences. Since $A\leqslant B$, we deduce from Lemma 2 and Proposition 3 that $A\to_{\nu_T} B\geqslant B$ and $A\diamond_{\nu_T} B$ forms a TL-fuzzy filter. This implies that $R_A\leqslant R_B\leqslant R_{A\to_{\nu_T} B}$.

For each $\lambda, \gamma \in X$, we obtain

$$\begin{aligned}(R_A \diamond_{\nu_T} R_B)(\lambda,\gamma) &= \bigwedge_{\zeta\in X} \nu_T\big(R_A(\lambda,\zeta), R_B(\zeta,\gamma)\big)\\ &\leqslant \nu_T\big(R_A(\lambda,\lambda), R_B(\lambda,\gamma)\big)\\ &= 1_L \to R_B(\lambda,\gamma)\\ &= R_B(\lambda,\gamma).\end{aligned}$$

This means that $R_A \diamond_{\nu_T} R_B \leqslant R_B \leqslant R_{A\to_{\nu_T}B}$.

Next, we need to show that $R_{A\to_{\nu_T}B} \leqslant R_A \diamond_{\nu_T} R_B$. In fact, for each $\lambda, \gamma, \zeta \in X$, we obtain

$$\lambda \to \gamma \leqslant (\zeta \to \lambda) \to (\zeta \to \gamma)$$

and

$$\gamma \to \lambda \leqslant (\lambda \to \zeta) \to (\gamma \to \zeta).$$

It can be deduced that

$$\begin{aligned}R_{A\to_{\nu_T}B}(\lambda,\gamma) &= T\big((A \to_{\nu_T} B)(\lambda\to\gamma), (A\to_{\nu_T} B)(\gamma\to\lambda)\big)\\ &= T\Big(\bigwedge_{\lambda\to\gamma\leqslant a\to b} \nu_T\big(A(a),B(b)\big), \bigwedge_{\gamma\to\lambda\leqslant m\to n} \nu_T\big(A(m),B(n)\big)\Big)\\ &\leqslant \bigwedge_{\lambda\to\gamma\leqslant a\to b,\gamma\to\lambda\leqslant m\to n} T\Big(\nu_T\big(A(a),B(b)\big), \nu_T\big(A(m),B(n)\big)\Big)\\ &\leqslant T\Big(\nu_T\big(A(\zeta\to\lambda),B(\zeta\to\gamma)\big), \nu_T\big(A(\lambda\to\zeta),B(\gamma\to\zeta)\big)\Big)\\ &\leqslant \nu_T\Big(T\big(A(\zeta\to\lambda),A(\lambda\to\zeta)\big), T\big(B(\zeta\to\gamma),B(\gamma\to\zeta)\big)\Big)\\ &= \nu_T\big(R_A(\lambda,\zeta),R_B(\zeta,\gamma)\big).\end{aligned}$$

By the arbitrariness of ζ, we obtain

$$R_{A\to_{\nu_T}B}(\lambda,\gamma) \leqslant \bigwedge_{\zeta\in X} \nu_T\big(R_A(\lambda,\zeta),R_B(\zeta,\gamma)\big) = R_A \diamond_{\nu_T} R_B(\lambda,\gamma).$$

This implies that $R_{A\to_{\nu_T}B} \leqslant R_A \diamond_{\nu_T} R_B$. Hence, $R_{A\to_{\nu_T}B} = R_A \diamond_{\nu_T} R_B$. The proof is completed. □

Proposition 12. *Suppose that R and S are TL-fuzzy congruences on X. Then, we obtain*

(1) *$R *_T S = S *_T R$ implies $A_{R*_TS} = A_R \otimes_T A_S$;*
(2) *$R \leqslant S$ implies $A_{R\diamond_{\nu_T}S} = A_R \to_{\nu_T} A_S$.*

Proof. (1) Let R and S be TL-fuzzy congruences on X. Since the commutativity $R *_T S = S *_T R$ holds, we deduce from Proposition 5 that $R *_T S$ is a TL-fuzzy congruence. Proposition 10 allows us to derive A_{R*_TS} is a TL-fuzzy filter of X. For each $\lambda \in X$, we obtain

$$A_{R*_TS}(\lambda) = (R *_T S)(\lambda, 1_X) = \bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,1_X)\big)$$

and

$$(A_R \otimes_T A_S)(\lambda) = \bigvee_{\lambda\geqslant m\otimes n} T\big(A_R(m),A_S(n)\big) = \bigvee_{\lambda\geqslant m\otimes n} T\big(R(m,1_X),S(n,1_X)\big).$$

In order to establish the validity of $A_{R*_TS} = A_R \otimes_T A_S$, it is necessary to prove that

$$\bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,1_X)\big) = \bigvee_{\lambda\geqslant a\otimes b} T\big(R(m,1_X), S(n,1_X)\big).$$

In fact, since R is a TL-fuzzy congruence, we establish from Definition 6 that

$$\begin{aligned} R(\zeta\to\lambda,1_X) &= R(\zeta\to\lambda,\zeta\to\zeta)\\ &\geqslant T\big(R(\zeta\to\lambda,\zeta\to\zeta), R(\lambda\to\zeta,\zeta\to\zeta)\big)\\ &\geqslant R(\lambda,\zeta). \end{aligned}$$

It is worth noting that for all $\lambda,\zeta\in X$, the relation $\lambda\geqslant(\zeta\to\lambda)\otimes\zeta$ holds, so we can infer that

$$\begin{aligned} &\nu_T\Big(\bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,1_X)\big), \bigvee_{\lambda\geqslant m\otimes n} T\big(R(m,1_X), S(n,1_X)\big)\Big)\\ =&\bigwedge_{\zeta\in X} \nu_T\Big(T\big(R(\lambda,\zeta), S(\zeta,1_X)\big), \bigvee_{\lambda\geqslant m\otimes n} T\big(R(m,1_X), S(n,1_X)\big)\Big)\\ \geqslant&\bigwedge_{\zeta\in X}\bigvee_{\lambda\geqslant m\otimes n} \nu_T\Big(T\big(R(\lambda,\zeta), S(\zeta,1_X)\big), T\big(R(m,1_X), S(n,1_X)\big)\Big) \ \text{(by Proposition 1(3))}\\ \geqslant&\bigwedge_{\zeta\in X}\bigvee_{\lambda\geqslant m\otimes n} T\Big(\nu_T\big(R(\lambda,\zeta), R(m,1_X)\big), \nu_T\big(S(\zeta,1_X), S(n,1_X)\big)\Big) \ \text{(by Proposition 1(6))}\\ \geqslant&\bigwedge_{\zeta\in X} T\Big(\nu_T\big(R(\lambda,\zeta), R(\zeta\to\lambda,1_X)\big), \nu_T\big(S(\zeta,1_X), S(\zeta,1_X)\big)\Big) \ \text{(since } \lambda\geqslant(\zeta\to\lambda)\otimes\zeta\text{)}\\ =&\bigwedge_{\zeta\in X} T\Big(\nu_T\big(R(\lambda,\zeta), R(\zeta\to\lambda,1_X)\big), 1_L\Big)\\ =&\bigwedge_{\zeta\in X} \nu_T\big(R(\lambda,\zeta), R(\zeta\to\lambda,1_X)\big)\\ =&1_L. \end{aligned}$$

This implies that

$$\begin{aligned} A_{R*_TS}(\lambda) &= \bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,1_X)\big)\\ &\leqslant \bigvee_{\lambda\geqslant m\otimes n} T\big(R(m,1_X), S(n,1_X)\big)\\ &= (A_R\otimes_T A_S)(\lambda), \end{aligned}$$

for all $\lambda\in X$. Hence, $A_{R*_TS}\leqslant A_R\otimes_T A_S$.

Next, we need to show that $A_{R*_TS}\geqslant A_R\otimes_T A_S$. In fact, since

$$\begin{aligned} A_{R*_TS}(\lambda) &= (R*_T S)(\lambda,1_X)\\ &= \bigvee_{\zeta\in X} T\big(R(\lambda,\zeta), S(\zeta,1_X)\big)\\ &\geqslant T\big(R(\lambda,1_X), S(1_X,1_X)\big)\\ &= R(\lambda,1_X)\\ &= A_R(\lambda). \end{aligned}$$

This means that $A_{R*_TS}\geqslant A_R$. Similarly, it can be shown that $A_{R*_TS}\geqslant A_S$.

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Notice that $A_{R*_T S}$ is a TL-fuzzy filter of X, so we can deduce that

$$\begin{aligned}(A_R \otimes_T A_S)(\lambda) &= \bigvee_{\lambda \geqslant m \otimes n} T\big(A_R(m), A_S(n)\big) \\ &\leqslant \bigvee_{\lambda \geqslant m \otimes n} T\big(A_{R*_T S}(m), A_{R*_T S}(n)\big) \\ &\leqslant \bigvee_{\lambda \geqslant m \otimes n} A_{R*_T S}(m \otimes n) \ \ \big(\text{by Definition 4(3)}\big) \\ &\leqslant A_{R*_T S}(\lambda) \ \ \big(\text{by Definition 4(2)}\big).\end{aligned}$$

This implies that $A_R \otimes_T A_S \leqslant A_{R*_T S}$. Therefore, $A_{R*_T S} = A_R \otimes_T A_S$.

(2) Let R and S be TL-fuzzy congruences on X. Since $R \leqslant S$, it follows from Propositions 6 and 10 that $R \diamond_{\nu_T} S$ is a TL-fuzzy congruence and $A_{R \diamond_{\nu_T} S}$ is a TL-fuzzy filter.

Notice that A_R, A_S are TL-fuzzy filters and $A_R \leqslant A_S$, so we obtain

$$A_S \leqslant A_R \to_{\nu_T} A_S$$

from Lemma 1.

For any $\lambda \in X$, since

$$\begin{aligned}A_{R \diamond_{\nu_T} S}(\lambda) &= (R \diamond_{\nu_T} S)(\lambda, 1_X) \\ &= \bigwedge_{\zeta \in X} \nu_T\big(R(\lambda, \zeta), S(\zeta, 1_X)\big) \\ &\leqslant \nu_T\big(R(\lambda, \lambda), S(\lambda, 1_X)\big) \\ &= \nu_T\big(1_L, S(\lambda, 1_X)\big) \\ &= S(\lambda, 1_X) \\ &= A_S(\lambda).\end{aligned}$$

This means that $A_{R \diamond_{\nu_T} S} \leqslant A_S$. Hence, we obtain $A_{R \diamond_{\nu_T} S} \leqslant A_R \to_{\nu_T} A_S$.

Next, we need to prove that

$$A_R \to_{\nu_T} A_S \leqslant A_{R \diamond_{\nu_T} S}.$$

In fact, since R is TL-fuzzy congruence, it follows from Definition 6 that

$$\begin{aligned}R(\lambda \to \zeta, 1_X) &= R(\lambda \to \zeta, \zeta \to \zeta) \\ &\geqslant T\big(R(\lambda \to \zeta, \zeta \to \zeta), R(\zeta \to \lambda, \zeta \to \zeta)\big) \ \ \big(\text{by Definition 2(T3) and (T4)}\big) \\ &\geqslant R(\lambda, \zeta), \qquad \big(\text{by Definition 6(FC5)}\big)\end{aligned}$$

for all $\lambda, \zeta \in X$.

Thus,

$$\nu_T\big(R(\lambda \to \zeta, 1_X), S(\zeta, 1_X)\big) \leqslant \nu_T\big(R(\lambda, \zeta), S(\zeta, 1_X)\big).$$

Notice that $\lambda \leqslant (\lambda \to \zeta) \to \zeta$, for all $\lambda, \zeta \in X$, so we can deduce that

$$\begin{aligned}&\nu_T\Big(\big(A_R \to_{\nu_T} A_S\big)(\lambda), A_{R \diamond_{\nu_T} S}(\lambda)\Big) \\ &= \nu_T\Big(\bigwedge_{\lambda \leqslant m \to n} \nu_T\big(A_R(m), A_S(n)\big), (R \diamond_{\nu_T} S)(\lambda, 1_X)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \nu_T\Big(\bigwedge_{\lambda\leqslant m\to n} \nu_T\big(R(m,1_X),S(n,1_X)\big), \bigwedge_{\zeta\in X} \nu_T\big(R(\lambda,\zeta),S(\zeta,1_X)\big)\Big)\\
&= \bigwedge_{\zeta\in X} \nu_T\Big(\bigwedge_{\lambda\leqslant m\to n} \nu_T\big(R(m,1_X),S(n,1_X)\big), \nu_T\big(R(\lambda,\zeta),S(\zeta,1_X)\big)\Big) \ \text{(by Proposition 1(9))}\\
&\geqslant \bigwedge_{\zeta\in X} \nu_T\Big(\nu_T\big(R(\lambda\to\zeta,1_X),S(\zeta,1_X)\big), \nu_T\big(R(\lambda,\zeta),S(\zeta,1_X)\big)\Big) \ \text{(since } \lambda\leqslant(\lambda\to\zeta)\to\zeta\text{)}\\
&= 1_L.
\end{aligned}
$$

It holds that $\nu_T\Big(\big(A_R\to_{\nu_T} A_S\big)(\lambda), A_{R\diamond_{\nu_T}S}(\lambda)\Big) = 1_L$. This means that

$$\big(A_R\to_{\nu_T} A_S\big)(\lambda)\leqslant A_{R\diamond_{\nu_T}S}(\lambda),$$

for all $\lambda\in X$. Hence, $A_R\to_{\nu_T} A_S\leqslant A_{R\diamond_{\nu_T}S}$. The proof is completed. □

6. Conclusions

As we all know, filters and congruences play vital roles in residuated lattices, as many of their important properties are built upon these two concepts. Currently, research on fuzzy filters and fuzzy congruences in residuated lattices primarily focuses on their equivalent characterizations, mutual induction, quotient structures induced by fuzzy congruences, and homomorphism theorems, with few researchers addressing their operational properties. In this paper, we mainly investigate the properties of TL-fuzzy filters and TL-fuzzy congruences on residuated lattices under two types of operations. Firstly, based on a left-continuous triangular norm on a complete lattice, we define the $\otimes_T$-operation and $\to_{\nu_T}$-operation for two L-fuzzy sets on a residuated lattice and study the relevant properties of TL-fuzzy filters under these two operations. Secondly, we discuss the preservation of congruence properties under the $*_T$-composition and $\diamond_{\nu_T}$-composition operations for TL-fuzzy congruences on residuated lattices, along with their operational laws. Finally, leveraging the characteristic that TL-fuzzy filters and TL-fuzzy congruences can mutually induce each other, we explore the operational properties between the $\otimes_T$-operation of TL-fuzzy filters and the $*_T$-composition of fuzzy congruences, as well as between the $\to_{\nu_T}$-operation of TL-fuzzy filters and the $\diamond_{\nu_T}$-composition of TL-fuzzy congruences, respectively. These results not only further refine the theory of fuzzy filters and fuzzy congruences in residuated lattices but also provide a new perspective for the study of residuated lattices and other logical algebras. Thanks to the results presented in this paper, we will consider the following problems in the future.

(1) Homomorphisms are powerful tools for investigating the properties of algebraic structures. This paper primarily focuses on two types of operational properties of TL-fuzzy filters and TL-fuzzy congruences, with no consideration given to their properties related to homomorphisms. Residuated lattices possess a rich set of operations, and thus the subsequent study of the homomorphic properties of TL-fuzzy filters and TL-fuzzy congruences will be of great significance.

(2) As a special class of residuated lattices, MV-algebras serve as the core models for fuzzy logic and non-classical logical algebras. Ideals and congruences are important substructures for describing and characterizing MV-algebras. In our future work, we will investigate the relevant operational properties of L-fuzzy ideals and L-fuzzy congruences in MV-algebras, including their homomorphic properties.

(3) In [18], by integrating the theories of residuated lattices and fuzzy rough sets, the authors investigated the basic properties of fuzzy rough approximation operators based on L-fuzzy filters. However, this work lacks further in-depth discussion. In our future

research, we will apply the results obtained in this paper to the relevant studies on fuzzy rough approximation operators.

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