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Finite Attainment and Rationality of Homothetic Covering Functionals

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Abstract

Let $K \subseteq \mathbb{R}^n$ be a convex body with the origin in its interior. For $W \subseteq \mathbb{R}^n$ and a positive integer m, let $\rho_m(W;K)$ be the infimum of $\lambda > 0$ such that m translates of λK cover W, and put $\Gamma_m(K) = \rho_m(K;K)$. The supremum of $\rho_m(W;K)$ over all nonempty finite subsets W of K equals $\Gamma_m(K)$, since $\Gamma_m(K) - \rho_m(W;K) \leq \delta$ if $\delta \geq 0$ and $K \subseteq W + \delta K$. The supremum is attained if and only if the least $|W|$ among finite $W \subseteq K$ with $\rho_m(W;K) > \eta$ remains bounded as $\eta \uparrow \Gamma_m(K)$. For a full-dimensional rational polytope $P \subseteq \mathbb{R}^n$, $\Gamma_m(P)$ is rational. The set of optimal center tuples for P is a finite union of rational polyhedra and contains a rational tuple. Every rational lower bound for $\Gamma_m(P)$ can be established using finitely many linear programs with rational data. No finite subset of a triangle T attains $\Gamma_6(T) = 1/2$. For the three-dimensional crosspolytope B_1^3, we give a 100-point set $W \subseteq 13B_1^3 \cap \mathbb{Z}^3$ and a binary set-covering certificate that $W/13$ attains the known value $\Gamma_{14}(B_1^3) = 7/13$.

Keywords: covering functional; finite attainment; rational polytope; set covering; integer programming; computer-assisted proof

MSC: 52C17; 52A20; 90C05; 90C10

1. Introduction

Throughout the paper, $n \in \mathbb{N}$, the *origin* of $\mathbb{R}^n$ is o, and $e_1, \dots, e_n$ are the standard basis vectors. For sets $A, B \subseteq \mathbb{R}^n$, we write

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\} \quad \text{and} \quad A - B = A + (-B).$$

Let K be a *convex body* (i.e., a compact convex set with nonempty interior) in $\mathbb{R}^n$. For $c \in \mathbb{R}^n$ and $\lambda > 0$, the set $c + \lambda K$ is a *positive homothet* of K. A *translate* of K is a set having the form $c + K$ with $c \in \mathbb{R}^n$. When $o \in \operatorname{int} K$, the *gauge* of K is defined by

$$p_K(y) := \inf\{\mu > 0 \mid y \in \mu K\}.$$

For $m \in \mathbb{N}$, a point $\mathbf{c} = (c_1, \dots, c_m) \in (\mathbb{R}^n)^m$ is called an *m-center tuple*.Throughout the paper, an m-center tuple denoted by $\mathbf{c}$ is always assumed to be $(c_1, \dots, c_m)$. We identify $(\mathbb{R}^n)^m$ with $\mathbb{R}^{mn}$ by listing the coordinates of $c_1, \dots, c_m$ in this order.

For a nonempty subset W of $\mathbb{R}^n$, set

$$\rho_m(W;K) := \inf\left\{\lambda > 0 \;\middle|\; \exists \mathbf{c} \in (\mathbb{R}^n)^m \text{ such that } W \subseteq \bigcup_{i=1}^{m} (c_i + \lambda K)\right\}.$$

We use the convention that $\inf \emptyset = +\infty$ and put $\rho_m(\emptyset;K) = 0$. By standard compactness arguments, every nonempty compact set W can be covered by m translates of $\rho_m(W;K)K$.

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Put $\Gamma_m(K) = \rho_m(K;K)$. Clearly, $\Gamma_m(K) \le 1$. We call $\Gamma_m(\cdot) : \mathcal{K}^n \to [0,1]$ the *m-th covering functional*, where $\mathcal{K}^n$ is the set of all convex bodies in $\mathbb{R}^n$. A center tuple $\mathbf{c}$ is *optimal for* K if

$$K \subseteq \bigcup_{i=1}^{m} (c_i + \Gamma_m(K)K).$$

The covering functional is *affine invariant*: $\Gamma_m(T(K)) = \Gamma_m(K)$ holds for every invertible affine map T of $\mathbb{R}^n$. More generally, if $T(x) = Ax + b$ is invertible, then $T(c + \lambda K) = Ac + (1-\lambda)b + \lambda T(K)$. Applying T to a cover of W by m translates of λK gives a cover of $T(W)$ by m translates of $\lambda T(K)$. Applying T^{-1} gives the converse, so $\rho_m(T(W);T(K)) = \rho_m(W;K)$.

The least number of smaller positive homothets of K needed to cover K is the least m for which $\Gamma_m(K) < 1$. Hadwiger's covering conjecture asserts that this number is at most 2^n, with equality precisely when K is affinely equivalent to a cube [1,2]. In Zong's quantitative formulation [3], the conjectured upper bound is equivalent to $\Gamma_{2^n}(K) < 1$ for every $K \in \mathcal{K}^n$.

If $o \in \operatorname{int} K$ and $W \subseteq \mathbb{R}^n$ is nonempty and finite, then

$$\rho_m(W;K) = \min_{\mathbf{c} \in (\mathbb{R}^n)^m} \max_{w \in W} \min_{1 \le i \le m} p_K(w - c_i).$$

For a center tuple $\mathbf{c}$, the number $\max_{w \in W} \min_{1 \le i \le m} p_K(w - c_i)$ is called the *covering radius* of W with respect to $\mathbf{c}$. For $\gamma \ge 0$, a nonempty finite set $W \subseteq K$ is called a *finite m-covering obstruction above* γ if $\rho_m(W;K) > \gamma$. When K, m, and γ are fixed, we abbreviate this to *finite obstruction*. It is not difficult to verify that

$$\rho_m(\eta W;K) = \eta \rho_m(W;K), \quad \forall \eta > 0. \tag{1}$$

Exact values of covering functionals have been determined for several classes of polytopes. For cubes, $\Gamma_{2^n}([-1,1]^n) = 1/2$; see Wu and He [4] (Section 4, Example 1).

For the standard simplex $\Delta_n = \operatorname{conv}\{o, e_1, \dots, e_n\}$, Yu, Gao, He, and Wu [5] (Theorem 10 and Corollary 12) proved that $\Gamma_{n+2}(\Delta_n) = n^2/(n^2+n+1)$ and $\Gamma_6(\Delta_3) = 2/3$.

Yu, Lv, Zhao, He, and Wu [6] (Theorem 1) bound the error when the points to be covered and the admissible centers are approximated separately. For a prescribed finite demand set, exact and approximation algorithms for Euclidean k-center, which minimizes the common radius of k covering balls, are given in [7]. Approximate core-sets, meaning subsets whose optimal radii approximate the radius of the given finite set, are studied in [8,9]. For containment of a compact set in one positive homothet of a convex body, Brandenberg and König [10] (Theorem 1.2) obtain bounds on the size of finite subsets determining or approximating the radius.

Wu, Fan, and He [11] relate coverings of a convex body to coverings of its extreme points. Here we ask whether a fixed finite subset preserves the optimal covering radius for the same prescribed number of homothets.

Upper bounds for covering functionals can be obtained by constructing coverings. Every covering of K also covers each finite set $W \subseteq K$, so $\rho_m(W;K) \le \Gamma_m(K)$. Thus, a covering of K by m translates of λK is optimal if there is a finite $W \subseteq K$ with $\rho_m(W;K) = \lambda$. For the three-dimensional crosspolytope K, Section 6 gives 100 points in K and an independently checkable set-covering certificate showing that fourteen translates of λK can cover those points only if $\lambda \ge 7/13$.

Proposition 1 below gives

$$\Gamma_m(K) = \sup_{\emptyset \ne W \subseteq K \text{ and } |W| < \infty} \rho_m(W;K)$$

and, when $o \in \operatorname{int} K$, the error estimate (5) for every nonempty finite $W \subseteq K$ and $\delta \ge 0$ satisfying $K \subseteq W + \delta K$. For $0 \le \eta < \Gamma_m(K)$, let $\tau_{m,\eta}(K)$ be the least cardinality of a finite $W \subseteq K$ with $\rho_m(W;K) > \eta$. By Theorem 1, a finite subset attains $\Gamma_m(K)$ if and only if $\tau_{m,\eta}(K)$ remains bounded as $\eta \uparrow \Gamma_m(K)$.

A *rational polytope* is the convex hull of finitely many points with rational coordinates. A *rational polyhedron* is an intersection of finitely many closed affine half-spaces defined by rational coefficients.

The covering functional of a singleton is zero.

Let $P \subseteq \mathbb{R}^d$ be a polytope of affine dimension $n \geq 1$. Choose affinely independent vertices $p_0, \ldots, p_n$ and let B be the matrix with columns $p_j - p_0$ for $1 \leq j \leq n$. Let $H = \operatorname{aff} P$ and define $\phi : H \to \mathbb{R}^n$ by $\phi(p_0 + Bt) = t$. Put $P' = \phi(P)$. If P is rational, ϕ and ϕ^{-1} preserve rational points because p_0 and B are rational and B has an invertible n-row submatrix. For two rational affine coordinate maps $\phi, \psi : H \to \mathbb{R}^n$, the rational affine isomorphism $\psi \circ \phi^{-1}$ maps $\phi(P)$ onto $\psi(P)$.

Put $L = H - p_0$. For an ambient translation vector c and $\lambda > 0$, the affine hull of $c + \lambda P$ is $p_0 + a + L$, where $a = c + (\lambda - 1)p_0$. If $a \notin L$, then $c + \lambda P$ is disjoint from H and can be omitted from a cover of P. If $a \in L$, write $a = Bs$. Then $\phi(c + \lambda P) = s + \lambda P'$. Conversely, the copy $s + \lambda P'$ in $\mathbb{R}^n$ corresponds to $c + \lambda P$ with $c = (1 - \lambda)p_0 + Bs$. A cover using fewer than m copies can be completed to m by repeating one of them.

For every rational polytope $P \in \mathcal{K}^n$, Theorem 2 proves that $\Gamma_m(P)$ is rational and that P admits an optimal center tuple in $(\mathbb{Q}^n)^m$. The set of all optimal center tuples of P is a finite union of rational polyhedra. If p is the centroid of a rational polytope $P \in \mathcal{K}^n$, then $P - p$ is rational and has o in its interior.

Theorem 2 and Corollary 3 give a finite exact procedure using linear programs with rational data. These programs arise from a subdivision of a compact set of center tuples and establish every rational lower bound. For the standard triangle Δ_2, Corollary 3 implies the existence of a finite rational certificate for $\Gamma_6(\Delta_2) \geq 1/2$. By Proposition 4, every finite $W \subseteq \Delta_2$ satisfies $\rho_6(W; \Delta_2) < 1/2$.

Write $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ with rational A and b. For $\lambda > 0$, a tuple $(c_1, \ldots, c_m)$ covers P at scale λ precisely when every $x \in P$ satisfies $A(x - c_i) \leq \lambda b$ for some i. Since x, the centers, and λ enter linearly, quantifier elimination in linear real arithmetic [12] (Sections 2 and 3) expresses feasibility by a Boolean combination of rational linear inequalities in the free variable λ. Hence the least feasible scale $\Gamma_m(P)$ is rational. At this scale, eliminating only x gives a Boolean combination of rational linear inequalities in the centers. For each $x \in P$, the tuples covering x form a closed set, so the set of optimal tuples is closed. It is the union of the closures of the relatively open cells it contains in the associated rational hyperplane arrangement. Thus, it is a finite union of rational polyhedra and contains a rational tuple. Cardinal, Goaoc, and Wajsbrot [13] (Sections 2 and 3) study continuous hitting and covering problems for affine families of polyhedra, with polynomial-time decision algorithms when the dimensions and the required cardinality are fixed. In Section 3 we use linear programming duality to construct a rational piecewise-affine representation of the covering radius and to give a dual certificate on each cell for every valid rational lower bound.

For a nonempty finite set $W \subseteq \mathbb{R}^n$ and a fixed $\lambda > 0$, only finitely many subsets of W occur as intersections with translates of λK. Retaining the inclusion-maximal intersections, namely those not properly contained in another such intersection, gives a binary set-covering program whose optimal value is the least number of translates covering W; see Theorem 3.

Write ℓ_1^n for $\mathbb{R}^n$ equipped with the norm $\|x\|_1 = \sum_{i=1}^n |x_i|$. For a nonempty finite set $S \subseteq \mathbb{R}^3$, Proposition 5 expresses the one-center radius in terms of the maxima $\max_{x \in S} \langle \sigma, x \rangle$ for $\sigma \in \{-1, 1\}^3$. A four-coordinate representation of the centers also gives the half-integrality results used in the application below.

Let $B_1^n = \{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$. For these crosspolytopes, Lyu, Chen, and Wu [14] (Theorem 2) proved that $\Gamma_{2n}(B_1^n) = \Gamma_{2n+1}(B_1^n) = \Gamma_{2n+2}(B_1^n) = (n-1)/n$ for $n \geq 3$. Joós [15]

(Theorem 5) gave the upper bound $\Gamma_{12}(B_1^4) \le 5/7$. In dimension three, Lyu, Chen, and Wu proved that

$$\Gamma_{14}(B_1^3) = \frac{7}{13} \tag{2}$$

in [14] (Theorem 1) and gave the explicit upper covering

$$13B_1^3 \subseteq 7B_1^3 + \big(\{\pm 6e_1, \pm 6e_2, \pm 6e_3\} \cup \pm\{(2,2,-2),(5,5,4),(-4,5,-5),(5,-4,-5)\}\big). \tag{3}$$

We show that the known value in (2) is attained by a fixed finite subset. More precisely, $\rho_{14}(W; B_1^3) = 7$ holds for a set $W \subseteq 13B_1^3$ consisting of 100 points; see Theorem 4. Lemma 3 and Corollary 4 give a complete finite set of centers at the scale needed for the lower bound. The resulting binary set-covering model is certified by rational inequalities on a finite tree. He, Lv, Martini, and Wu [16] use geometric branch-and-bound methods to estimate covering functionals by subdividing continuous center domains. Here the tree rules out every cover of W using at most fourteen columns of the finite model, and its rational inequalities can be checked without an optimization solver.

The triangle in Proposition 4 and the crosspolytope in Theorem 4 lead to the following open question. *For which pairs (P, m), where P is a polytope of positive affine dimension and m is a positive integer, does there exist a finite set $W \subseteq P$ such that $\rho_m(W; P) = \Gamma_m(P)$?*

2. Finite Approximation and Attainment

Throughout this section, $m \in \mathbb{N}$, $K \in \mathcal{K}^n$, and $o \in \operatorname{int} K$. Write vol for n-dimensional Lebesgue volume. If K can be covered by m translates of λK for some $\lambda > 0$, then

$$m\lambda^n \operatorname{vol}(K) \ge \operatorname{vol}(K).$$

Consequently,

$$0 < m^{-1/n} \le \Gamma_m(K).$$

Proposition 1. *For every positive integer m,*

$$\Gamma_m(K) = \sup\{\rho_m(W; K) \mid \emptyset \ne W \subseteq K \text{ and } |W| < \infty\}. \tag{4}$$

Moreover, if $W \subseteq K$ is nonempty and finite, $\delta \ge 0$, and $K \subseteq W + \delta K$, then

$$\rho_m(W; K) \le \Gamma_m(K) \le \rho_m(W; K) + \delta. \tag{5}$$

Proof. For $\delta > 0$, the upper bound in (5) follows from [6] (Theorem 1) by taking $p = m$, $S = W$, $V = P_K = \mathbb{R}^n$, and $\epsilon_1 = \delta$ in the notation of that paper. The condition on the centers then holds for every $\epsilon_2 > 0$. We give a direct proof. The lower bound follows from $W \subseteq K$. Suppose that

$$W \subseteq \bigcup_{i=1}^{m} (c_i + \rho_m(W; K)K).$$

Then

$$K \subseteq W + \delta K \subseteq \bigcup_{i=1}^{m} \big(c_i + (\rho_m(W; K) + \delta)K\big),$$

from which the second inequality follows.

For every $\delta > 0$, the set

$$\{w + \delta \operatorname{int} K \mid w \in K\}$$

is an open cover of K. Compactness gives a finite $W_\delta \subseteq K$ such that $K \subseteq W_\delta + \delta K$. Hence

$$\Gamma_m(K) - \delta \le \rho_m(W_\delta; K) \le \Gamma_m(K), \ \forall \delta > 0.$$

Thus (4) holds. □

One can use the following criterion to get the exact value of $\Gamma_m(K)$.

Proposition 2. *Let $W \subseteq K$ be finite and $\gamma > 0$. Suppose that W cannot be covered by m translates of $\mathrm{int}(\gamma K)$, whereas K can be covered by m translates of γK. Then*

$$\rho_m(W;K) = \Gamma_m(K) = \gamma.$$

Proof. The inclusion $W \subseteq K$ gives $\rho_m(W;K) \leq \Gamma_m(K)$. The assumed cover of K gives $\Gamma_m(K) \leq \gamma$. If $\rho_m(W;K) < \gamma$, then there exist a center tuple $\mathbf{c}$ and $\lambda \in (0,\gamma)$ such that

$$W \subseteq \bigcup_{i=1}^{m} (c_i + \lambda K).$$

Since $o \in \mathrm{int}\, K$, we have $\lambda K \subseteq \mathrm{int}(\gamma K)$ and therefore

$$W \subseteq \bigcup_{i=1}^{m} (c_i + \mathrm{int}(\gamma K)),$$

a contradiction. □

For $\eta \in [0, \Gamma_m(K))$, set

$$\tau_{m,\eta}(K) := \min\{|W| \mid W \subseteq K \text{ is finite and } \rho_m(W;K) > \eta\}.$$

Proposition 1 ensures the existence of the subset W in the definition of $\tau_{m,\eta}(K)$. Thus, $\tau_{m,\eta}(K)$ is well-defined. The function $\eta \mapsto \tau_{m,\eta}(K)$ is nondecreasing. For right-continuity, fix $0 \leq \eta < \Gamma_m(K)$ and choose a finite $W \subseteq K$ with $|W| = \tau_{m,\eta}(K)$ and $r = \rho_m(W;K) > \eta$. For $\eta \leq t < r$, the inequality $\rho_m(W;K) > t$ gives $\tau_{m,t}(K) \leq |W|$. Monotonicity gives the reverse bound $\tau_{m,t}(K) \geq \tau_{m,\eta}(K) = |W|$.

Corollary 1. *If $\eta \in [0, \Gamma_m(K))$ and $\delta \in (0, \Gamma_m(K) - \eta)$, then*

$$\tau_{m,\eta}(K) \leq \min\{|W| \mid W \subseteq K \textit{ is finite and } K \subseteq W + \delta K\}.$$

Proof. Let $W \subseteq K$ be finite with $K \subseteq W + \delta K$. If $\rho_m(W;K) \leq \eta$, then Proposition 1 gives

$$\Gamma_m(K) \leq \eta + \delta < \Gamma_m(K),$$

a contradiction. Thus, $\tau_{m,\eta}(K) \leq |W|$ for every such W. □

Theorem 1. *The following are equivalent.*

(i) *There is a finite set $W \subseteq K$ such that $\rho_m(W;K) = \Gamma_m(K)$.*
(ii) *The quantities $\tau_{m,\eta}(K)$ remain bounded as $\eta \uparrow \Gamma_m(K)$.*

Proof. Assume (i) and choose a finite $W \subseteq K$ with $\rho_m(W;K) = \Gamma_m(K)$. Then W is a finite obstruction above every $\eta \in [0, \Gamma_m(K))$. Thus, $\tau_{m,\eta}(K) \leq |W|$, and (ii) holds.

Conversely, suppose that there exists $N \in \mathbb{N}$ such that $\tau_{m,\eta}(K) \leq N$ whenever $\eta \in [0, \Gamma_m(K))$. Choose a sequence $\{\eta_j\}_{j=1}^{\infty}$ in $[0, \Gamma_m(K))$ increasing to $\Gamma_m(K)$. For each $j \in \mathbb{N}$, let W_j be a subset of K with at most N points satisfying $\rho_m(W_j;K) > \eta_j$. List their elements, with repetitions if necessary, as

$$w_{j,1}, \ldots, w_{j,N}.$$

Since K^N is compact, by choosing subsequences if necessary, we may assume that $w_{j,\ell} \to w_\ell$, $\forall\, \ell \in \{1, \ldots, N\}$. Put $W = \{w_1, \ldots, w_N\}$. Then $W \subseteq K$ and $\rho_m(W;K) \leq \Gamma_m(K)$.

For any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$w_{j,\ell} - w_\ell \in \varepsilon(K \cap (-K)),\ \forall j \geq N_0,\ \forall \ell \in \{1, \dots, N\}.$$

We have

$$W_j \subseteq W + \varepsilon(K \cap (-K)) \quad \text{and} \quad W \subseteq W_j + \varepsilon(K \cap (-K)),\ \forall j \geq N_0.$$

If a center tuple $\mathbf{c}$ satisfies

$$W_j \subseteq \bigcup_{i=1}^{m} (c_i + \rho_m(W_j; K)K),$$

then

$$W \subseteq \bigcup_{i=1}^{m} (c_i + (\rho_m(W_j; K) + \varepsilon)K).$$

Hence $\rho_m(W; K) \leq \rho_m(W_j; K) + \varepsilon,\ \forall j \geq N_0$. In a similar way, one can show that $\rho_m(W_j; K) \leq \rho_m(W; K) + \varepsilon,\ \forall j \geq N_0$. It follows that

$$\rho_m(W; K) = \lim_j \rho_m(W_j; K) \geq \lim_j \eta_j = \Gamma_m(K). \quad \square$$

The monotonicity of $\tau_{m,\eta}(K)$ in η ensures the existence of the limit

$$N = \lim_{\eta \uparrow \Gamma_m(K)} \tau_{m,\eta}(K) \in \mathbb{N} \cup \{+\infty\}.$$

If a finite $W \subseteq K$ attains $\Gamma_m(K)$, then $\tau_{m,\eta}(K) \leq |W|$ for all $0 \leq \eta < \Gamma_m(K)$, so $N \leq |W|$. If N is finite, the compactness argument in the proof of Theorem 1 gives an attaining set of at most N points. Thus, N is the minimum cardinality of a finite subset attaining $\Gamma_m(K)$ whenever such a subset exists. If no finite subset attains $\Gamma_m(K)$, then $\tau_{m,\eta}(K)$ tends to infinity as $\eta \uparrow \Gamma_m(K)$.

Let $B_2^n = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\}$ be the Euclidean unit ball.

Proposition 3. *Let $2 \leq m \leq n$. Then $\Gamma_m(B_2^n) = 1$, and $\rho_m(W; B_2^n) < 1$ for every finite set $W \subseteq B_2^n$.*

Proof. Given $c_1, \dots, c_m \in \mathbb{R}^n$, choose a unit vector u orthogonal to $c_1, \dots, c_{m-1}$, which is possible since $m - 1 < n$. Choose the sign of u so that $\langle u, c_m \rangle \leq 0$. Thus, $\|u - c_i\|_2^2 = 1 + \|c_i\|_2^2 - 2\langle u, c_i \rangle \geq 1$ for all i. Since B_2^n covers itself, $\Gamma_m(B_2^n) = 1$.

Let $W \subseteq B_2^n$ be finite and nonempty. Choose a unit vector v outside the finitely many hyperplanes orthogonal to the points of $W \cap \partial B_2^n$. For $w \in W$ and $\varepsilon > 0$,

$$\min_{\sigma \in \{-1,1\}} \|w - \sigma\varepsilon v\|_2^2 = \|w\|_2^2 - 2\varepsilon|\langle w, v \rangle| + \varepsilon^2.$$

If $\|w\|_2 < 1$, the right-hand side is less than 1 for all sufficiently small $\varepsilon > 0$ by continuity at 0. If $\|w\|_2 = 1$, then $\langle w, v \rangle \neq 0$ and the right-hand side is less than 1 whenever $0 < \varepsilon < 2|\langle w, v \rangle|$. Since W is finite, we may choose a single $\varepsilon > 0$ for which every $w \in W$ is at distance less than 1 from the nearer of $\pm\varepsilon v$. The maximum of these distances is less than 1, so $\rho_m(W; B_2^n) < 1$ for $m \geq 2$. The empty set has covering radius zero. $\square$

For the standard triangle Δ_2 we use the barycentric representation

$$T = \{x \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = 1,\ x_0, x_1, x_2 \geq 0\}.$$

The affine coordinates (x_1, x_2) identify T with Δ_2. If g is the barycenter of Δ_2, then $\Delta_2 - g$ contains the origin in its interior, as required by Theorem 1. Translating Δ_2 and a finite $W \subseteq \Delta_2$ by $-g$ preserves $\Gamma_6(\Delta_2)$, $\rho_6(W; \Delta_2)$, and $|W|$.

Proposition 4. *For every finite set* $W \subseteq \Delta_2$, $\rho_6(W;\Delta_2) < \Gamma_6(\Delta_2) = 1/2$.

Proof. The equality $\Gamma_6(\Delta_2) = 1/2$ is recorded by Brzozowska and Fudali [17] (Equation (1.1)). For $i \in \{0,1,2\}$, with indices taken modulo 3, put

$$U_i = \{x \in T \mid x_i > 1/2\},$$
$$V_i = \{x \in T \mid x_i > 1/4,\ x_{i+1} \geq 1/4\}.$$

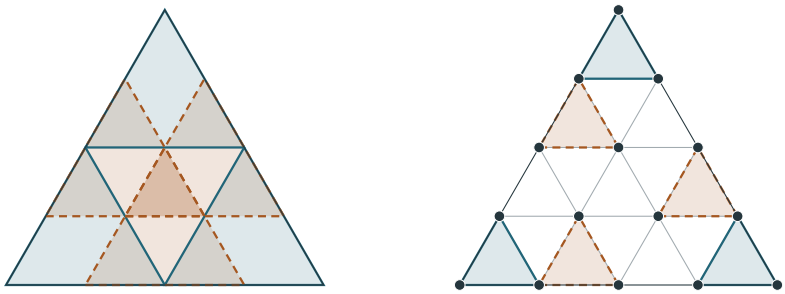
These six regions cover T. Indeed, given $x \in T$, choose i with $x_i = \max\{x_0, x_1, x_2\}$. If $x_i > 1/2$, then $x \in U_i$. Otherwise $1/3 \leq x_i \leq 1/2$. If $x_{i+1} \geq 1/4$, then $x \in V_i$. In the remaining case $x_{i-1} = 1 - x_i - x_{i+1} > 1/4$, so $x \in V_{i-1}$.

Let $W \subseteq T$ be finite and nonempty. Assign each $w \in W$ to one region containing w. Let $i(w)$ be the index of the chosen region. Set $a_w = 1/2$ if the region is $U_{i(w)}$ and $a_w = 1/4$ if it is $V_{i(w)}$. In either case $w_{i(w)} > a_w$. Put

$$\varepsilon = \min\left\{\frac{1}{4},\ \min_{w \in W}(w_{i(w)} - a_w)\right\} > 0.$$

In each of the six regions, replace the strict lower bound $x_i > a$ by $x_i \geq a + \varepsilon$ and retain the other lower bounds. Since $w_{i(w)} \geq a_w + \varepsilon$, each $w \in W$ still belongs to its assigned region after this change. For each modified region let $b = (b_0, b_1, b_2)$ be the vector of coordinate lower bounds, with $b_j = 0$ for each coordinate constrained only by $x_j \geq 0$. Since $b_0 + b_1 + b_2 = 1/2 + \varepsilon$ and $x_0 + x_1 + x_2 = 1$, this region is $b + (1/2 - \varepsilon)T$. The six translates cover W, so $\rho_6(W;T) \leq 1/2 - \varepsilon$. For $W = \emptyset$ the radius is zero. □

For the grid $\{x \in T \mid 4x \in \mathbb{Z}^3\}$ every positive difference $w_{i(w)} - a_w$ in the proof is at least $1/4$. Taking $\varepsilon = 1/4$ gives the covering in Figure 1b.



(**a**) $\lambda = 1/2$ (**b**) $\lambda = 1/4$

Figure 1. (**a**) Six translates of $(1/2)T$ cover the triangle T. The solid teal and dashed ochre outlines mark the closures of U_i and V_i, respectively, for $i = 0, 1, 2$. (**b**) Six translates of $(1/4)T$ cover the 15 black dots, which represent all points of T whose barycentric coordinates are multiples of $1/4$.

3. Rationality and Optimal Centers

A *face* of a polyhedron is the set where an affine functional attains its finite maximum. The polyhedron itself and the empty set are also defined to be faces. A *vertex* of a polyhedron is a point that is not a nontrivial convex combination of two points of the polyhedron.

Let $d \in \mathbb{N}$ and $D \subseteq \mathbb{R}^d$ be a rational polyhedron. A *rational polyhedral subdivision of* D is a finite collection of rational polyhedra, called *cells*, whose union is D, such that every face of a cell belongs to the collection and the intersection of any two cells is a face of both. A function $f : D \to \mathbb{R}$ is *rational piecewise affine* if there is a rational polyhedral subdivision of D such that, on each cell, f agrees with an affine function on $\mathbb{R}^d$ having rational coefficients.

For a rational polytope $P \subseteq \mathbb{R}^n$ with $o \in \operatorname{int} P$ and $m \in \mathbb{N}$, set

$$F(\mathbf{c}) := \max_{x \in P} \min_{1 \leq i \leq m} p_P(x - c_i), \quad \forall \mathbf{c} \in (\mathbb{R}^n)^m.$$

A center tuple $\mathbf{c}$ is optimal for P if and only if $F(\mathbf{c}) = \Gamma_m(P)$. We use linear-programming duality in the form of [18] (Theorem 2.6).

In Figure 2, the body is $P = [-1, 1]$ and the two centers are $c_1 = -1/2$ and $c_2 = b \in [0, 1]$. For fixed b, the distance from $x \in P$ to the nearer center equals $|x + 1/2|$ to the left of the midpoint $(b - 1/2)/2$ and $|x - b|$ to the right. This distance is convex on each of the two subintervals, so its maximum over $x \in P$ is attained at -1, 1, or the midpoint. Hence

$$F(-1/2, b) = \max\left\{\frac{1}{2},\ 1 - b,\ \frac{b + 1/2}{2}\right\}.$$

For $0 \le b < 1/2$ only $x = 1$ attains the maximum over $x \in P$. For $1/2 < b \le 1$ the unique maximizer is the midpoint $x = (b - 1/2)/2$. Thus, the maximizing point varies with b even on an interval where $b \mapsto F(-1/2, b)$ is affine.

Lemma 1. *Let $p, q \in \mathbb{N}$, $A \in \mathbb{Q}^{p \times q}$, and $b \in \mathbb{Q}^p$. Suppose that*

$$D = \{x \in \mathbb{R}^q \mid Ax = b \text{ and } x \ge 0\}$$

is nonempty. There is a finite set $V \subseteq D \cap \mathbb{Q}^q$ such that every linear functional on $\mathbb{R}^q$ attaining a minimum or maximum on D attains that extremum at a point of V.

Proof. Let V be a set consisting of the points $x \in D$ for which the columns of A indexed by the positive coordinates of x are linearly independent. Each subset of columns determines at most one such point, whose positive coordinates are the unique solution of a system of rational linear equations. Thus, V is finite and consists of rational points.

Let ℓ be a linear functional attaining a minimum on D. Choose a minimizer x with the fewest positive coordinates. If the columns of A indexed by the positive coordinates of x were linearly dependent, there would be a nonzero vector $h \in \mathbb{R}^q$ with $Ah = o$ and $h_i = 0$ whenever $x_i = 0$. Put

$$\varepsilon_0 = \min\left\{\frac{x_i}{|h_i|} \,\middle|\, i \in \{1, \dots, q\} \text{ and } h_i \ne 0\right\} > 0.$$

Both $x + \varepsilon_0 h$ and $x - \varepsilon_0 h$ belong to D. Since x minimizes ℓ, the two feasible points imply $\ell(h) = 0$. Both points are therefore minimizers. By the choice of ε_0, at least one has fewer positive coordinates than x, a contradiction. Hence $x \in V$. The assertion for maxima follows by applying the minimum assertion to $-\ell$. □

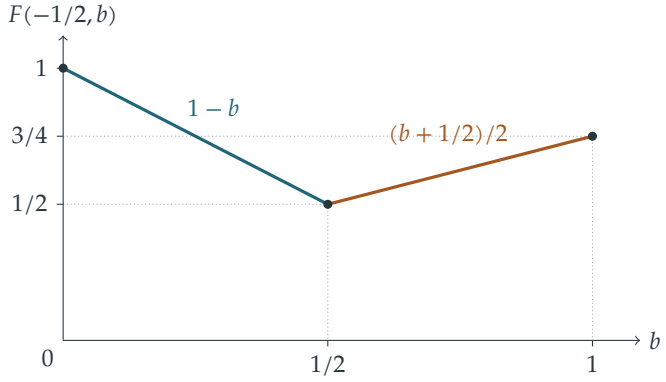


Figure 2. The covering radius $F(-1/2, b)$ of $[-1, 1]$ as the second center b varies from 0 to 1 with the first center fixed at $-1/2$. When $b = 1/2$, the distance to the nearer center attains its maximum $1/2$ at $x = -1, 0, 1$.

Theorem 2. *Let $P \in \mathcal{K}^n$ be a rational polytope. For every $m \in \mathbb{N}$, $\Gamma_m(P)$ is rational, and P admits an optimal center tuple in $(\mathbb{Q}^n)^m$.*

Proof. We may assume, without loss of generality, that $o \in \operatorname{int} P$. There exist $f \in \mathbb{N}$ and $a_1, \dots, a_f \in \mathbb{Q}^n$ such that

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; a_j^{\mathsf{T}} x \le 1,\ \forall j \in \{1, \dots, f\} \right\}.$$

Then $p_P(y) = \max_{1 \le j \le f} a_j^{\mathsf{T}} y$. For every real $m \times f$ matrix (ζ_{ij}), we have

$$\min_{1 \le i \le m} \max_{1 \le j \le f} \zeta_{ij} = \max_{\sigma \in \{1, \dots, f\}^m} \min_{1 \le i \le m} \zeta_{i,\sigma_i}. \tag{6}$$

For each σ, the minimum of the selected entries is at most the minimum of the row maxima. Choosing each σ_i to index a maximal entry in row i gives equality. For a center tuple $\mathbf{c}$, applying (6) for each $x \in P$ and then interchanging the maxima gives

$$F(\mathbf{c}) = \max_{x \in P} \max_{\sigma \in \{1, \dots, f\}^m} \min_{1 \le i \le m} a_{\sigma_i}^{\mathsf{T}} (x - c_i) = \max_{\sigma \in \{1, \dots, f\}^m} \psi_\sigma(\mathbf{c}),$$

where $\psi_\sigma(\mathbf{c})$ is the optimal value of

$$\begin{aligned} &\text{maximize } \tau, \\ &\text{subject to } a_j^{\mathsf{T}} x \le 1, && \forall j \in \{1, \dots, f\}, \\ &\qquad \tau \le a_{\sigma_i}^{\mathsf{T}} (x - c_i), && \forall i \in \{1, \dots, m\}. \end{aligned}$$

The dual program is

$$\begin{aligned} \text{minimize } & \sum_{j=1}^{f} \alpha_j - \sum_{i=1}^{m} \beta_i a_{\sigma_i}^{\mathsf{T}} c_i, \\ \text{subject to } & \sum_{j=1}^{f} \alpha_j a_j = \sum_{i=1}^{m} \beta_i a_{\sigma_i}, \\ & \sum_{i=1}^{m} \beta_i = 1, \\ & \alpha_j \ge 0, \quad \forall j \in \{1, \dots, f\}, \\ & \beta_i \ge 0, \quad \forall i \in \{1, \dots, m\}. \end{aligned}$$

For each fixed $\mathbf{c}$, the primal program is feasible and has a finite optimal value because P is compact. The dual is feasible: take $\beta_1 = \alpha_{\sigma_1} = 1$ and all other variables zero. Strong linear-programming duality therefore gives an optimal dual solution. The dual feasible region is independent of $\mathbf{c}$ and is defined by rational linear equations and nonnegativity. For each σ, Lemma 1 gives a finite set of rational dual feasible points such that, for every $\mathbf{c}$, the dual minimum is attained at one of those points. The objective associated with each of these points is a rational affine function of $\mathbf{c}$. Thus, every ψ_σ is the minimum of finitely many rational affine functions of $\mathbf{c}$. Formula (6) expresses F by finitely many maxima and minima of these functions. Comparing all pairs of these affine functions subdivides $(\mathbb{R}^n)^m$ into rational polyhedra on which F agrees with one of these affine functions. Including all faces gives a rational polyhedral subdivision of $(\mathbb{R}^n)^m$. Thus, F is rational piecewise affine on $(\mathbb{R}^n)^m$.

Let $\mathbf{c}$ be an optimal center tuple. For each $i \in \{1, \dots, m\}$, if

$$(c_i + \Gamma_m(P)P) \cap P \neq \emptyset,$$

then $c_i \in P - \Gamma_m(P)P \subseteq P - P$; otherwise, we may replace c_i with o. Hence the minimization of F may be restricted to the rational polytope $(P-P)^m$. Intersecting the cells of the subdivision of $(\mathbb{R}^n)^m$ with $(P-P)^m$ and including all faces gives a rational polyhedral subdivision of $(P-P)^m$ on whose cells F is affine. Every nonempty cell is a rational polytope, so the minimum of F on that cell is attained at a vertex. The coordinates of that vertex are rational because they are determined by a full-rank system of rational active equalities. Hence the global minimum is rational and is attained at a center tuple in $(\mathbb{Q}^n)^m$. □

The rational piecewise-affine representation of F on $(\mathbb{R}^n)^m$ also describes the set of all optimal center tuples of P.

Corollary 2. *Under the hypotheses of Theorem 2, the set of optimal center tuples of P is a finite union of rational polyhedra. If $o \in \operatorname{int} P$, its intersection with $(P-P)^m$ is a nonempty finite union of rational polytopes.*

Proof. It suffices to consider the case when $o \in \operatorname{int} P$. Consider the rational polyhedral subdivision of $(\mathbb{R}^n)^m$ constructed in the proof of Theorem 2. The optimal center tuples of P in a cell Q form

$$\{\mathbf{c} \in Q \mid F(\mathbf{c}) = \Gamma_m(P)\}.$$

Since F has a rational affine expression on Q and $\Gamma_m(P)$ is rational, this level set is a rational polyhedron, possibly empty. Taking the union over the finitely many cells proves the first assertion. Intersecting with the rational polytope $(P-P)^m$ makes every member of the union a rational polytope, and nonemptiness follows from the compact center restriction in the proof of Theorem 2. □

Corollary 3. *Let $P \subseteq \mathbb{R}^n$ be a rational polytope with $o \in \operatorname{int} P$, let m be a positive integer, and let $\lambda \in \mathbb{Q}$. The assertion*

$$\Gamma_m(P) \geq \lambda,$$

when true, can be established using finitely many linear programs with rational data. More precisely, it can be verified using a rational polyhedral subdivision of $(P-P)^m$ and, on each nonempty cell, a rational feasible solution of the dual linear program for minimizing F on that cell, with an objective value of at least λ.

Proof. Suppose that $\Gamma_m(P) \geq \lambda$. By the proof of Theorem 2,

$$\Gamma_m(P) = \min_{\mathbf{c} \in (P-P)^m} F(\mathbf{c}).$$

On every cell of the rational polyhedral subdivision of $(P-P)^m$ constructed in that proof, F agrees with an affine function having rational coefficients.

Let Q be a nonempty cell. For some positive integer s, write

$$Q = \{\mathbf{c} \in \mathbb{R}^{mn} \mid A\mathbf{c} \leq b\} \quad \text{and} \quad F(\mathbf{c}) = u^{\mathsf{T}}\mathbf{c} + \nu, \quad \forall \mathbf{c} \in Q,$$

where $A \in \mathbb{Q}^{s \times mn}$, $b \in \mathbb{Q}^s$, $u \in \mathbb{Q}^{mn}$, and $\nu \in \mathbb{Q}$. Equalities defining Q are written as pairs of inequalities. The linear program

$$\min_{\mathbf{c} \in Q} \left(u^{\mathsf{T}}\mathbf{c} + \nu\right)$$

has the dual

$$\max\left\{\nu - b^{\mathsf{T}}\eta \mid \eta \in \mathbb{R}^s,\ A^{\mathsf{T}}\eta = -u,\ \text{and } \eta \geq 0\right\}.$$

Since Q is nonempty and compact, the primal optimal value is finite and is at least $\Gamma_m(P) \geq \lambda$. The inequality $-u^{\mathsf{T}}\mathbf{c} \leq \nu - \lambda$ holds for every $\mathbf{c} \in Q$, so [18] (Corollary 2.5b) gives a feasible dual solution. Strong linear-programming duality then gives an optimal dual solution whose objective value equals the primal optimal value.

Applying Lemma 1 to $A^{\mathsf{T}}\eta = -u$, $\eta \geq 0$, and the objective $-b^{\mathsf{T}}\eta$ gives a rational optimal dual solution η. In particular,

$$\eta \geq 0, \quad A^{\mathsf{T}}\eta = -u, \quad \text{and} \quad \nu - b^{\mathsf{T}}\eta \geq \lambda.$$

For every $\mathbf{c} \in Q$, these relations give

$$F(\mathbf{c}) = \nu - \eta^{\mathsf{T}} A\mathbf{c} \geq \nu - \eta^{\mathsf{T}} b \geq \lambda.$$

The cells, their affine expressions for F, and one such rational multiplier for each nonempty cell establish $F \geq \lambda$ on $(P-P)^m$. Hence $\Gamma_m(P) \geq \lambda$. □

Let $P = \Delta_2 - g$, where g is the barycenter of Δ_2. Since $\Gamma_6(P) = \Gamma_6(\Delta_2) = 1/2$, Corollary 3 guarantees the existence of a finite rational certificate for $\Gamma_6(P) \geq 1/2$. Such a certificate consists of a rational polyhedral subdivision of $(P-P)^6$, an affine expression for F on each cell, and rational dual multipliers establishing $F \geq 1/2$ on that cell. For each finite $W \subseteq \Delta_2$, Proposition 4 gives a ratio $0 < \lambda < 1/2$, depending on W, for which six translates of $\lambda\Delta_2$ cover W.

4. Finite Coverings and Set Cover

For a finite set and a finite family of sets whose union contains that set, the *set-covering problem* is to select a covering subfamily of least cardinality.

Let $K \in \mathcal{K}^n$, $W \subseteq \mathbb{R}^n$ be nonempty and finite, and $\lambda > 0$. Put

$$\mathcal{A}_\lambda(W;K) := \{W \cap (c + \lambda K) \mid c \in \mathbb{R}^n \text{ and } W \cap (c + \lambda K) \neq \emptyset\}.$$

The family $\mathcal{A}_\lambda(W;K)$ is finite because its members are subsets of W. Every point of W belongs to a translate of λK, so these members cover W. A member is called *inclusion-maximal* if it is not properly contained in another member of the family. List the inclusion-maximal members as $A_1, \ldots, A_N$. For each $j \in \{1, \ldots, N\}$, choose a center c_j with $A_j = W \cap (c_j + \lambda K)$.

Theorem 3. *The least number of translates of λK needed to cover W equals the optimal value of the binary integer program*

$$\begin{aligned} &\textit{minimize} \sum_{j=1}^{N} x_j, \\ &\textit{subject to} \sum_{\substack{1 \leq j \leq N \\ w \in A_j}} x_j \geq 1, \quad \forall w \in W, \\ &\qquad x_j \in \{0,1\}, \qquad \forall j \in \{1, \ldots, N\}. \end{aligned} \tag{7}$$

Proof. Let q_* be the least number of translates of λK covering W and p_* be the optimal value of (7). Both minima exist because the finite family $A_1, \ldots, A_N$ covers W.

Choose $d_1, \ldots, d_{q_*} \in \mathbb{R}^n$ such that

$$W \subseteq \bigcup_{i=1}^{q_*} (d_i + \lambda K).$$

For each $i \in \{1, \ldots, q_*\}$, the intersection $B_i := W \cap (d_i + \lambda K)$ is nonempty by minimality of q_*. Choose $j(i) \in \{1, \ldots, N\}$ such that $B_i \subseteq A_{j(i)}$, and put

$$J := \{j(i) \mid 1 \leq i \leq q_*\} \quad \text{and} \quad x_j := \begin{cases} 1, & j \in J, \\ 0, & j \notin J, \end{cases} \quad \forall j \in \{1, \ldots, N\}.$$

Since

$$W = \bigcup_{i=1}^{q_*} B_i \subseteq \bigcup_{j \in J} A_j \subseteq W,$$

we have

$$\sum_{\substack{1\le j\le N\\ w\in A_j}} x_j = \left|\left\{j\in J \;\middle|\; w\in A_j\right\}\right| \ge 1, \quad \forall w\in W.$$

Thus x is feasible in (7), and

$$p_* \le \sum_{j=1}^{N} x_j = |J| \le q_*.$$

Let x^* be an optimal solution of (7). Its covering constraints and $A_j = W\cap(c_j+\lambda K)$ give

$$W = \bigcup_{\substack{1\le j\le N\\ x_j^*=1}} A_j \subseteq \bigcup_{\substack{1\le j\le N\\ x_j^*=1}} (c_j+\lambda K).$$

Since $x_j^*\in\{0,1\}$ for every $j\in\{1,\dots,N\}$,

$$q_* \le \left|\left\{j\in\{1,\dots,N\} \;\middle|\; x_j^*=1\right\}\right| = \sum_{j=1}^{N} x_j^* = p_*.$$

Hence $p_* = q_*$. □

To apply Theorem 3, one needs the inclusion-maximal members of $\mathcal{A}_\lambda(W;K)$. Section 6 determines these members for the point set and radius considered there by enumerating a finite set of centers.

5. Covering Radii in ℓ_1^3

Plotnikov and Krivulin [19] (Corollary 2) gave an explicit formula for the optimal value of the additively weighted one-center problem in ℓ_1^3. Here a four-sign representation is used to derive the half-integrality statements needed for the reduction to a finite set of centers.

For a nonempty finite set $S\subseteq\mathbb{R}^3$, put

$$M_\sigma(S) := \max_{x\in S}\langle\sigma,x\rangle, \quad \forall\sigma\in\{-1,1\}^3.$$

We use the four sign vectors

$$\mathcal{J}_+ := \{(-1,-1,1),(-1,1,-1),(1,-1,-1),(1,1,1)\}$$

and put $\mathcal{J}_- := -\mathcal{J}_+$.

The sum of the vectors in $\mathcal{J}_+$ is o, and $\{-1,1\}^3 = \mathcal{J}_+\cup\mathcal{J}_-$. Moreover,

$$\|y\|_1 = \max_{\sigma\in\mathcal{J}_+}|\langle\sigma,y\rangle|, \quad \forall y\in\mathbb{R}^3.$$

Lemma 2. *Let $S\subseteq\mathbb{R}^3$ be nonempty and finite and $\lambda\ge 0$. Put*

$$\alpha_\sigma := M_\sigma(S)-\lambda \quad \textit{and} \quad \beta_\sigma := \lambda - M_{-\sigma}(S), \quad \forall\sigma\in\mathcal{J}_+.$$

Then S is contained in a translate of λB_1^3 if and only if

$$\alpha_\sigma\le\beta_\sigma, \quad \forall\sigma\in\mathcal{J}_+ \quad \textit{and} \quad \sum_{\sigma\in\mathcal{J}_+}\alpha_\sigma \le 0 \le \sum_{\sigma\in\mathcal{J}_+}\beta_\sigma. \tag{8}$$

Moreover, if $\xi_\sigma\in[\alpha_\sigma,\beta_\sigma]$ for every $\sigma\in\mathcal{J}_+$ and $\sum_{\sigma\in\mathcal{J}_+}\xi_\sigma = 0$, then

$$S\subseteq c+\lambda B_1^3,$$

where

$$c := \frac{1}{4}\sum_{\sigma\in\mathcal{J}_+}\xi_\sigma\sigma.$$

Proof. *Necessity.* Suppose that $S \subseteq c + \lambda B_1^3$ for some $c \in \mathbb{R}^3$. Put $\xi_\sigma := \langle \sigma, c \rangle$ for $\sigma \in \mathcal{J}_+$. Then

$$\sum_{\sigma \in \mathcal{J}_+} \xi_\sigma = \left\langle \sum_{\sigma \in \mathcal{J}_+} \sigma, c \right\rangle = 0.$$

Fix $x \in S$ and $\sigma \in \mathcal{J}_+$. Since $\|x - c\|_1 \le \lambda$, we have

$$|\langle \sigma, x \rangle - \xi_\sigma| = |\langle \sigma, x - c \rangle| \le \max_{\sigma' \in \mathcal{J}_+} |\langle \sigma', x - c \rangle| = \|x - c\|_1 \le \lambda.$$

Thus $-\lambda \le \langle \sigma, x \rangle - \xi_\sigma \le \lambda$, or, equivalently,

$$\langle \sigma, x \rangle - \lambda \le \xi_\sigma \le \langle \sigma, x \rangle + \lambda.$$

For every $\sigma \in \mathcal{J}_+$,

$$\min_{x \in S} \langle \sigma, x \rangle = -\max_{x \in S} \langle -\sigma, x \rangle = -M_{-\sigma}(S).$$

Taking extrema over all $x \in S$ therefore gives

$$\alpha_\sigma = \max_{x \in S} \langle \sigma, x \rangle - \lambda \le \xi_\sigma \le \min_{x \in S} \langle \sigma, x \rangle + \lambda = \beta_\sigma, \quad \forall \sigma \in \mathcal{J}_+.$$

Consequently, (8) holds.

Sufficiency. Assume (8) and put

$$\eta := \sum_{\sigma \in \mathcal{J}_+} \alpha_\sigma \le 0 \quad \text{and} \quad \zeta := \sum_{\sigma \in \mathcal{J}_+} \beta_\sigma \ge 0.$$

If $\eta = \zeta$, then $\eta = \zeta = 0$; put $\theta := 0$. If $\eta < \zeta$, set

$$\theta := \frac{-\eta}{\zeta - \eta} \in [0, 1].$$

In both cases,

$$(1 - \theta)\eta + \theta\zeta = \eta + \theta(\zeta - \eta) = 0.$$

Put $\xi_\sigma := (1 - \theta)\alpha_\sigma + \theta\beta_\sigma$ for $\sigma \in \mathcal{J}_+$. Since $\alpha_\sigma \le \beta_\sigma$, we have

$$\xi_\sigma - \alpha_\sigma = \theta(\beta_\sigma - \alpha_\sigma) \ge 0 \quad \text{and} \quad \beta_\sigma - \xi_\sigma = (1 - \theta)(\beta_\sigma - \alpha_\sigma) \ge 0, \quad \forall \sigma \in \mathcal{J}_+,$$

and

$$\sum_{\sigma \in \mathcal{J}_+} \xi_\sigma = (1 - \theta) \sum_{\sigma \in \mathcal{J}_+} \alpha_\sigma + \theta \sum_{\sigma \in \mathcal{J}_+} \beta_\sigma = (1 - \theta)\eta + \theta\zeta = 0.$$

Thus a zero-sum family in the prescribed intervals exists.

For every zero-sum family $(\xi_\sigma)_{\sigma \in \mathcal{J}_+}$ in $\prod_{\sigma \in \mathcal{J}_+} [\alpha_\sigma, \beta_\sigma]$, put $c := \frac{1}{4} \sum_{\sigma \in \mathcal{J}_+} \xi_\sigma \sigma$. Fix $\sigma' \in \mathcal{J}_+$. The definition of $\mathcal{J}_+$ gives

$$\langle \sigma', \sigma' \rangle = 3 \quad \text{and} \quad \langle \sigma', \sigma \rangle = -1, \quad \forall \sigma \in \mathcal{J}_+ \setminus \{\sigma'\}.$$

Hence

$$\langle \sigma', c \rangle = \frac{1}{4} \left(3\xi_{\sigma'} - \sum_{\substack{\sigma \in \mathcal{J}_+ \\ \sigma \ne \sigma'}} \xi_\sigma \right) = \frac{1}{4}(3\xi_{\sigma'} + \xi_{\sigma'}) = \xi_{\sigma'}.$$

Fix $x \in S$ and $\sigma \in \mathcal{J}_+$. The interval bounds give

$$\langle \sigma, x - c \rangle = \langle \sigma, x \rangle - \xi_\sigma \le M_\sigma(S) - \alpha_\sigma = \lambda,$$

and

$$-\langle \sigma, x - c \rangle = -\langle \sigma, x \rangle + \xi_\sigma \le M_{-\sigma}(S) + \beta_\sigma = \lambda.$$

Consequently,

$$\max_{x \in S} \|x - c\|_1 = \max_{x \in S} \max_{\sigma \in \mathcal{J}_+} |\langle \sigma, x - c \rangle| \le \lambda,$$

which implies that $S \subseteq c + \lambda B_1^3$. □

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Proposition 5. *For every nonempty finite $S \subseteq \mathbb{R}^3$,*

$$\rho_1(S;B_1^3) = \max\left\{\max_{\sigma\in\mathcal{J}_+}\frac{M_\sigma(S)+M_{-\sigma}(S)}{2}, \frac{1}{4}\sum_{\sigma\in\mathcal{J}_-}M_\sigma(S), \frac{1}{4}\sum_{\sigma\in\mathcal{J}_+}M_\sigma(S)\right\}. \tag{9}$$

Consequently, for every $\tau \in \mathbb{R}$, if $\rho_1(S;B_1^3) \geq \tau$, then S contains a nonempty subset E with $|E| \leq 4$ and $\rho_1(E;B_1^3) \geq \tau$.

Proof. Let μ denote the right-hand side of (9). Since $M_\sigma(S)+M_{-\sigma}(S) \geq 0$ for every $\sigma \in \mathcal{J}_+$, $\mu \geq 0$. For $\lambda \geq 0$, let α_σ and β_σ be defined as in Lemma 2. Their definitions give

$$\alpha_\sigma \leq \beta_\sigma \iff \lambda \geq \frac{M_\sigma(S)+M_{-\sigma}(S)}{2}, \quad \forall \sigma \in \mathcal{J}_+,$$
$$\sum_{\sigma\in\mathcal{J}_+}\alpha_\sigma = \sum_{\sigma\in\mathcal{J}_+}M_\sigma(S) - 4\lambda, \quad \text{and} \quad \sum_{\sigma\in\mathcal{J}_+}\beta_\sigma = 4\lambda - \sum_{\sigma\in\mathcal{J}_-}M_\sigma(S).$$

Hence

$$\exists c \in \mathbb{R}^3 \quad \text{s.t.} \quad S \subseteq c + \lambda B_1^3 \iff \lambda \geq \mu.$$

Thus $\rho_1(S;B_1^3) = \mu$.

Suppose that $\rho_1(S;B_1^3) \geq \tau$. By (9), choose H such that

$$H \in \left\{\{\sigma,-\sigma\} \,\middle|\, \sigma \in \mathcal{J}_+\right\} \cup \{\mathcal{J}_-, \mathcal{J}_+\} \quad \text{and} \quad \frac{1}{|H|}\sum_{\sigma\in H}M_\sigma(S) = \rho_1(S;B_1^3).$$

Choose $x_\sigma \in S$ with $\langle\sigma, x_\sigma\rangle = M_\sigma(S)$ for $\sigma \in H$. Put $E := \{x_\sigma \mid \sigma \in H\}$. Then

$$1 \leq |E| \leq |H| \leq 4 \quad \text{and} \quad M_\sigma(S) = \langle\sigma, x_\sigma\rangle \leq M_\sigma(E) \leq M_\sigma(S), \quad \forall \sigma \in H.$$

By (9),

$$\rho_1(E;B_1^3) \geq \frac{1}{|H|}\sum_{\sigma\in H}M_\sigma(E) = \frac{1}{|H|}\sum_{\sigma\in H}M_\sigma(S) = \rho_1(S;B_1^3) \geq \tau. \quad \square$$

For every nonempty finite $S \subseteq \mathbb{R}^3$ and every convex body K in $\mathbb{R}^3$, the zero-error case of Brandenberg and König [10] (Theorem 1.2) gives a subset $E \subseteq S$ with $|E| \leq 4$ and $\rho_1(E;K) = \rho_1(S;K)$. For $K = B_1^3$, choose one average attaining the maximum in Equation (9). The directions in that average form either a pair $\{\sigma,-\sigma\}$ or one of the sets $\mathcal{J}_+$ and $\mathcal{J}_-$. For each direction σ in the chosen average, take one point of S attaining $M_\sigma(S)$. These points form a set E with $|E| \leq 4$ and $\rho_1(E;B_1^3) = \rho_1(S;B_1^3)$.

Lemma 3. *Let $S \subseteq \mathbb{Z}^3$ be nonempty and finite and $\lambda \in \frac{1}{2}\mathbb{Z} \cap [0,\infty)$. If S is contained in a translate of λB_1^3, then there exists $c \in \frac{1}{2}\mathbb{Z}^3$ such that $S \subseteq c + \lambda B_1^3$.*

Proof. Let α_σ and β_σ be defined as in Lemma 2 and write $\mathcal{J}_+ = \{\sigma_1,\dots,\sigma_4\}$. The hypothesis and (8) give

$$\alpha_\sigma, \beta_\sigma \in \lambda + \mathbb{Z} \quad \text{and} \quad \beta_\sigma - \alpha_\sigma \in \mathbb{Z}_{\geq 0}, \quad \forall \sigma \in \mathcal{J}_+.$$

Put $b_i := \beta_{\sigma_i} - \alpha_{\sigma_i}$ for each $i \in \{1,\dots,4\}$. Then

$$r := -\sum_{\sigma\in\mathcal{J}_+}\alpha_\sigma = 4\lambda - \sum_{\sigma\in\mathcal{J}_+}M_\sigma(S) \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad r \leq \sum_{i=1}^{4} b_i.$$

Define

$$r_0 := r, \quad k_i := \min\{r_{i-1}, b_i\}, \quad \text{and} \quad r_i := r_{i-1} - k_i, \quad \forall i \in \{1,\dots,4\}.$$

The recursion gives

$$r_i = \max\{0, r_{i-1} - b_i\} = \max\left\{0, r - \sum_{h=1}^{i} b_h\right\}, \quad \forall i \in \{1,\dots,4\} \quad \text{and} \quad r_4 = 0.$$

Thus

$$k_i \in [0, b_i] \cap \mathbb{Z}, \quad \forall i \in \{1, \dots, 4\} \quad \text{and} \quad \sum_{i=1}^{4} k_i = r_0 - r_4 = r.$$

Put $\xi_{\sigma_i} := \alpha_{\sigma_i} + k_i$ for each $i \in \{1, \dots, 4\}$. Then

$$\xi_\sigma \in [\alpha_\sigma, \beta_\sigma] \cap (\lambda + \mathbb{Z}), \quad \forall \sigma \in \overset{\frown}{J}_+ \quad \text{and} \quad \sum_{\sigma \in \overset{\frown}{J}_+} \xi_\sigma = \sum_{\sigma \in \overset{\frown}{J}_+} \alpha_\sigma + r = 0.$$

By Lemma 2, $S \subseteq c + \lambda B_1^3$, where

$$c := \frac{1}{4} \sum_{\sigma \in \overset{\frown}{J}_+} \xi_\sigma \sigma.$$

For every $j \in \{1, 2, 3\}$,

$$\left|\left\{\sigma \in \overset{\frown}{J}_+ \,\middle|\, \sigma_j = 1\right\}\right| = 2 \quad \text{and} \quad (2c)_j = \frac{1}{2} \sum_{\sigma \in \overset{\frown}{J}_+} \xi_\sigma \sigma_j = \sum_{\substack{\sigma \in \overset{\frown}{J}_+ \\ \sigma_j = 1}} \xi_\sigma \in 2\lambda + \mathbb{Z} = \mathbb{Z}.$$

Thus $c \in \frac{1}{2}\mathbb{Z}^3$. □

Corollary 4. *Let $W \subseteq \mathbb{Z}^3$ be nonempty and finite, and suppose that the coordinate sums of its points all have the same parity. For every positive integer m, we have*

$$\rho_m(W; B_1^3) \in \frac{1}{2}\mathbb{Z}.$$

Proof. Let $\varepsilon \in \{0, 1\}$ be the common parity of the coordinate sums. For nonempty $S \subseteq W$,

$$\langle \sigma, x \rangle \equiv x_1 + x_2 + x_3 \equiv \varepsilon \pmod 2, \quad \forall \sigma \in \{-1, 1\}^3, \ \forall x \in S,$$

whence

$$M_\sigma(S) \in 2\mathbb{Z} + \varepsilon, \quad \frac{M_\sigma(S) + M_{-\sigma}(S)}{2} \in \mathbb{Z}, \quad \forall \sigma \in \{-1, 1\}^3,$$

$$\frac{1}{4} \sum_{\sigma \in \overset{\frown}{J}_+} M_\sigma(S) \in \frac{1}{2}\mathbb{Z}, \quad \text{and} \quad \frac{1}{4} \sum_{\sigma \in \overset{\frown}{J}_-} M_\sigma(S) \in \frac{1}{2}\mathbb{Z}.$$

By (9),

$$\rho_1(S; B_1^3) \in \frac{1}{2}\mathbb{Z}, \quad \forall \emptyset \neq S \subseteq W.$$

Fix $m \in \mathbb{N}$. For $f : W \to \{1, \dots, m\}$, write $W_i(f) := f^{-1}(\{i\})$. Define

$$\mu := \min_{f: W \to \{1, \dots, m\}} \max_{\substack{1 \le i \le m \\ W_i(f) \neq \emptyset}} \rho_1(W_i(f); B_1^3) \in \frac{1}{2}\mathbb{Z}.$$

Put $\lambda := \rho_m(W; B_1^3)$. There exist $c_1, \dots, c_m \in \mathbb{R}^3$ with $W \subseteq \bigcup_{i=1}^m (c_i + \lambda B_1^3)$. Define

$$f(w) := \min\{i \in \{1, \dots, m\} \mid \|w - c_i\|_1 \le \lambda\}, \quad \forall w \in W.$$

Then

$$\mu \le \max_{\substack{1 \le i \le m \\ W_i(f) \neq \emptyset}} \rho_1(W_i(f); B_1^3) \le \lambda = \rho_m(W; B_1^3).$$

Conversely, choose f^* attaining μ. For every $i \in \{1, \dots, m\}$ with $W_i(f^*) \neq \emptyset$, choose d_i such that

$$W_i(f^*) \subseteq d_i + \rho_1(W_i(f^*); B_1^3) B_1^3 \subseteq d_i + \mu B_1^3.$$

Consequently,

$$W = \bigcup_{\substack{1 \le i \le m \\ W_i(f^*) \neq \emptyset}} W_i(f^*) \subseteq \bigcup_{\substack{1 \le i \le m \\ W_i(f^*) \neq \emptyset}} (d_i + \mu B_1^3),$$

and therefore

$$\rho_m(W; B_1^3) \le \mu \le \rho_m(W; B_1^3),$$

which implies that $\rho_m(W; B_1^3) = \mu \in \frac{1}{2}\mathbb{Z}$. □

6. Finite Attainment for the Three-Dimensional Crosspolytope

Theorem 4. *There is an explicitly listed set* $W \subseteq 13B_1^3 \cap \mathbb{Z}^3$ *with* $|W| = 100$ *such that*

$$\rho_{14}(W; B_1^3) = 7.$$

Consequently,

$$\rho_{14}(W/13; B_1^3) = \Gamma_{14}(B_1^3) = \frac{7}{13}.$$

The upper bound follows from (3). For the lower bound, we reduce the covering problem to a binary set-covering program and rule out every cover of W using at most fourteen columns by rational inequalities on a finite tree.

6.1. A Finite Set of Centers

Let $W = \{w_1, \dots, w_{100}\}$ be the set in Appendix A. The points in W are distinct integer points with

$$\|w_i\|_1 \in \{3, 7, 13\}, \quad \forall i \in \{1, \dots, 100\}.$$

Thus $W \subseteq 13B_1^3$, and every point of W has an odd coordinate sum. By Corollary 4,

$$\rho_{14}(W; B_1^3) < 7 \iff \rho_{14}(W; B_1^3) \le \frac{13}{2}.$$

By attainment for finite sets, $\rho_{14}(W; B_1^3) \le 13/2$ is equivalent to a cover of W by at most fourteen closed balls of radius $13/2$.

Put

$$G := \frac{1}{2}\mathbb{Z}^3 \cap [-13, 13]^3.$$

Then $|G| = 53^3 = 148877$. For $c \in \frac{1}{2}\mathbb{Z}^3$, define $c' \in G$ by

$$c'_d := \min\{13, \max\{-13, c_d\}\}, \quad \forall d \in \{1, 2, 3\}. \tag{10}$$

Since every coordinate of a point of W belongs to $[-13, 13]$,

$$\|w - c'\|_1 = \sum_{d=1}^{3} |w_d - c'_d| \le \sum_{d=1}^{3} |w_d - c_d| = \|w - c\|_1, \quad \forall w \in W.$$

For $c \in \mathbb{R}^3$, put

$$A(c) := W \cap \left(c + \frac{13}{2}B_1^3\right).$$

Clearly,

$$w_i \in A(c) \iff \|2w_i - 2c\|_1 \le 13, \quad \forall i \in \{1, \dots, 100\},\ \forall c \in G.$$

The family $\{A(c) \mid c \in G\}$ has 4662 distinct nonempty members, of which 484 are inclusion-maximal. Including the empty set gives 4663 distinct members.

Fix $c \in \mathbb{R}^3$ with $A(c) \neq \emptyset$. By Lemma 3, there exists $b \in \frac{1}{2}\mathbb{Z}^3$ such that $A(c) \subseteq b + \frac{13}{2}B_1^3$. Define $g \in G$ by (10) with $c = b$ and $c' = g$. The preceding distance estimate gives

$$\|w - g\|_1 \le \|w - b\|_1 \le \frac{13}{2}, \quad \forall w \in A(c).$$

Thus $A(c) \subseteq A(g)$. Since $G \subseteq \mathbb{R}^3$, the families $\{A(c) \mid c \in G\}$ and $\{A(c) \mid c \in \mathbb{R}^3\}$ have the same inclusion-maximal members. For $K = B_1^3$ and $\lambda = 13/2$, the sets $A_1, \dots, A_N$ of Section 4 are therefore the $N = 484$ inclusion-maximal members of $\{A(c) \mid c \in G\}$.

By Theorem 3, the optimal value of (7) with $A_1, \dots, A_{484}$ equals the least number of radius-13/2 balls covering W. The program has 484 binary variables and 100 covering

inequalities, one for each point of W. Selecting column j means setting $x_j = 1$. It remains to show that the additional constraint

$$\sum_{j=1}^{484} x_j \le 14$$

makes the program infeasible. A cover by fifteen balls of radius 13/2 is also supplied in Supplementary File S1. As shown in Section 6.2, each of the six axial vertices of $13B_1^3$ belongs to a unique column A_j, the singleton containing that vertex.

6.2. An Integer Lower Bound

At a node of a finite tree of binary choices for the covering program, let k be the number of variables already fixed to one, let $J \subseteq \{1, \dots, 484\}$ index the variables not yet fixed, and let $R \subseteq W$ consist of the points not covered by the selected columns. The root has state

$$(k, J, R) = (0, \{1, \dots, 484\}, W).$$

Branching on $j \in J$ gives the states

$$x_j = 0 : \ (k, J \setminus \{j\}, R) \quad \text{and} \quad x_j = 1 : \ (k+1, J \setminus \{j\}, R \setminus A_j).$$

If a point of R belongs to exactly one set A_j with $j \in J$, its covering constraint forces $x_j = 1$. In that case, use the $x_j = 1$ state without a branch.

Table 1 counts one node per branching or leaf record in the certificate. The root record contains all six forced selections as entries within that node.

Table 1. Counts for the grid G, the covered subsets $A(c)$ with $c \in G$, and the certificate tree excluding a cover of W by fourteen closed ℓ_1-balls of radius 13/2.

Quantity	Count
Grid centers in G	148,877
Distinct nonempty sets $A(c)$ with $c \in G$	4662
Inclusion-maximal sets A_j	484
Forced selections at the root	6
Binary branching nodes	80
Leaves with rational dual weights	81
Total nodes	161

Lemma 4. *Suppose that a finite tree is constructed by these operations and contains both children of every binary branch. Suppose that, at every leaf, rational weights η_w, $w \in R$, satisfy*

$$\eta_w \ge 0, \quad \forall w \in R \quad \text{and} \quad \sum_{w \in R \cap A_j} \eta_w \le 1, \quad \forall j \in J,$$

and

$$k + \sum_{w \in R} \eta_w > 14.$$

Then the covering program has no feasible binary vector with at most fourteen selected columns.

Proof. Let x be a feasible binary vector for (7). The vector x determines a path from the root to a leaf and satisfies every forced selection along that path. At that leaf,

$$\sum_{\substack{j \in J \\ w \in A_j}} x_j \ge 1, \quad \forall w \in R.$$

Multiplying by the nonnegative weights and summing gives

$$\sum_{w \in R} \eta_w \le \sum_{w \in R} \eta_w \sum_{\substack{j \in J \\ w \in A_j}} x_j = \sum_{j \in J} x_j \sum_{w \in R \cap A_j} \eta_w \le \sum_{j \in J} x_j.$$

Consequently,

$$\sum_{j=1}^{484} x_j = k + \sum_{j \in J} x_j \geq k + \sum_{w \in R} \eta_w > 14. \quad \square$$

If $w_0 \in R$ belongs to no A_j with $j \in J$, the covering constraint for w_0 has left-hand side zero. Such a node satisfies the leaf conditions of Lemma 4 with $\eta_{w_0} = 15$ and $\eta_w = 0$ for $w \in R \setminus \{w_0\}$. Every available column has weight sum zero and $k + \sum_{w \in R} \eta_w = k + 15 > 14$. The checker `check_bnb.py` accepts three leaf types, `dual`, `budget`, and `no_column`. All 81 leaves in the supplied certificate are of type `dual`.

Proof of Theorem 4. For every $a \in \{\pm 13e_1, \pm 13e_2, \pm 13e_3\}$, the coordinates in Appendix A give

$$\|a - w\|_1 \geq 14, \quad \forall w \in W \setminus \{a\}.$$

This inequality is also checked in Supplementary File S1. A ball of radius 13/2 has diameter 13, so $a \in A(c)$ implies $A(c) = \{a\}$ for every $c \in \mathbb{R}^3$. The six singleton sets are therefore forced at the root. After their selection,

$$k = 6, \quad |R| = 100 - 6 = 94, \quad \text{and} \quad |J| = 484 - 6 = 478.$$

At each leaf, the supplied rational weights have denominator 10^9 and satisfy the inequalities in Lemma 4.

At the leaf with index 105 in the zero-based node list in `certificate.json`, seven columns have been selected ($k = 7$). Write $\eta_w = p_w/D$ with $D = 10^9$. The nonnegative integers p_w at this leaf satisfy

$$\sum_{w \in R} p_w = 7{,}005{,}279{,}004, \qquad \max_{j \in J} \sum_{w \in R \cap A_j} p_w = 999{,}999{,}999 < D.$$

Hence $7D + \sum_{w \in R} p_w = 14{,}005{,}279{,}004 > 14D$. Among all 81 leaves, the lower bound $k + \sum_{w \in R} \eta_w$ is smallest at leaf 105. Thus, at every leaf

$$k + \sum_{w \in R} \eta_w \geq 14 + \frac{1{,}319{,}751}{250{,}000{,}000} > 14.$$

Lemma 4 excludes a binary cover using at most fourteen columns. By Theorem 3 applied to $A_1, \ldots, A_{484}$, W has no cover by at most fourteen balls of radius 13/2.

Suppose that fourteen open balls of radius 7 cover W. Since W is finite, the largest distance of a point of W from its nearest center is strictly less than 7. Corollary 4 then gives $\rho_{14}(W; B_1^3) \leq 13/2$. Since W is finite, an optimal center tuple would cover W by fourteen closed balls of radius 13/2. By Theorem 3 this would give a cover by at most fourteen columns, contrary to Lemma 4. Scaling by 1/13 shows that fourteen open balls of radius 7/13 cannot cover $W/13$. Dividing the cover (3) of Lyu, Chen, and Wu [14] (Lemma 3) by 13 gives a cover of B_1^3 by fourteen closed balls of radius 7/13. Proposition 2 applied to $W/13 \subseteq B_1^3$ yields

$$\rho_{14}(W/13; B_1^3) = \Gamma_{14}(B_1^3) = 7/13.$$

By (1), $\rho_{14}(W; B_1^3) = 7$. $\square$

Corollary 3 guarantees the existence of finite rational certificates for $\Gamma_{14}(B_1^3) \geq 7/13$ and $\Gamma_6(\Delta_2) \geq 1/2$. Supplementary File S1 supplies the set-covering certificate used in the proof of Theorem 4. For the crosspolytope, the 100 points of $W/13$ give $\tau_{14,\eta}(B_1^3) \leq 100$ for every $0 \leq \eta < 7/13$. For the triangle, Proposition 4 and Theorem 1 give $\lim_{\eta \uparrow 1/2} \tau_{6,\eta}(\Delta_2) = +\infty$.

Supplementary File S1 contains the point coordinates, the sets $A_1, \ldots, A_{484}$ with representative centers, the tree, the rational weights, and the checking programs. Running

```
python3 -B verify_all.py
```

in the Supplementary File S1 directory computes $A(c)$ for all 148,877 centers $c \in G$ and verifies the 484 inclusion-maximal members and the binary program. The program checks every branch, forced selection, and leaf inequality in Lemma 4. It also verifies the supplied fifteen-ball cover at radius 13/2 and the fourteen-ball cover of W at radius 7. All checks use only the Python standard library and exact integer and rational arithmetic.

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/math14234364/s1, Supplementary File S1: Exact data, certificates, and verification programs. The ZIP archive contains the point coordinates, maximal covered subsets with representative centers, the binary program, the complete rational tree certificate, upper-bound witnesses, mathematical documentation, and Python checking programs.

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Appendix A. Coordinates of the Point Set

The following table lists the set $W = \{w_1, \ldots, w_{100}\}$ used in Section 6. Write $w_i = (x_i, y_i, z_i)$. There are 7, 22, and 71 points of ℓ_1-norm 3, 7, and 13, respectively. Since $w_4 = (-6, -5, -2) \in W$ and $-w_4 = (6, 5, 2) \notin W$, inversion in the origin does not preserve W.

No.	(x_i, y_i, z_i)	No.	(x_i, y_i, z_i)	No.	(x_i, y_i, z_i)	No.	(x_i, y_i, z_i)	No.	(x_i, y_i, z_i)
1	$(-13, 0, 0)$	21	$(-5, 2, -6)$	41	$(-1, -3, -3)$	61	$(1, 3, 3)$	81	$(5, -2, -6)$
2	$(-6, -6, -1)$	22	$(-5, 2, 6)$	42	$(-1, -1, -1)$	62	$(1, 6, -6)$	82	$(5, -2, 6)$
3	$(-6, -6, 1)$	23	$(-5, 6, -2)$	43	$(-1, -1, 1)$	63	$(1, 6, 6)$	83	$(5, 2, -6)$
4	$(-6, -5, -2)$	24	$(-5, 6, 2)$	44	$(-1, 1, -1)$	64	$(2, -6, -5)$	84	$(5, 2, 6)$
5	$(-6, -5, 2)$	25	$(-3, -3, -1)$	45	$(-1, 1, 1)$	65	$(2, -6, 5)$	85	$(5, 4, 4)$
6	$(-6, -3, -4)$	26	$(-3, -3, 1)$	46	$(-1, 3, -3)$	66	$(2, -5, 6)$	86	$(5, 6, -2)$
7	$(-6, -2, -5)$	27	$(-3, -1, -3)$	47	$(-1, 3, 3)$	67	$(2, 5, -6)$	87	$(5, 6, 2)$
8	$(-6, -2, 5)$	28	$(-3, -1, 3)$	48	$(-1, 6, -6)$	68	$(2, 5, 6)$	88	$(6, -6, -1)$
9	$(-6, -1, -6)$	29	$(-3, 1, -3)$	49	$(-1, 6, 6)$	69	$(2, 6, -5)$	89	$(6, -6, 1)$
10	$(-6, -1, 6)$	30	$(-3, 1, 3)$	50	$(0, -13, 0)$	70	$(3, -3, -1)$	90	$(6, -5, 2)$
11	$(-6, 1, -6)$	31	$(-3, 3, -1)$	51	$(0, 0, -13)$	71	$(3, -3, 1)$	91	$(6, -2, -5)$
12	$(-6, 1, 6)$	32	$(-3, 3, 1)$	52	$(0, 0, 13)$	72	$(3, -1, -3)$	92	$(6, -2, 5)$
13	$(-6, 2, -5)$	33	$(-2, -6, -5)$	53	$(0, 13, 0)$	73	$(3, -1, 3)$	93	$(6, -1, -6)$
14	$(-6, 2, 5)$	34	$(-2, -6, 5)$	54	$(1, -6, -6)$	74	$(3, 1, -3)$	94	$(6, -1, 6)$
15	$(-6, 5, -2)$	35	$(-2, -5, 6)$	55	$(1, -6, 6)$	75	$(3, 1, 3)$	95	$(6, 1, -6)$
16	$(-6, 5, 2)$	36	$(-2, 5, -6)$	56	$(1, -3, -3)$	76	$(3, 3, -1)$	96	$(6, 1, 6)$
17	$(-6, 6, -1)$	37	$(-2, 5, 6)$	57	$(1, -1, -1)$	77	$(3, 3, 1)$	97	$(6, 2, 5)$
18	$(-6, 6, 1)$	38	$(-2, 6, -5)$	58	$(1, -1, 1)$	78	$(3, 6, -4)$	98	$(6, 6, -1)$
19	$(-5, -6, -2)$	39	$(-1, -6, -6)$	59	$(1, 1, -1)$	79	$(5, -6, -2)$	99	$(6, 6, 1)$
20	$(-5, -6, 2)$	40	$(-1, -6, 6)$	60	$(1, 3, -3)$	80	$(5, -4, -4)$	100	$(13, 0, 0)$

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