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The Euler Sombor Index and the Average Sombor Index of the Cozero-Divisor Graph over the Ring $\mathbb{Z}_n$

Amal S. Alali ¹, Muzibur Rahman Mozumder ² and Mohd Salman Ali ^{2,*}¹ Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalali@pnu.edu.sa² Department of Mathematics, Faculty of Science, Aligarh Muslim University, Aligarh 202002, India; muzibamu81@gmail.com

* Correspondence: salmanmath17@gmail.com

Abstract

Let $\mathcal{Z}(\mathcal{R})'$ denote the set of all elements in the ring $\mathcal{R}$ that are neither zero nor units, where $\mathcal{R}$ is assumed to be a commutative ring with a multiplicative identity satisfying $1 \neq 0$. Two distinct vertices w and κ are defined to be adjacent if and only if κ does not lie in the ideal generated by w in $\mathcal{R}$, that is, $\kappa \notin w\mathcal{R}$, and simultaneously, w does not lie in the ideal generated by κ in $\mathcal{R}$, that is, $w \notin \kappa\mathcal{R}$. The cozero-divisor graph of $\mathcal{R}$, denoted by $\Gamma'(\mathcal{R})$, is an undirected graph in which the vertices are given by the set $\mathcal{Z}(\mathcal{R})'$. This article presents a comprehensive evaluation of both the Euler Sombor index and the average Sombor index for the graphs $\Gamma'(\mathbb{Z}_n)$ corresponding to various values of n .

Keywords: cozero-divisor graph; the Euler Sombor index; the average Sombor index; ring of integer modulo n

MSC: 05C12; 05C25; 05C50; 15A18

1. Introduction

Unless stated otherwise, the notation $\mathcal{R}$ shall denote a commutative ring with identity, where $1 \neq 0$, throughout this paper. The ideal generated by an element κ is denoted by $\kappa\mathcal{R} = \{\kappa c : c \in \mathcal{R}\}$. The set of all non-zero and non-unit elements of $\mathcal{R}$ is represented as $\mathcal{Z}(\mathcal{R})'$. For any positive integer n , $\mathbb{Z}_n$ refers to the ring of integers modulo n .

The graph $\mathbb{G}$ is denoted by the pair $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ denotes the vertex set and $\mathbb{E}$ denotes the edge set of $\mathbb{G}$. In the graph $\mathbb{G}$, if y_1 and y_2 are distinct vertices, the notation $y_1 \sim y_2$ signifies that y_1 is adjacent to y_2 . A complete graph with m vertices is denoted by K_m . The collection of vertices in $\mathbb{G}$ that are connected to the vertex y is called its neighbourhood and is denoted by $\mathcal{N}_{\mathbb{G}}(y)$. The notation $\deg(y)$ denotes the *degree* of vertex y , which specifies how many edges are connected to the vertex $y \in \mathbb{V}$. If vertex y has degree zero, it is referred to as an *isolated* vertex. If every vertex in the graph $\mathbb{G}$ specifically has the same number of connections, k then the graph $\mathbb{G}$ is known as *k-regular*.

The Sombor index, denoted as $SO(\mathbb{G})$, is a recently introduced topological index by Gutman [1]. It is defined for a graph $\mathbb{G}$ as the sum, over all pairs of adjacent vertices v_i and v_j , of the square root of the sum of the squares of their degrees:

$$SO(\mathbb{G}) = \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2}.$$



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In simple terms, to compute the Sombor index, take every edge in the graph, look at the degrees of its two endpoints, calculate the square root of the sum of their squared degrees, and then add up these values for all edges in the graph.

Further information about the Sombor index is available in references [1–8]. The Euler Sombor index, written as $EUS(\mathbb{G})$, and the average Sombor index, denoted $SO_{avr}(\mathbb{G})$, were recently introduced by Gutman. The Euler Sombor index is defined for a graph $\mathbb{G}$ by adding up, over all edges, the square root of the sum the squares of the degrees of the two endpoints plus their product:

$$EUS(\mathbb{G}) = \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)}.$$

Meanwhile, the average Sombor index $SO_{avr}(\mathbb{G})$ is calculated by taking every edge, subtracting the average degree $\frac{2m'}{n'}$ from each endpoint's degree, computing the square root of the sum, and adding these values for all edges:

$$SO_{avr} = SO_{avr}(\mathbb{G}) = \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2}$$

where m', n' denotes cardinality of edges, cardinality of vertices of graph $\mathbb{G}$, respectively.

In other words, these two indices capture new ways of measuring graph structure, by combining the degrees of adjacent vertices through geometric formulas.

Afkhami et al. [9] was the first to introduce the concept of the cozero-divisor graph for commutative rings in their work. The cozero-divisor graph of $\mathcal{R}$, denoted as $\mathcal{Z}(\mathcal{R})'$, has vertices from the set $\mathcal{Z}(\mathcal{R})'$. Two distinct vertices w and κ are joined by an edge only if w is not contained in $\kappa\mathcal{R}$, i.e., $w \notin \kappa\mathcal{R}$ and κ is not contained in $w\mathcal{R}$, i.e., $\kappa \notin w\mathcal{R}$. For more details on the cozero-divisor graph, readers may consult references [10–15].

Gursoy and collaborators [16] computed the Sombor index for the zero-divisor graph of the ring $\mathbb{Z}_n$ for a range of values of n in their study. Section 2 provides an overview of essential ideas that form the basis for the main results. In Section 3, we calculate the reduced Sombor index of $\Gamma'(\mathbb{Z}_n)$ in the cases where $n = \zeta_1\zeta_2, \zeta_1^2\zeta_2, \zeta_1^2\zeta_2^2$ and $\zeta_1\zeta_2\zeta_3$, with ζ_1, ζ_2 and ζ_3 being distinct prime numbers. Section 4 presents the computation of the Albertson index for $\Gamma'(\mathbb{Z}_n)$, in the cases where n takes the forms $n = \zeta_1\zeta_2, \zeta_1^2\zeta_2, \zeta_1^2\zeta_2^2$, and $\zeta_1\zeta_2\zeta_3$, with ζ_1, ζ_2 and ζ_3 being distinct primes.

2. Preliminaries

We begin by introducing the definition of a generalized join graph and outlining several established results that form the foundation for our main findings.

Definition 1. Consider the disjoint graphs $H_k(\mathbb{V}_k, \mathbb{E}_k)$ for $1 \leq k \leq m$, each having order m_k . Let $\mathbb{G}(\mathbb{V}, \mathbb{E})$ be a graph of order m whose vertex set is $\{v_1, v_2, \dots, v_m\}$. The generalized join graph $\mathbb{G}[H_1, H_2, \dots, H_m]$ is constructed by connecting the graphs $H_1, H_2, \dots, H_m$ together. Whenever k and l are adjacent in $\mathbb{G}$, every vertex from H_k is connected to every vertex from H_l .

For a positive integer n , the number of positive divisor of n is given by $\tau(n)$. We write $x|n$ to denotes x divides n and (x, n) denotes the gcd of x and n . An integer x divides n for $1 \leq x \leq n$ if and only if x is a proper divisor of n . Euler's phi function, $\phi(n)$ is defined as the set of positive integers less than or equal to n that are co-prime to n , i.e., $\phi(n) = \{\kappa \in \mathbb{Z}^+ | \kappa \leq n \text{ and } (\kappa, n) = 1\}$. The prime factorization of n is $n = c_1^{h_1} c_2^{h_2} \dots c_k^{h_k}$, where the integers $h_1, h_2, \dots, h_k$ are positive and $c_1, c_2, \dots, c_k$ are distinct primes.

Denote the proper divisors of n by $x_1, x_2, \dots, x_k$. Let $A_{x_r} = \{s \in \mathbb{Z}_n : (s, n) = x_r\}$ for all r satisfying $1 \leq r \leq k$. It is observed that $A_{x_r} \cap A_{x_s} = \emptyset$, whenever $r \neq s$. Hence, the sets $A_{x_1}, A_{x_2}, \dots, A_{x_k}$ are pairwise disjoint, and therefore

$$\mathbb{V}(\Gamma'(\mathbb{Z}_n)) = A_{x_1} \cup A_{x_2} \cup \dots \cup A_{x_k}.$$

Lemma 1 ([17] Lemma 3.3). Suppose $a \in A_{c_i}$ and $b \in A_{c_j}$ for some $i, j \in \{1, 2, \dots, k\}$. Then, in $\Gamma'(\mathbb{Z}_n)$, the vertices a and b are adjacent, i.e., $a \sim b$ if and only if neither c_i divides c_j , i.e., $c_i \nmid c_j$ nor c_j divides c_i , i.e., $c_j \nmid c_i$.

Define δ_n as the simple graph whose vertices are the unique proper divisors of n , denoted by $c_1, c_2, \dots, c_k$. Two distinct vertices c_i and c_j in δ_n are adjacent precisely when neither c_i divides c_j nor c_j divides c_i . If n has prime factorization $n = \zeta_1^{n_1} \zeta_2^{n_2} \dots \zeta_r^{n_r}$, then the number of vertices of the graph δ_n is given by

$$|\mathbb{V}(\delta_n)| = \prod_{i=1}^r (n_i + 1) - 2.$$

Lemma 2 ([17] Lemma 3.6). Let $c_1, c_2, \dots, c_k$ be the proper divisors of n . Then, $\Gamma'(\mathbb{Z}_n) = \delta_n[\Gamma'(A_{c_1}), \Gamma'(A_{c_2}), \dots, \Gamma'(A_{c_k})]$.

The following lemma determines the cardinality of A_{c_i} :

Lemma 3 ([15]). For each divisor c_i of n the set A_{c_i} has cardinality $\phi(\frac{n}{c_i})$, for $1 \leq i \leq k$.

Remark 1. Let $c_1, c_2, \dots, c_k$ be the prime divisors of n . For $1 \leq i \leq k$, throughout the paper, we consider the following sets $A_{c_i} = \{x \in \mathbb{Z}_n : \gcd(x, n) = c_i\}$.

Corollary 1 ([17] Corollary 3.4). The following statements hold:

- (1) For all i in the range 1 to k , the induced subgraph $\Gamma'(A_{c_i})$ within $\Gamma'(\mathbb{Z}_n)$ is isomorphic to $\overline{K}_{\phi(\frac{n}{c_i})}$, where $\overline{K}_{\phi(\frac{n}{c_i})}$ is a complement of the complete graph $K_{\phi(\frac{n}{c_i})}$.
- (2) For any distinct indices $i, j, i \in \{1, 2, \dots, k\}$, every vertex in $\Gamma'(A_{c_i})$ is adjacent to either all vertices in $\Gamma'(A_{c_j})$ or none of them.

3. The Euler Sombor Index of Cozero-Divisor Graph $\Gamma'(\mathbb{Z}_n)$

This section is devoted to computing the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$.

Theorem 1. For distinct prime numbers ζ_1 and ζ_2 , the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2})$ is given as

$$EUS(\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2})) = (\zeta_1 - 1)(\zeta_2 - 1) \sqrt{(\zeta_1 - 1)^2 + (\zeta_2 - 1)^2 + (\zeta_1 - 1)(\zeta_2 - 1)}.$$

Proof. Consider $n = \zeta_1 \zeta_2$, where ζ_1 and ζ_2 are distinct primes; in this case, the numbers ζ_1 and ζ_2 form the proper divisors of n . By Lemma 1, $\zeta_1 \sim \zeta_2$ and in view of Lemma 2, we have $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2}) = \delta_{\zeta_1 \zeta_2}[\Gamma'(A_{\zeta_1}), \Gamma'(A_{\zeta_2})]$. By Lemma 3, $|A_{\zeta_1}| = \phi(\zeta_2)$ and $|A_{\zeta_2}| = \phi(\zeta_1)$. Here, $|A_{\zeta_1}| |A_{\zeta_2}|$ is the number of edges. For $v_i \in A_{\zeta_1}$, $\deg(v_i) = \zeta_1 - 1$ and for $v_j \in A_{\zeta_2}$, $\deg(v_j) = \zeta_2 - 1$. We have

$$\begin{aligned}
EUS(\Gamma'(\mathbb{Z}_{\xi_1\xi_2})) &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
&= \sum_{v_i \sim v_j} \sqrt{(\xi_1 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1 - 1)(\xi_2 - 1)} \\
&= |A_{\xi_1}| |A_{\xi_2}| \sqrt{(\xi_1 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1 - 1)(\xi_2 - 1)} \\
&= (\xi_1 - 1)(\xi_2 - 1) \sqrt{(\xi_1 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1 - 1)(\xi_2 - 1)}.
\end{aligned}$$

□

Example 1. Let $n = 6 = 2 \cdot 3$, where 2 and 3 are distinct primes. Here, 2 and 3 are the proper divisors of n . By Lemma 1, $\delta_6 : 2 \sim 3$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_6) = \delta_6[\Gamma'(A_2), \Gamma'(A_3)]$.

By Lemma 3, $|A_2| = \phi(\frac{6}{2}) = 2$ and $|A_3| = \phi(\frac{6}{3}) = 1$. If $2 \in A_2$, $2 \sim 3$, then $\deg(2) = |A_3| = 1$ and if $3 \in A_3$, $3 \sim 2$, then $\deg(3) = |A_2| = 2$. Thus, the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_6)$ is:

$$\begin{aligned}
EUS(\Gamma'(\mathbb{Z}_6)) &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
&= \sum_{2 \sim 3} \sqrt{(\deg(2))^2 + (\deg(3))^2 + \deg(2) \deg(3)} \\
&= \sum_{2 \sim 3} \sqrt{(1)^2 + (2)^2 + 1 \cdot 2} \\
&= |A_2| |A_3| \sqrt{1 + 4 + 2} \\
&= 2\sqrt{7}.
\end{aligned}$$

Given two distinct prime numbers ξ_1 and ξ_2 , the subsequent theorem presents the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$:

Theorem 2. For two distinct prime numbers ξ_1 and ξ_2 , it follows that the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$ is

$$\begin{aligned}
EUS(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})) &= (\xi_1 - 1)(\xi_2 - 1) \\
&\quad \left[(\xi_1^2 - \xi_1) \sqrt{(\xi_1^2 - \xi_1)^2 + (\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - \xi_1)(\xi_1\xi_2 - \xi_1)} \right. \\
&\quad + \xi_1 \sqrt{(\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1\xi_2 - \xi_1)(\xi_1^2 - 1)} \\
&\quad \left. + \sqrt{(\xi_1^2 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1^2 - 1)(\xi_2 - 1)} \right].
\end{aligned}$$

Proof. If $n = \xi_1^2\xi_2$ with distinct primes ξ_1 and ξ_2 , then its proper divisors are ξ_1, ξ_2, ξ_1^2 and $\xi_1\xi_2$. So, by Lemma 1, $\delta_{\xi_1^2\xi_2} : \xi_1 \sim \xi_2 \sim \xi_1^2 \sim \xi_1\xi_2$ and by Lemma 2, we have $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2}) = \delta_{\xi_1^2\xi_2}[\Gamma'(A_{\xi_1}), \Gamma'(A_{\xi_2}), \Gamma'(A_{\xi_1^2}), \Gamma'(A_{\xi_1\xi_2})]$. According to Corollary 1, we obtain

$$\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2}) = \delta_{\xi_1^2\xi_2}[\overline{K}_{\phi(\xi_1\xi_2)}, \overline{K}_{\phi(\xi_1^2)}, \overline{K}_{\phi(\xi_2)}, \overline{K}_{\phi(\xi_1)}].$$

By Lemma 3, $|A_{\xi_1}| = (\xi_1 - 1)(\xi_2 - 1)$, $|A_{\xi_2}| = \xi_1^2 - \xi_1$, $|A_{\xi_1^2}| = \xi_2 - 1$, $|A_{\xi_1\xi_2}| = \xi_1 - 1$.

If $v \in A_{\xi_1}$, $\xi_1 \sim \xi_2$, then $\deg(v)$ is the cardinality of A_{ξ_2} . i.e., $\deg(v) = |A_{\xi_2}|$, similarly if $v \in A_{\xi_2}$, $\xi_1 \sim \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_1^2}|$, if $v \in A_{\xi_1^2}$, $\xi_2 \sim \xi_1^2 \sim \xi_1\xi_2$, then

$\deg(v) = |A_{\xi_2}| + |A_{\xi_1\xi_2}|$, if $v \in A_{\xi_1\xi_2}$, $\xi_1^2 \sim \xi_1\xi_2$, then $\deg(v) = |A_{\xi_1^2}|$, so the degree of vertex v is defined as

$$\deg(v) = \begin{cases} |A_{\xi_2}|, & v \in A_{\xi_1} \\ |A_{\xi_1}| + |A_{\xi_1^2}|, & v \in A_{\xi_2} \\ |A_{\xi_2}| + |A_{\xi_1\xi_2}|, & v \in A_{\xi_1^2} \\ |A_{\xi_1^2}|, & v \in A_{\xi_1\xi_2}. \end{cases}$$

For any edge ε in $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$, the following cases are possible:

Case 1: If $v_i \in A_{\xi_1}$ and $v_j \in A_{\xi_2}$ s.t. $v_iv_j = \varepsilon$. Here, $|A_{\xi_1}||A_{\xi_2}|$ is the number of edges and $\deg(v_i) = \xi_1^2 - \xi_1$, $\deg(v_j) = \xi_1\xi_2 - \xi_1$. We have

$$\begin{aligned} & \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\ &= |A_{\xi_1}||A_{\xi_2}| \sqrt{(\xi_1^2 - \xi_1)^2 + (\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - \xi_1)(\xi_1\xi_2 - \xi_1)}. \end{aligned}$$

Case 2: If $v_i \in A_{\xi_2}$ and $v_j \in A_{\xi_1^2}$ s.t. $v_iv_j = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1^2}|$ is the number of edges and $\deg(v_i) = \xi_1\xi_2 - \xi_1$, $\deg(v_j) = \xi_1^2 - 1$. We have

$$\begin{aligned} & \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\ &= |A_{\xi_2}||A_{\xi_1^2}| \sqrt{(\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1\xi_2 - \xi_1)(\xi_1^2 - 1)}. \end{aligned}$$

Case 3: If $v_i \in A_{\xi_1^2}$ and $v_j \in A_{\xi_1\xi_2}$ s.t. $v_iv_j = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(v_i) = \xi_1^2 - 1$, $\deg(v_j) = \xi_2 - 1$. We have

$$\begin{aligned} & \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\ &= |A_{\xi_1^2}||A_{\xi_1\xi_2}| \sqrt{(\xi_1^2 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1^2 - 1)(\xi_2 - 1)}. \end{aligned}$$

From all above cases, it follows that the Euler Sombor index of $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$ can be written as

$$\begin{aligned} EUS(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})) &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\ &= |A_{\xi_1}||A_{\xi_2}| \sqrt{(\xi_1^2 - \xi_1)^2 + (\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - \xi_1)(\xi_1\xi_2 - \xi_1)} \\ &\quad + |A_{\xi_2}||A_{\xi_1^2}| \sqrt{(\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1\xi_2 - \xi_1)(\xi_1^2 - 1)} \\ &\quad + |A_{\xi_1^2}||A_{\xi_1\xi_2}| \sqrt{(\xi_1^2 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1^2 - 1)(\xi_2 - 1)} \\ &= (\xi_1 - 1)(\xi_2 - 1)(\xi_1^2 - \xi_1) \sqrt{(\xi_1^2 - \xi_1)^2 + (\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - \xi_1)(\xi_1\xi_2 - \xi_1)} \\ &\quad + (\xi_1^2 - \xi_1)(\xi_2 - 1) \sqrt{(\xi_1\xi_2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1\xi_2 - \xi_1)(\xi_1^2 - 1)} \\ &\quad + (\xi_1 - 1)(\xi_2 - 1) \sqrt{(\xi_1^2 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1^2 - 1)(\xi_2 - 1)}. \end{aligned}$$

$$\begin{aligned}
&= (\xi_1 - 1)(\xi_2 - 1) \left[(\xi_1^2 - \xi_1) \sqrt{(\xi_1^2 - \xi_1)^2 + (\xi_1 \xi_2 - \xi_1)^2 + (\xi_1^2 - \xi_1)(\xi_1 \xi_2 - \xi_1)} \right. \\
&\quad + \xi_1 \sqrt{(\xi_1 \xi_2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1 \xi_2 - \xi_1)(\xi_1^2 - 1)} \\
&\quad \left. + \sqrt{(\xi_1^2 - 1)^2 + (\xi_2 - 1)^2 + (\xi_1^2 - 1)(\xi_2 - 1)} \right].
\end{aligned}$$

□

Example 2. Let $n = 12 = 2^2 \cdot 3$, where 2 and 3 are distinct primes. Here, 2, 3, 4 and 6 are the proper divisors of n . So, by Lemma 1, $\delta_{12} : 2 \sim 3 \sim 4 \sim 6$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_{12}) = \delta_{12}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_4), \Gamma'(A_6)]$. By Corollary 1, we obtain $\Gamma'(\mathbb{Z}_{12}) = \delta_{12}[\overline{K}_{\phi(6)}, \overline{K}_{\phi(4)}, \overline{K}_{\phi(3)}, \overline{K}_{\phi(2)}]$.

By Lemma 3, $|A_2| = 2, |A_3| = 2, |A_4| = 2, |A_6| = 1$. If $2 \in A_2, 2 \sim 3$, then $\deg(2) = |A_3| = 2$, if $3 \in A_3, 2 \sim 3 \sim 4$, then $\deg(3) = |A_2| + |A_4| = 4$, if $4 \in A_4, 3 \sim 4 \sim 6$, then $\deg(4) = |A_3| + |A_6| = 3$, if $6 \in A_6, 4 \sim 6$, then $\deg(6) = |A_4| = 2$. Thus, the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{12})$ is:

$$\begin{aligned}
EUS(\Gamma'(\mathbb{Z}_{12})) &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
&= |A_2||A_3| \sqrt{(\deg(2))^2 + (\deg(3))^2 + \deg(2) \deg(3)} \\
&\quad + |A_3||A_4| \sqrt{(\deg(3))^2 + (\deg(4))^2 + \deg(3) \deg(4)} \\
&\quad + |A_4||A_6| \sqrt{(\deg(4))^2 + (\deg(6))^2 + \deg(4) \deg(6)} \\
&= 2 \cdot 2 \sqrt{(2)^2 + (4)^2 + 2 \cdot 4} + 2 \cdot 2 \sqrt{(4)^2 + (3)^2 + 4 \cdot 3} + 2 \cdot 1 \sqrt{(3)^2 + (2)^2 + 3 \cdot 2} \\
&= 4\sqrt{28} + 4\sqrt{37} + 2\sqrt{19}.
\end{aligned}$$

The following theorem determines the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$ for distinct primes ξ_1 and ξ_2 :

Theorem 3. For distinct prime numbers ξ_1 and ξ_2 , the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$ is given as

$$\begin{aligned}
EUS(\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})) &= \xi_1 \xi_2 (\xi_1 - 1)(\xi_2 - 1) \\
&\quad \left[(\xi_1 - 1)(\xi_2 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1 \xi_2)} \right. \\
&\quad + (\xi_1 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
&\quad + (\xi_2 - 1) \sqrt{(\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)(\xi_1^2 \xi_2 - \xi_2)} \\
&\quad + \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
&\quad + \frac{(\xi_2 - 1)}{\xi_1} \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
&\quad \left. + \frac{1}{\xi_1} \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_2^2 - 1)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(\xi_1 - 1)}{\xi_2} \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + \frac{1}{\xi_2} \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 - 1)} \\
& + \frac{1}{\xi_1 \xi_2} \sqrt{(\xi_1^2 - 1)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 - 1)(\xi_2^2 - 1)} \Big].
\end{aligned}$$

Proof. Let $n = \xi_1^2 \xi_2^2$, where ξ_1 and ξ_2 are distinct primes. Then $\xi_1, \xi_2, \xi_1^2, \xi_1 \xi_2, \xi_2^2, \xi_1^2 \xi_2$ and $\xi_1 \xi_2^2$ are the proper divisors of n . So, by Lemma 1, $\delta_{\xi_1^2 \xi_2^2} : \xi_2 \sim \xi_1 \sim \xi_2^2 \sim \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2 \sim \xi_1^2 \sim \xi_1 \xi_2, \xi_1^2 \sim \xi_2^2 \sim \xi_1 \xi_2, \xi_2 \sim \xi_1^2$ and by Lemma 2, we have $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2}) = \delta_{\xi_1^2 \xi_2^2}[\Gamma'(A_{\xi_1}), \Gamma'(A_{\xi_2}), \Gamma'(A_{\xi_1^2}), \Gamma'(A_{\xi_1 \xi_2}), \Gamma'(A_{\xi_2^2}), \Gamma'(A_{\xi_1^2 \xi_2}), \Gamma'(A_{\xi_1 \xi_2^2})]$. Utilizing Corollary 1, we get

$$\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2}) = \delta_{\xi_1^2 \xi_2^2}[\bar{K}_{\phi(\xi_1 \xi_2^2)}, \bar{K}_{\phi(\xi_1^2 \xi_2)}, \bar{K}_{\phi(\xi_2^2)}, \bar{K}_{\phi(\xi_1 \xi_2)}, \bar{K}_{\phi(\xi_1^2)}, \bar{K}_{\phi(\xi_2)}, \bar{K}_{\phi(\xi_1)}].$$

By Lemma 3, $|A_{\xi_1}| = \xi_2(\xi_1 - 1)(\xi_2 - 1), |A_{\xi_2}| = \xi_1(\xi_1 - 1)(\xi_2 - 1), |A_{\xi_1^2}| = \xi_2(\xi_2 - 1), |A_{\xi_1 \xi_2}| = (\xi_1 - 1)(\xi_2 - 1), |A_{\xi_2^2}| = \xi_1(\xi_1 - 1), |A_{\xi_1^2 \xi_2}| = (\xi_2 - 1), |A_{\xi_1 \xi_2^2}| = (\xi_1 - 1)$.

If $v \in A_{\xi_1}, \xi_2 \sim \xi_1 \sim \xi_2^2$, then $\deg(v)$ is the cardinality of A_{ξ_2} and $A_{\xi_2^2}$. i.e., $\deg(v) = |A_{\xi_2}| + |A_{\xi_2^2}|$ similarly if $v \in A_{\xi_2}, \xi_2 \sim \xi_1, \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_1^2}|$, if $v \in A_{\xi_1^2}, \xi_1 \xi_2^2 \sim \xi_1^2 \sim \xi_1 \xi_2, \xi_1^2 \sim \xi_2^2, \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_2}| + |A_{\xi_2^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1 \xi_2^2}|$, if $v \in A_{\xi_1 \xi_2}, \xi_1^2 \sim \xi_1 \xi_2, \xi_2^2 \sim \xi_1 \xi_2$, then $\deg(v) = |A_{\xi_1^2}| + |A_{\xi_2^2}|$, if $v \in A_{\xi_2^2}, \xi_1 \sim \xi_2^2 \sim \xi_1^2 \xi_2, \xi_1^2 \sim \xi_2^2 \sim \xi_1 \xi_2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1^2 \xi_2}|$, if $v \in A_{\xi_1^2 \xi_2}, \xi_2^2 \sim \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2$, then $\deg(v) = |A_{\xi_2^2}| + |A_{\xi_1 \xi_2^2}|$, if $v \in A_{\xi_1 \xi_2^2}, \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_2^2}| + |A_{\xi_1^2 \xi_2}|$, so the degree of vertex v can be expressed as follows:

$$\deg(v) = \begin{cases} |A_{\xi_2}| + |A_{\xi_2^2}|, & v \in A_{\xi_1} \\ |A_{\xi_1}| + |A_{\xi_1^2}|, & v \in A_{\xi_2} \\ |A_{\xi_2}| + |A_{\xi_2^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1 \xi_2^2}|, & v \in A_{\xi_1^2} \\ |A_{\xi_1^2}| + |A_{\xi_2^2}|, & v \in A_{\xi_1 \xi_2} \\ |A_{\xi_1}| + |A_{\xi_2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1^2 \xi_2}|, & v \in A_{\xi_2^2} \\ |A_{\xi_2^2}| + |A_{\xi_1 \xi_2^2}|, & v \in A_{\xi_1^2 \xi_2} \\ |A_{\xi_2^2}| + |A_{\xi_1^2 \xi_2}|, & v \in A_{\xi_1 \xi_2^2}. \end{cases}$$

Within the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$, each edge ε falls into one of the following categories:

Case 1: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}| |A_{\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2 \xi_2 - \xi_1 \xi_2, \deg(v) = \xi_1 \xi_2^2 - \xi_1 \xi_2$. We have

$$|A_{\xi_1}| |A_{\xi_2}| \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1 \xi_2)}.$$

Case 2: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}| |A_{\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2 \xi_2 - \xi_1 \xi_2, \deg(v) = \xi_1 \xi_2^2 - \xi_1$. We have

$$|A_{\xi_1}| |A_{\xi_2^2}| \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1)}.$$

Case 3: If $u \in A_{\xi_2}$ and $v \in A_{\xi_1^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1^2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1\xi_2$, $\deg(v) = \xi_1^2\xi_2 - \xi_2$. We have

$$|A_{\xi_2}||A_{\xi_1^2}|\sqrt{(\xi_1\xi_2^2 - \xi_1\xi_2)^2 + (\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1\xi_2^2 - \xi_1\xi_2)(\xi_1^2\xi_2 - \xi_2)}.$$

Case 4: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_1\xi_2^2 - \xi_1$. We have

$$|A_{\xi_1^2}||A_{\xi_2}|\sqrt{(\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1\xi_2^2 - \xi_1)^2 + (\xi_1^2\xi_2 - \xi_2)(\xi_1\xi_2^2 - \xi_1)}.$$

Case 5: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_1^2 + \xi_2^2 - \xi_1 - \xi_2$. We have

$$|A_{\xi_1^2}||A_{\xi_1\xi_2}|\sqrt{(\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1^2\xi_2 - \xi_2)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)}.$$

Case 6: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_1\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_2^2 - 1$. We have

$$|A_{\xi_1^2}||A_{\xi_1\xi_2^2}|\sqrt{(\xi_1^2\xi_2 - \xi_2)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2\xi_2 - \xi_2)(\xi_2^2 - 1)}.$$

Case 7: If $u \in A_{\xi_2^2}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1$, $\deg(v) = \xi_1^2 + \xi_2^2 - \xi_1 - \xi_2$. We have

$$|A_{\xi_2^2}||A_{\xi_1\xi_2}|\sqrt{(\xi_1\xi_2^2 - \xi_1)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1\xi_2^2 - \xi_1)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)}.$$

Case 8: If $u \in A_{\xi_2^2}$ and $v \in A_{\xi_1^2\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2^2}||A_{\xi_1^2\xi_2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1$, $\deg(v) = \xi_1^2 - 1$. We have

$$|A_{\xi_2^2}||A_{\xi_1^2\xi_2}|\sqrt{(\xi_1\xi_2^2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1\xi_2^2 - \xi_1)(\xi_1^2 - 1)}.$$

Case 9: If $u \in A_{\xi_1^2\xi_2}$ and $v \in A_{\xi_1\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2\xi_2}||A_{\xi_1\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2 - 1$, $\deg(v) = \xi_2^2 - 1$. We have

$$|A_{\xi_1^2\xi_2}||A_{\xi_1\xi_2^2}|\sqrt{(\xi_1^2 - 1)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 - 1)(\xi_2^2 - 1)}.$$

Combining all the previously discussed cases, we see that the Euler Sombor index of $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2^2})$, is expressed as

$$\begin{aligned} &EUS(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2^2})) \\ &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i)\deg(v_j)} \\ &= |A_{\xi_1}||A_{\xi_2}|\sqrt{(\xi_1^2\xi_2 - \xi_1\xi_2)^2 + (\xi_1\xi_2^2 - \xi_1\xi_2)^2 + (\xi_1^2\xi_2 - \xi_1\xi_2)(\xi_1\xi_2^2 - \xi_1\xi_2)} \\ &\quad + |A_{\xi_1}||A_{\xi_2^2}|\sqrt{(\xi_1^2\xi_2 - \xi_1\xi_2)^2 + (\xi_1\xi_2^2 - \xi_1)^2 + (\xi_1^2\xi_2 - \xi_1\xi_2)(\xi_1\xi_2^2 - \xi_1)} \\ &\quad + |A_{\xi_2}||A_{\xi_1^2}|\sqrt{(\xi_1\xi_2^2 - \xi_1\xi_2)^2 + (\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1\xi_2^2 - \xi_1\xi_2)(\xi_1^2\xi_2 - \xi_2)} \\ &\quad + |A_{\xi_1^2}||A_{\xi_2}|\sqrt{(\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1\xi_2^2 - \xi_1)^2 + (\xi_1^2\xi_2 - \xi_2)(\xi_1\xi_2^2 - \xi_1)} \\ &\quad + |A_{\xi_1^2}||A_{\xi_1\xi_2}|\sqrt{(\xi_1^2\xi_2 - \xi_2)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1^2\xi_2 - \xi_2)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \end{aligned}$$

$$\begin{aligned}
& + |A_{\xi_1^2}| |A_{\xi_1 \xi_2^2}| \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_2^2 - 1)} \\
& + |A_{\xi_2^2}| |A_{\xi_1 \xi_2}| \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + |A_{\xi_2^2}| |A_{\xi_1^2 \xi_2}| \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 - 1)} \\
& + |A_{\xi_1^2 \xi_2}| |A_{\xi_1 \xi_2^2}| \sqrt{(\xi_1^2 - 1)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 - 1)(\xi_2^2 - 1)}. \\
& = \xi_1 \xi_2 (\xi_1 - 1)^2 (\xi_2 - 1)^2 \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1 \xi_2)} \\
& + \xi_1 \xi_2 (\xi_1 - 1)^2 (\xi_2 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
& + \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1)^2 \sqrt{(\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)(\xi_1^2 \xi_2 - \xi_2)} \\
& + \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
& + \xi_2 (\xi_1 - 1) (\xi_2 - 1)^2 \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + \xi_2 (\xi_1 - 1) (\xi_2 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_2^2 - 1)} \\
& + \xi_1 (\xi_1 - 1)^2 (\xi_2 - 1) \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + \xi_1 (\xi_1 - 1) (\xi_2 - 1) \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 - 1)} \\
& + (\xi_1 - 1) (\xi_2 - 1) \sqrt{(\xi_1^2 - 1)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 - 1)(\xi_2^2 - 1)}. \\
& = \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1) \\
& \left[(\xi_1 - 1) (\xi_2 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1 \xi_2)} \right. \\
& + (\xi_1 - 1) \sqrt{(\xi_1^2 \xi_2 - \xi_1 \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_1 \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
& + (\xi_2 - 1) \sqrt{(\xi_1 \xi_2^2 - \xi_1 \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1 \xi_2)(\xi_1^2 \xi_2 - \xi_2)} \\
& + \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1 \xi_2^2 - \xi_1)} \\
& + \frac{(\xi_2 - 1)}{\xi_1} \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + \frac{1}{\xi_1} \sqrt{(\xi_1^2 \xi_2 - \xi_2)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 \xi_2 - \xi_2)(\xi_2^2 - 1)} \\
& + \frac{(\xi_1 - 1)}{\xi_2} \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2)} \\
& + \frac{1}{\xi_2} \sqrt{(\xi_1 \xi_2^2 - \xi_1)^2 + (\xi_1^2 - 1)^2 + (\xi_1 \xi_2^2 - \xi_1)(\xi_1^2 - 1)} \\
& \left. + \frac{1}{\xi_1 \xi_2} \sqrt{(\xi_1^2 - 1)^2 + (\xi_2^2 - 1)^2 + (\xi_1^2 - 1)(\xi_2^2 - 1)} \right].
\end{aligned}$$

□

Example 3. Let $n = 36 = 2^2 \cdot 3^2$, where 2 and 3 are distinct primes. Here, 2, 3, 4, 6, 9, 12 and 18 are the proper divisors of n . So, by Lemma 1, $\delta_{36} : 3 \sim 2 \sim 9 \sim 12 \sim 18 \sim 4 \sim 6, 4 \sim 9 \sim 6, 3 \sim 4$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_{36}) = \delta_{36}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_4), \Gamma'(A_6), \Gamma'(A_9), \Gamma'(A_{12}), \Gamma'(A_{18})]$. By Corollary 1, we get $\Gamma'(\mathbb{Z}_{36}) = \delta_{36}[\bar{K}_{\phi(18)}, \bar{K}_{\phi(12)}, \bar{K}_{\phi(9)}, \bar{K}_{\phi(6)}, \bar{K}_{\phi(4)}, \bar{K}_{\phi(3)}, \bar{K}_{\phi(2)}]$.

By Lemma 3, $|A_2| = 6, |A_3| = 4, |A_4| = 6, |A_6| = 2, |A_9| = 2, |A_{12}| = 2, |A_{18}| = 1$. If $2 \in A_2, 3 \sim 2 \sim 9$, then $\deg(2) = |A_3| + |A_9| = 6$, if $3 \in A_3, 3 \sim 2, 3 \sim 4$, then $\deg(3) = |A_2| + |A_4| = 12$, if $4 \in A_4, 18 \sim 4 \sim 6, 4 \sim 9, 3 \sim 4$, then $\deg(4) = |A_{18}| + |A_6| + |A_9| + |A_3| = 9$, if $6 \in A_6, 4 \sim 6, 9 \sim 6$, then $\deg(6) = |A_4| + |A_9| = 8$, if $9 \in A_9, 2 \sim 9 \sim 12, 4 \sim 9 \sim 6$, then $\deg(9) = |A_2| + |A_{12}| + |A_4| + |A_6| = 16$, if $12 \in A_{12}$,

$9 \sim 12 \sim 18$, then $\deg(12) = |A_9| + |A_{18}| = 3$, if $18 \in A_{18}$, $12 \sim 18 \sim 4$, then $\deg(18) = |A_{12}| + |A_4| = 8$. Thus, the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{36})$ is:

$$\begin{aligned}
 EUS(\Gamma'(\mathbb{Z}_{36})) &= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
 &= |A_2||A_3| \sqrt{(\deg(2))^2 + (\deg(3))^2 + \deg(2) \deg(3)} \\
 &\quad + |A_2||A_9| \sqrt{(\deg(2))^2 + (\deg(9))^2 + \deg(2) \deg(9)} \\
 &\quad + |A_3||A_4| \sqrt{(\deg(3))^2 + (\deg(4))^2 + \deg(3) \deg(4)} \\
 &\quad + |A_4||A_9| \sqrt{(\deg(4))^2 + (\deg(9))^2 + \deg(4) \deg(9)} \\
 &\quad + |A_4||A_6| \sqrt{(\deg(4))^2 + (\deg(6))^2 + \deg(4) \deg(6)} \\
 &\quad + |A_4||A_{18}| \sqrt{(\deg(4))^2 + (\deg(18))^2 + \deg(4) \deg(18)} \\
 &\quad + |A_9||A_6| \sqrt{(\deg(9))^2 + (\deg(6))^2 + \deg(9) \deg(6)} \\
 &\quad + |A_9||A_{12}| \sqrt{(\deg(9))^2 + (\deg(12))^2 + \deg(9) \deg(12)} \\
 &\quad + |A_{12}||A_{18}| \sqrt{(\deg(12))^2 + (\deg(18))^2 + \deg(12) \deg(18)} \\
 &= 6.4 \sqrt{(6)^2 + (12)^2} + 6.12 + 6.2 \sqrt{(6)^2 + (16)^2} + 6.16 \\
 &\quad + 4.6 \sqrt{(12)^2 + (9)^2} + 12.9 + 6.2 \sqrt{(9)^2 + (16)^2} + 9.16 \\
 &\quad + 6.2 \sqrt{(9)^2 + (8)^2} + 9.8 + 6.1 \sqrt{(9)^2 + (8)^2} + 9.8 \\
 &\quad + 2.2 \sqrt{(16)^2 + (8)^2} + 16.8 + 2.2 \sqrt{(16)^2 + (3)^2} + 16.3 \\
 &\quad + 2.1 \sqrt{(3)^2 + (8)^2} + 3.8 \\
 &= 24\sqrt{252} + 12\sqrt{388} + 24\sqrt{333} + 12\sqrt{481} + 12\sqrt{217} + 6\sqrt{217} \\
 &\quad + 4\sqrt{448} + 4\sqrt{313} + 2\sqrt{97}.
 \end{aligned}$$

Given three distinct prime numbers ξ_1, ξ_2 and ξ_3 , the subsequent theorem presents the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1 \xi_2 \xi_3})$:

Theorem 4. For three distinct prime numbers ξ_1, ξ_2 and ξ_3 , it follows that the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1 \xi_2 \xi_3})$ is

$$\begin{aligned}
 EUS(\Gamma'(\mathbb{Z}_{\xi_1 \xi_2 \xi_3})) &= (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \\
 &\quad \left[(\xi_3 - 1) \sqrt{(\xi_1 \xi_2 + \xi_1 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1 \xi_2 + \xi_2 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \right. \\
 &\quad \left. + (\xi_2 - 1) \sqrt{(\xi_1 \xi_2 + \xi_1 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1 \xi_3 + \xi_2 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \right. \\
 &\quad \left. + \sqrt{(\xi_1 \xi_2 + \xi_1 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_2 \xi_3 - 1)^2} \right] \\
 &\quad + (\xi_1 \xi_2 + \xi_1 \xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_2 \xi_3 - 1)
 \end{aligned}$$

$$\begin{aligned}
& + (\zeta_1 - 1) \sqrt{(\zeta_1 \zeta_2 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2 + (\zeta_1 \zeta_3 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2} \\
& + \sqrt{(\zeta_1 \zeta_2 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2 + (\zeta_1 \zeta_3 - 1)^2} \\
& + \sqrt{(\zeta_1 \zeta_3 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2 + (\zeta_1 \zeta_2 - 1)^2} \\
& + \frac{1}{(\zeta_2 - 1)} \sqrt{(\zeta_1 \zeta_2 - 1)^2 + (\zeta_2 \zeta_3 - 1)^2 + (\zeta_1 \zeta_2 - 1)(\zeta_2 \zeta_3 - 1)} \\
& + \frac{1}{(\zeta_1 - 1)} \sqrt{(\zeta_1 \zeta_2 - 1)^2 + (\zeta_1 \zeta_3 - 1)^2 + (\zeta_1 \zeta_2 - 1)(\zeta_1 \zeta_3 - 1)} \\
& + \frac{1}{(\zeta_3 - 1)} \sqrt{(\zeta_2 \zeta_3 - 1)^2 + (\zeta_1 \zeta_3 - 1)^2 + (\zeta_2 \zeta_3 - 1)(\zeta_1 \zeta_3 - 1)} \Big].
\end{aligned}$$

Proof. Consider $n = \zeta_1 \zeta_2 \zeta_3$, where ζ_1, ζ_2 and ζ_3 are distinct primes; in this case, the numbers $\zeta_1, \zeta_2, \zeta_3, \zeta_1 \zeta_2, \zeta_2 \zeta_3$ and $\zeta_1 \zeta_3$ form the proper divisors of n . So, by Lemma 1, $\delta_{\zeta_1 \zeta_2 \zeta_3} : \zeta_1 \zeta_2 \sim \zeta_3 \sim \zeta_1 \sim \zeta_2 \sim \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3, \zeta_1 \sim \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3, \zeta_2 \sim \zeta_3$ and by Lemma 2, we have $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3}) = \delta_{\zeta_1 \zeta_2 \zeta_3}[\Gamma'(A_{\zeta_1}), \Gamma'(A_{\zeta_2}), \Gamma'(A_{\zeta_3}), \Gamma'(A_{\zeta_1 \zeta_2}), \Gamma'(A_{\zeta_2 \zeta_3}), \Gamma'(A_{\zeta_1 \zeta_3})]$. By Corollary 1, it follows that

$$\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3}) = \delta_{\zeta_1 \zeta_2 \zeta_3}[\overline{K}_{\phi(\zeta_2 \zeta_3)}, \overline{K}_{\phi(\zeta_1 \zeta_3)}, \overline{K}_{\phi(\zeta_1 \zeta_2)}, \overline{K}_{\phi(\zeta_3)}, \overline{K}_{\phi(\zeta_1)}, \overline{K}_{\phi(\zeta_2)}].$$

By Lemma 3, $|A_{\zeta_1}| = (\zeta_2 - 1)(\zeta_3 - 1), |A_{\zeta_2}| = (\zeta_1 - 1)(\zeta_3 - 1), |A_{\zeta_3}| = (\zeta_1 - 1)(\zeta_2 - 1), |A_{\zeta_1 \zeta_2}| = (\zeta_3 - 1), |A_{\zeta_2 \zeta_3}| = (\zeta_1 - 1), |A_{\zeta_1 \zeta_3}| = (\zeta_2 - 1)$.

If $v \in A_{\zeta_1}, \zeta_3 \sim \zeta_1 \sim \zeta_2, \zeta_1 \sim \zeta_2 \zeta_3$, then $\deg(v)$ is the cardinality of A_{ζ_2}, A_{ζ_3} and $A_{\zeta_2 \zeta_3}$. i.e., $\deg(v) = |A_{\zeta_2}| + |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}|$, similarly if $v \in A_{\zeta_2}, \zeta_1 \sim \zeta_2 \sim \zeta_1 \zeta_3, \zeta_2 \sim \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_3}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_3}, \zeta_1 \zeta_2 \sim \zeta_3 \sim \zeta_1, \zeta_2 \sim \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}|$, if $v \in A_{\zeta_1 \zeta_2}, \zeta_1 \zeta_2 \sim \zeta_3, \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3$, then $\deg(v) = |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_2 \zeta_3}, \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3, \zeta_1 \sim \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_1 \zeta_3}, \zeta_2 \sim \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2, \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3$, then $\deg(v) = |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_2 \zeta_3}|$, so the degree of vertex v is defined as

$$\deg(v) = \begin{cases} |A_{\zeta_2}| + |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}|, & v \in A_{\zeta_1} \\ |A_{\zeta_1}| + |A_{\zeta_3}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_2} \\ |A_{\zeta_1}| + |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}|, & v \in A_{\zeta_3} \\ |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_1 \zeta_2} \\ |A_{\zeta_1}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_2 \zeta_3} \\ |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_2 \zeta_3}|, & v \in A_{\zeta_1 \zeta_3}. \end{cases}$$

In the graph $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3})$, every edge ε can be classified according to the situations below:

Case 1: If $u \in A_{\zeta_1}$ and $v \in A_{\zeta_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\zeta_1}| |A_{\zeta_2}|$ is the number of edges and $\deg(u) = (\zeta_1 - 1)(\zeta_2 + \zeta_3 - 1), \deg(v) = (\zeta_2 - 1)(\zeta_1 + \zeta_3 - 1)$. We have

$$|A_{\zeta_1}| |A_{\zeta_2}| \sqrt{(\zeta_1 \zeta_2 + \zeta_1 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2 + (\zeta_1 \zeta_2 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)^2} \\
+ (\zeta_1 \zeta_2 + \zeta_1 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1)(\zeta_1 \zeta_2 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 + 1).$$

Case 2: If $u \in A_{\xi_1}$ and $v \in A_{\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}||A_{\xi_3}|$ is the number of edges and $\deg(u) = (\xi_1 - 1)(\xi_2 + \xi_3 - 1)$, $\deg(v) = (\xi_3 - 1)(\xi_1 + \xi_2 - 1)$. We have

$$|A_{\xi_1}||A_{\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)}.$$

Case 3: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}||A_{\xi_2\xi_3}|$ is the number of edges and $\deg(u) = (\xi_1 - 1)(\xi_2 + \xi_3 - 1)$, $\deg(v) = \xi_2\xi_3 - 1$. We have

$$|A_{\xi_1}||A_{\xi_2\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_2\xi_3 - 1)^2 + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_2\xi_3 - 1)}.$$

Case 4: If $u \in A_{\xi_2}$ and $v \in A_{\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_3}|$ is the number of edges and $\deg(u) = (\xi_2 - 1)(\xi_1 + \xi_3 - 1)$, $\deg(v) = (\xi_3 - 1)(\xi_1 + \xi_2 - 1)$. We have

$$|A_{\xi_2}||A_{\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)}.$$

Case 5: If $u \in A_{\xi_2}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = (\xi_2 - 1)(\xi_1 + \xi_3 - 1)$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_2}||A_{\xi_1\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 - 1)}.$$

Case 6: If $u \in A_{\xi_3}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_3}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = (\xi_3 - 1)(\xi_1 + \xi_2 - 1)$, $\deg(v) = \xi_1\xi_2 - 1$. We have

$$|A_{\xi_3}||A_{\xi_1\xi_2}| \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 - 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_2 - 1)}.$$

Case 7: If $u \in A_{\xi_1\xi_2}$ and $v \in A_{\xi_2\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1\xi_2}||A_{\xi_2\xi_3}|$ is the number of edges and $\deg(u) = \xi_1\xi_2 - 1$, $\deg(v) = \xi_2\xi_3 - 1$. We have

$$|A_{\xi_1\xi_2}||A_{\xi_2\xi_3}| \sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_2\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_2\xi_3 - 1)}.$$

Case 8: If $u \in A_{\xi_1\xi_2}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1\xi_2}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = \xi_1\xi_2 - 1$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_1\xi_2}||A_{\xi_1\xi_3}| \sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_1\xi_3 - 1)}.$$

Case 9: If $u \in A_{\xi_2\xi_3}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2\xi_3}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = \xi_2\xi_3 - 1$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_2\xi_3}||A_{\xi_1\xi_3}| \sqrt{(\xi_2\xi_3 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_2\xi_3 - 1)(\xi_1\xi_3 - 1)}.$$

By aggregating all the previously examined cases, the Euler Sombor index of $\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})$, can be determined as follows:

$$\begin{aligned}
& EUS(\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})) \\
&= \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
&= |A_{\xi_1}| |A_{\xi_2}| \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + |A_{\xi_1}| |A_{\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + |A_{\xi_1}| |A_{\xi_2\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_2\xi_3 - 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_2\xi_3 - 1) \\
&\quad + |A_{\xi_2}| |A_{\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + |A_{\xi_2}| |A_{\xi_1\xi_3}| \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 - 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 - 1) \\
&\quad + |A_{\xi_3}| |A_{\xi_1\xi_2}| \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 - 1)^2} \\
&\quad + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_2 - 1) \\
&\quad + |A_{\xi_1\xi_2}| |A_{\xi_2\xi_3}| \sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_2\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_2\xi_3 - 1)} \\
&\quad + |A_{\xi_1\xi_2}| |A_{\xi_1\xi_3}| \sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_1\xi_3 - 1)} \\
&\quad + |A_{\xi_2\xi_3}| |A_{\xi_1\xi_3}| \sqrt{(\xi_2\xi_3 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_2\xi_3 - 1)(\xi_1\xi_3 - 1)}. \\
&= (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)^2 \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)^2(\xi_3 - 1) \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_2\xi_3 - 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_2\xi_3 - 1) \\
&\quad + (\xi_1 - 1)^2(\xi_2 - 1)(\xi_3 - 1) \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1) \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 - 1)^2} \\
&\quad + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_3 - 1) \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 - 1)^2} \\
&\quad + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)(\xi_1\xi_2 - 1)
\end{aligned}$$

$$\begin{aligned}
& + (\xi_1 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_2\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_2\xi_3 - 1)} \\
& + (\xi_2 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_1\xi_3 - 1)} \\
& + (\xi_1 - 1)(\xi_2 - 1)\sqrt{(\xi_2\xi_3 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_2\xi_3 - 1)(\xi_1\xi_3 - 1)} \\
& = (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \\
& \left[(\xi_3 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \right. \\
& \quad + (\xi_2 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
& \quad + \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_2\xi_3 - 1)^2} \\
& \quad + (\xi_1 - 1)\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2} \\
& \quad + \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_3 - 1)^2} \\
& \quad + \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 + 1)^2 + (\xi_1\xi_2 - 1)^2} \\
& \quad + \frac{1}{(\xi_2 - 1)}\sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_2\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_2\xi_3 - 1)} \\
& \quad + \frac{1}{(\xi_1 - 1)}\sqrt{(\xi_1\xi_2 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_1\xi_2 - 1)(\xi_1\xi_3 - 1)} \\
& \quad \left. + \frac{1}{(\xi_3 - 1)}\sqrt{(\xi_2\xi_3 - 1)^2 + (\xi_1\xi_3 - 1)^2 + (\xi_2\xi_3 - 1)(\xi_1\xi_3 - 1)} \right].
\end{aligned}$$

□

Example 4. Let $n = 30 = 2 \cdot 3 \cdot 5$, where 2, 3 and 5 are distinct primes. Here, 2, 3, 5, 6, 10 and 15 are the proper divisors of n . So, by Lemma 1, $\delta_{30} : 6 \sim 5 \sim 2 \sim 3 \sim 10 \sim 6 \sim 15, 2 \sim 15 \sim 10, 3 \sim 5$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_{30}) = \delta_{30}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_5), \Gamma'(A_6), \Gamma'(A_{15}), \Gamma'(A_{10})]$. By Corollary 1, it follows that $\Gamma'(\mathbb{Z}_{30}) = \delta_{30}[\bar{K}_{\phi(15)}, \bar{K}_{\phi(10)}, \bar{K}_{\phi(6)}, \bar{K}_{\phi(5)}, \bar{K}_{\phi(2)}, \bar{K}_{\phi(3)}]$.

By Lemma 3, $|A_2| = 8, |A_3| = 4, |A_5| = 2, |A_6| = 4, |A_{15}| = 1, |A_{10}| = 2$. If $2 \in A_2$, $5 \sim 2 \sim 3, 2 \sim 15$, then $\deg(2) = |A_5| + |A_3| + |A_{15}| = 7$, if $3 \in A_3, 2 \sim 3 \sim 10, 3 \sim 5$, then $\deg(3) = |A_2| + |A_{10}| + |A_5| = 12$, if $5 \in A_5, 6 \sim 5 \sim 2, 3 \sim 5$, then $\deg(5) = |A_6| + |A_2| + |A_3| = 16$, if $6 \in A_6, 6 \sim 5, 10 \sim 6 \sim 15$, then $\deg(6) = |A_5| + |A_{10}| + |A_{15}| = 5$, if $15 \in A_{15}, 6 \sim 15, 2 \sim 15 \sim 10$, then $\deg(15) = |A_6| + |A_2| + |A_{10}| = 14$, if $10 \in A_{10}, 3 \sim 10 \sim 6, 15 \sim 10$, then $\deg(10) = |A_3| + |A_6| + |A_{15}| = 9$. Thus, the Euler Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{30})$ is:

$$\begin{aligned}
& EUS(\Gamma'(\mathbb{Z}_{30})) \\
& = \sum_{v_i \sim v_j} \sqrt{(\deg(v_i))^2 + (\deg(v_j))^2 + \deg(v_i) \deg(v_j)} \\
& = |A_2||A_3|\sqrt{(\deg(2))^2 + (\deg(3))^2 + \deg(2) \deg(3)}
\end{aligned}$$

$$\begin{aligned}
& + |A_2||A_5|\sqrt{(\deg(2))^2 + (\deg(5))^2 + \deg(2)\deg(5)} \\
& + |A_2||A_{15}|\sqrt{(\deg(2))^2 + (\deg(15))^2 + \deg(2)\deg(15)} \\
& + |A_3||A_5|\sqrt{(\deg(3))^2 + (\deg(5))^2 + \deg(3)\deg(5)} \\
& + |A_3||A_{10}|\sqrt{(\deg(3))^2 + (\deg(10))^2 + \deg(3)\deg(10)} \\
& + |A_5||A_6|\sqrt{(\deg(5))^2 + (\deg(6))^2 + \deg(5)\deg(6)} \\
& + |A_6||A_{15}|\sqrt{(\deg(6))^2 + (\deg(15))^2 + \deg(6)\deg(15)} \\
& + |A_6||A_{10}|\sqrt{(\deg(6))^2 + (\deg(10))^2 + \deg(6)\deg(10)} \\
& + |A_{15}||A_{10}|\sqrt{(\deg(15))^2 + (\deg(10))^2 + \deg(15)\deg(10)} \\
& = 8.4\sqrt{(7)^2 + (12)^2 + 7.12} + 8.2\sqrt{(7)^2 + (16)^2 + 7.16} \\
& + 8.1\sqrt{(7)^2 + (14)^2 + 7.14} + 4.2\sqrt{(12)^2 + (16)^2 + 12.16} \\
& + 4.2\sqrt{(12)^2 + (9)^2 + 12.9} + 2.4\sqrt{(16)^2 + (5)^2 + 16.5} \\
& + 4.1\sqrt{(5)^2 + (14)^2 + 5.14} + 4.2\sqrt{(5)^2 + (9)^2 + 5.9} \\
& + 1.2\sqrt{(14)^2 + (9)^2 + 14.9} \\
& = 32\sqrt{277} + 16\sqrt{417} + 8\sqrt{343} + 8\sqrt{592} + 8\sqrt{333} + 8\sqrt{361} \\
& + 4\sqrt{291} + 8\sqrt{151} + 2\sqrt{403}.
\end{aligned}$$

4. The Average Sombor Index of Cozero-Divisor Graph $\Gamma'(\mathbb{Z}_n)$

This section is dedicated to determining the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_n)$.

Theorem 5. For distinct prime numbers ξ_1 and ξ_2 , the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1\xi_2})$ is given as

$$SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1\xi_2})) = (\xi_1 - 1)(\xi_2 - 1)\sqrt{(\xi_1 - 2)^2 + (\xi_2 - 2)^2}.$$

Proof. Suppose $n = \xi_1\xi_2$, with p_1 and p_2 being distinct primes. In this case, ξ_1 and ξ_2 are proper divisors of n . From Lemma 1, it follows that $\xi_1 \sim \xi_2$ and using Lemma 2, we obtain $\Gamma'(\mathbb{Z}_{\xi_1\xi_2}) = \delta_{\xi_1\xi_2}[\Gamma'(A_{\xi_1}), \Gamma'(A_{\xi_2})]$. By Lemma 3, $|A_{\xi_1}| = \phi(\xi_2)$ and $|A_{\xi_2}| = \phi(\xi_1)$. Here, $|A_{\xi_1}||A_{\xi_2}|$ is the number of edges. For each $v_i \in A_{\xi_1}$, we have $\deg(v_i) = \xi_1 - 1$, and for each $v_j \in A_{\xi_2}$, $\deg(v_j) = \xi_2 - 1$. Here, $m' = 1, n' = 2$, then $\frac{2m'}{n'} = 1$. Thus, we obtain

$$\begin{aligned}
SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1\xi_2})) &= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\
&= \sum_{v_i \sim v_j} \sqrt{((\xi_1 - 1) - 1)^2 + ((\xi_2 - 1) - 1)^2} \\
&= |A_{\xi_1}||A_{\xi_2}|\sqrt{(\xi_1 - 2)^2 + (\xi_2 - 2)^2} \\
&= (\xi_1 - 1)(\xi_2 - 1)\sqrt{(\xi_1 - 2)^2 + (\xi_2 - 2)^2}.
\end{aligned}$$

□

Example 5. Let $n = 6 = 2 \cdot 3$, where 2 and 3 are distinct primes. Here, 2 and 3 are the proper divisors of n . By Lemma 1, $\delta_6 : 2 \sim 3$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_6) = \delta_6[\Gamma'(A_2), \Gamma'(A_3)]$.

By Lemma 3, $|A_2| = \phi(\frac{6}{2}) = 2$ and $|A_3| = \phi(\frac{6}{3}) = 1$. If $2 \in A_2$, $2 \sim 3$, then $\deg(2) = |A_3| = 1$ and if $3 \in A_3$, $3 \sim 2$, then $\deg(3) = |A_2| = 2$.

Here $m' = 1$, $n' = 2$, then $\frac{2m'}{n'} = 1$. Thus, the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_6)$ is:

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_6)) &= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\ &= \sum_{2 \sim 3} \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(3) - \frac{2m'}{n'}\right)^2} \\ &= |A_2||A_3|\sqrt{(1-1)^2 + (2-1)^2} \\ &= 2.1\sqrt{0+1} \\ &= 2. \end{aligned}$$

Considering distinct primes ξ_1 and ξ_2 , we state the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$ in the theorem below:

Theorem 6. For two distinct prime numbers ξ_1 and ξ_2 , it follows that the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$ is

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})) &= (\xi_1 - 1)(\xi_2 - 1) \left[(\xi_1^2 - \xi_1) \sqrt{\left(\xi_1^2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2} \right. \\ &\quad \left. + \xi_1 \sqrt{\left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1^2 - \frac{5}{2}\right)^2} + \sqrt{\left(\xi_1^2 - \frac{5}{2}\right)^2 + \left(\xi_2 - \frac{5}{2}\right)^2} \right]. \end{aligned}$$

Proof. Suppose $n = \xi_1^2\xi_2$, with ξ_1 and ξ_2 as distinct primes. The set of proper divisors of n includes ξ_1, ξ_2, ξ_1^2 and $\xi_1\xi_2$. Lemma 1 implies $\delta_{\xi_1^2\xi_2}$ follows the adjacency $\xi_1 \sim \xi_2 \sim \xi_1^2 \sim \xi_1\xi_2$ while, according to Lemma 2, $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2}) = \delta_{\xi_1^2\xi_2}[\Gamma'(A_{\xi_1}), \Gamma'(A_{\xi_2}), \Gamma'(A_{\xi_1^2}), \Gamma'(A_{\xi_1\xi_2})]$. Applying Corollary 1, we obtain

$$\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2}) = \delta_{\xi_1^2\xi_2}[\bar{K}_{\phi(\xi_1\xi_2)}, \bar{K}_{\phi(\xi_1^2)}, \bar{K}_{\phi(\xi_2)}, \bar{K}_{\phi(\xi_1)}].$$

By Lemma 3, $|A_{\xi_1}| = (\xi_1 - 1)(\xi_2 - 1)$, $|A_{\xi_2}| = \xi_1^2 - \xi_1$, $|A_{\xi_1^2}| = \xi_2 - 1$, and $|A_{\xi_1\xi_2}| = \xi_1 - 1$. If $v \in A_{\xi_1}$, $\xi_1 \sim \xi_2$, then $\deg(v)$ is the cardinality of A_{ξ_2} , i.e., $\deg(v) = |A_{\xi_2}|$, similarly if $v \in A_{\xi_2}$, $\xi_1 \sim \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_1^2}|$, if $v \in A_{\xi_1^2}$, $\xi_2 \sim \xi_1^2 \sim \xi_1\xi_2$, then $\deg(v) = |A_{\xi_2}| + |A_{\xi_1\xi_2}|$, if $v \in A_{\xi_1\xi_2}$, $\xi_1^2 \sim \xi_1\xi_2$, then $\deg(v) = |A_{\xi_1^2}|$, so the degree of a vertex v is given by

$$\deg(v) = \begin{cases} |A_{\xi_2}|, & v \in A_{\xi_1} \\ |A_{\xi_1}| + |A_{\xi_1^2}|, & v \in A_{\xi_2} \\ |A_{\xi_2}| + |A_{\xi_1\xi_2}|, & v \in A_{\xi_1^2} \\ |A_{\xi_1^2}|, & v \in A_{\xi_1\xi_2}. \end{cases}$$

Here, $m' = 3$, $n' = 4$, then $\frac{2m'}{n'} = \frac{3}{2}$.

For any edge ε in the graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$, the possibilities are as follows:

Case 1: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}||A_{\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2 - \xi_1$, $\deg(v) = \xi_1\xi_2 - \xi_1$. We have

$$|A_{\xi_1}||A_{\xi_2}|\sqrt{\left(\xi_1^2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2}. \quad (1)$$

Case 2: If $u \in A_{\xi_2}$ and $v \in A_{\xi_1^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1^2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2 - \xi_1$, $\deg(v) = \xi_1^2 - 1$. We have

$$|A_{\xi_2}||A_{\xi_1^2}|\sqrt{\left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1^2 - \frac{5}{2}\right)^2}. \quad (2)$$

Case 3: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2 - 1$, $\deg(v) = \xi_2 - 1$. We have

$$|A_{\xi_1^2}||A_{\xi_1\xi_2}|\sqrt{\left(\xi_1^2 - \frac{5}{2}\right)^2 + \left(\xi_2 - \frac{5}{2}\right)^2}. \quad (3)$$

Utilizing Equations (1)–(3), we find that the average Sombor index of the graph $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})$ can be formulated as

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2})) &= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\ &= |A_{\xi_1}||A_{\xi_2}|\sqrt{\left(\xi_1^2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2} \\ &\quad + |A_{\xi_2}||A_{\xi_1^2}|\sqrt{\left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1^2 - \frac{5}{2}\right)^2} \\ &\quad + |A_{\xi_1^2}||A_{\xi_1\xi_2}|\sqrt{\left(\xi_1^2 - \frac{5}{2}\right)^2 + \left(\xi_2 - \frac{5}{2}\right)^2} \\ &= (\xi_1 - 1)(\xi_2 - 1)(\xi_1^2 - \xi_1)\sqrt{\left(\xi_1^2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2} \\ &\quad + (\xi_1^2 - \xi_1)(\xi_2 - 1)\sqrt{\left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1^2 - \frac{5}{2}\right)^2} \\ &\quad + (\xi_1 - 1)(\xi_2 - 1)\sqrt{\left(\xi_1^2 - \frac{5}{2}\right)^2 + \left(\xi_2 - \frac{5}{2}\right)^2} \\ &= (\xi_1 - 1)(\xi_2 - 1) \left[(\xi_1^2 - \xi_1)\sqrt{\left(\xi_1^2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2} \right. \\ &\quad \left. + \xi_1\sqrt{\left(\xi_1\xi_2 - \xi_1 - \frac{3}{2}\right)^2 + \left(\xi_1^2 - \frac{5}{2}\right)^2} + \sqrt{\left(\xi_1^2 - \frac{5}{2}\right)^2 + \left(\xi_2 - \frac{5}{2}\right)^2} \right]. \end{aligned}$$

□

Example 6. Let $n = 12 = 2^2 \cdot 3$, where 2 and 3 are distinct primes. Here, 2, 3, 4 and 6 are the proper divisors of n . So, by Lemma 1, $\delta_{12} : 2 \sim 3 \sim 4 \sim 6$. By Lemma 2, we

have $\Gamma'(\mathbb{Z}_{12}) = \delta_{12}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_4), \Gamma'(A_6)]$. By Corollary 1, we obtain $\Gamma'(\mathbb{Z}_{12}) = \delta_{12}[\overline{K}_{\phi(6)}, \overline{K}_{\phi(4)}, \overline{K}_{\phi(3)}, \overline{K}_{\phi(2)}]$.

By Lemma 3, $|A_2| = 2, |A_3| = 2, |A_4| = 2, |A_6| = 1$. If $2 \in A_2, 2 \sim 3$, then $\deg(2) = |A_3| = 2$, if $3 \in A_3, 2 \sim 3 \sim 4$, then $\deg(3) = |A_2| + |A_4| = 4$, if $4 \in A_4, 3 \sim 4 \sim 6$, then $\deg(4) = |A_3| + |A_6| = 3$, if $6 \in A_6, 4 \sim 6$, then $\deg(6) = |A_4| = 2$.

Here $m' = 3, n' = 4$, then $\frac{2m'}{n'} = \frac{3}{2}$. Thus, the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{12})$ is:

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{12})) &= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\ &= |A_2||A_3| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(3) - \frac{2m'}{n'}\right)^2} \\ &\quad + |A_3||A_4| \sqrt{\left(\deg(3) - \frac{2m'}{n'}\right)^2 + \left(\deg(4) - \frac{2m'}{n'}\right)^2} \\ &\quad + |A_4||A_6| \sqrt{\left(\deg(4) - \frac{2m'}{n'}\right)^2 + \left(\deg(6) - \frac{2m'}{n'}\right)^2} \\ &= 2.2 \sqrt{\left(2 - \frac{3}{2}\right)^2 + \left(4 - \frac{3}{2}\right)^2} + 2.2 \sqrt{\left(4 - \frac{3}{2}\right)^2 + \left(3 - \frac{3}{2}\right)^2} \\ &\quad + 2.1 \sqrt{\left(3 - \frac{3}{2}\right)^2 + \left(2 - \frac{3}{2}\right)^2} \\ &= 2\sqrt{26} + 2\sqrt{34} + \sqrt{10}. \end{aligned}$$

For distinct primes ξ_1 and ξ_2 , we provide the average Sombor index of a cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$ in the following theorem:

Theorem 7. For distinct prime numbers ξ_1 and ξ_2 , the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$ is given as

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})) &= \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1) \\ &\quad \left[(\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2} \right. \\ &\quad + (\xi_1 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\ &\quad + (\xi_2 - 1) \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2} \\ &\quad + \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\ &\quad + \frac{(\xi_2 - 1)}{\xi_1} \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\ &\quad + \frac{1}{\xi_1} \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \\ &\quad \left. + \frac{(\xi_1 - 1)}{\xi_2} \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \right] \end{aligned}$$

$$+ \frac{1}{\xi_2} \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 - \frac{25}{7}\right)^2} \\ + \frac{1}{\xi_1 \xi_2} \sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \Big].$$

Proof. Suppose $n = \xi_1^2 \xi_2^2$, with ξ_1 and ξ_2 as distinct primes. The set of proper divisors of n includes $\xi_1, \xi_2, \xi_1^2, \xi_1 \xi_2, \xi_2^2, \xi_1^2 \xi_2$ and $\xi_1 \xi_2^2$. Lemma 1 implies $\delta_{\xi_1^2 \xi_2^2}$ follows the adjacency $\xi_2 \sim \xi_1 \sim \xi_2^2 \sim \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2 \sim \xi_1^2 \sim \xi_1 \xi_2, \xi_1^2 \sim \xi_2^2 \sim \xi_1 \xi_2, \xi_2 \sim \xi_1^2$ while, according to Lemma 2, $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2}) = \delta_{\xi_1^2 \xi_2^2}[\Gamma'(A_{\xi_1}), \Gamma'(A_{\xi_2}), \Gamma'(A_{\xi_1^2}), \Gamma'(A_{\xi_1 \xi_2}), \Gamma'(A_{\xi_2^2}), \Gamma'(A_{\xi_1^2 \xi_2}), \Gamma'(A_{\xi_1 \xi_2^2})]$. Applying Corollary 1, we obtain

$$\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2}) = \delta_{\xi_1^2 \xi_2^2}[\bar{K}_{\phi(\xi_1 \xi_2^2)}, \bar{K}_{\phi(\xi_1^2 \xi_2)}, \bar{K}_{\phi(\xi_2^2)}, \bar{K}_{\phi(\xi_1 \xi_2)}, \bar{K}_{\phi(\xi_1^2)}, \bar{K}_{\phi(\xi_2)}, \bar{K}_{\phi(\xi_1)}].$$

By Lemma 3, $|A_{\xi_1}| = \xi_2(\xi_1 - 1)(\xi_2 - 1), |A_{\xi_2}| = \xi_1(\xi_1 - 1)(\xi_2 - 1), |A_{\xi_1^2}| = \xi_2(\xi_2 - 1), |A_{\xi_1 \xi_2}| = (\xi_1 - 1)(\xi_2 - 1), |A_{\xi_2^2}| = \xi_1(\xi_1 - 1), |A_{\xi_1^2 \xi_2}| = (\xi_2 - 1)$, and $|A_{\xi_1 \xi_2^2}| = (\xi_1 - 1)$.

If $v \in A_{\xi_1}, \xi_2 \sim \xi_1 \sim \xi_2^2$, then $\deg(v)$ is the cardinality of A_{ξ_2} and $A_{\xi_2^2}$, i.e., $\deg(v) = |A_{\xi_2}| + |A_{\xi_2^2}|$, similarly if $v \in A_{\xi_2}, \xi_2 \sim \xi_1, \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_1^2}|$, if $v \in A_{\xi_1^2}, \xi_1 \xi_2^2 \sim \xi_1^2 \sim \xi_1 \xi_2, \xi_1^2 \sim \xi_2^2, \xi_2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_2}| + |A_{\xi_2^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1 \xi_2^2}|$, if $v \in A_{\xi_1 \xi_2}, \xi_1^2 \sim \xi_1 \xi_2, \xi_2^2 \sim \xi_1 \xi_2$, then $\deg(v) = |A_{\xi_1^2}| + |A_{\xi_2^2}|$, if $v \in A_{\xi_2^2}, \xi_1 \sim \xi_2^2 \sim \xi_1^2 \xi_2, \xi_1^2 \sim \xi_2^2 \sim \xi_1 \xi_2$, then $\deg(v) = |A_{\xi_1}| + |A_{\xi_1^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1^2 \xi_2}|$, if $v \in A_{\xi_1^2 \xi_2}, \xi_2^2 \sim \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2$, then $\deg(v) = |A_{\xi_2^2}| + |A_{\xi_1 \xi_2^2}|$, if $v \in A_{\xi_1 \xi_2^2}, \xi_1^2 \xi_2 \sim \xi_1 \xi_2^2 \sim \xi_1^2$, then $\deg(v) = |A_{\xi_1^2}| + |A_{\xi_1^2 \xi_2}|$. Therefore,

$$\deg(v) = \begin{cases} |A_{\xi_2}| + |A_{\xi_2^2}|, & v \in A_{\xi_1} \\ |A_{\xi_1}| + |A_{\xi_1^2}|, & v \in A_{\xi_2} \\ |A_{\xi_2}| + |A_{\xi_2^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1 \xi_2^2}|, & v \in A_{\xi_1^2} \\ |A_{\xi_1^2}| + |A_{\xi_2^2}|, & v \in A_{\xi_1 \xi_2} \\ |A_{\xi_1}| + |A_{\xi_1^2}| + |A_{\xi_1 \xi_2}| + |A_{\xi_1^2 \xi_2}|, & v \in A_{\xi_2^2} \\ |A_{\xi_2^2}| + |A_{\xi_1 \xi_2^2}|, & v \in A_{\xi_1^2 \xi_2} \\ |A_{\xi_1^2}| + |A_{\xi_1^2 \xi_2}|, & v \in A_{\xi_1 \xi_2^2}. \end{cases}$$

Here, $m' = 9, n' = 7$, then $\frac{2m'}{n'} = \frac{18}{7}$.

Any edge ε in $\Gamma'(\mathbb{Z}_{\xi_1^2 \xi_2^2})$, must fall into one of the following categories:

Case 1: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}| |A_{\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2 \xi_2 - \xi_1 \xi_2, \deg(v) = \xi_1 \xi_2^2 - \xi_1 \xi_2$. We have

$$|A_{\xi_1}| |A_{\xi_2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2}.$$

Case 2: If $u \in A_{\xi_1}$ and $v \in A_{\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1}| |A_{\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2 \xi_2 - \xi_1 \xi_2, \deg(v) = \xi_1 \xi_2^2 - \xi_1$. We have

$$|A_{\xi_1}| |A_{\xi_2^2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2}.$$

Case 3: If $u \in A_{\xi_2}$ and $v \in A_{\xi_1^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1^2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1\xi_2$, $\deg(v) = \xi_1^2\xi_2 - \xi_2$. We have

$$|A_{\xi_2}||A_{\xi_1^2}|\sqrt{\left(\xi_1\xi_2^2 - \xi_1\xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2\xi_2 - \xi_2 - \frac{18}{7}\right)^2}.$$

Case 4: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_1\xi_2^2 - \xi_1$. We have

$$|A_{\xi_1^2}||A_{\xi_2^2}|\sqrt{\left(\xi_1^2\xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1\xi_2^2 - \xi_1 - \frac{18}{7}\right)^2}.$$

Case 5: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_1^2 + \xi_2^2 - \xi_1 - \xi_2$. We have

$$|A_{\xi_1^2}||A_{\xi_1\xi_2}|\sqrt{\left(\xi_1^2\xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2}.$$

Case 6: If $u \in A_{\xi_1^2}$ and $v \in A_{\xi_1\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2}||A_{\xi_1\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2\xi_2 - \xi_2$, $\deg(v) = \xi_2^2 - 1$. We have

$$|A_{\xi_1^2}||A_{\xi_1\xi_2^2}|\sqrt{\left(\xi_1^2\xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2}.$$

Case 7: If $u \in A_{\xi_2^2}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2^2}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1$, $\deg(v) = \xi_1^2 + \xi_2^2 - \xi_1 - \xi_2$. We have

$$|A_{\xi_2^2}||A_{\xi_1\xi_2}|\sqrt{\left(\xi_1\xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2}.$$

Case 8: If $u \in A_{\xi_2^2}$ and $v \in A_{\xi_1^2\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2^2}||A_{\xi_1^2\xi_2}|$ is the number of edges and $\deg(u) = \xi_1\xi_2^2 - \xi_1$, $\deg(v) = \xi_1^2 - 1$. We have

$$|A_{\xi_2^2}||A_{\xi_1^2\xi_2}|\sqrt{\left(\xi_1\xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 - \frac{25}{7}\right)^2}.$$

Case 9: If $u \in A_{\xi_1^2\xi_2}$ and $v \in A_{\xi_1\xi_2^2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1^2\xi_2}||A_{\xi_1\xi_2^2}|$ is the number of edges and $\deg(u) = \xi_1^2 - 1$, $\deg(v) = \xi_2^2 - 1$. We have

$$|A_{\xi_1^2\xi_2}||A_{\xi_1\xi_2^2}|\sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2}.$$

Compiling the earlier analyzed cases enables us to present the average Sombor index of $\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2^2})$ as follows:

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1^2\xi_2^2})) \\ = \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \end{aligned}$$

$$\begin{aligned}
&= |A_{\xi_1}| |A_{\xi_2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_1}| |A_{\xi_2^2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_2}| |A_{\xi_1^2}| \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_1^2}| |A_{\xi_2^2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_1^2}| |A_{\xi_1 \xi_2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_1^2}| |A_{\xi_1 \xi_2^2}| \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \\
&\quad + |A_{\xi_2^2}| |A_{\xi_1 \xi_2}| \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + |A_{\xi_2^2}| |A_{\xi_1^2 \xi_2}| \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 - \frac{25}{7}\right)^2} \\
&\quad + |A_{\xi_1^2 \xi_2}| |A_{\xi_1 \xi_2^2}| \sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2}. \\
&= \xi_1 \xi_2 (\xi_1 - 1)^2 (\xi_2 - 1)^2 \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + \xi_1 \xi_2 (\xi_1 - 1)^2 (\xi_2 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\
&\quad + \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1)^2 \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\
&\quad + \xi_2 (\xi_1 - 1) (\xi_2 - 1)^2 \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + \xi_2 (\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \\
&\quad + \xi_1 (\xi_1 - 1)^2 (\xi_2 - 1) \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
&\quad + \xi_1 (\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 - \frac{25}{7}\right)^2} \\
&\quad + (\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2}. \\
&= \xi_1 \xi_2 (\xi_1 - 1) (\xi_2 - 1) \\
&\quad \left[(\xi_1 - 1) (\xi_2 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2} \right. \\
&\quad \left. + (\xi_1 - 1) \sqrt{\left(\xi_1^2 \xi_2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \right. \\
&\quad \left. + (\xi_2 - 1) \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2} \right. \\
&\quad \left. + \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \right. \\
&\quad \left. + \sqrt{\left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \right. \\
&\quad \left. + \sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \right].
\end{aligned}$$

$$\begin{aligned}
& + (\xi_2 - 1) \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2} \\
& + \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2} \\
& + \frac{(\xi_2 - 1)}{\xi_1} \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
& + \frac{1}{\xi_1} \sqrt{\left(\xi_1^2 \xi_2 - \xi_2 - \frac{18}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2} \\
& + \frac{(\xi_1 - 1)}{\xi_2} \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 + \xi_2^2 - \xi_1 - \xi_2 - \frac{18}{7}\right)^2} \\
& + \frac{1}{\xi_2} \sqrt{\left(\xi_1 \xi_2^2 - \xi_1 - \frac{18}{7}\right)^2 + \left(\xi_1^2 - \frac{25}{7}\right)^2} \\
& + \frac{1}{\xi_1 \xi_2} \sqrt{\left(\xi_1^2 - \frac{25}{7}\right)^2 + \left(\xi_2^2 - \frac{25}{7}\right)^2}.
\end{aligned}$$

□

Example 7. Let $n = 36 = 2^2 \cdot 3^2$, where 2 and 3 are distinct primes. Here, 2, 3, 4, 6, 9, 12 and 18 are the proper divisors of n . So, by Lemma 1, $\delta_{36} : 3 \sim 2 \sim 9 \sim 12 \sim 18 \sim 4 \sim 6, 4 \sim 9 \sim 6, 3 \sim 4$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_{36}) = \delta_{36}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_4), \Gamma'(A_6), \Gamma'(A_9), \Gamma'(A_{12}), \Gamma'(A_{18})]$. By Corollary 1, we get $\Gamma'(\mathbb{Z}_{36}) = \delta_{36}[\bar{K}_{\phi(18)}, \bar{K}_{\phi(12)}, \bar{K}_{\phi(9)}, \bar{K}_{\phi(6)}, \bar{K}_{\phi(4)}, \bar{K}_{\phi(3)}, \bar{K}_{\phi(2)}]$.

By Lemma 3, $|A_2| = 6, |A_3| = 4, |A_4| = 6, |A_6| = 2, |A_9| = 2, |A_{12}| = 2, |A_{18}| = 1$. If $2 \in A_2, 3 \sim 2 \sim 9$, then $\deg(2) = |A_3| + |A_9| = 6$, if $3 \in A_3, 3 \sim 2, 3 \sim 4$, then $\deg(3) = |A_2| + |A_4| = 12$, if $4 \in A_4, 18 \sim 4 \sim 6, 4 \sim 9, 3 \sim 4$, then $\deg(4) = |A_{18}| + |A_6| + |A_9| + |A_3| = 9$, if $6 \in A_6, 4 \sim 6, 9 \sim 6$, then $\deg(6) = |A_4| + |A_9| = 8$, if $9 \in A_9, 2 \sim 9 \sim 12, 4 \sim 9 \sim 6$, then $\deg(9) = |A_2| + |A_{12}| + |A_4| + |A_6| = 16$, if $12 \in A_{12}, 9 \sim 12 \sim 18$, then $\deg(12) = |A_9| + |A_{18}| = 3$, if $18 \in A_{18}, 12 \sim 18 \sim 4$, then $\deg(18) = |A_{12}| + |A_4| = 8$.

Here $m' = 9, n' = 7$, then $\frac{2m'}{n'} = \frac{18}{7}$. Thus, the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{36})$ is:

$$\begin{aligned}
& SO_{avr}(\Gamma'(\mathbb{Z}_{36})) \\
& = \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\
& = |A_2||A_3| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(3) - \frac{2m'}{n'}\right)^2} \\
& \quad + |A_2||A_9| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(9) - \frac{2m'}{n'}\right)^2} \\
& \quad + |A_3||A_4| \sqrt{\left(\deg(3) - \frac{2m'}{n'}\right)^2 + \left(\deg(4) - \frac{2m'}{n'}\right)^2} \\
& \quad + |A_4||A_9| \sqrt{\left(\deg(4) - \frac{2m'}{n'}\right)^2 + \left(\deg(9) - \frac{2m'}{n'}\right)^2} \\
& \quad + |A_4||A_6| \sqrt{\left(\deg(4) - \frac{2m'}{n'}\right)^2 + \left(\deg(6) - \frac{2m'}{n'}\right)^2}
\end{aligned}$$

$$\begin{aligned}
& + |A_4||A_{18}|\sqrt{\left(\deg(4) - \frac{2m'}{n'}\right)^2 + \left(\deg(18) - \frac{2m'}{n'}\right)^2} \\
& + |A_9||A_6|\sqrt{\left(\deg(9) - \frac{2m'}{n'}\right)^2 + \left(\deg(6) - \frac{2m'}{n'}\right)^2} \\
& + |A_9||A_{12}|\sqrt{\left(\deg(9) - \frac{2m'}{n'}\right)^2 + \left(\deg(12) - \frac{2m'}{n'}\right)^2} \\
& + |A_{12}||A_{18}|\sqrt{\left(\deg(12) - \frac{2m'}{n'}\right)^2 + \left(\deg(18) - \frac{2m'}{n'}\right)^2}. \\
& = 6.4\sqrt{\left(6 - \frac{18}{7}\right)^2 + \left(12 - \frac{18}{7}\right)^2} + 6.2\sqrt{\left(6 - \frac{18}{7}\right)^2 + \left(12 - \frac{18}{7}\right)^2} \\
& + 4.6\sqrt{\left(12 - \frac{18}{7}\right)^2 + \left(9 - \frac{18}{7}\right)^2} + 6.2\sqrt{\left(9 - \frac{18}{7}\right)^2 + \left(16 - \frac{18}{7}\right)^2} \\
& + 6.2\sqrt{\left(9 - \frac{18}{7}\right)^2 + \left(8 - \frac{18}{7}\right)^2} + 6.1\sqrt{\left(9 - \frac{18}{7}\right)^2 + \left(8 - \frac{18}{7}\right)^2} \\
& + 2.2\sqrt{\left(16 - \frac{18}{7}\right)^2 + \left(8 - \frac{18}{7}\right)^2} + 2.2\sqrt{\left(16 - \frac{18}{7}\right)^2 + \left(3 - \frac{18}{7}\right)^2} \\
& + 2.1\sqrt{\left(3 - \frac{18}{7}\right)^2 + \left(8 - \frac{18}{7}\right)^2}. \\
& = 24\frac{\sqrt{4932}}{7} + 12\frac{\sqrt{9412}}{7} + 24\frac{\sqrt{6381}}{7} + 12\frac{\sqrt{10861}}{7} + 12\frac{\sqrt{3469}}{7} \\
& + 6\frac{\sqrt{3469}}{7} + 4\frac{\sqrt{10280}}{7} + 4\frac{\sqrt{8845}}{7} + 2\frac{\sqrt{1453}}{7}.
\end{aligned}$$

The average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})$ for three distinct primes ξ_1, ξ_2 and ξ_3 is stated in the following theorem:

Theorem 8. Let the prime numbers ξ_1, ξ_2 and ξ_3 be distinct. So the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})$ is

$$\begin{aligned}
& SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})) \\
& = (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \\
& \quad \left[(\xi_3 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \right. \\
& \quad + (\xi_2 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
& \quad + \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_2\xi_3 - 4)^2} \\
& \quad + (\xi_1 - 1)\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
& \quad + \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 - 4)^2} \\
& \quad \left. + \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 - 4)^2} \right]
\end{aligned}$$

$$+ \frac{1}{(\zeta_2 - 1)} \sqrt{(\zeta_1 \zeta_2 - 4)^2 + (\zeta_2 \zeta_3 - 4)^2} + \frac{1}{(\zeta_1 - 1)} \sqrt{(\zeta_1 \zeta_2 - 4)^2 + (\zeta_1 \zeta_3 - 4)^2} \\ + \frac{1}{(\zeta_3 - 1)} \sqrt{(\zeta_2 \zeta_3 - 4)^2 + (\zeta_1 \zeta_3 - 4)^2} \Big].$$

Proof. Let $n = \zeta_1 \zeta_2 \zeta_3$, where ζ_1, ζ_2 and ζ_3 are distinct primes. Then the proper divisors of n are $\zeta_1, \zeta_2, \zeta_3, \zeta_1 \zeta_2, \zeta_2 \zeta_3$ and $\zeta_1 \zeta_3$. So, by Lemma 1, $\delta_{\zeta_1 \zeta_2 \zeta_3} : \zeta_1 \zeta_2 \sim \zeta_3 \sim \zeta_1 \sim \zeta_2 \sim \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3, \zeta_1 \sim \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3, \zeta_2 \sim \zeta_3$ and by Lemma 2, we have $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3}) = \delta_{\zeta_1 \zeta_2 \zeta_3}[\Gamma'(A_{\zeta_1}), \Gamma'(A_{\zeta_2}), \Gamma'(A_{\zeta_3}), \Gamma'(A_{\zeta_1 \zeta_2}), \Gamma'(A_{\zeta_2 \zeta_3}), \Gamma'(A_{\zeta_1 \zeta_3})]$. By applying Corollary 1, it follows that

$$\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3}) = \delta_{\zeta_1 \zeta_2 \zeta_3}[\bar{K}_{\phi(\zeta_2 \zeta_3)}, \bar{K}_{\phi(\zeta_1 \zeta_3)}, \bar{K}_{\phi(\zeta_1 \zeta_2)}, \bar{K}_{\phi(\zeta_3)}, \bar{K}_{\phi(\zeta_1)}, \bar{K}_{\phi(\zeta_2)}].$$

By Lemma 3, $|A_{\zeta_1}| = (\zeta_2 - 1)(\zeta_3 - 1), |A_{\zeta_2}| = (\zeta_1 - 1)(\zeta_3 - 1), |A_{\zeta_3}| = (\zeta_1 - 1)(\zeta_2 - 1), |A_{\zeta_1 \zeta_2}| = (\zeta_3 - 1), |A_{\zeta_2 \zeta_3}| = (\zeta_1 - 1), |A_{\zeta_1 \zeta_3}| = (\zeta_2 - 1)$.

If $v \in A_{\zeta_1}, \zeta_3 \sim \zeta_1 \sim \zeta_2, \zeta_1 \sim \zeta_2 \zeta_3$ then $\deg(v)$ is the cardinality of A_{ζ_2}, A_{ζ_3} and $A_{\zeta_2 \zeta_3}$, i.e., $\deg(v) = |A_{\zeta_2}| + |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}|$, similarly if $v \in A_{\zeta_2}, \zeta_1 \sim \zeta_2 \sim \zeta_1 \zeta_3, \zeta_2 \sim \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_3}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_3}, \zeta_1 \zeta_2 \sim \zeta_3 \sim \zeta_1, \zeta_2 \sim \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}|$, if $v \in A_{\zeta_1 \zeta_2}, \zeta_1 \zeta_2 \sim \zeta_3, \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3$, then $\deg(v) = |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_2 \zeta_3}, \zeta_1 \zeta_2 \sim \zeta_2 \zeta_3, \zeta_1 \sim \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3$, then $\deg(v) = |A_{\zeta_1}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_1 \zeta_3}|$, if $v \in A_{\zeta_1 \zeta_3}, \zeta_2 \sim \zeta_1 \zeta_3 \sim \zeta_1 \zeta_2, \zeta_2 \zeta_3 \sim \zeta_1 \zeta_3$ then $\deg(v) = |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_2 \zeta_3}|$, so the degree of v for a vertex v can be written as

$$\deg(v) = \begin{cases} |A_{\zeta_2}| + |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}|, & v \in A_{\zeta_1} \\ |A_{\zeta_1}| + |A_{\zeta_3}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_2} \\ |A_{\zeta_1}| + |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}|, & v \in A_{\zeta_3} \\ |A_{\zeta_3}| + |A_{\zeta_2 \zeta_3}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_1 \zeta_2} \\ |A_{\zeta_1}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_1 \zeta_3}|, & v \in A_{\zeta_2 \zeta_3} \\ |A_{\zeta_2}| + |A_{\zeta_1 \zeta_2}| + |A_{\zeta_2 \zeta_3}|, & v \in A_{\zeta_1 \zeta_3}. \end{cases}$$

Here, $m' = 9, n' = 6$, then $\frac{2m'}{n'} = 3$.

For any edge ε in $\Gamma'(\mathbb{Z}_{\zeta_1 \zeta_2 \zeta_3})$, the following possibilities arise:

Case 1: If $u \in A_{\zeta_1}$ and $v \in A_{\zeta_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\zeta_1}| |A_{\zeta_2}|$ is the number of edges and $\deg(u) = (\zeta_1 - 1)(\zeta_2 + \zeta_3 - 1), \deg(v) = (\zeta_2 - 1)(\zeta_1 + \zeta_3 - 1)$. We have

$$|A_{\zeta_1}| |A_{\zeta_2}| \sqrt{(\zeta_1 \zeta_2 + \zeta_1 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 - 2)^2 + (\zeta_1 \zeta_2 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 - 2)^2}.$$

Case 2: If $u \in A_{\zeta_1}$ and $v \in A_{\zeta_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\zeta_1}| |A_{\zeta_3}|$ is the number of edges and $\deg(u) = (\zeta_1 - 1)(\zeta_2 + \zeta_3 - 1), \deg(v) = (\zeta_3 - 1)(\zeta_1 + \zeta_2 - 1)$. We have

$$|A_{\zeta_1}| |A_{\zeta_3}| \sqrt{(\zeta_1 \zeta_2 + \zeta_1 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 - 2)^2 + (\zeta_1 \zeta_3 + \zeta_2 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 - 2)^2}.$$

Case 3: If $u \in A_{\zeta_1}$ and $v \in A_{\zeta_2 \zeta_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\zeta_1}| |A_{\zeta_2 \zeta_3}|$ is the number of edges and $\deg(u) = (\zeta_1 - 1)(\zeta_2 + \zeta_3 - 1), \deg(v) = \zeta_2 \zeta_3 - 1$. We have

$$|A_{\zeta_1}| |A_{\zeta_2 \zeta_3}| \sqrt{(\zeta_1 \zeta_2 + \zeta_1 \zeta_3 - \zeta_1 - \zeta_2 - \zeta_3 - 2)^2 + (\zeta_2 \zeta_3 - 4)^2}.$$

Case 4: If $u \in A_{\xi_2}$ and $v \in A_{\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_3}|$ is the number of edges and $\deg(u) = (\xi_2 - 1)(\xi_1 + \xi_3 - 1)$, $\deg(v) = (\xi_3 - 1)(\xi_1 + \xi_2 - 1)$. We have

$$|A_{\xi_2}||A_{\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2}.$$

Case 5: If $u \in A_{\xi_2}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = (\xi_2 - 1)(\xi_1 + \xi_3 - 1)$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_2}||A_{\xi_1\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 - 4)^2}.$$

Case 6: If $u \in A_{\xi_3}$ and $v \in A_{\xi_1\xi_2}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_3}||A_{\xi_1\xi_2}|$ is the number of edges and $\deg(u) = (\xi_3 - 1)(\xi_1 + \xi_2 - 1)$, $\deg(v) = \xi_1\xi_2 - 1$. We have

$$|A_{\xi_3}||A_{\xi_1\xi_2}|\sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 - 4)^2}.$$

Case 7: If $u \in A_{\xi_1\xi_2}$ and $v \in A_{\xi_2\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1\xi_2}||A_{\xi_2\xi_3}|$ is the number of edges and $\deg(u) = \xi_1\xi_2 - 1$, $\deg(v) = \xi_2\xi_3 - 1$. We have

$$|A_{\xi_1\xi_2}||A_{\xi_2\xi_3}|\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_2\xi_3 - 4)^2}.$$

Case 8: If $u \in A_{\xi_1\xi_2}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_1\xi_2}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = \xi_1\xi_2 - 1$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_1\xi_2}||A_{\xi_1\xi_3}|\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_1\xi_3 - 4)^2}.$$

Case 9: If $u \in A_{\xi_2\xi_3}$ and $v \in A_{\xi_1\xi_3}$ s.t. $uv = \varepsilon$. Here, $|A_{\xi_2\xi_3}||A_{\xi_1\xi_3}|$ is the number of edges and $\deg(u) = \xi_2\xi_3 - 1$, $\deg(v) = \xi_1\xi_3 - 1$. We have

$$|A_{\xi_2\xi_3}||A_{\xi_1\xi_3}|\sqrt{(\xi_2\xi_3 - 4)^2 + (\xi_1\xi_3 - 4)^2}.$$

By gathering all of the previously considered cases, we can express the average Sombor index of $\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})$ as follows:

$$\begin{aligned} SO_{avr}(\Gamma'(\mathbb{Z}_{\xi_1\xi_2\xi_3})) &= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\ &= |A_{\xi_1}||A_{\xi_2}|\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\ &\quad + |A_{\xi_1}||A_{\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\ &\quad + |A_{\xi_1}||A_{\xi_2\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_2\xi_3 - 4)^2} \\ &\quad + |A_{\xi_2}||A_{\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\ &\quad + |A_{\xi_2}||A_{\xi_1\xi_3}|\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 - 4)^2} \\ &\quad + |A_{\xi_3}||A_{\xi_1\xi_2}|\sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 - 4)^2} \\ &\quad + |A_{\xi_1\xi_2}||A_{\xi_2\xi_3}|\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_2\xi_3 - 4)^2} \\ &\quad + |A_{\xi_1\xi_2}||A_{\xi_1\xi_3}|\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_1\xi_3 - 4)^2} \\ &\quad + |A_{\xi_2\xi_3}||A_{\xi_1\xi_3}|\sqrt{(\xi_2\xi_3 - 4)^2 + (\xi_1\xi_3 - 4)^2}. \end{aligned}$$

$$\begin{aligned}
&= (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)^2 \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)^2(\xi_3 - 1) \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_2\xi_3 - 4)^2} \\
&\quad + (\xi_1 - 1)^2(\xi_2 - 1)(\xi_3 - 1) \\
&\quad \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 - 4)^2} \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 - 4)^2} \\
&\quad + (\xi_1 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_2\xi_3 - 4)^2} \\
&\quad + (\xi_2 - 1)(\xi_3 - 1)\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_1\xi_3 - 4)^2} \\
&\quad + (\xi_1 - 1)(\xi_2 - 1)\sqrt{(\xi_2\xi_3 - 4)^2 + (\xi_1\xi_3 - 4)^2}. \\
&= (\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1) \\
&\quad \left[(\xi_3 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \right. \\
&\quad + (\xi_2 - 1)\sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
&\quad + \sqrt{(\xi_1\xi_2 + \xi_1\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_2\xi_3 - 4)^2} \\
&\quad + (\xi_1 - 1)\sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2} \\
&\quad + \sqrt{(\xi_1\xi_2 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_3 - 4)^2} \\
&\quad + \sqrt{(\xi_1\xi_3 + \xi_2\xi_3 - \xi_1 - \xi_2 - \xi_3 - 2)^2 + (\xi_1\xi_2 - 4)^2} \\
&\quad + \frac{1}{(\xi_2 - 1)}\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_2\xi_3 - 4)^2} + \frac{1}{(\xi_1 - 1)}\sqrt{(\xi_1\xi_2 - 4)^2 + (\xi_1\xi_3 - 4)^2} \\
&\quad \left. + \frac{1}{(\xi_3 - 1)}\sqrt{(\xi_2\xi_3 - 4)^2 + (\xi_1\xi_3 - 4)^2} \right].
\end{aligned}$$

□

Example 8. Let $n = 30 = 2 \cdot 3 \cdot 5$, where 2, 3 and 5 are distinct primes. Here, 2, 3, 5, 6, 10 and 15 are the proper divisors of n . So, by Lemma 1, $\delta_{30} : 6 \sim 5 \sim 2 \sim 3 \sim 10 \sim 6 \sim 15, 2 \sim 15 \sim 10, 3 \sim 5$. By Lemma 2, we have $\Gamma'(\mathbb{Z}_{30}) = \delta_{30}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_5), \Gamma'(A_6), \Gamma'(A_{15}), \Gamma'(A_{10})]$. By Corollary 1, it follows that $\Gamma'(\mathbb{Z}_{30}) = \delta_{30}[\bar{K}_{\phi(15)}, \bar{K}_{\phi(10)}, \bar{K}_{\phi(6)}, \bar{K}_{\phi(5)}, \bar{K}_{\phi(2)}, \bar{K}_{\phi(3)}]$.

By Lemma 3, $|A_2| = 8, |A_3| = 4, |A_5| = 2, |A_6| = 4, |A_{15}| = 1, |A_{10}| = 2$. If $2 \in A_2, 5 \sim 2 \sim 3, 2 \sim 15$, then $\deg(2) = |A_5| + |A_3| + |A_{15}| = 7$, if $3 \in A_3, 2 \sim 3 \sim 10, 3 \sim 5$, then $\deg(3) = |A_2| + |A_{10}| + |A_5| = 12$, if $5 \in A_5, 6 \sim 5 \sim 2, 3 \sim 5$, then $\deg(5) = |A_6| + |A_2| + |A_3| = 16$, if $6 \in A_6, 6 \sim 5, 10 \sim 6 \sim 15$, then $\deg(6) = |A_5| + |A_{10}| + |A_{15}| = 5$, if $15 \in A_{15}, 6 \sim 15, 2 \sim 15 \sim 10$, then $\deg(15) = |A_6| + |A_2| + |A_{10}| = 14$, if $10 \in A_{10}, 3 \sim 10 \sim 6, 15 \sim 10$, then $\deg(10) = |A_3| + |A_6| + |A_{15}| = 9$.

Here, $m' = 9, n' = 6$, then $\frac{2m'}{n'} = 3$. Thus, the average Sombor index of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{30})$ is:

$$\begin{aligned}
& SO_{avr}(\Gamma'(\mathbb{Z}_{30})) \\
&= \sum_{v_i \sim v_j} \sqrt{\left(\deg(v_i) - \frac{2m'}{n'}\right)^2 + \left(\deg(v_j) - \frac{2m'}{n'}\right)^2} \\
&= |A_2||A_3| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(3) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_2||A_5| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(5) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_2||A_{15}| \sqrt{\left(\deg(2) - \frac{2m'}{n'}\right)^2 + \left(\deg(15) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_3||A_5| \sqrt{\left(\deg(3) - \frac{2m'}{n'}\right)^2 + \left(\deg(5) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_3||A_{10}| \sqrt{\left(\deg(3) - \frac{2m'}{n'}\right)^2 + \left(\deg(10) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_5||A_6| \sqrt{\left(\deg(5) - \frac{2m'}{n'}\right)^2 + \left(\deg(6) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_6||A_{15}| \sqrt{\left(\deg(6) - \frac{2m'}{n'}\right)^2 + \left(\deg(15) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_6||A_{10}| \sqrt{\left(\deg(6) - \frac{2m'}{n'}\right)^2 + \left(\deg(10) - \frac{2m'}{n'}\right)^2} \\
&\quad + |A_{15}||A_{10}| \sqrt{\left(\deg(15) - \frac{2m'}{n'}\right)^2 + \left(\deg(10) - \frac{2m'}{n'}\right)^2}. \\
&= 8.4\sqrt{(7-3)^2 + (12-3)^2} + 8.2\sqrt{(7-3)^2 + (16-3)^2} \\
&\quad + 8.1\sqrt{(7-3)^2 + (14-3)^2} + 4.2\sqrt{(12-3)^2 + (16-3)^2} \\
&\quad + 4.2\sqrt{(12-3)^2 + (9-3)^2} + 2.4\sqrt{(16-3)^2 + (5-3)^2} \\
&\quad + 4.1\sqrt{(5-3)^2 + (14-3)^2} + 4.2\sqrt{(5-3)^2 + (9-3)^2} \\
&\quad + 1.2\sqrt{(14-3)^2 + (9-3)^2}. \\
&= 32\sqrt{97} + 16\sqrt{185} + 8\sqrt{137} + 8\sqrt{250} + 8\sqrt{117} \\
&\quad + 8\sqrt{173} + 4\sqrt{125} + 8\sqrt{40} + 2\sqrt{157}.
\end{aligned}$$

5. Conclusions and Further Work

In our work, we have explicitly calculated both the Euler Sombor index and the average Sombor index for the cozero-divisor graph defined over the integers modulo n . These results offer concrete formulas that can guide further exploration in this area. In fact, building on this foundation, one could naturally extend our approach to analyze cases where n takes the form $\xi_1^{\mathcal{M}} \xi_2^{\mathcal{N}}$, with positive integers $\mathcal{M}, \mathcal{N}$, and with ξ_1, ξ_2 being distinct primes. Exploring such generalizations may lead to new formulas and a richer understanding of how the arithmetic structure of n influences graph-theoretical properties. This opens up exciting directions for future research, connecting the combinatorial intricacies of cozero-divisor graphs to deeper algebraic insights, and encouraging others to investigate even more complex prime power decompositions.

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