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Subordination Properties of Meromorphic Functions Associated with the Hadamard Inverse of the Legendre Chi Function via the Mittag-Leffler Identity

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Abstract

This paper contributes to geometric function theory through the introduction and application of a new linear operator, to study subclasses of meromorphic functions via differential subordinations. The operator is a modified version of an existing operator, based on a Hadamard product with a kernel whose coefficients combine the coefficients of the Hadamard inverse of the Legendre Chi function with the coefficients related to Mittag-Leffler function. The classes defined through the use of this operator and by differential subordinations to a convex function are built on the arithmetic mean of the meromorphic functions. The primary class, denoted by $\Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, is formulated as a broad generalization within this setting. Under suitable parameter selection, this class reduces to some known classes as the class of starlike meromorphic functions and the class of close-to-convex functions with respect to a specific starlike function, thereby revealing its unifying potential. In addition, three other classes, $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, $\wp^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \delta; \mu)$ and $\Re^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \delta; \mu)$, are introduced and analyzed, further extending the applicability of the proposed operator. Using Briot–Bouquet-type lemmas and convexity property of the dominant function, we obtained results such as the closure under averaging and, respectively, under a Bernardi-type integral operator and inclusion properties related to consecutive values of some of the parameters. The results presented here provide a broader analytical framework and open avenues for further developments in the study of meromorphic function classes.

Keywords: analytic function; differential subordination; meromorphic function; Legendre Chi Function; Mittag-Leffler function

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1. Introduction

Geometric function theory (GFT) is a prominent branch of complex analysis that investigates analytic functions through their geometric behavior and establishes a deep connection between analytic properties and geometric features such as growth, distortion, rotation, and covering theorems. Among its central themes are the study of univalent (injective) and multivalent functions, both of which have been extensively explored due to their rich theoretical structure and wide applicability. These classes of functions continue

to attract significant attention, not only for their intrinsic mathematical elegance but also for their relevance in various applied domains.

The theory of univalent functions, in particular, occupies a foundational position in complex analysis, serving as a cornerstone for developments in both single-variable and multi-variable settings. The classic monographs by Duren [1] and Goodman [2] provide comprehensive treatments of this subject, highlighting its fundamental concepts and key results. Over time, numerous subclasses of analytic and meromorphic functions have been introduced and investigated, each contributing to a deeper understanding of geometric properties and functional behavior.

An essential tool in GFT is the concept of subordination, which provides a powerful method to compare analytic functions based on their mapping properties. This notion was first introduced by Littlewood [3] and later formalized and systematically developed by Rogosinski [4], who established several foundational results. Subordination has since become a central technique in the study of analytic function classes, enabling the derivation of inclusion relationships, coefficient bounds, and structural properties.

The scope of subordination theory was significantly broadened with the introduction of differential subordination by Miller and Mocanu [5], which extends classical subordination to include relations involving derivatives of analytic functions. This framework has proven to be highly effective in addressing a wide range of problems in GFT.

In recent years, there has been growing interest in integrating operator theory, convolution structures, and special functions into the study of analytic and meromorphic function classes. Such approaches have led to the development of new subclasses and the establishment of refined inclusion and characterization results. A considerable body of literature reflects these advances, as seen in studies such as [6–9], where various geometric properties and operator-based techniques have been explored in depth. For example, Yi et al. [10] studied a subclass of close-to-convex meromorphic functions built on the product of k-symmetric point transformations, and Tayyah et al. [11] used a convolution operator to study third-order differential subordinations for p-valent meromorphic functions.

Motivated by these developments, the present work aims to further extend the theory by incorporating a novel operator framework into the study of meromorphic functions defined via subordination. This approach not only enriches existing theory but also provides new perspectives for analyzing function classes and their interrelationships.

Let Σ_κ denote the class of all meromorphic functions $\mathfrak{f}$ of the form:

$$\mathfrak{f}(\xi) = \xi^{-1} + \sum_{n=\kappa}^{\infty} a_n \xi^{n-1} \quad (\kappa \in \mathbb{N} = \{1, 2, 3, \ldots\},\ \xi \in \mathfrak{D}^* = \mathfrak{D} \setminus \{0\}), \tag{1}$$

which are analytic in the punctured open unit disc $\mathfrak{D}^*$, where $\mathfrak{D} = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$.

We say that $\mathfrak{f}$ and $\mathfrak{l}$ are analytic in $\mathfrak{D}$, $\mathfrak{f}$ is *subordinate* to $\mathfrak{l}$, denoted $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$, for all $\xi \in \mathfrak{D}$, such that $\mathfrak{f}(\xi) = \mathfrak{l}(\varpi(\xi))$, $\xi \in \mathfrak{D}$. If the function $\mathfrak{l}$ is univalent in $\mathfrak{D}$ (cf., e.g., [12,13]; see also [14]), we reduce $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$ to:

$$\mathfrak{f}(0) = \mathfrak{l}(0) \text{ and } \mathfrak{f}(\mathfrak{D}) \subset \mathfrak{l}(\mathfrak{D}).$$

For functions $\mathfrak{f}_j \in \Sigma_\kappa$, given by

$$\mathfrak{f}_j(\xi) = \xi^{-1} + \sum_{n=\kappa}^{\infty} a_{n,j} \xi^{n-1} \quad (j = 1, 2), \tag{2}$$

we define the Hadamard product (or convolution) of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ by (see, e.g., [1], p. 246)

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(\xi) = \xi^{-1} + \sum_{n=\kappa}^{\infty} a_{n,1} a_{n,2} \xi^{n-1} = (\mathfrak{f}_2 * \mathfrak{f}_1)(\xi). \tag{3}$$

In this context, the present investigation used an analytical framework that combines the Hadamard inverse of the Legendre Chi function with the Mittag-Leffler function (see [11]). This combination provides a unified and flexible approach for generating new results, allowing for a broader generalization of several existing findings. The interaction between these special functions enhances the analytical machinery available for studying complex function classes and offers a refined perspective on problems involving differential inequalities.

The methodology adopted in this work is rooted in the well-established theory of differential subordination which has proven to be an effective tool in geometric function theory. By utilizing these principles, we construct a new linear operator defined through the Hadamard inverse of the Legendre Chi and Mittag-Leffler functions. This operator facilitates the derivation of various structural properties of the function classes under consideration.

Moreover, the incorporation of the Hadamard product (convolution) plays a crucial role in linking analytic properties with algebraic structures, by enabling a systematic formulation of the results. The combined use of convolution techniques, operator theory, and subordination methods ensures a rigorous and comprehensive treatment of the problem. Consequently, the framework developed in this study not only extends several known results but also allows new avenues for further research in the theory of analytic and meromorphic functions and, due to the operator structure, lets the path open to fractional calculus.

In what follows, we present the special functions used to build the convolution-based operator.

Srivastava and Tomovski [15] introduced the Mittag-Leffler function as follows:

$$\mathcal{E}_{\alpha,\varepsilon}(\xi) = \frac{1}{\Gamma(\varepsilon)} + \sum_{n=\kappa}^{\infty} \frac{\xi^n}{\Gamma(\alpha n + \varepsilon)}, \quad (\alpha, \varepsilon \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \varepsilon \neq 0). \tag{4}$$

The Legendre Chi function is a special function, defined as (see [16])

$$\chi_\rho(\sqrt{\xi}) = \sum_{n=0}^{\infty} \frac{\xi^{n+\frac{1}{2}}}{(2n+1)^\rho}, \quad (\mathrm{Re}(\rho) > 1). \tag{5}$$

Now, we introduce a new linear operator, $\mathcal{L}_\varepsilon^\rho(\alpha) : \Sigma_\kappa \to \Sigma_\kappa$, by using a similar but different transformation compared with that used by Tayyah et al. in [11]:

$$(\mathcal{L}_\varepsilon^\rho(\alpha)\mathfrak{f}(\xi)) * (\xi^{-(\frac{3}{2})}\chi_\rho(\sqrt{\xi})) = \xi^{-1}\Gamma(\varepsilon)\mathcal{E}_{\alpha,\varepsilon}(\xi) * \mathfrak{f}(\xi).$$

Some standard computations lead to the formula below:

$$\mathcal{L}_\varepsilon^\rho(\alpha)\mathfrak{f}(\xi) = \xi^{-1} + \sum_{n=\kappa}^{\infty} \frac{(2n+1)^\rho \Gamma(\varepsilon)}{\Gamma(\alpha n + \varepsilon)} a_n \xi^{n-1} \quad (\alpha, \rho \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \varepsilon \in \mathbb{C}/\{0, -1, -2, \ldots\}). \tag{6}$$

We note that, due to how the two special functions and their parameters combine in the structure of the operator, the conditions imposed on the parameters for having a well-defined operator need to be slightly modified, compared with those requested directly by the Mittag-Leffler and Legendre Chi functions.

Further, let us also remark that, although the Mittag-Leffler function is not defined at $\alpha = 0$, if α is real and α approaches 0 from the right, we have (in terms of limits):

(1) $\mathcal{L}_\varepsilon^0(0)\mathfrak{f}(\xi) = \mathfrak{f}(\xi)$;
(2) $\mathcal{L}_\varepsilon^1(0)\mathfrak{f}(\xi) = 2\xi\mathfrak{f}'(\xi) + 3\mathfrak{f}(\xi)$.

From (6) it is clear that

$$\alpha\xi(\mathcal{L}_{\varepsilon+1}^\rho(\alpha)\mathfrak{f}(\xi))' = \varepsilon\mathcal{L}_\varepsilon^\rho(\alpha)\mathfrak{f}(\xi) - (\varepsilon + \alpha)\mathcal{L}_{\varepsilon+1}^\rho(\alpha)\mathfrak{f}(\xi), \tag{7}$$

$$\xi(\mathcal{L}_\varepsilon^\rho(\alpha)\mathfrak{f}(\xi))' = \frac{1}{2}\mathcal{L}_\varepsilon^{\rho+1}(\alpha)\mathfrak{f}(\xi) - \frac{3}{2}\mathcal{L}_\varepsilon^\rho(\alpha)\mathfrak{f}(\xi). \tag{8}$$

The structure of such an operator combines the coefficients of Mittag-Leffler function with the coefficients of Hadamard inverse of the Legendre Chi function, to obtain a general operator that acts both as a bridge to fractional calculus and as an umbrella for known operators. The parameters involved control the type of operator (differential or integral, depending on the sign of the parameter of Legendre Chi function). In addition, this combined structure leads to a stable analytic mapping in the punctured unit disk (through the Gamma function and the parameter α).

While Tayyah et al. [11] applied their operator to study third-order differential subordination for individual p-valent meromorphic functions, in the next section, through our modified operator, based on a different transformation of meromorphic functions with a simple pole in origin, we will build some new classes and analyze the joint differential behavior of a multi-function family relative to its arithmetic mean as well as other subordination properties. Also, meromorphic starlike functions class and some subclasses of meromorphic close-to-convex functions class will be recovered from ours for appropriate specialization of the parameters.

Particular cases:

1. When $\alpha \to 0^+$ and $\varepsilon \to 0^+$ the operator yields a well-defined algebraic multiplier transformation. Asymptotically, the Gamma ratio reduces to a constant factor, dropping the fractional character of the kernel and yielding the classical meromorphic Sălăgean differential structure.
2. For $\rho = 0$ and $\alpha = 1$, our operator relates to the structure of the meromorphic Ruscheweyh derivative operator.
3. For negative values of ρ, the operator behaves as a generalized fractional integral operator of the Srivastava–Attiya type.

Using the operator $\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)$, we investigate several classes of analytic functions, on the punctured open unit disk, where $\mathfrak{f}(\xi) = \frac{1}{\xi} + \sum_{n=\kappa}^{\infty} a_n \xi^{n-1}$,

$$\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi) = \mathfrak{f}(\xi) * \left\{ \frac{1}{\xi} + \sum_{n=\kappa}^{\infty} \frac{(2n+1)^{\rho}\Gamma(\varepsilon+1)}{\Gamma(\alpha n+\varepsilon+1)} \xi^{n-1} \right\} \tag{9}$$

$$= (\mathfrak{f} * \mathfrak{K}^{\rho}_{\varepsilon+1}(\alpha))(\xi),$$

and

$$\mathfrak{K}^{\rho}_{\varepsilon+1}(\alpha)(\xi) = \frac{1}{\xi} + \sum_{n=\kappa}^{\infty} \frac{(2n+1)^{\rho}\Gamma(\varepsilon+1)}{\Gamma(\alpha n+\varepsilon+1)} \xi^{n-1}. \tag{10}$$

2. Main Results

The following lemmas are used to demonstrate our results:

Lemma 1 ([17]; see also ([11], Theorem 3.2a. p. 81)). *Let $\beta, \sigma \in \mathbb{C}, \beta \neq 0$, μ be convex in $\mathfrak{D}$ with $\mathrm{Re}(\beta\mu(\xi) + \sigma) > 0$. If p is analytic in $\mathfrak{D}$ with $p(0) = \mu(0)$, and*

$$p(\xi) + \frac{\xi p'(\xi)}{\beta p(\xi) + \sigma} \prec \mu(\xi),$$

then $p(\xi) \prec \mu(\xi)$.

Lemma 2 ([18]). *Let $\beta, \sigma \in \mathbb{C}, \beta \neq 0$. Also let μ be convex univalent in $\mathfrak{D}$ with $\mu(0) = 1$ and $\mathrm{Re}(\beta\mu(\xi) + \sigma) > 0\, (\xi \in \mathfrak{D})$, and let q be analytic with $q(0) = 1$ and $q(\xi) \prec \mu(\xi)\, (\xi \in \mathfrak{D})$. If $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ is analytic in $\mathfrak{D}$, and*

$$p(\xi) + \frac{\xi p'(\xi)}{\beta q(\xi) + \sigma} \prec \mu(\xi) \qquad (\xi \in \mathfrak{D}),$$

then

$$p(\xi) \prec \mu(\xi).$$

To simplify further calculations throughout this paper, unless otherwise mentioned, we set the following requirements: $\alpha, \varepsilon, \rho \in \mathbb{R}$, $\alpha, \varepsilon, \rho > 0$, $\xi \in \mathfrak{D}^*$, $\vartheta \in \mathbb{N}$ and $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\}$, $\mathfrak{f}_i \in \Sigma_\kappa$. Also, let Φ denote the class of analytic and univalent convex functions in $\mathfrak{D}$, satisfying the properties $\mu(0) = 1$ and $\mathrm{Re}(\mu(\xi)) > 0$, for all $\xi \in \mathfrak{D}$.

2.1. The Class $\Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$

Definition 1. *Let $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\}$, $\mathfrak{f}_i \in \Sigma_\kappa, 1 \le i \le \vartheta$, satisfy the subordination*

$$\frac{-\xi\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)} \prec \mu(\xi), \qquad (\xi \in \mathfrak{D}; i = 1, 2, \dots, \vartheta),$$

where $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) \neq 0$ in $\mathfrak{D}$, and $\mu \in \Phi$. Then we say that $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$.

Remark 1. *When $\alpha \to 0^+$, we note that $\Xi^{0}_{\varepsilon+1}(0, 1; \frac{1-\xi}{1+\xi}) = \Xi^*$; consequently, the already known class of meromorphic starlike functions, studied by Clunie [19], Pommerenke [20] and others, is recovered asymptotically by the newly introduced class.*

Theorem 1. *Let $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ and $\mathcal{F}(\xi) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mathfrak{f}_i(\xi)$. Then $\mathcal{F}(\xi)$ satisfies the condition*

$$\frac{-\xi\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)\right)'}{\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)} \prec \mu(\xi), \qquad (\xi \in \mathfrak{D}) \tag{11}$$

Proof. Let $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$. Then for any $\xi_0 \in \mathfrak{D}$, we have

$$\frac{-\xi_0\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi_0)} \in \mu(\mathfrak{D})$$

and hence is equal to $\mu(w_i)$ for some $w_i \in \mathfrak{D}, i = 1, 2, \dots, \vartheta$. So

$$-\frac{\sum_{i=1}^{\vartheta}\xi_0\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi_0)} = \sum_{i=1}^{\vartheta}\mu(w_i).$$

Considering (10), we can write

$$\frac{-\xi_0\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi_0)\right)'}{\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi_0)} = \frac{-\xi_0[\mathfrak{K}^{\rho}_{\varepsilon+1}(\alpha)(\xi_0) * \sum_{i=1}^{\vartheta}\mathfrak{f}_i(\xi_0)]'}{\mathfrak{K}^{\rho}_{\varepsilon+1}(\alpha)(\xi_0) * \sum_{j=1}^{\vartheta}\mathfrak{f}_j(\xi_0)}.$$

Since

$$\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\sum_{j=1}^{\vartheta}\mathfrak{f}_j(\xi) = \sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi),$$

we have

$$\frac{-\xi_0\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi_0)\right)'}{\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi_0)} = \frac{1}{\vartheta}\left[\frac{-\xi_0\sum_{i=1}^{\vartheta}\left(\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi_0)}\right]$$

$$= \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i) = \mu(w_0),$$

for some $w_0 \in \mathfrak{D}$, since μ is convex in $\mathfrak{D}$. This completes the proof. □

Remark 2. *If* $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ *and* $\mu(\xi) = \frac{1-\xi}{1+\xi}$, *then Theorem 1 shows that if* $\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)$ *are close-to-convex meromorphic functions, then* $\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)$ *is meromorphic starlike function, where* $\mathcal{F}(\xi) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mathfrak{f}_i(\xi)$ *(see* [21], *also* [10]*).*

Now we define $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_\vartheta\}$ where $\mathcal{F}_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^{\xi} t^{\varsigma+1}\mathfrak{f}_i(t)dt \quad (\varsigma > 0; i = 1, 2, \dots, \vartheta)$ and verify if $\mathcal{F} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, whenever $\mathfrak{f} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ with $(\varsigma + 2) > \mathrm{Re}(\mu(\xi))$.

Theorem 2. *Let* $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ *such that the functions*

$$\mathcal{F}_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^{\xi} t^{\varsigma+1}\mathfrak{f}_i(t)dt \quad (\varsigma > 0; i = 1, 2, \dots, \vartheta)$$

satisfy $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi) \neq 0$. *If* $(\varsigma+2) > \mathrm{Re}(\mu(\xi))$, *for all* $\xi \in \mathfrak{D}$, *then* $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$.

Proof. From the definition of $\mathcal{F}_i(\xi)$, it follows that

$$\xi^{\varsigma+2}\mathcal{F}_i(\xi) = (\varsigma+1)\int_0^{\xi} t^{\varsigma+1}\mathfrak{f}_i(t)dt \quad (\varsigma > 0; i = 1, 2, \dots, \vartheta).$$

Differentiating both sides and multiplying by ξ, we get

$$\xi\mathcal{F}_i'(\xi) + (\varsigma+2)\mathcal{F}_i(\xi) = (\varsigma+1)\mathfrak{f}_i(\xi),$$

and further, we obtain

$$\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi)]' + (\varsigma+2)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi) = (\varsigma+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi), \quad i = 1, 2, \dots, \vartheta. \tag{12}$$

Let

$$p_i(\xi) = -\frac{\vartheta\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi)]'}{\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi)}. \tag{13}$$

Since $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi) \neq 0$, the function p_i is well defined and analytic in $\mathfrak{D}$. From (12) and (13), we have

$$-\frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi) + (\varsigma+2)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi) = (\varsigma+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi). \tag{14}$$

Now differentiating (14) with respect to ξ, we get

$$-\frac{p_i'(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi) - \frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_j(\xi)]'$$

$$+(\varsigma+2)[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi)]' = (\varsigma+1)[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)]'.$$

From (13), we have

$$-p_i'(\xi)\frac{\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}{\vartheta}+\frac{p_i(\xi)}{\vartheta}\frac{\sum_{i=1}^{\vartheta}p_i(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}{\vartheta\xi}$$

$$-(\varsigma+2)\frac{p_i(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}{\vartheta\xi}=(\varsigma+1)[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'.$$

Hence

$$-p_i'(\xi)+\frac{p_i(\xi)}{\vartheta\xi}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)\frac{p_i(\xi)}{\xi}=\frac{(\varsigma+1)[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}.$$

Then

$$\frac{-\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}+p_i(\xi)=\frac{(\varsigma+1)\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}.\frac{1}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}$$

$$=\frac{(\varsigma+1)\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{-1}{\vartheta}\left\{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi).\sum_{i=1}^{\vartheta}p_i(\xi)+(\varsigma+2)\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)\right\}}.$$

From (14), we have

$$\frac{-\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}+p_i(\xi)=\frac{(\varsigma+1)\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{-1}{\vartheta}(\varsigma+1)\sum_{i=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_i(\xi)}\prec\mu(\xi),\tag{15}$$

since $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_\vartheta\}\in\Xi_{\varepsilon+1}^{\rho}(\alpha,\vartheta;\mu)$. Now we can write for any $\xi_0\in\mathfrak{D}$,

$$\frac{-\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0)-(\varsigma+2)}+\frac{1}{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i),$$

for some $w_i\in\mathfrak{D}$. This is true for $i=1,2,\ldots,\vartheta$. Since μ is convex, there exists a $w_0\in\mathfrak{D}$ such that

$$\frac{\xi_0 Q'(\xi_0)}{-Q(\xi_0)+(\varsigma+2)}+Q(\xi_0)=\mu(w_0),$$

where $Q(\xi)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)$. Hence

$$\frac{\xi Q'(\xi)}{-Q(\xi)+(\varsigma+2)}+Q(\xi)\prec\mu(\xi).\tag{16}$$

Further, if we apply Lemma 1 for $p=Q$ (Q analytic, $Q(0)=1$, satisfying (16)), μ (μ convex, $\mu(0)=1$), $\beta=-1$, $\sigma=\varsigma+2$ (and $(\varsigma+2)>\mathrm{Re}(\mu(\xi))$), we obtain $Q(\xi)\prec\mu(\xi)$ ($\xi\in\mathfrak{D}$). From (15), we have

$$\frac{\xi[p_i(\xi)]'}{-Q(\xi)+(\varsigma+2)}+p_i(\xi)\prec\mu(\xi),\tag{17}$$

where $Q(\xi)\prec\mu(\xi)$. Now, if in Lemma 2 we take $q=Q$ (Q analytic, $Q(0)=1$), $p=p_i$ (p_i analytic, $p_i(0)=1$), μ (μ convex, $\mu(0)=1$), $\beta=-1$, $\sigma=\varsigma+2$ (and $(\varsigma+2)>\mathrm{Re}(\mu(\xi))$), since we have (17), we obtain $p_i(\xi)\prec\mu(\xi)$ ($\xi\in\mathfrak{D}$), $i=1,2,\ldots,\vartheta$, that is

$$\frac{-\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathcal{F}_j(\xi)}\prec\mu(\xi).$$

Thus $\mathcal{F}=\{\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_\vartheta\}\in\Xi_{\varepsilon+1}^{\rho}(\alpha,\vartheta;\mu)$. □

The general non-vanishing condition from the theorem hypothesis can be refined as $\mathrm{Re}(g(\xi)) > 0$, where $\mathfrak{g}(\xi) = \xi \sum_{i=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)$. We note that, for every i, $1 \le i \le \vartheta$, we have

$$\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma+2}} \int_0^{\xi} t^{\varsigma+1}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(t)dt,$$

and hence

$$\begin{aligned}\sum_{i=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi) &= \frac{\varsigma+1}{\xi^{\varsigma+2}} \int_0^{\xi} t^{\varsigma+1} \sum_{i=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(t)dt \\ &= \frac{\varsigma+1}{\xi^{\varsigma+2}} \int_0^{\xi} t^{\varsigma}\mathfrak{g}(t)dt.\end{aligned}$$

If we define

$$\Omega(\xi) = \sum_{n=1}^{\infty} \frac{\varsigma+1}{\varsigma+n}\xi^{n-1}, \qquad \varsigma > 0,$$

a straightforward calculation leads to

$$\xi \sum_{i=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}_i(\xi) = (\Omega * \mathfrak{g})(\xi) \neq 0 \quad (\mathrm{Re}(g(\xi)) > 0).$$

Now, before stating the next result, let us observe that the definition of the class $\Xi^{\rho}_{\varepsilon+}(\alpha, \vartheta; \mu)$ can be translated into the definition of the class $\Xi^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$ if in the definition subordination we use the operator $\mathcal{L}^{\rho}_{\varepsilon}(\alpha)$ and the non-vanishing condition $\xi \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_j(\xi) \neq 0$.

Theorem 3. *If* $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$ *and* $\xi \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) \neq 0$, *then* $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ *holds for* $(\frac{\varepsilon}{\alpha} + 1) > \mathrm{Re}(\mu(\xi))$ *in* $\mathfrak{D}$.

Proof. Let

$$p_i(\xi) = -\frac{\vartheta\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)]'}{\sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)} \qquad (\xi \in \mathfrak{D}; i = 1, 2, \ldots, \vartheta), \tag{18}$$

where $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$. From the hypothesis we have $\xi \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) \neq 0$; hence the function p_i is well defined and analytic in $\mathfrak{D}$. From (7) and (18), we have

$$-\frac{1}{\vartheta}p_i(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) = \frac{\varepsilon}{\alpha}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_i(\xi) - (\frac{\varepsilon}{\alpha} + 1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi). \tag{19}$$

Differentiating (19) with respect to ξ, we get

$$\begin{aligned}&-\frac{\xi}{\vartheta}p_i'(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) - \frac{\xi}{\vartheta}p_i(\xi) \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)]' \\ = \;& \frac{\varepsilon}{\alpha}\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_i(\xi)]' - (\frac{\varepsilon}{\alpha} + 1)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)]'.\end{aligned}$$

Using (18), we obtain

$$\begin{aligned}&-\frac{\xi}{\vartheta}p_i'(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi) - p_i(\xi)\Bigg[\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)]' \\ &+ \frac{(\frac{\varepsilon}{\alpha} + 1)}{\vartheta} \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)\Bigg] = \frac{\varepsilon}{\alpha}\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_i(\xi)]'.\end{aligned}$$

Then

$$\frac{-\frac{\xi}{\vartheta}p_i'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)} - p_i(\xi)$$
$$= \frac{\frac{\varepsilon}{\alpha}\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)}.$$

Using (7), we have

$$\frac{-\frac{\xi}{\vartheta}p_i'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}_j(\xi)} - p_i(\xi) = \frac{\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_j(\xi)}. \tag{20}$$

Here we note that the expression on the right side is well defined since $\xi\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_j(\xi) \neq 0$, from $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi_{\varepsilon}^{\rho}(\alpha, \vartheta; \mu)$. Using (18) in (20), we have

$$\frac{-\xi p_i'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} - p_i(\xi) = \frac{\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_j(\xi)}.$$

Further, since $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi_{\varepsilon}^{\rho}(\alpha, \vartheta; \mu)$, based on the class definition, we get

$$\frac{\xi p_i'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p_i(\xi) = \frac{-\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}_j(\xi)} \prec \mu(\xi), \quad i = 1, 2, \ldots, \vartheta. \tag{21}$$

Therefore for any $\xi_0 \in \mathfrak{D}$, we have

$$\frac{\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0) + (\frac{\varepsilon}{\alpha}+1)} + \frac{1}{\vartheta}p_i(\xi_0) = \frac{1}{\vartheta}\mu(w_i)$$

for some $w_i \in \mathfrak{D}$. Since μ is convex, there exists a $w_0 \in \mathfrak{D}$, such that

$$\frac{\frac{\xi_0}{\vartheta}\sum_{i=1}^{\vartheta}p_i'(\xi_0)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0) + (\frac{\varepsilon}{\alpha}+1)} + \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi_0) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i) = \mu(w_0).$$

Setting $Q(\xi) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)$, we have

$$\frac{\xi Q'(\xi)}{-Q(\xi) + (\frac{\varepsilon}{\alpha}+1)} + Q(\xi) \prec \mu(\xi), \tag{22}$$

Now, if we apply Lemma 1 for Q (Q analytic, $Q(0) = 1$, satisfying (22)) and $\beta = -1$, $\sigma = \frac{\varepsilon}{\alpha} + 1$ (with $(\frac{\varepsilon}{\alpha}+1) > \mathrm{Re}(\mu(\xi))$), we obtain $Q(\xi) \prec \mu(\xi)$. The subordination (21) gives us

$$\frac{\xi p_i'(\xi)}{-Q(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p_i(\xi) \prec \mu(\xi). \tag{23}$$

Further, by Lemma 2 applied for Q and p_i (also p_i analytic, $p_i(0) = 1$), from (23) we obtain $p_i(\xi) \prec \mu(\xi)$, which implies that $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta\} \in \Xi_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$. □

2.2. The Class $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$

Definition 2. *Let* $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$ *denote the class of functions* $\mathfrak{f}\in\Sigma_{\kappa}$ *which satisfy*

$$\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}\prec\mu(\xi)\qquad(\xi\in\mathfrak{D}),$$

where $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)\neq 0$, $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_{\vartheta}\}\in\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$, *and* $\mu\in\Phi$. *We call* $\mathfrak{g}$, *the associated reference tuple function of* $\mathfrak{f}$.

Let us observe that the definition of the class $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$ can be translated into the definition of the class $\mathfrak{C}^{\rho}_{\varepsilon}(\alpha,\vartheta;\mu)$ if in the definition subordination we use the operator $\mathcal{L}^{\rho}_{\varepsilon}(\alpha)$, the associated reference tuple function $\mathfrak{g}$ of $\mathfrak{f}$ from the class $\Xi^{\rho}_{\varepsilon}(\alpha,\vartheta;\mu)$ and the non-vanishing condition $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{g}_j(\xi)\neq 0$.

Remark 3. *When* $\alpha\to 0^{+}$, $\rho=0$ *and* $\mu(\xi)=\frac{1-\xi}{1+\xi}$, *then* $\mathfrak{C}^{0}_{\varepsilon+1}(0,1;\frac{1-\xi}{1+\xi})=\mathfrak{C}$, *the close-to-convex meromorphic functions, which is asymptotically recovered by the newly introduced class (see* [21] *and* [10]*).*

Theorem 4. *Let* $\mathfrak{f}\in\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$, *such that the functions*

$$G_i(\xi)=\frac{v+1}{\xi^{v+2}}\int_0^{\xi}t^{v+1}\mathfrak{g}_i(t)dt\qquad(v>0;i=1,2,\dots,\vartheta)$$

satisfy $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)G_j(\xi)\neq 0$, *where* $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_{\vartheta}\}\in\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$ *is the associated reference tuple function of* $\mathfrak{f}$. *If* $(v+2)>\mathrm{Re}(\mu(\xi))$ *in* $\mathfrak{D}$, *then*

$$\mathcal{F}(\xi)=\frac{v+1}{\xi^{v+2}}\int_0^{\xi}t^{v+1}\mathfrak{f}(t)dt\qquad(\xi\in\mathfrak{D};v>0),$$

also belongs to $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$.

Proof. Let $\mathfrak{f}\in\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$. Then from the class definition and the hypothesis, we have

$$\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}\prec\mu(\xi),\ \mu\in\Phi\qquad(\xi\in\mathfrak{D}).$$

and $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)G_j(\xi)\neq 0$, where

$$G_i(\xi)=\frac{v+1}{\xi^{v+2}}\int_0^{\xi}t^{v+1}\mathfrak{g}_i(t)dt\qquad(v>0),$$

and $\mathfrak{g}=\{\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_{\vartheta}\}\in\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$ (with $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)\neq 0$), the associated reference tuple function of $\mathfrak{f}$. Then by Theorem 2, we have $G=\{G_1,G_2,\dots,G_{\vartheta}\}\in\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$. Now, let

$$p(\xi)=-\frac{\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)G_j(\xi)}\qquad(\xi\in\mathfrak{D}),\tag{24}$$

which, due to the hypothesis condition $\xi \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi) \neq 0$, is a well-defined analytic function. Now, from the definitions of G_i and $\mathcal{F}$, we have

$$\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_i(\xi)]' + (v+2)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_i(\xi) = (v+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_i(\xi), \tag{25}$$

and

$$\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)]' + (v+2)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi) = (v+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi). \tag{26}$$

From (24) and (26), we have

$$-\frac{1}{\vartheta} p(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi) + (v+2)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi) = (v+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi). \tag{27}$$

Differentiating (27) with respect to ξ, and multiplying the resulting equation by ξ, we have

$$-\frac{\xi}{\vartheta} p'(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi) - \frac{\xi}{\vartheta} p(\xi) \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)]'$$

$$+(v+2)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathcal{F}(\xi)]' = (v+1)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'. \tag{28}$$

From (24) into (28), we have

$$-\frac{\xi}{\vartheta} p'(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi) - \frac{\xi}{\vartheta} p(\xi) \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)]'$$

$$-(v+2)\frac{p(\xi)}{\vartheta} \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi) = (v+1)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'.$$

Hence, we get

$$\frac{-\frac{\xi}{\vartheta} p'(\xi) \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)}{\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)]' + \frac{(v+2)}{\vartheta} \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)} - p(\xi)$$

$$= \frac{(v+1)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)]' + \frac{(v+2)}{\vartheta} \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)}.$$

From the above, we have

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta} \sum_{j=1}^{\vartheta} Q_j(\xi) + (v+2)} + p(\xi) = \frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta} \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} \prec \mu(\xi),$$

where $Q_j(\xi) = \dfrac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_j(\xi)]'}{\frac{1}{\vartheta} \sum_{i=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha) G_i(\xi)}$. Now, from Theorem 1, since $G = \{G_1, G_2, \ldots, G_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, we have $Q(\xi) = \frac{1}{\vartheta} \sum_{j=1}^{\vartheta} Q_j(\xi) \prec \mu(\xi)$. Since μ is convex with $(v+2) > \mathrm{Re}(\mu(\xi))$, applying Lemma 2 for the analytic functions Q and p, and for $\beta = -1$, $\sigma = v+2$, we get $p(\xi) \prec \mu(\xi)$; hence $\mathcal{F} \in \mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$. □

Theorem 5. *If* $\mathfrak{f} \in \mathfrak{C}^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$ *and* $\xi \sum_{j=1}^{\vartheta} \mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi) \neq 0$, *where* $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_\vartheta\}$ *is the associated reference tuple function of* $\mathfrak{f}$ *in the class* $\mathfrak{C}^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$ *, then* $\mathfrak{f} \in \mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ *holds for* $(\frac{\varepsilon}{\alpha} + 1) > \mathrm{Re}(\mu(\xi))$ *in* $\mathfrak{D}$.

Proof. Let

$$p(\xi) = \frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} \quad (\xi \in \mathfrak{D}), \tag{29}$$

where $\mathfrak{f} \in \mathfrak{C}^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$ and $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_\vartheta\} \in \Xi^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$, is the associated reference tuple function of $\mathfrak{f}$ in the class $\mathfrak{C}^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$.

On the other hand, from the hypothesis we have $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi) \neq 0$; hence the function p is well defined and analytic in $\mathfrak{D}$.

Now let us note that since $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_\vartheta\} \in \Xi^{\rho}_{\varepsilon}(\alpha, \vartheta; \mu)$, from Theorem 3, we also have $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$.

From (7) and (29), we have

$$-\frac{1}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi) = \frac{\varepsilon}{\alpha}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi) - (\frac{\varepsilon}{\alpha}+1)\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi). \tag{30}$$

Differentiating (30) with respect to ξ, we get

$$\begin{aligned} &-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi) - \frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)]' \\ = \;& \frac{\varepsilon}{\alpha}\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi)]' - (\frac{\varepsilon}{\alpha}+1)\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'. \end{aligned}$$

Using (29), we obtain

$$\begin{aligned} &-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi) - p(\xi)\Bigg[\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)]' \\ &+ \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)\Bigg] = \frac{\varepsilon}{\alpha}\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi)]'. \end{aligned}$$

Then

$$\begin{aligned} &\frac{-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} - p(\xi) \\ = \;& \frac{\frac{\varepsilon}{\alpha}\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi)]'}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}. \end{aligned}$$

Using (7), we have

$$\frac{-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)]' + \frac{(\frac{\varepsilon}{\alpha}+1)}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} - p(\xi) = \frac{\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{g}_j(\xi)} \tag{31}$$

and further

$$\frac{-\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}q_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} - p(\xi) = \frac{\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{g}_j(\xi)}, \tag{32}$$

where

$$q_i(\xi) = \frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}, \quad (i = 1, 2, \dots, \vartheta).$$

Note that from the hypothesis we have $\xi \sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_j(\xi) \neq 0$; hence the function q_i is well defined and analytic in $\mathfrak{D}$. Moreover, in the beginning of the proof we showed that $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_\vartheta\} \in \Xi_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$, hence

$$q_i(\xi) = \frac{-\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_j(\xi)} \prec \mu(\xi), \quad (i = 1, 2, \dots, \vartheta).$$

and further, setting $Q(\xi) = \frac{1}{\vartheta}\sum_{i=1}^{\vartheta} q_i(\xi)$, from Theorem 1, we get $Q(\xi) \prec \mu(\xi)$.

Since $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \mathfrak{C}_{\varepsilon}^{\rho}(\alpha, \vartheta; \mu)$, then, based on (32), we have

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p(\xi) = \frac{-\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{g}_j(\xi)} \prec \mu(\xi). \tag{33}$$

Finally, we can write,

$$\frac{\xi p'(\xi)}{-Q(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p(\xi) \prec \mu(\xi). \tag{34}$$

Using this subordination in Lemma 2, with μ convex and p and Q analytic, $\beta = -1$, $\sigma = \frac{\varepsilon}{\alpha}+1$, we obtain $p(\xi) \prec \mu(\xi)$, which implies that $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_\vartheta\} \in \mathfrak{C}_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$. □

2.3. The Class $\wp_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \delta; \mu)$

Definition 3. *Let $\wp_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \delta; \mu)$, $\delta \geq 0$, denote the class of functions $\mathfrak{f} \in \Sigma_\kappa$ satisfying*

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_\vartheta)(\xi) = -\left\{\delta \frac{\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{g}_j(\xi)} + (1-\delta)\frac{\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_j(\xi)}\right\} \prec \mu(\xi) \ (\xi \in \mathfrak{D}),$$

where the associated reference tuple function, $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_{2,\dots,}\mathfrak{g}_\vartheta\}$, belongs to $\Xi_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$, with the conditions $\xi \sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{g}_j(\xi) \neq 0$, $\xi \sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_j(\xi) \neq 0$ in $\mathfrak{D}$, and $\mu \in \Phi$.

Remark 4. *We note that $\wp_{\varepsilon+1}^{\rho}(\alpha, \vartheta; 0; \mu) = \mathfrak{C}_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$.*

Theorem 6. *If $\mathfrak{f} \in \wp_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \delta; \mu)$, then $\mathfrak{f} \in \mathfrak{C}_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$ holds for $(\frac{\varepsilon}{\alpha}+1) > \mathrm{Re}(\mu(\xi))$.*

Proof. For $\delta = 0$, the theorem is trivial; hence we can presume that $\delta \neq 0$. Let

$$p(\xi) = \frac{-\xi[\mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon+1}^{\rho}(\alpha)\mathfrak{g}_j(\xi)} \quad (\xi \in \mathfrak{D}),$$

where $\mathfrak{f}$ belongs to $\wp_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \delta; \mu)$ and $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_{2,\dots,}\mathfrak{g}_\vartheta\} \in \Xi_{\varepsilon+1}^{\rho}(\alpha, \vartheta; \mu)$ is the associated tuple function of $\mathfrak{f}$. Based on non-vanishing condition from the class definition, the function p is well-defined and analytic.

By using (7) and differentiation on both sides we obtain

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p(\xi) = \frac{-\xi[\mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathcal{L}_{\varepsilon}^{\rho}(\alpha)\mathfrak{g}_j(\xi)},$$

with $q_i(\xi) = \dfrac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)}$, and further

$$\mathfrak{J}(\delta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi) = \frac{\delta\xi p'(\xi)}{-\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p(\xi).$$

Let us note that, due to the non-vanishing condition from the class definition, the function q_i, $(i = 1, 2, \dots, \vartheta)$, is also well-defined and analytic. Moreover, since $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_{2,\dots,}\mathfrak{g}_\vartheta\} \in \Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$, we have $q_i(\xi) \prec \mu(\xi)$, $(i = 1, 2, \dots, \vartheta)$ and further, based on Theorem 1, we have $Q(\xi) = \frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) \prec \mu(\xi)$.

On the other hand, since $\mathfrak{f} \in \wp^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$, we get

$$\mathfrak{J}(\delta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi) = \frac{\delta\xi p'(\xi)}{-Q(\xi) + (\frac{\varepsilon}{\alpha}+1)} + p(\xi) \prec \mu(\xi).$$

Now we can apply Lemma 2 for analytic functions p and Q, with $\sigma = \frac{(\frac{\varepsilon}{\alpha}+1)}{\delta}$ and $\beta = -\frac{1}{\delta}$ to obtain $p(\xi) \prec \mu(\xi)$ which completes the proof. □

Theorem 7. *For* $\delta > \lambda \geq 0$, $\wp^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu) \subset \wp^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\lambda;\mu)$ *holds for* $(\frac{\varepsilon}{\alpha}+1) > \mathrm{Re}(\mu(\xi))$.

Proof. The case $\lambda = 0$ was treated in the previous theorem. Hence we assume that $\lambda \neq 0$. Suppose that $\mathfrak{f} \in \wp^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$. Then

$$\mathfrak{J}(\delta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi) \prec \mu(\xi). \tag{35}$$

Further, from Theorem 6, we have $\mathfrak{f} \in \mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$; hence

$$\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} \prec \mu(\xi). \tag{36}$$

Let ξ_1 be any arbitrary point in $\mathfrak{D}$. Then From (35) and (36) it follows that

$$\frac{-\xi_1[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi_1)} \in \mu(\mathfrak{D})$$

and

$$-\left(\delta\frac{\xi_1[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{g}_j(\xi_1)} + (1-\delta)\frac{\xi_1[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi_1)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi_1)}\right) \in \mu(\mathfrak{D}).$$

Since

$$\mathfrak{J}(\lambda;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi) = (1-\frac{\lambda}{\delta})\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{g}_j(\xi)} + \frac{\lambda}{\delta}\mathfrak{J}(\delta;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi),$$

and $\mu(\mathfrak{D})$ is convex, $\frac{\lambda}{\delta} < 1$, then we have $\mathfrak{J}(\lambda;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi_1) \in \mu(\mathfrak{D})$, hence $\mathfrak{J}(\lambda;\mathfrak{f};\mathfrak{g}_1,\mathfrak{g}_2,\dots,\mathfrak{g}_\vartheta)(\xi) \prec \mu(\xi)$, and we get $\mathfrak{f} \in \wp^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\lambda;\mu)$. □

2.4. The Class $\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$

Definition 4. *Let* $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_\vartheta\}$, $\mathfrak{f}_i\in\Sigma_\kappa, 1\le i\le\vartheta$ *satisfying*

$$\mathfrak{J}(\delta;\mathfrak{f};\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_\vartheta)(\xi)=-\left\{\delta\frac{\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_j(\xi)}+(1-\delta)\frac{\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)}\right\}\prec\mu(\xi)\ (\xi\in\mathfrak{D}),$$

where $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_j(\xi)\neq 0$ *and* $\xi\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)\neq 0$ *in* $\mathfrak{D}$, *and* $\mu\in\Phi$, *then we say that* $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_\vartheta\}\in\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$.

Remark 5. *We note that* $\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;0;\mu)=\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$.

Theorem 8. *If* $\mathfrak{f}\in\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$, *then* $\mathfrak{f}\in\Xi^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\mu)$ *holds for* $(\frac{\varepsilon}{\alpha}+1)>\mathrm{Re}(\mu(\xi))$.

Proof. For $\delta=0$, the theorem is trivial; hence we can presume that $\delta\neq 0$. Let

$$p_i(\xi)=\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon+1}(\alpha)\mathfrak{f}_j(\xi)}\qquad(\xi\in\mathfrak{D}).$$

where $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_\vartheta\}\in\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$. We can see that due to the non-vanishing condition from the class definition, the function p_i, $(i=1,2,\dots,\vartheta)$, is well-defined and analytic. Then by using (7) and differentiation on both sides we obtain

$$\frac{\xi p_i'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi)+(\frac{\varepsilon}{\alpha}+1)}+p_i(\xi)=\frac{-\xi[\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathcal{L}^{\rho}_{\varepsilon}(\alpha)\mathfrak{f}_j(\xi)},$$

Since $\mathfrak{f}\in\mathfrak{R}^{\rho}_{\varepsilon+1}(\alpha,\vartheta;\delta;\mu)$, we have

$$J(\delta;\mathfrak{f};\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_\vartheta)(\xi)=\frac{\delta\xi p_i'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi)+(\frac{\varepsilon}{\alpha}+1)}+p_i(\xi)\prec\mu(\xi).\tag{37}$$

Hence, for any $\xi_0\in\mathfrak{D}$, we have

$$\frac{\frac{1}{\vartheta}\delta\xi_0 p_i'(\xi_0)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi)+(\frac{\varepsilon}{\alpha}+1)}+\frac{1}{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i)$$

for some $w_i\in\mathfrak{D}$. Since μ is convex, there exists an $w_0\in\mathfrak{D}$, such that

$$\frac{\frac{\xi_0}{\vartheta}\delta\sum_{i=1}^{\vartheta}p_i'(\xi_0)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0)+(\frac{\varepsilon}{\alpha}+1)}+\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i)=\mu(w_0).$$

Setting $Q(\xi)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)$, which is also analytic, we have

$$\frac{\xi Q'(\xi)}{\frac{-1}{\delta}Q(\xi)+\frac{(\frac{\varepsilon}{\alpha}+1)}{\delta}}+Q(\xi)\prec\mu(\xi),$$

Therefore, by Lemma 1, applied for the analytic function Q, with $\sigma=\frac{(\frac{\varepsilon}{\alpha}+1)}{\delta}$ and $\beta=-\frac{1}{\delta}$, we get $Q(\xi)\prec\mu(\xi)$. Further, based on (37), we apply Lemma 2 for the analytic functions p_i and Q, with $\sigma=\frac{(\frac{\varepsilon}{\alpha}+1)}{\delta}$ and $\beta=-\frac{1}{\delta}$, obtaining $p_i(\xi)\prec\mu(\xi)$ which completes the proof. □

Example 1. *For* $\vartheta = 1$ *and* $\mu(\xi) = \frac{1-\xi}{1+\xi}$, *we have*

$$\mathfrak{f}(\xi) = \frac{1}{\xi} + \frac{\Gamma(\alpha n + \varepsilon + 1)}{(n-1)(2n+1)^{\rho}\Gamma(\varepsilon+1)}\xi^{n-1} \in \Xi^{\rho}_{\varepsilon+1}(\alpha, 1; \frac{1-\xi}{1+\xi}) \quad (n \geq \kappa \geq 2).$$

This example shows that the class $\Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$ is not empty. Considering how the last three classes are defined, this example is also a proof that all the classes are not empty.

3. Conclusions

The present study introduces a novel linear operator constructed through the Hadamard inverse of the Legendre Chi function and the Mittag–Leffler function, both of which play a significant role in fractional calculus. Consequently, besides the results from this paper, owing to its structural flexibility, the operator could have potential relevance across diverse applied mathematical fields where fractional calculus is applied. Regarding the current results, the operator constitutes an effective framework for establishing closure properties and inclusion relationships for some classes of functions, within analytic function theory. The introductory section outlines the research background, emphasizing the motivation behind selecting these analytical tools, and displays the foundational concept of the convolution operator $\mathcal{L}^{\rho}_{\varepsilon}(\alpha)$. Building on this groundwork, the study applies the proposed operator to define and analyze several subclasses of meromorphic functions via differential subordination techniques. The principal results are systematically presented through the investigation of the newly introduced function classes $\Xi^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, $\mathfrak{C}^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \mu)$, $\wp^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \delta; \mu)$ and $\Re^{\rho}_{\varepsilon+1}(\alpha, \vartheta; \delta; \mu)$. These classes enrich the existing literature by extending the applicability of differential subordination methods. Future research may focus on deriving coefficient bounds and further geometric properties for these classes, particularly through the use of alternative operators and in the setting of p-valent meromorphic functions, thereby broadening the scope and impact of the present results.

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