mathematics

MDPI

Article

Qualitative Analysis and Oscillation Criteria for Third-Order Semi-Canonical Differential Equations via Canonical Reduction

Asma Al-Jaser [1], Faizah Alharbi [2], Insaf F. Ben Saoud [3] and Belgees Qaraad [4,*]

[1] Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; ajaljaser@pnu.edu.sa
[2] Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah 24227, Saudi Arabia; fmharbi@uqu.edu.sa
[3] Department of Mathematics, Faculty of Education, University of Benghazi, Benghazi 1308, Libya; insaf.bensaud@uob.edu.ly
[4] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: belgeesmath2016@gmail.com

Abstract

This paper investigates the oscillatory behavior of third-order nonlinear semi-canonical neutral differential equations with multiple delayed arguments, treating advanced and delayed neutral arguments as alternative settings. Through a transformation of the dependent variable and associated coefficients, the original semi-canonical structure is converted into an equivalent canonical framework, allowing the analysis to be organized around two classes of eventually positive canonical functions. For advanced neutral arguments, comparison principles, and integral averaging yield sufficient oscillation criteria. A separate delayed-neutral analysis develops class-specific comparison estimates, complemented by a Riccati criterion for a specified subclass of equations. Combining the resulting class exclusions provides further sufficient conditions for oscillation. Three examples illustrate the criteria, including explicit sufficient parameter bounds and applications of both canonical transformations.

Keywords: oscillation criteria; third-order; semi-canonical; neutral differential equations

MSC: 34C10; 34K11

1. Introduction

Oscillation theory constitutes a fundamental branch of the qualitative analysis of functional differential equations, providing effective tools for describing the asymptotic behavior of solutions without requiring explicit solution formulas. Its importance is particularly evident for equations involving memory effects, hereditary phenomena, or state-dependent interactions, which arise naturally in a wide range of applications including population dynamics, control theory, engineering, mathematical biology, and physics. Consequently, the development of effective oscillation criteria has become an active research topic in the qualitative theory of differential equations; see, for example, refs. [1–4] and the references cited therein.

Recent advances in numerical methods have also contributed to the study of nonlinear differential equations. In this context, Li and Qu [5] developed a high-precision barycentric interpolation collocation method for the extended Fisher–Kolmogorov equation.

Among functional differential equations, neutral differential equations constitute one of the most challenging classes because the highest-order derivative depends not only on

check for updates

Academic Editor: Fengping Yao

Received: 25 September 2026
Revised: 14 November 2026
Accepted: 16 November 2026
Published: 20 November 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

the current state but also on delayed or advanced values of the unknown function. This structural feature substantially increases the complexity of the qualitative analysis and often prevents the direct application of techniques developed for ordinary delay differential equations. The difficulty becomes even more pronounced for third-order equations, whose solutions exhibit richer asymptotic structures and require more delicate analytical arguments. Furthermore, the coexistence of delayed deviating arguments and advanced neutral terms introduces additional mathematical challenges in establishing oscillation criteria, particularly in the semi-canonical setting where standard canonical methods are no longer directly applicable. These challenges have motivated considerable research devoted to extending oscillation theory for higher-order neutral differential equations under increasingly general assumptions; see, for instance, refs. [6–9], and related contributions.

This paper is devoted to the investigation of the oscillatory behavior of solutions to the following class of third-order semi-canonical neutral differential equations:

$$L_3 z(\top) + \sum_{i=1}^{n} \mu_i(\top) h(x(\sigma_i(\top))) = 0, \quad \top \in J_0, \tag{1}$$

where n is a positive integer,

$$L_3 z = \left(b_2 \left(b_1 z'\right)'\right)'. \tag{2}$$

Moreover, the neutral term z is defined by

$$z(\top) = x(\top) + d(\top) x(\tau(\top)).$$

Throughout this paper, the following assumptions are imposed:

(A_1) $b_1, b_2 \in C(J_0, (0,\infty))$, $\tau, \sigma_i \in C^1(J_0, \mathbb{R})$, $d, \mu_i \in C(J_0, [0,\infty))$, $i = 1, 2, \ldots, n$, where $J_0 := [\top_0, \infty)$.

(A_2) $\sigma_i(\top) < \top$, $\sigma_i'(\top) > 0$, $\tau'(\top) > 0$, $i = 1, 2, \ldots, n$, $\lim_{\top\to\infty} \sigma_i(\top) = \infty, \sigma_-(\top) := \min_{1\le i\le n} \sigma_i(\top)$, $\sigma_+(\top) := \max_{1\le i\le n} \sigma_i(\top)$. Moreover, $\mu_i(\top) > 0$, $i = 1, 2, \ldots, n$, for all sufficiently large $\top$, and there exists a constant $d_0 \in [0, 1)$ such that

$$0 \le d(\top) \le d_0, \ \top \in J_0. \tag{3}$$

(A_3) $h \in C(\mathbb{R}, \mathbb{R})$ is odd, $uh(u) > 0$ for $u \neq 0$, and there exist a constant $k > 0$ and a ratio α of two positive odd integers such that $h(u)/u^\alpha \ge k, u \neq 0$.

(A_4) $\Pi_1(\top_0) := \int_{\top_0}^{\infty} \frac{1}{b_1(\top)} d\top = \infty$, $\Pi_2(\top_0) := \int_{\top_0}^{\infty} \frac{1}{b_2(\top)} d\top < \infty$.

In addition to (A_1)–(A_4), we consider the following two alternative settings:

(A_{5a}) $\tau(\top) > \top$ for all sufficiently large $\top$.

Alternatively,

(A_{5b}) $\tau(\top) \le \top$, $\tau'(\top) \ge \tau_0 > 0$, $\tau(\sigma_i(\top)) = \sigma_i(\tau(\top))$, $i = 1, 2, \ldots, n$, for all sufficiently large $\top$, where τ_0 is a positive constant.

The assumptions (A_{5a}) and (A_{5b}) are alternative hypotheses and are not imposed simultaneously. Each result below explicitly specifies the setting under which it is established.

For convenience, we introduce the operators

$$L_0 z = z, \quad L_1 z = b_1 z', \quad L_2 z = b_2 \left(b_1 z'\right)'. \tag{4}$$

Definition 1. *By a solution of (1) we mean a function $x \in C([I_x, \infty), \mathbb{R})$ such that $L_0 z, L_1 z$, and $L_2 z$ belong to $C^1([I_x, \infty), \mathbb{R})$, and there exists $\top_x \ge \top_0$ such that $\sigma_i(\top) \ge I_x$ for all $\top \ge \top_x$ and $i = 1, 2, \ldots, n$, with x satisfying Equation (1) for all $\top \ge \top_x$. We consider only those solutions $x(\top)$ of (1) satisfying $\sup\{|x(\top)| : \top \ge I\} > 0$ for every $I \ge I_x$. A solution of (1) is said to be*

oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Remark 1. *The definition 1 specifies the regularity and domain of the solutions considered. Our oscillation results apply to every nontrivial solution defined on a half-line, whenever such a solution exists, and do not require uniqueness. Existence and uniqueness for prescribed data require a separate analysis, particularly in the advanced neutral setting* (A_{5a})*, and are not asserted under the present assumptions. For background on neutral functional differential equations, see [10]; for well-posedness questions concerning equations with both advanced and retarded arguments, see [11].*

Depending on the convergence or divergence of the integrals $\Pi_1(t_0)$ and $\Pi_2(\top_0)$, the associated differential operator can be classified as canonical, fully noncanonical, or semi-canonical. More precisely, the canonical configuration is characterized by

$$\Pi_1(\top_0) = \infty, \ \Pi_2(\top_0) = \infty, \tag{5}$$

whereas the fully noncanonical configuration is defined by

$$\Pi_1(\top_0) < \infty, \ \Pi_2(\top_0) < \infty, \tag{6}$$

The semi-canonical case occurs when exactly one of these two integrals diverges, giving rise to the two complementary configurations

$$\Pi_1(\top_0) < \infty, \ \Pi_2(\top_0) = \infty, \tag{7}$$

and

$$\Pi_1(\top_0) = \infty, \ \Pi_2(\top_0) < \infty. \tag{8}$$

A substantial body of the literature on third-order differential equations has been devoted to equations that are either special cases of, or structurally related to, the equation investigated in the present paper. Nevertheless, most available oscillation results have been established under additional assumptions concerning the deviating arguments, the neutral coefficient, or the canonical structure of the differential operator. Among the most frequently imposed restrictions is the commutativity condition.

$$\sigma_i \circ \tau = \tau \circ \sigma_i, \tag{9}$$

the requirement that the neutral argument be delayed,

$$\tau(\top) \leq \top, \tag{10}$$

and restrictions on the neutral coefficient, such as

$$d(\top) \in [1, \infty). \tag{11}$$

Against this background, the related literature may be organized according to the structure of the differential operator, the neutral term, and the type of deviating arguments. For equations involving multiple discrete deviating arguments, the authors in [12] investigated the third-order neutral differential equation

$$\left(b_2(\top)\left(z''(\top)\right)^{\gamma}\right)' + \sum_{i=1}^{n} \mu_i(\top) x^{\gamma}(\sigma_i(\top)) = 0,$$

where γ is a quotient of two positive odd integers. Their analysis was conducted under the canonical condition

$$\int_{\top_0}^{\infty} \frac{1}{b_2^{1/\gamma}(\top)} \mathrm{d}\top = \infty \tag{12}$$

together with assumptions (10) and (11). Under these hypotheses, the authors derived sufficient oscillation criteria guaranteeing that every solution of the equation is oscillatory. A related class of third-order neutral differential equations involving several delayed arguments was studied in [13] in the form

$$\left(b_2\left(z''(\top)\right)^{\gamma}\right)' + \sum_{i=1}^{n} \mu_i(\top) x^{\beta_i}(\sigma_i(\top)) = 0,$$

where γ and each β_i are the quotients of two positive odd integers. The analysis was carried out under the canonical condition (12) along with condition (10) and

$$d(\top) \in (-1, 0]. \tag{13}$$

Several contributions have also been devoted to structurally related equations involving a single deviating term, some of which can be recovered through particular reductions of the present setting. In the nonneutral setting, the authors in [14] examined the third-order differential equation

$$\left(b_2(\top)\left(x'(\top)\right)^{\gamma}\right)'' + \mu(\top) x(\sigma(\top)) = 0,$$

where α is a quotient of two positive odd integers. Their analysis was performed under the canonical condition (12) and the advanced-argument assumption

$$\sigma(\top) \geq \top.$$

A closely related problem was investigated by Baculíková and Džurina in [15], who considered the equation

$$\left(b_2(\top) x''(\top)\right)' + \mu(\top) x(\top) = 0,$$

under the condition

$$\Pi_2(\top_0) = \infty.$$

The semi-canonical arrangement (8) was later addressed in [16] for the third-order nonlinear delay differential equation

$$\left(b_2(\top)\left(b_1(\top)(x(\top))'\right)'\right)' + \mu(\top) x^{\alpha}(\sigma(\top)) = 0.$$

Baculíková et al. [17] investigated the oscillatory behavior of a class of third-order neutral differential equations with an advanced neutral argument $\tau(\top) > \top$ of the form

$$\left(b_2(\top)\left(b_1(\top)\left((x(\top) + d(\top) x(\tau(\top)))'\right)^{\alpha}\right)'\right)' + \mu(\top) x^{\alpha}(\top) = 0,$$

under the assumptions

$$\int_{\top_0}^{\infty} \frac{1}{b_1(\top)} \mathrm{d}\top = \int_{\top_0}^{\infty} \frac{1}{b_1^{1/\alpha}(\top)} \mathrm{d}\top = \infty.$$

Chatzarakis et al. [18] established sufficient conditions for the oscillation of all solutions of the third-order Emden–Fowler neutral differential equation

$$(x(\top) + d(\top)x(\tau(\top)))''' + \mu(\top)x^{\alpha}(\sigma(\top)) = 0,$$

under assumptions (10) and (11). A further class of third-order neutral differential equations was considered in [19], namely

$$\left(b_2(\top)\left(b_1(\top)(x(\top) + d(\top)x^{\alpha}(\tau(\top)))'\right)'\right)' + \mu(\top)x(\sigma(\top)) = 0,$$

the oscillatory behavior of its solutions was investigated under assumption (10) with both cases (5) and (8).

Among the contributions devoted to third-order neutral differential equations, the authors in [20] considered

$$\left(b_2(\top)\left(\left(b_1(\top)(x(\top) + d(\top)x(\tau(\top)))'\right)'\right)^{\gamma}\right)' + \mu(\top)x^{\beta}(\sigma(\top)) = 0. \tag{14}$$

In the particular case $\beta = \gamma = 1$, their analysis was carried out under the canonical configuration (5) with (9) and (10). They established criteria excluding Kneser-type solutions and, by combining these results with previously available oscillation theorems, obtained sufficient conditions ensuring the oscillation of all solutions.

Feng and Han [8] also investigated Equation (14) under the semi-canonical configuration (8). However, their results additionally required assumptions (3), (9) and (10). More recently, Saranya et al. [21] extended this line of research to the complementary coefficient range described by (11). Further work addressed other configurations of the differential operator. In [22], Equation (14) was studied under the fully noncanonical configuration (6), together with the delayed-neutral-argument condition (10).

Vidhyaa et al. [23] established established oscillation criteria for third-order semi-canonical equations with an advanced neutral argument and a single delay using canonical reduction, comparison arguments, and integral averaging. The present study shares this methodological foundation and considers multiple delayed terms under two alternative neutral-argument settings.

Accordingly, we investigate Equation (1) under the semi-canonical configuration (8) and the neutral coefficient condition (3). In the advanced setting, we establish oscillation criteria without requiring commutativity. In the delayed setting, under commutativity and the stated argument assumptions, we derive class-specific estimates allowing the transformed neutral coefficient to satisfy $0 \le a(\top) \le a_0 < \infty$, without requiring $a_0 < 1$. Criteria excluding N_0 are complemented, when $\alpha = 1$ and $b_1 \equiv 1$, by a Riccati criterion excluding N_2. Combining these exclusions yields sufficient conditions for oscillation. Canonical reduction provides the common analytical framework for both settings.

2. Canonical Transformation and Auxiliary Results

The analysis begins by converting the semi-canonical operator into a canonical form under the two possible configurations. The resulting sign classification of eventually positive solutions provides the basis for subsequent comparison, integral, and Riccati arguments, through which sufficient oscillation criteria are established under the corresponding assumptions.

To streamline the presentation of the subsequent results, we introduce the following auxiliary functions. Let L_3z be the semi-canonical operator introduced in (2). Set

$$A_2(\top) := \int_{\top}^{\infty} \frac{1}{b_2(s)}\mathrm{d}s \quad \text{and} \quad A_{12}(\top) := \int_{\top}^{\infty} \frac{A_2(s)}{b_1(s)}\mathrm{d}s,$$

and define Ω by

$$\Omega(\top) := \begin{cases} 1, & A_{12}(\top_0) = \infty, \\ A_{12}(\top), & A_{12}(\top_0) < \infty. \end{cases}$$

Define

$$\begin{aligned} p_1(\top) &: = b_1(\top)\Omega^2(\top)/A_2(\top), \\ p_2(\top) &: = b_2(\top)A_2^2(\top)/\Omega(\top), \\ P_i(\top) &: = \frac{1}{p_i(\top)}, \ i = 1, 2. \end{aligned}$$

We distinguish between the following two cases:

$$\begin{aligned} &(C_1) \ A_{12}(\top_0) = \infty; \\ &(C_2) \ A_{12}(\top_0) < \infty. \end{aligned}$$

Under (C_1), $\Omega \equiv 1$, so $p_1 = b_1/A_2$ and $p_2 = b_2A_2^2$. Under (C_2), $\Omega = A_{12}$, so $p_1 = b_1A_{12}^2/A_2$ and $p_2 = b_2A_2^2/A_{12}$. In both cases, p_1 and p_2 are positive. The divergence of the integrals of their reciprocals is established separately for each case in the proof of Theorem 1.

For a fixed $\top_* \geq \top_0$, set

$$B_i(\top) = \int_{\top_*}^{\top} P_i(s)\mathrm{d}s, \ i = 1, 2,$$

and

$$B_{12}(\top) = \int_{\top_*}^{\top} P_1(s) \int_{\top_*}^{s} P_2(s_1)\mathrm{d}s_1\mathrm{d}s, \ \top \geq \top_*.$$

Furthermore, define

$$E(\top) := \Omega(\top)(1 - d_0).$$

For $\top > \top_*$ with $\tau(\top) \geq \top$, define

$$E_1(\top) := \Omega(\top)\left(1 - d_0\frac{B_{12}(\tau(\top))}{B_{12}(\top)}\right).$$

Finally, for $v \geq u \geq \top_0$, set

$$R(v, u) := \int_u^v P_1(s)\left(\int_s^v P_2(r)\mathrm{d}r\right)\mathrm{d}s.$$

The following result provides the canonical representation of L_3z in each case.

Theorem 1 ([24] (Theorem 2.1) and [25] (Theorem 2)). *Let L_3z be the semi-canonical operator defined in (2). Then A_2L_3z can be written in the canonical form*

(i) *If (C_1) holds, then*

$$A_2L_3z = \left(\left(b_2A_2^2\left(\frac{b_1}{A_2}z'\right)'\right)\right)';$$

(*ii*) *If* (C_2) *holds, then*

$$A_2L_3z = \left(\frac{b_2A_2^2}{A_{12}}\left(\frac{A_{12}^2b_1}{A_2}\left(\frac{1}{A_{12}}z\right)'\right)'\right)'.$$

Proof. Since $A_2' = -1/b_2$, we obtain

$$\begin{aligned}[A_2L_2z + L_1z]' &= -L_2z/b_2 + A_2L_3z + (L_1z)' \\ &= A_2L_3z. \end{aligned} \tag{15}$$

Case (C_1). Here $\Omega = 1$ and $y = z$. Consequently,

$$p_1y' = L_1z/A_2,$$

and

$$\begin{aligned}p_2(p_1y')' &= b_2A_2^2(L_1z/A_2)' \\ &= A_2L_2z + L_1z.\end{aligned}$$

Differentiating and using (15) gives

$$\left[p_2(p_1y')'\right]' = A_2L_3z.$$

To verify canonicality, fix $\top \geq \top_0$. By (C_1),

$$\int_{\top}^{\infty} P_1(s)\mathrm{d}s = \int_{\top}^{\infty} A_2(s)/b_1(s)\mathrm{d}s = \infty.$$

Moreover,

$$P_2 = \frac{1}{b_2A_2^2} = \left(\frac{1}{A_2}\right)'.$$

Since $A_2(\top) \to 0$,

$$\int_{\top}^{\top} P_2(s)\mathrm{d}s = \frac{1}{A_2(\top)} - \frac{1}{A_2(\top)} \to \infty.$$

Case (C_2). Here $\Omega = A_{12}$ and $y = z/A_{12}$. Using $(A_{12})' = -A_2/b_1$, we find

$$p_1y' = \left(b_1A_{12}^2/A_2\right)(z/A_{12})' = (A_{12}/A_2)L_1z + z.$$

Since $(A_{12}/A_2)' = -1/b_1 + A_{12}/\left(b_2A_2^2\right)$ and $z' = L_1z/b_1$, differentiation yields

$$(p_1y')' = (A_{12}/A_2)(L_1z)' + \left[A_{12}/\left(b_2A_2^2\right)\right]L_1z.$$

Therefore,

$$p_2(p_1y')' = A_2L_2z + L_1z.$$

Differentiating and applying (15) again gives

$$\left[p_2(p_1y')'\right]' = A_2L_3z.$$

For canonicality, observe that

$$P_1 = \frac{A_2}{b_1 A_{12}^2} = \left(\frac{1}{A_{12}}\right)'.$$

Under (C_2), A_{12} is a finite tail integral of A_2/b_1, hence, $A_{12}(\top) \to 0$ as $\top \to \infty$. Consequently,

$$\int_{\top}^{\top} P_1(s)\mathrm{d}s = \frac{1}{A_{12}(\top)} - \frac{1}{A_{12}(\top)} \to \infty.$$

Furthermore,

$$P_2 = \frac{A_{12}}{b_2 A_2^2} = \left(\frac{A_{12}}{A_2}\right)' + \frac{1}{b_1}.$$

Hence,

$$\begin{aligned} \int_{\top}^{\top} P_2(s)\mathrm{d}s &= \frac{A_{12}(\top)}{A_2(\top)} - \frac{A_{12}(\top)}{A_2(\top)} + \int_{\top}^{\top} \frac{1}{b_1(s)}\mathrm{d}s \\ &\geq -\frac{A_{12}(\top)}{A_2(\top)} + \int_{\top}^{\top} \frac{1}{b_1(s)}\mathrm{d}s \to \infty. \end{aligned}$$

Here, the inequality follows from $A_{12}/A_2 > 0$, and the divergence follows from (A_4). Thus, both transformed coefficients satisfy the required divergence conditions, and the operator is canonical in either case. This completes the proof. □

In view of Theorem 1, Equation (1) can be transformed into an equivalent canonical equation. To this end, define

$$\begin{aligned} y(\top) &= z(\top) \text{ if } (C_1) \text{ holds,} \\ y(\top) &= \frac{1}{A_{12}(\top)} z(\top) \text{ if } (C_2) \text{ holds.} \end{aligned}$$

Introduce the operators

$$F_0 y = y, \ \ F_i y = p_i (F_{i-1} y)', \ \ i = 1,2, \ \ F_3 y = (F_2 y)'.$$

By Theorem 1 and the definitions above

$$F_3 y(\top) = A_2(\top) L_3 z(\top).$$

Then one obtains

$$F_3 y(\top) + A_2(\top) \sum_{i=1}^{n} \mu_i(\top) h(x(\sigma_i(\top))) = 0. \tag{16}$$

Equation (16), together with $\Omega(\top) y(\top) = x(\top) + d(\top) x(\tau(\top))$, is equivalent to (1) within the prescribed solution class, since $A_2 > 0$. This transformation requires neither an ordering of $\tau(\top)$ relative to $\top$ nor commutativity between τ and σ_i.

The canonicality established in the proof of Theorem 1 gives

$$\int_{\top_0}^{\infty} P_1(\top)\mathrm{d}\top = \int_{\top_0}^{\infty} P_2(\top)\mathrm{d}\top = \infty. \tag{17}$$

Corollary 1. *The semi-canonical Equation (1) admits a solution if and only if its canonical counterpart (16) admits a corresponding solution.*

Corollary 2. *The semi-canonical differential Equation (1) admits a positive solution $x(\top)$ if and only if its canonical differential Equation (16) admits the corresponding positive solution.*

The following sign classification of eventually positive solutions is a standard consequence of the generalized Kiguradze lemma; see, for example, refs. [26,27].

Corollary 3. *Let $x(\top) > 0$ is a solution of (1), and let z be its corresponding neutral function. Then, for all sufficiently large $\top$, z belongs to exactly one of the following classes:*

$$\begin{aligned} S_0 &= \{z : z > 0,\ L_1 z < 0,\ L_2 z > 0,\ L_3 z < 0\}, \\ S_1 &= \{z : z > 0,\ L_1 z > 0,\ L_2 z < 0,\ L_3 z < 0\}, \\ S_2 &= \{z : z > 0,\ L_1 z > 0,\ L_2 z > 0,\ L_3 z < 0\}. \end{aligned}$$

Thus, the class S of neutral functions corresponding to eventually positive solutions of (1) admits the decomposition

$$S = S_0 \cup S_1 \cup S_2.$$

By passing to the canonical Equation (16), the analysis reduces to only two eventual behaviors. This follows from the generalized Kiguradze lemma; see [26], under condition (17).

Lemma 1. *Assume that $x(\top) > 0$ is a solution of Equation (16). Then, for some $\top_1 \geq \top_0$, the corresponding function $y(\top)$ satisfies*

$$y > 0,\ F_2 y > 0,\ F_3 y < 0.$$

Moreover, one of the following two cases holds:

$$y \in N_0 \Longleftrightarrow F_1 y < 0,$$

or

$$y \in N_2 \Longleftrightarrow F_1 y > 0.$$

Therefore, the set N of nonoscillatory solutions of (16) can be decomposed as

$$N = N_0 \cup N_2.$$

Lemma 2. *Assume that $y \in N_2$. Then, for $\top > \top_*$, the following properties hold:*

$$\begin{aligned} &(a)\ F_1 y(\top) - B_2(\top) F_2 y(\top) \geq 0 \text{ and } \left(\frac{1}{B_2(\top)} F_1 y(\top)\right)' \leq 0, \\ &(b)\ \left(\frac{1}{B_{12}(\top)} y(\top)\right)' \leq 0, \\ &(c)\ y(\top) - B_{12}(\top) F_2 y(\top) \geq 0. \end{aligned}$$

Proof. Since $y \in N_2$, by increasing $\top_*$, if necessary, we may assume that

$$F_1 y > 0, F_2 y(\top) > 0, F_3 y(\top) < 0 \text{ for all } \top \geq \top_*.$$

Since $F_3 y(\top) = (F_2 y(\top))' < 0$, the function $F_2 y$ is nonincreasing. Therefore, for $\top \geq s \geq \top_*$,

$$F_2 y(s) \geq F_2 y(\top).$$

Using $(F_1y(\top))' = P_2(\top)F_2y(\top)$, we obtain

$$\begin{aligned} F_1y(\top) &= F_1y(\top_*) + \int_{\top_*}^{\top} P_2(s)F_2y(s)\mathrm{d}s \\ &\geq F_2y(\top)\int_{\top_*}^{\top} P_2(s)\mathrm{d}s = B_2(\top)F_2y(\top). \end{aligned} \tag{18}$$

Moreover,

$$\left(\frac{1}{B_2(\top)}F_1y(\top)\right)' = \frac{1}{B_2^2(\top)}P_2(\top)(B_2(\top)F_2y(\top) - F_1y(\top)) \leq 0.$$

Hence, (a) holds.
It follows from (a) that, for $\top \geq s > \top_*$,

$$\frac{F_1y(s)}{B_2(s)} \geq \frac{F_1y(\top)}{B_2(\top)}.$$

Using $y'(\top) = P_1(\top)F_1y(\top)$, we obtain

$$\begin{aligned} y(\top) &= y(\top_*) + \int_{\top_*}^{\top} \frac{1}{p_1(s)}F_1y(s)\mathrm{d}s \\ &= y(\top_*) + \int_{\top_*}^{\top} \frac{B_2(s)}{B_2(s)p_1(s)}F_1y(s)\mathrm{d}s \geq \frac{B_{12}(\top)}{B_2(\top)}F_1y(\top). \end{aligned} \tag{19}$$

Consequently,

$$\left(\frac{1}{B_{12}(\top)}y(\top)\right)' = \frac{1}{B_{12}^2(\top)}P_1(\top)(B_{12}(\top)F_1y(\top) - B_2(\top)y(\top)) \leq 0.$$

The last inequality proves (b).
Finally, combining (18) and (19) yields

$$\frac{y(\top)}{B_{12}(\top)} \geq F_2y(\top).$$

Which proves (c). This completes the proof. □

3. Qualitative Properties and Oscillation Criteria

The analysis distinguishes between advanced and delayed neutral arguments, establishing class-specific estimates and oscillation criteria for each setting. Under the respective assumptions, these criteria are derived by excluding the two possible classes of eventually positive canonical solutions, N_0 and N_2.

3.1. Qualitative Properties for an Advanced Neutral Argument

We establish estimates relating an eventually positive solution to its corresponding canonical function y when the neutral argument is advanced. These estimates are used in the subsequent analysis of the classes N_0 and N_2.

Lemma 3. *Assume that (A_1)–(A_4) and (A_{5a}) hold. Suppose that $x(\top) > 0$ is a solution of (1), and let $y(\top)$ be its corresponding function. Then, for all sufficiently large $\top$, the following inequalities hold:*

(i) *If $y \in N_0$, then*

$$x(\top) - E(\top)y(\top) \geq 0. \tag{20}$$

(ii) *If $y \in N_2$ and the condition*

$$\frac{d_0 B_{12}(\tau(\top))}{B_{12}(\top)} < 1, \tag{21}$$

holds eventually, then

$$x(\top) - E_1(\top)y(\top) \geq 0. \tag{22}$$

Proof. Let $x(\top) > 0$ be a positive solution of (16). Choose $t_1 \geq \top_0$ sufficiently large such that

$$x(\top) > 0, x(\tau(\top)) > 0, x(\sigma_i(\top)) > 0, \ \top \geq \top_1.$$

We prove the two assertions separately.

(i) Assume that $y \in N_0$. Since $F_1 y < 0$, the function y is decreasing on $[\top_1, \infty)$. Hence,

$$y(\tau(\top)) < y(\top), \ \top \geq \top_1.$$

First, suppose that (C_1) holds. Then,

$$\begin{aligned} x(\top) &= y(\top) - d(\top)x(\tau(\top)) \\ &\geq y(\top) - d_0 y(\tau(\top)). \\ &\geq (1 - d_0)y(\top). \end{aligned} \tag{23}$$

Next, suppose that (C_2) holds. Then $z(\top) = A_{12}(\top)y(\top)$, and hence

$$\begin{aligned} x(\top) &= A_{12}(\top)y(\top) - d(\top)x(\tau(\top)) \\ &\geq A_{12}(\top)y(\top) - d_0 A_{12}(\tau(\top))y(\tau(\top)) \\ &\geq A_{12}(\top)\left(1 - \frac{d_0 A_{12}(\tau(\top))}{A_{12}(\top)}\right)y(\top). \end{aligned} \tag{24}$$

Since $\tau(\top) > \top$ and $A_{12}(\top)$ is decreasing, we obtain

$$A_{12}(\tau(\top)) \leq A_{12}(\top),$$

that is

$$x(\top) \geq A_{12}(\top)(1 - d_0)y(\top). \tag{25}$$

From (23) and (25), we get (20).

(ii) Assume that $y \in N_2$ and that condition (21) holds. By Lemma 2, $y(\top)/B_{12}(\top)$ is nonincreasing. Consequently,

$$y(\tau(\top)) \leq \frac{B_{12}(\tau(\top))}{B_{12}(\top)}y(\top). \tag{26}$$

If (C_1) holds, then

$$x(\top) \geq y(\top) - d_0 y(\tau(\top)).$$

Using (26) yields

$$\begin{aligned} x(\top) &\geq y(\top) - d_0 B_{12}(\tau(\top))\frac{y(\tau(\top))}{B_{12}(\tau(\top))} \\ &\geq \left(1 - \frac{d_0 B_{12}(\tau(\top))}{B_{12}(\top)}\right)y(\top). \end{aligned} \tag{27}$$

According to (21), we obtain

$$1 - d_0 \frac{B_{12}(\tau(\top))}{B_{12}(\top)} > 0.$$

Therefore,

$$E_1(\top) = \left(1 - d_0 \frac{B_{12}(\tau(\top))}{B_{12}(\top)}\right) > 0.$$

Next, suppose that (C_2) holds. Again, from the definition of y, we obtain

$$x(\tau(\top)) \leq A_{12}(\tau(\top))y(\tau(\top)).$$

By (26) and $\tau(\top) > \top$, we have

$$\begin{aligned} x(\top) &\geq A_{12}(\top)y(\top) - d_0 A_{12}(\tau(\top))y(\tau(\top)) \\ &\geq A_{12}(\top)\left(1 - \frac{d_0 A_{12}(\tau(\top))B_{12}(\tau(\top))}{A_{12}(\top)B_{12}(\top)}\right)y(\top) \\ &\geq A_{12}(\top)\left(1 - \frac{d_0 B_{12}(\tau(\top))}{B_{12}(\top)}\right)y(\top). \end{aligned} \tag{28}$$

Using (21), we get

$$E_1(\top) = A_{12}(\top)\left(1 - \frac{d_0 B_{12}(\tau(\top))}{B_{12}(\top)}\right) > 0.$$

Thus, from (27) and (28), we conclude that

$$x(\top) \geq E_1(\top)y(\top).$$

This proves (ii) and completes the proof. □

Remark 2. *Condition (21) ensures $E_1(\top) > 0$ in the lower estimate $x(\top) \geq E_1(\top)y(\top)$ for the class N_2. If $B_{12}(\tau(\top))/B_{12}(\top) \to L < \infty$, the condition holds eventually when $d_0 L < 1$ and fails eventually when $d_0 L > 1$; the borderline case $d_0 L = 1$ cannot be decided from the limit alone. For $\tau(\top) = \lambda\top$, $\lambda > 1$, and $B_{12}(\lambda\top)/B_{12}(\top) \to \lambda^p$, $p > 0$, the inequality $d_0\lambda^p < 1$ is sufficient. In Examples 2 and 3, $d_0 B_{12}(\tau(\top))/B_{12}(\top) \to 1/2$. Failure of this condition means that positivity of this lower bound is no longer guaranteed; it does not imply nonoscillation of the original equation.*

Lemma 4. *Assume that (A_1)–(A_4) and (A_{5a}) hold. Let $x(\top) > 0$ is a solution of (1), and let y be its corresponding canonical function. Then, for all sufficiently large $\top$:*

(i) *If $y \in N_0$, then*

$$\digamma_3 y(\top) + kA_2(\top)\left(\sum_{i=1}^{n} \mu_i(\top)(E(\sigma_i(\top)))^{\alpha}\right) y^{\alpha}(\sigma_+(\top)) \leq 0. \tag{29}$$

(ii) *If $y \in N_2$ and (21) holds eventually, then*

$$\digamma_3 y(\top) + kA_2(\top)\left(\sum_{i=1}^{n} \mu_i(\top)(E_1(\sigma_i(\top)))^{\alpha}\right) y^{\alpha}(\sigma_-(\top)) \leq 0. \tag{30}$$

Proof. By (A_3), Equation (16), and estimate (20), we have, when $y \in N_0$,

$$\digamma_3 y(\top) + kA_2(\top)\sum_{i=1}^{n} \mu_i(\top)E^{\alpha}(\sigma_i(\top))y^{\alpha}(\sigma_i(\top)) \leq 0.$$

Since y is decreasing and $\sigma_i(\top) \le \sigma_+(\top)$, inequality (29) follows.
Similarly, if $y \in N_2$, then (A_3), Equation (16), and estimate (22) give

$$F_3 y(\top) + kA_2(\top) \sum_{i=1}^{n} \mu_i(\top) E_1^{\alpha}(\sigma_i(\top)) y^{\alpha}(\sigma_i(\top)) \le 0.$$

Since y is increasing and $\sigma_i(\top) \ge \sigma_-(\top)$, we obtain (30). This completes the proof. □

3.2. Qualitative Properties for a Delayed Neutral Argument

For a delayed neutral argument, we derive auxiliary estimates and comparison criteria excluding eventually positive solutions whose corresponding canonical functions belong to N_0 or N_2.

Define

$$u(\top) := \frac{x(\top)}{\Omega(\top)}, \quad a(\top) = d(\top)\frac{\Omega(\tau(\top))}{\Omega(\top)},$$

so that

$$y(\top) = u(\top) + a(\top)u(\tau(\top)). \tag{31}$$

Assume that there exists a constant $a_0 \ge 0$ such that

$$0 \le a(\top) \le a_0 < \infty \text{ for all sufficiently large } \top. \tag{32}$$

For $i = 1, \ldots, n$, set

$$\begin{aligned} q_i(\top) &: = kA_2(\top)\mu_i(\top)\Omega^{\alpha}(\sigma_i(\top)), \\ Q_i(\top) &: = \min\{q_i(\top), q_i(\tau(\top))\}, \end{aligned}$$

and define

$$C_{\alpha} := \max\left\{1, 2^{\alpha-1}\right\}, \quad M_{\alpha} := 1 + \frac{a_0^{\alpha}}{\tau_0}.$$

Theorem 2. *Assume that (A_1)–(A_4), (A_{5b}) and (32) hold. Suppose that there exists a continuous function ζ, defined on a sufficiently large half-line, satisfying*

$$\max_{1 \le i \le n} \sigma_i(\top) < \zeta(\top) < \tau(\top), \quad \lim_{\top \to \infty} \zeta(\top) = \infty. \tag{33}$$

Define

$$g(\top) := \tau^{-1}(\zeta(\top)),$$

and

$$\varpi_{\alpha}(\top) := \frac{\sum_{i=1}^{n} Q_i(\top) R^{\alpha}(\zeta(\top), \sigma_i(\top))}{C_{\alpha} M_{\alpha}^{\alpha}}. \tag{34}$$

If the first-order delay differential inequality

$$\omega'(\top) + \varpi_{\alpha}(\top)\omega^{\alpha}(g(\top)) \le 0 \tag{35}$$

has no eventually positive C^1 solution, then Equation (1) admits no eventually positive solution whose corresponding canonical function y belongs to N_0.

Proof. Suppose, to the contrary, that x is an eventually positive solution of (1) and $y \in N_0$. All subsequent estimates hold for sufficiently large arguments. Set

$$V(\top) := F_2 y(\top).$$

Since $y \in N_0$, we have $y > 0$, $F_1 y(\top) < 0, V > 0$, and $V' < 0$. By (A_3) and the canonical equation, we have

$$V'(\top) + \sum_{i=1}^{n} q_i(\top) u^{\alpha}(\sigma_i(\top)) \leq 0. \tag{36}$$

Evaluating (36) at $\tau(\top)$ and using $\tau'(\top) \geq \tau_0$ and $V' < 0$ yields

$$\frac{a_0^{\alpha}}{\tau_0} \frac{\mathrm{d}}{\mathrm{d}\top} V(\tau(\top)) + a_0^{\alpha} \sum_{i=1}^{n} q_i(\tau(\top)) u^{\alpha}(\sigma_i(\tau(\top))) \leq 0. \tag{37}$$

Indeed,

$$\frac{a_0^{\alpha}}{\tau_0} \frac{\mathrm{d}}{\mathrm{d}\top} V(\tau(\top)) = \frac{a_0^{\alpha} \tau'(\top)}{\tau_0} V'(\tau(\top)) \leq a_0^{\alpha} V'(\tau(\top)).$$

Define

$$W(\top) := V(\top) + \frac{a_0^{\alpha}}{\tau_0} V(\tau(\top)) > 0.$$

Combining (36) and (37), and using $\tau(\sigma_i(\top)) = \sigma_i(\tau(\top))$, gives

$$W'(\top) + \sum_{i=1}^{n} Q_i(\top) [u^{\alpha}(\sigma_i(\top)) + a_0^{\alpha} u^{\alpha}(\tau(\sigma_i(\top)))] \leq 0.$$

For $r, s \geq 0$, we have

$$(r+s)^{\alpha} \leq C_{\alpha}(r^{\alpha} + s^{\alpha}). \tag{38}$$

Applying (38) to (31) at $\sigma_i(\top)$ gives

$$\begin{aligned} y^{\alpha}(\sigma_i(\top)) &= [u(\sigma_i(\top)) + a(\sigma_i(\top)) u(\tau(\sigma_i(\top)))]^{\alpha} \\ &\leq C_{\alpha}[u^{\alpha}(\sigma_i(\top)) + a^{\alpha}(\sigma_i(\top)) u^{\alpha}(\tau(\sigma_i(\top)))]. \end{aligned}$$

Consequently, using (32), we obtain

$$W'(\top) + \frac{1}{C_{\alpha}} \sum_{i=1}^{n} Q_i(\top) y^{\alpha}(\sigma_i(\top)) \leq 0. \tag{39}$$

For sufficiently large $v \geq \varrho$ and $s \in [\varrho, v]$, the signs defining N_0 and the monotonicity of V give

$$-F_1 y(s) = -F_1 y(v) + \int_s^v P_2(r) V(r) \mathrm{d}r \geq V(v) \int_s^v P_2(r) \mathrm{d}r.$$

Multiplying by $P_1(s)$ and integrating over $[\varrho, v]$, we obtain

$$y(\varrho) - y(v) = \int_{\varrho}^{v} P_1(s)(-F_1 y(s)) \mathrm{d}s \geq R(v, \varrho) V(v).$$

Since $y(v) > 0$, it follows that

$$y(\varrho) \geq R(v, \varrho) V(v). \tag{40}$$

Taking $\varrho = \sigma_i(\top)$ and $v = \zeta(\top)$, we get

$$y^{\alpha}(\sigma_i(\top)) \geq R^{\alpha}(\zeta(\top), \sigma_i(\top)) V^{\alpha}(\zeta(\top)). \tag{41}$$

Moreover, since V is decreasing and $\tau(s) \leq s$,

$$W(s) = V(s) + \frac{a_0^\alpha}{\tau_0} V(\tau(s)) \leq M_\alpha V(\tau(s)).$$

Taking $s = g(\top)$, we find

$$V^\alpha(\zeta(\top)) \geq M_\alpha^{-\alpha} W^\alpha(g(\top)). \tag{42}$$

Substituting (41) and (42) into (39) yields

$$W'(\top) + \varpi_\alpha(\top) W^\alpha(g(\top)) \leq 0.$$

Thus, W is an eventually positive C^1 solution of (35), which is a contradiction, and the proof is complete. □

Corollary 4. *Assume that* (A_1)–(A_4), (A_{5b}) *and* (32) *hold, and retain the notation of Theorem 2. Suppose that* ζ *satisfies* (33) *and* $0 < \alpha \leq 1$. *If either*

(i) $0 < \alpha < 1$ *and*

$$\int_\top^\infty \varpi_\alpha(\top) \mathrm{d}\top = \infty \tag{43}$$

for some sufficiently large $\top$, *or*

(ii) $\alpha = 1$, *the function* g *is eventually nondecreasing, and*

$$\liminf_{\top \to \infty} \int_{g(\top)}^{\top} \sum_{i=1}^{n} Q_i(s) R(\zeta(s), \sigma_i(s)) ds > \frac{M_1}{e}, \tag{44}$$

then Equation (1) admits no eventually positive solution whose corresponding canonical function belongs to N_0.

Proof. By Theorem 2, it suffices to show that (35) admits no eventually positive solution.

In case (i), suppose that ω is such a solution. Since $\varpi_\alpha \geq 0$ and $g(\top) \to \infty$, ω is eventually nonincreasing. As $g(\top) < \top$, we have $\omega(g(\top)) \geq \omega(\top)$ for all sufficiently large $\top$. Consequently,

$$\left(\omega^{1-\alpha}(\top)\right)' \leq -(1-\alpha)\varpi_\alpha(\top). \tag{45}$$

Integrating (45) from a sufficiently large $\top_1$ to $\top$ yields

$$\omega^{1-\alpha}(\top) \leq \omega^{1-\alpha}(t_1) - (1-\alpha) \int_{\top_1}^{\top} \varpi_\alpha(s) \mathrm{d}s.$$

Since $1 - \alpha > 0$ and the integral diverges by (43), the right-hand side eventually becomes negative, a contradiction.

In case (ii), setting $\alpha = 1$ in (35) gives

$$\omega'(\top) + \varpi_1(\top)\omega(g(\top)) \leq 0. \tag{46}$$

Since $C_1 = 1$ and $M_1 := 1 + \frac{a_0}{\tau_0}$, condition (44) implies

$$\liminf_{\top \to \infty} \int_{g(\top)}^{\top} \varpi_1(s) \mathrm{d}s > \frac{1}{e}.$$

The classical criterion recalled in [28] (Condition (4)) shows that every solution of the corresponding delay equation oscillates. Moreover, $\varpi_1 > 0$ eventually; hence, by [28]

(Lemma 1) the inequality (46) admits no eventually positive solution. This completes the proof. □

Corollary 5. *Assume that* (A_1)–(A_4), (A_{5b}) *and (32) hold, with* $\alpha > 1$, *and retain the notation of Theorem 2. Suppose that* ζ *satisfies (33), is continuously differentiable, and* $\zeta'(\top) > 0$ *eventually. If there exists a continuously differentiable function* φ, *defined on a sufficiently large half-line, such that*

$$\varphi'(\top) > 0, \quad \varphi(\top) \to \infty, \tag{47}$$

$$\limsup_{\top\to\infty} \frac{\alpha\varphi'(g(\top))g'(\top)}{\varphi'(\top)} < 1,$$

and

$$\liminf_{\top\to\infty} \frac{\varpi_\alpha(\top)e^{-\varphi(\top)}}{\varphi'(\top)} > 0,$$

then Equation (1) admits no eventually positive solution whose corresponding canonical function belongs to N_0.

Proof. The assumptions imply that g is continuously differentiable, $g'(\top) > 0, g(\top) < \top$, and $g(\top) \to \infty$. By [28] (Theorem 1), every solution of

$$\omega'(\top) + \varpi_\alpha(\top)\omega^\alpha(g(\top)) = 0$$

oscillates. Since $\varpi_\alpha(\top) > 0$ eventually, the nonvanishing condition of [28] (Lemma 1) is satisfied. Hence, the corresponding differential inequality (35) admits no eventually positive C^1 solution. The proof is complete. □

Theorem 3. *Assume that* (A_1)–(A_4), (A_{5b}) *and (32) hold. Suppose that there exists a continuous function* η, *defined on a sufficiently large half-line, such that*

$$\eta(\top) < \top, \quad \lim_{\top\to\infty} \eta(\top) = \infty, \quad \max_{1\le i\le n} \sigma_i(\top) < \tau(\eta(\top))$$

for all sufficiently large $\top$. *Define*

$$\mathcal{L}_\alpha(\top) := \sum_{i=1}^{n} R^\alpha(\tau(\eta(\top)), \sigma_i(\top)) \int_{\eta(\top)}^{\top} Q_i(s)\,\mathrm{d}s.$$

$$\mathcal{L}_\alpha(\top) := \sum_{i=1}^{n} R^\alpha(\tau(\eta(\top)), \sigma_i(\top)) \int_{\eta(\top)}^{\top} Q_i(s)\,\mathrm{d}s.$$

If either

$$0 < \alpha < 1, \quad \limsup_{\top\to\infty} \mathcal{L}_\alpha(\top) = \infty, \tag{48}$$

or

$$\alpha = 1, \quad \limsup_{\top\to\infty} \mathcal{L}_1(\top) > 1 + \frac{a_0}{\tau_0}, \tag{49}$$

then Equation (1) admits no eventually positive solution whose corresponding canonical function belongs to N_0.

Proof. Suppose, to the contrary, that x is an eventually positive solution of (1) whose corresponding canonical function y belongs to N_0. Set

$$V(\top) := \digamma_2 y(\top), \quad W(\top) := V(\top) + \frac{a_0^\alpha}{\tau_0} V(\tau(\top)).$$

Then $V > 0, V' < 0$, and $W > 0$. By (39) in the proof of Theorem 2, with $C_\alpha = 1$, we obtain

$$W'(\top) + \sum_{i=1}^{n} Q_i(\top) y^\alpha(\sigma_i(\top)) \le 0. \tag{50}$$

Integrating (50) from $\eta(\top)$ to $\top$ yields

$$W(\eta(\top)) \ge W(\top) + \sum_{i=1}^{n} \int_{\eta(\top)}^{\top} Q_i(s) y^\alpha(\sigma_i(s)) \mathrm{d}s.$$

Since each σ_i is increasing and y is decreasing, it follows that

$$W(\eta(\top)) \ge \sum_{i=1}^{n} y^\alpha(\sigma_i(\top)) \int_{\eta(\top)}^{\top} Q_i(s) \mathrm{d}s. \tag{51}$$

Moreover, $\tau(\eta(\top)) \le \eta(\top)$ and the monotonicity of V imply

$$W(\eta(\top)) \le M_\alpha V(\tau(\eta(\top))). \tag{52}$$

Applying (40) with $\varrho = \sigma_i(\top),\ v = \tau(\eta(\top))$ gives

$$y(\sigma_i(\top)) \ge R(\tau(\eta(\top)), \sigma_i(\top)) V(\tau(\eta(\top))). \tag{53}$$

Combining (51)–(53), we obtain

$$\mathcal{L}_\alpha(\top) \le M_\alpha V^{1-\alpha}(\tau(\eta(\top))). \tag{54}$$

If $0 < \alpha < 1$, choose $\top$ sufficiently large that V is positive and decreasing on $[\top, \infty)$. Since $\tau(\eta(\top)) \to \infty$, (54) implies, eventually,

$$\mathcal{L}_\alpha(\top) \le M_\alpha V^{1-\alpha}(\top) < \infty,$$

contrary to condition (48).

If $\alpha = 1$, inequality (54) reduces to

$$\mathcal{L}_1(\top) \le M_1 = 1 + \frac{a_0}{\tau_0},$$

contrary to condition (49). This completes the proof. □

Corollary 6. *Assume that* (A_1)–(A_4), (A_{5b}) *and* (32) *hold, with* $0 < \alpha \le 1$. *Suppose that*

$$\tau(\top) < \top, \quad \max_{1 \le i \le n} \sigma_i(\top) < \tau(\tau(\top)).$$

Let $\mathcal{L}_\alpha$ *be as defined in Theorem* 3, *with* $\eta = \tau$. *If either*

$$0 < \alpha < 1, \quad \limsup_{\top \to \infty} \mathcal{L}_\alpha(\top) = \infty,$$

or

$$\alpha = 1, \quad \limsup_{\top \to \infty} \mathcal{L}_1(\top) > 1 + \frac{a_0}{\tau_0}, \tag{55}$$

then Equation (1) *admits no eventually positive solution whose corresponding canonical function belongs to* N_0.

For the following results, let $\alpha = 1$ and set

$$Q(\top) := \sum_{i=1}^{n} Q_i(\top), \quad M := M_1 = 1 + \frac{a_0}{\tau_0}, \quad B_2(v, \top) := \int_{\top}^{v} P_2(r)\mathrm{d}r,$$

$v \geq \top$. In particular, $B_2(v) = B_2(v, \top_*)$.

The following result adapts the Riccati approach in [29] (Theorem 1) to Equation (1) with multiple delayed arguments.

Theorem 4 ([29] (Theorem 1)). *Assume that* (A_1)–(A_4), (A_{5b}) *and (32) hold, with* $b_1 \equiv 1$ *and* $\alpha = 1$. *Suppose that there exists a continuously differentiable function* ξ, *defined on a sufficiently large half-line, satisfying*

$$\xi'(\top) > 0, \xi(\top) \leq \min\{\tau(\top), \sigma_1(\top), \ldots, \sigma_n(\top)\}, \quad \lim_{\top \to \infty} \xi(\top) = \infty$$

for all sufficiently large $\top$. *If there exists a positive continuously differentiable function* ρ, *defined on a sufficiently large half-line, such that, for every sufficiently large* $\top$, *there exists* $\top_2 > \top$ *with* $\xi(s) > \top$ *for all* $s \geq \top_2$ *and*

$$\limsup_{\top \to \infty} \int_{\top_2}^{\top} \left[\rho(s)Q(s) - \frac{M\Omega^2(\xi(s))\left(\rho'(s)_+\right)^2}{4\rho(s)A_2(\xi(s))B_2(\xi(s), \top)\xi'(s)} \right] \mathrm{d}s = \infty, \tag{56}$$

then Equation (1) admits no eventually positive solution whose corresponding canonical function belongs to N_2.

Here, $\rho'(s)_+ := \max\{0, \rho'(s)\}$.

Proof. Suppose, to the contrary, that (1) admits an eventually positive solution whose corresponding canonical function $y \in N_2$. Set

$$V(\top) := F_2 y(\top), \quad W(\top) := V(\top) + \frac{a_0}{\tau_0} V(\tau(\top)).$$

Then V is positive and decreasing, and W is positive. The estimate (39) in the proof of Theorem 2 remains valid for $y \in N_2$. Setting $\alpha = 1$ and $C_1 = 1$, we obtain

$$W'(\top) + \sum_{i=1}^{n} Q_i(\top) y(\sigma_i(\top)) \leq 0.$$

Since y is increasing and $\xi(\top) \leq \sigma_i(\top)$, it follows that

$$W'(\top) + Q(\top) y(\xi(\top)) \leq 0. \tag{57}$$

Choose $\top$ sufficiently large so that the signs defining N_2 hold on $[\top, \infty)$. Since V is decreasing,

$$F_1 y(v) \geq B_2(v, \top) V(v), \; v \geq \top. \tag{58}$$

Moreover, $\xi(\top) \leq \tau(\top) \leq \top$ implies

$$W(\top) \leq MV(\tau(\top)) \leq MV(\xi(\top)). \tag{59}$$

Choose $\top_2$ as in the theorem, increasing it if necessary so that all subsequent estimates hold for $\top \geq \top_2$. Since $b_1 \equiv 1$, inequalities (58) and (59) yield

$$y'(\xi(\top))\xi'(\top) \geq \frac{A_2(\xi(\top))B_2(\xi(\top), \top)\xi'(\top)}{M\Omega^2(\xi(\top))} W(\top). \tag{60}$$

Define

$$Y(\top) := \frac{\rho(\top)W(\top)}{y(\xi(\top))} > 0.$$

By (57) and (60), we get

$$Y'(\top) \leq -\rho(\top)Q(\top) + \frac{\rho'(\top)}{\rho(\top)}Y(\top) - \frac{A_2(\xi(\top))B_2(\xi(\top),\top)\xi'(\top)}{M\rho(\top)\Omega^2(\xi(\top))}Y^2(\top).$$

Using $Y > 0$ and completing the square yields

$$Y'(\top) \leq -\rho(\top)Q(\top) + \frac{M\Omega^2(\xi(\top))\left(\rho'(\top)_+\right)^2}{4\rho(\top)A_2(\xi(\top))B_2(\xi(\top),\top)\xi'(\top)}. \tag{61}$$

Integrating (61) from $\top_2$ to $\top$ gives

$$\int_{\top_2}^{\top}\left(\rho(s)Q(s) - \frac{M\Omega^2(\xi(s))\left(\rho'(s)_+\right)^2}{4\rho(s)A_2(\xi(s))B_2(\xi(s),\top)\xi'(s)}\right)\mathrm{d}s \leq Y(\top_2),$$

contradicting (56). This completes the proof. □

3.3. Oscillation Criteria for an Advanced Neutral Argument

Using the preceding estimates, we establish sufficient conditions for the oscillation of all solutions of Equation (1) when the neutral argument is advanced. For a function ρ satisfying the requirements specified below, set

$$\begin{aligned}\Psi_1(\top) &:= P_1(\top)\int_{\top}^{\rho(\top)} P_2(s)\int_s^{\rho(s)} A_2(u)\sum_{i=1}^{n}\mu_i(u)E^{\alpha}(\sigma_i(u))\mathrm{d}u\mathrm{d}s,\\ \Psi_2(\top) &:= A_2(\top)B_{12}^{\alpha}(\sigma_-(\top))\sum_{i=1}^{n}\mu_i(\top)E_1^{\alpha}(\sigma_i(\top)).\end{aligned}$$

For $\eta(\top) = \sigma_+(\rho(\rho(\top)))$, define

$$\begin{aligned}m_1(\top) &:= k\int_{\eta(\top)}^{\top}\Psi_1(s)\mathrm{d}s,\\ m_2(\top) &:= k\int_{\sigma_-(\top)}^{\top}\Psi_2(s)\mathrm{d}s.\end{aligned}$$

Theorem 5. *Assume that* (A_1)–(A_4), (A_{5a}) *hold, and that condition (21) is satisfied eventually. Suppose that there exists a nondecreasing function* $\rho \in C^1(J_0,\mathbb{R})$ *such that*

$$\rho(\top) > \top,\ \eta(\top) < \top \tag{62}$$

for all sufficiently large $\top$. *If the differential equations*

$$H'(\top) + k\Psi_1(\top)H^{\alpha}(\eta(\top)) = 0, \tag{63}$$

and

$$V'(\top) + k\Psi_2(\top)V^{\alpha}(\sigma_-(\top)) = 0 \tag{64}$$

are oscillatory, then every solution of Equation (1) oscillates.

Proof. Suppose, to the contrary, that $x(\top)$ is a nonoscillatory solution of (1). By the oddness of h, we may assume that x is eventually positive. The corresponding canonical function

y satisfies the transformed equation and, by Lemma 1, belongs to either N_0 or N_2. All subsequent inequalities hold for sufficiently large arguments.

First, assume that $y \in N_0$. By (29), we have

$$F_3 y(\top) \leq -kA_2(\top) y^{\alpha}(\sigma_+(\top)) \sum_{i=1}^{n} \mu_i(\top) E^{\alpha}(\sigma_i(\top)). \tag{65}$$

Integrating (65) over $[s,\rho(s)]$, and using $F_2 y(\rho(s)) > 0$ together with the monotonicity of y and σ_+, yields

$$F_2 y(s) \geq k y^{\alpha}(\sigma_+(\rho(s))) \int_s^{\rho(s)} A_2(u) \sum_{i=1}^{n} \mu_i(u) E^{\alpha}(\sigma_i(u)) \mathrm{d}u.$$

Moreover, because $(F_1 y)' = P_2 F_2 y$ and $F_1 y < 0$,

$$-F_1 y(\top) \geq \int_{\top}^{\rho(\top)} P_2(s) F_2 y(s) \mathrm{d}s.$$

For $\top \leq s \leq \rho(\top)$, the monotonicity of ρ and σ_+ implies

$$\sigma_+(\rho(s)) \leq \sigma_+(\rho(\rho(\top))) = \eta(\top).$$

Consequently, the monotonicity of y and the definition of Ψ_1 give

$$-y'(\top) = P_1(\top)(-F_1 y(\top)) \geq k\Psi_1(\top) y^{\alpha}(\eta(\top)). \tag{66}$$

Integrating (66) from $\top$ to $v > \top$, we find

$$y(\top) \geq y(v) + k \int_{\top}^{v} \Psi_1(s) y^{\alpha}(\eta(s)) \mathrm{d}s.$$

Since y is positive and decreasing, the preceding inequality implies that the improper integral defining H converges. Define

$$H(\top) := k \int_{\top}^{\infty} \Psi_1(s) y^{\alpha}(\eta(s)) \mathrm{d}s.$$

Then, $0 < H(\top) \leq y(\top)$. Since $\eta(\top) \to \infty$, we obtain, for all sufficiently large $\top$,

$$H'(\top) = -k\Psi_1(\top) y^{\alpha}(\eta(\top)) \leq -k\Psi_1(\top) H^{\alpha}(\eta(\top)).$$

It follows that

$$H'(\top) + k\Psi_1(\top) H^{\alpha}(\eta(\top)) \leq 0.$$

By the comparison result in [30] (Theorem 1), Equation (63) admits an eventually positive solution, contradicting its assumed oscillation.

Next, assume that $y(\top) \in N_2$. By (30), we have

$$F_3 y(\top) \leq -kA_2(\top) y^{\alpha}(\sigma_-(\top)) \sum_{i=1}^{n} \mu_i(\top) E_1^{\alpha}(\sigma_i(\top)). \tag{67}$$

Lemma 2-(c) gives

$$y(\sigma_-(\top)) \geq B_{12}(\sigma_-(\top)) F_2 y(\sigma_-(\top)). \tag{68}$$

Therefore, from (67), (68), and the definition of Ψ_2, we have

$$(F_2 y(\top))' + k\Psi_2(\top)(F_2 y(\sigma_-(\top)))^{\alpha} \leq 0. \tag{69}$$

Set $V(\top) := F_2 y(\top) > 0$. Then inequality (69) becomes

$$V'(\top) + k\Psi_2(\top)V^{\alpha}(\sigma_-(\top)) \leq 0.$$

By the comparison result in [30] (Theorem 1), Equation (64) admits an eventually positive solution, contradicting its assumed oscillation. This completes the proof. □

Remark 3. *A practical choice of ρ can be obtained by testing a simple parameterized family satisfying $\rho(\top) > \top$ and $\sigma_+(\rho(\rho(\top))) < \top$ eventually. For proportional delays $\sigma_+(\top) = c\top$, $0 < c < 1$, the choice $\rho(\top) = r\top$ is admissible whenever $1 < r < 1/\sqrt{c}$. For constant delays $\sigma_+(\top) = \top - \delta$, $\delta > 0$, one may take $\rho(\top) = \top + a$ with $0 < a < \delta/2$. Both choices are continuously differentiable and increasing. Within either family, increasing the parameter enlarges the integration intervals defining Ψ_1, but also increases η and shortens the interval defining m_1. Thus, admissible parameters should be compared using the resulting oscillation condition. For $\alpha = 1$, as in Corollary 7, one may compare $\liminf_{\top\to\infty} m_1(\top)$ over the admissible parameter range. No universally optimal choice of ρ is asserted.*

Corollary 7. *Assume the hypotheses of Theorem 5 and retain its notation, with $\alpha = 1$. If*

$$\liminf_{\top\to\infty} m_1(\top) > \frac{1}{e} \tag{70}$$

and

$$\liminf_{\top\to\infty} m_2(\top) > \frac{1}{e}, \tag{71}$$

then Equation (1) is oscillatory.

Proof. Applying [31] (Theorem 1) under Conditions (70) and (71) shows that Equations (63) and (64) are oscillatory. Therefore, the result now follows immediately from Theorem 5. This completes the proof. □

Corollary 8. *Assume that (A_1)–(A_4), (A_{5a}) hold, and that (21) is satisfied eventually. Suppose that either:*

(i) $0 < \alpha < 1$, *condition (62) holds and, for some sufficiently large $\top$,*

$$\int_{\top}^{\infty} \Psi_1(s)\mathrm{d}s = \infty \text{ and } \int_{\top}^{\infty} \Psi_2(s)\mathrm{d}s = \infty; \tag{72}$$

or

(ii) $\alpha > 1$, *and, eventually,*

$$\sigma_-(\top) = \theta_-\top, \ \sigma_+(\top) = \theta_+\top, \ \rho(\top) = \theta_3\top,$$

where $0 < \theta_- \leq \theta_+ < 1$ and $\theta_3 > 1$. Set $\theta_2 = \theta_+\theta_3^2 < 1$, so that $\eta(\top) = \theta_2\top$. Suppose that there exist $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \ln(1/\theta_2) > \ln\alpha, \text{ and } \lambda_2 \ln(1/\theta_-) > \ln\alpha,$$

and

$$\liminf_{\top\to\infty}\left[\Psi_1(\top)\exp\left(-\top^{\lambda_1}\right)\right] > 0, \tag{73}$$

$$\liminf_{\top\to\infty}\left[\Psi_2(\top)\exp\left(-\top^{\lambda_2}\right)\right] > 0. \tag{74}$$

Then, every solution of (1) is oscillatory.

Proof. We show that both comparison Equations (63) and (64) are oscillatory.

In case (i), suppose that either equation admits an eventually positive solution ω. Since its coefficient is eventually positive, ω is eventually decreasing. Its delayed argument tends to infinity; hence, for all sufficiently large $\top$,

$$\left(\omega^{1-\alpha}(\top)\right)' \leq -(1-\alpha)k\Psi_j(\top),$$

where $j = 1$ or 2, respectively. Integration contradicts (72). By symmetry, eventually negative solutions are also excluded.

In case (ii), suppose that $\alpha > 1$. By applying [28] (Theorem 4) under the assumptions in (73) and (74), we conclude that Equations (63) and (64) are oscillatory. This completes the proof. □

Corollary 9. *Assume that* (A_1)–(A_4), (A_{5a}) *hold, with* $\alpha > 1$, *and that (21) is satisfied eventually. Retain the preceding notation and suppose that, for all sufficiently large* $\top$,

$$\sigma_-(\top) = \top - \delta_-, \ \ \sigma_+(\top) = \top - \delta_+,$$

where $0 < \delta_+ \leq \delta_-$. *Let*

$$\rho(\top) = \top + \theta_3, \ 0 < 2\theta_3 < \delta_+,$$

and set $\theta_2 := \delta_+ - 2\theta_3$, *so that* $\eta(\top) = \top - \theta_2$. *If there exist constants* $\lambda_1, \lambda_2 > 0$ *satisfying*

$$\lambda_1\theta_2 > \ln\alpha, \ \lambda_2\delta_- > \ln\alpha,$$

and

$$\liminf_{\top\to\infty} \frac{\Psi_1(\top)}{\exp\left(e^{\lambda_1\top}\right)} > 0, \tag{75}$$

$$\liminf_{\top\to\infty} \frac{\Psi_2(\top)}{\exp\left(e^{\lambda_2\top}\right)} > 0, \tag{76}$$

then every solution of Equation (1) is oscillatory.

Proof. Applying [28] (Theorem 3) under conditions (75) and (76) yields that Equations (63) and (64) are oscillatory. Therefore, Theorem 5 implies that Equation (1) is oscillatory. This completes the proof. □

Theorem 6. *Assume that* (A_1)–(A_4), (A_{5a}). *Suppose that (21) holds. If* $\alpha < 1$, *and*

$$\min\{G_1, G_2\} = \infty, \tag{77}$$

where

$$G_1 = \limsup_{\top\to\infty} \int_{\sigma_+(\top)}^{\top} A_2(s)R^{\alpha}(\sigma_+(\top), \sigma_+(s)) \sum_{i=1}^{n} \mu_i(s)E^{\alpha}(\sigma_i(s))\mathrm{d}s,$$

and

$$G_2 = \limsup_{\top\to\infty} B_{12}^{\alpha}(\sigma_-(\top)) \int_{\top}^{\infty} A_2(s) \sum_{i=1}^{n} \mu_i(s)E_1(\sigma_i^{\alpha}(s))\mathrm{d}s,$$

then Equation (1) is oscillatory.

Proof. Assume, to the contrary, that Equation (1) has a positive solution $x(\top)$. Then, there exists $\top_1 \geq \top_0$ such that

$$x(\top) > 0, x(\tau(\top)) > 0, x(\sigma_i(\top)) > 0 \text{ for all } \top \geq \top_1.$$

By Corollary 2, $x(\top) > 0$ is a nonoscillatory solution of (16). Consequently, from Lemma 1, we see that $y(\top)$ belongs to either N_0 or N_2. First, suppose that $y \in N_0$. Then $y(\top)$ satisfies (29). Since $(F_2 y(\top))' < 0$, for $j \geq u$, we obtain

$$\begin{aligned} -F_1 y(u) &\geq \int_u^j \frac{(F_1 y(s))'}{p_2(s)} p_2(s) \mathrm{d}s \\ &\geq F_2 y(j) \int_u^j P_2(s) \mathrm{d}s. \end{aligned}$$

Moreover,

$$y(u) \geq F_2 y(j) \int_u^j P_1(s) \int_s^j P_2(s_1) \mathrm{d}s_1 \mathrm{d}s = R(j,u) F_2 y(j). \tag{78}$$

Setting $u = \sigma_+(s)$ and $j = \sigma_+(\top)$ in (78), we obtain

$$y(\sigma_+(s)) \geq R(\sigma_+(\top), \sigma_+(s)) F_2 y(\sigma_+(\top)) \tag{79}$$

for $s \in [\sigma_+(\top), \top]$. Integrating (29) over the interval $[\sigma_+(\top), \top]$ and using (79), we get

$$\begin{aligned} F_2 y(\sigma(\top)) &\geq k \left(\int_{\sigma(\top)}^{\top} A_2(s) \sum_{i=1}^{n} \mu_i(s) y^{\alpha}(\sigma_i(s)) E^{\alpha}(\sigma_i(s)) \mathrm{d}s \right) \\ &\geq k (F_2 y(\sigma_+(\top)))^{\alpha} \int_{\sigma_+(\top)}^{\top} A_2(s) R^{\alpha}(\sigma_+(\top), \sigma_+(s)) \sum_{i=1}^{n} \mu_i(s) E^{\alpha}(\sigma_i(s)) \mathrm{d}s. \end{aligned}$$

That is

$$(F_2 y(\sigma_+(\top)))^{1-\alpha} \geq k \int_{\sigma(\top)}^{\top} A_2(s) R^{\alpha}(\sigma_+(\top), \sigma_+(s)) \sum_{i=1}^{n} \mu_i(s) E^{\alpha}(\sigma_i(s)) \mathrm{d}s. \tag{80}$$

This contradicts condition (77).

Next, suppose that $y \in N_2$. Then $y(\top)$ satisfies (30). Since $F_2 y(\top)$ is decreasing, (c) in Lemma 2 gives

$$\begin{aligned} y(\sigma_-(s)) &\geq y(\sigma_-(\top)) \geq B_{12}(\sigma_-(\top)) F_2 y(\sigma_-(\top)) \\ &\geq B_{12}(\sigma_-(\top)) F_2 y(\top). \end{aligned}$$

Integrating (30) from $\top$ to ∞, we obtain

$$\begin{aligned} F_2 y(\top) &\geq k \int_{\top}^{\infty} A_2(s) \sum_{i=1}^{n} \mu_i(s) y^{\alpha}(\sigma_i(s)) E_1^{\alpha}(\sigma_i(s)) \mathrm{d}s \\ &\geq k B_{12}^{\alpha}(\sigma_-(\top)) (F_2 y(\sigma_-(\top)))^{\alpha} \int_{\top}^{\infty} A_2(s) \sum_{i=1}^{n} \mu_i(s) E_1^{\alpha}(\sigma_i(s)) \mathrm{d}s. \end{aligned}$$

Since $F_2 y(\top) > 0$, we have

$$(F_2 y(\top))^{1-\alpha} \geq k B_{12}^{\alpha}(\sigma_-(\top)) \int_{\top}^{\infty} A_2(s) \sum_{i=1}^{n} \mu_i(s) E_1^{\alpha}(\sigma_i(s)) \mathrm{d}s. \tag{81}$$

This again contradicts condition (77). This completes the proof. □

Theorem 7. *Assume that* (A_1)–(A_4) *and* (A_{5a}) *hold, with* $\alpha = 1$, *and that (21) is satisfied eventually. If*

$$\min\{G_3, G_4\} > \frac{1}{k}, \tag{82}$$

where

$$G_3 = \limsup_{\top\to\infty} \int_{\sigma_+(\top)}^{\top} A_2(s)R(\sigma_+(\top),\sigma_+(s))\sum_{i=1}^{n}\mu_i(s)E(\sigma_i(s))\mathrm{d}s$$

and

$$G_4 = \limsup_{\top\to\infty} B_{12}(\sigma_-(\top))\int_{\top}^{\infty} A_2(s)\sum_{i=1}^{n}\mu_i(s)E_1(\sigma_i(s))\mathrm{d}s,$$

then Equation (1) is oscillatory.

Proof. Assume, to the contrary, that Equation (1) has a positive nonoscillatory solution $x(\top)$. Then, there exists $\top_1 \geq t_0$ such that

$$x(\top) > 0, x(\tau(\top)) > 0, x(\sigma_i(\top)) > 0 \text{ for all } \top \geq \top_1.$$

In view of Corollary 2, $x(\top) > 0$ is a nonoscillatory solution of (16). Hence, Lemma 1 shows that $y(\top)$ belongs to either N_0 or N_2.

Proceeding as in the proof of Theorem 6, when $y(\top) \in N_0$, we obtain

$$1 \geq k\int_{\sigma(\top)}^{\top} A_2(s)R(\sigma_+(\top),\sigma_+(s))\sum_{i=1}^{n}\mu_i(s)E(\sigma_i(s))\mathrm{d}s, \tag{83}$$

and when $y(\top) \in N_2$, we obtain

$$1 \geq kB_{12}(\sigma_-(\top))\int_{\top}^{\infty} A_2(s)\sum_{i=1}^{n}\mu_i(s)E_1(\sigma_i(s))\mathrm{d}s. \tag{84}$$

Taking the upper limit yields $G_3 \leq \frac{1}{k}$ or $G_4 \leq \frac{1}{k}$, respectively. Both alternatives contradict condition (82). This completes the proof. □

3.4. Oscillation Criteria for a Delayed Neutral Argument

We derive criteria excluding eventually positive solutions of Equation (1) whose corresponding canonical functions y belong to N_0 or N_2. In the linear case, combining Theorem 4 with Corollary 4-(ii) and Corollary 6, respectively, yields the oscillation criteria.

Theorem 8. *Assume that* (A_1)–(A_4), (A_{5b}), *and (32) hold, with* $b_1 \equiv 1$ *and* $\alpha = 1$. *Let* ξ *and* ρ *satisfy the conditions of Theorem 4, including (56). Suppose that there exists a continuous function* ζ, *defined on a sufficiently large half-line, satisfying*

$$\max_{1\leq i\leq n}\sigma_i(\top) < \zeta(\top) < \tau(\top),\ \zeta(\top)\to\infty.$$

If the corresponding function g *is eventually nondecreasing and (44) holds, then every solution of Equation (1) oscillates.*

Theorem 9. *Assume that* (A_1)–(A_4),(A_{5b}), *and (32) hold, with* $b_1 \equiv 1$ *and* $\alpha = 1$. *Let* ξ *and* ρ *satisfy the conditions of Theorem 4, including (56). Suppose that*

$$\max_{1\leq i\leq n}\sigma_i(\top) < \tau(\tau(\top)),\ \tau(\top) < \top$$

for all sufficiently large $\top$. *If (55) holds, then every solution of Equation (1) oscillates.*

Remark 4. *The established criteria provide sufficient conditions for oscillation. The resulting thresholds are not claimed to be optimal, since sharpness of an auxiliary comparison criterion need not be preserved through the intermediate estimates.*

4. Examples

We present three examples illustrating the established oscillation criteria. The first treats proportional arguments in both the delayed and advanced neutral settings, while the second considers noncommuting advanced neutral and delayed arguments; both fall under (C_1). The third treats a nonlinear equation with multiple delayed arguments under (C_2). Thus, both canonical transformations in Theorem 1 are illustrated.

Example 1. *Consider the third-order neutral differential equation*

$$\left[\top^2(x(\top)+dx(\tau\top))''\right]'+\frac{\mu_0}{\top}\sum_{i=1}^{n}x(\sigma_i\top)=0,\ \ \top\geq 1, \tag{85}$$

where $\mu_0>0, 0\leq d<1, \tau>0$, *and* $0<\sigma_i<1$ *for* $i=1,\ldots,n$. *Here,* τ *and* σ_i *are constants, and the corresponding argument functions are* $\tau(\top)=\tau\top$ *and* $\sigma_i(\top)=\sigma_i\top$. *Set*

$$\sigma_-:=\min_{1\leq i\leq n}\sigma_i\ ,\sigma_+:=\max_{1\leq i\leq n}\sigma_i.$$

For Equation (85),

$$b_1(\top)=1,\ b_2(\top)=\top^2,\ \mu_i(\top)=\frac{\mu_0}{\top},\ h(u)=u,$$

with $k=\alpha=1$. *Direct calculation gives*

$$A_2(\top)=\frac{1}{\top},\ A_{12}(\top)=\infty,\ \Omega(\top)=1.$$

Thus, $y(\top)=x(\top)+dx(\tau\top)$, *and the canonical equation is*

$$(\top y'(\top))''+\frac{\mu_0}{\top^2}\sum_{i=1}^{n}x(\sigma_i\top)=0.$$

Moreover,

$$P_1(\top)=\frac{1}{\top},\ P_2(\top)=1,\ B_2(v,\top)=v-T,$$

and

$$R(v,u)=v\ln\frac{v}{u}-v+u,\ \ v\geq u.$$

We consider the delayed and advanced neutral arguments separately.

(a) *Delayed neutral argument:* $0<\tau<1$. *The argument functions commute, and (32) holds with* $a_0=d$. *Taking* $\tau_0=\tau$, *we obtain*

$$M=1+\frac{d}{\tau},\ Q_i(\top)=\frac{\mu_0}{\top^2},\ Q(\top)=\frac{n\mu_0}{\top^2},$$

To verify condition (56), choose

$$\xi(\top)=\sigma_-\top,\ \ \rho(\top)=\top.$$

Under either of the argument restrictions imposed below, $\sigma_- < \tau$, so ξ satisfies the required argument conditions. For sufficiently large $\top$, choose $\top_2 > \top/\sigma_-$. Then,

$$\begin{aligned}
&\int_{\top_2}^{\top}\left[\rho(s)Q(s) - \frac{M\Omega^2(\xi(s))\left(\rho'(s)_+\right)^2}{4\rho(s)A_2(\xi(s))B_2(\xi(s),\top)\xi'(s)}\right]\mathrm{d}s\\
=\ &\int_{\top_2}^{\top}\left[\frac{n\mu_0}{s} - \frac{M}{4\sigma_-(\sigma_- s - \top)}\right]\mathrm{d}s\\
=\ &\left(n\mu_0 - \frac{M}{4\sigma_-^2}\right)\ln\top + O(1).
\end{aligned}$$

Consequently, condition (56) holds whenever

$$\mu_0 > \frac{M}{4n\sigma_-^2} = \frac{\tau + d}{4n\tau\sigma_-^2}.$$

Define

$$F(v) = \sum_{i=1}^{n}(v\ln(v/\sigma_i) - v + \sigma_i),\ \ v > \sigma_+.$$

First, assume that $\sigma_+ < \tau$, and set

$$c := \frac{\sigma_+ + \tau}{2},\ \ \zeta(\top) = ct,\ \ g(\top) = \frac{c}{\tau}\top.$$

Then $\sigma_i\top < \zeta(\top) < \tau\top$, and g is increasing. Furthermore,

$$\int_{g(\top)}^{\top}\sum_{i=1}^{n}Q_i(s)R(\zeta(s),\sigma_i(s))\mathrm{d}s = \mu_0 F(c)\ln\frac{\tau}{c}.$$

Therefore, Theorem 8 implies that every solution of (85) oscillates if

$$\mu_0 > \max\left\{\frac{\tau + d}{4n\tau\sigma_-^2}, \frac{\tau + d}{e\tau F(c)\ln(\tau/c)}\right\}.$$

Alternatively, assume that $\sigma_+ < \tau^2$. Since $\tau(\tau(\top)) = \tau^2\top$, we have

$$\sum_{i=1}^{n}R(\tau(\tau(\top)),\sigma_i(\top))\int_{\tau\top}^{\top}Q_i(s)\mathrm{d}s = \mu_0\frac{1-\tau}{\tau}F\left(\tau^2\right).$$

Hence, Theorem 9 implies that every solution of (85) oscillates if

$$\mu_0 > \max\left\{\frac{\tau + d}{4n\tau\sigma_-^2}, \frac{\tau + d}{(1-\tau)F(\tau^2)}\right\}.$$

(*b*) *Advanced neutral argument. Retain the same coefficients and let $\tau(\top) = \tau\top$, where $\tau > 1$ and $d\tau < 1$. Since $\Omega(\top) = 1$, we have $E(\top) = 1 - d$. For a fixed sufficiently large $\top_* > 0$,*

$$B_{12}(\top) = \int_{\top_*}^{\top}\frac{(s - \top_*)}{s}\mathrm{d}s = \top - \top_* - \top_*\ln(\top/\top_*),\ \ \top \geq \top_*.$$

Thus, $\frac{B_{12}(\top)}{\top} \to 1$, and consequently,

$$\frac{dB_{12}(\tau\top)}{B_{12}(\top)} \to d\tau < 1.$$

Hence, condition (21) holds eventually, and

$$E_1(\top) = 1 - d\frac{B_{12}(\tau\top)}{B_{12}(\top)} \to 1 - d\tau > 0.$$

Choose a constant r satisfying

$$1 < r < \frac{1}{\sqrt{\sigma_+}},$$

and set $\rho(\top) = rt$. *Then* ρ *is continuously differentiable and increasing, with* $\rho(\top) > \top$, *and*

$$\eta(\top) = \sigma_+(\rho(\rho(\top))) = \sigma_+ r^2 \top < \top.$$

Therefore, condition (62) is satisfied. Substituting the coefficients of Equation (85) into the previously defined functions Ψ_1 *and* Ψ_2, *with* $\alpha = k = 1$, *gives*

$$\begin{aligned}\Psi_1(\top) &= \frac{n\mu_0(1-d)}{\top}\int_{\top}^{r\top}\int_{s}^{rs} u^{-2}\mathrm{d}u\mathrm{d}s \\ &= \frac{n\mu_0(1-d)(1-1/r)\ln(r)}{\top},\end{aligned}$$

and

$$\Psi_2(\top) = \frac{\mu_0 B_{12}(\sigma_-\top)}{\top^2}\sum_{i=1}^{n} E_1(\sigma_i\top).$$

It follows that

$$\begin{aligned}m_1(\top) &= \int_{\eta(\top)}^{\top}\Psi_1(s)\mathrm{d}s \\ &= n\mu_0(1-d)(1-1/r)\ln(r)\ln\Big(1/\Big(\sigma_+ r^2\Big)\Big).\end{aligned}$$

Moreover, since $\frac{B_{12}(\sigma_-\top)}{\top} \to \sigma_-$ *and* $E_1(\sigma_i\top) \to 1 - d\tau$ *for every i,*

$$t\Psi_2(\top) \to n\mu_0\sigma_-(1-d\tau).$$

Consequently,

$$m_2(\top) = \int_{\sigma_-\top}^{\top}\frac{s\Psi_2(s)}{s}\mathrm{d}s \to n\mu_0\sigma_-(1-d\tau)\ln(1/\sigma_-).$$

All factors in these limiting expressions are positive. Therefore, Corollary 7 implies that every solution of (85) oscillates if

$$\mu_0 > \max\left\{\frac{1}{en(1-d)(1-1/r)\ln(r)\ln(1/(\sigma_+ r^2))}, \frac{1}{en\sigma_-(1-d\tau)\ln(1/\sigma_-)}\right\},$$

for some $r \in \left(1, \frac{1}{\sqrt{\sigma_+}}\right)$.

Example 2. *Consider the third-order neutral differential equation*

$$\left[\top^3\left(\top^{-2}\left(x(\top) + \frac{1}{16}x(2\top)\right)'\right)'\right]' + 3\top^3 x\left(\frac{\sqrt{\top}}{2}\right) + 5\top^3 x\left(\frac{\sqrt{\top}}{2}\right) = 0, \quad \top \geq 1, \quad (86)$$

The coefficients and argument functions are

$$b_1(\top)=\top^{-2}, b_2(\top)=\top^3, \mu_1(\top)=3\top^3, \mu_2(\top)=5\top^3, d(\top)=\frac{1}{16}, \tau(\top)=2\top, \sigma_1(\top)=\sigma_2(\top)=\frac{\sqrt{\top}}{2}, h(u)=u. \quad (87)$$

Assumptions (A_1)–(A_4) *and* (A_{5a}) *are satisfied with* $\alpha = k = 1$ *and* $d_0 = \frac{1}{16}$. *Indeed, the delayed arguments are strictly increasing and tend to infinity,* $\tau(\top) > \top$. *Furthermore,*

$$A_2(\top)=\frac{1}{2\top^2}, A_{12}(\top)=\infty.$$

Thus, case (C_1) *applies, and* $\Omega(\top)=1$ *and* $y(\top)=z(\top)$. *The canonical coefficients are*

$$p_1(\top)=2,\ p_2(\top)=1/(4\top).$$

Consequently,

$$P_1(\top)=\frac{1}{2}\ \text{and}\ P_2(\top)=4t.$$

By the canonical transformation theorem, Equation (86) is equivalent to

$$\left(y''(\top)/(2\top)\right)'+(3\top/2)x\left(\sqrt{\top}/2\right)+(5t/2)x\left(\sqrt{\top}/2\right)=0.$$

For a fixed base point $a=\top_* \geq 1$, *direct integration gives*

$$B_1(\top)=(\top-a)/2,\ B_2(\top)=2\left(\top^2-a^2\right),\ \text{and}\ B_{12}(\top)=\top^3/3-a^2\top+2a^3/3.$$

Hence, $B_{12}(\top)/\top^3 \to 1/3$, *and therefore*

$$\frac{B_{12}(\tau(\top))}{B_{12}(\top)}=\frac{B_{12}(2\top)}{B_{12}(\top)}\to 8.$$

It follows that

$$d_0\frac{B_{12}(\tau(\top))}{B_{12}(\top)}\to 1/2<1.$$

Thus, condition (21) holds for all sufficiently large $\top$. *Moreover,*

$$E(\top)=15/16\ \text{and}\ E_1(\top)=1-B_{12}(2\top)/(16B_{12}(\top))\to 1/2.$$

Hence, E_1 *is eventually positive. Choose* $\rho(\top)=2\top$. *Clearly,* $\rho\in C^1$,$\rho'(\top)=2>0$, *and* $\rho(\top)>\top$ *for* $\top\geq 1$. *Since* $\sigma_-(\top)=\sigma_+(\top)=\sqrt{\top}/2$, *we obtain*

$$\eta(\top)=\sigma_+(\rho(\rho(\top)))=\sqrt{\top}<\top,\ \ \top>1.$$

Hence, condition (62) is satisfied.
Using the definition of Ψ_1 *and retaining both coefficient contributions, we obtain*

$$\begin{aligned} & A_2(u)[\mu_1(u)E(\sigma_1(u))+\mu_2(u)E(\sigma_2(u))] \\ = & \left[1/\left(2u^2\right)\right]\left(3u^3+5u^3\right)(15/16) \\ = & 15u/4. \end{aligned}$$

Consequently,

$$\begin{aligned}\Psi_1(\top) &= (1/2)\int_{\top}^{2\top} 4s\left[\int_s^{2s} 15u/4\mathrm{d}u\right]\mathrm{d}s\\ &= (1/2)\int_{\top}^{2\top} 4s\left(45s^2/8\right)\mathrm{d}s\\ &= (675/16)\top^4.\end{aligned}$$

Since $k = 1$, *it follows that*

$$m_1(\top) := \int_{\sqrt{\top}}^{\top} \Psi_1(s)\mathrm{d}s = (135/16)\left(\top^5 - \top^{5/2}\right) \to \infty.$$

Now, put $v = \sqrt{\top}/2$. *Since the two delayed arguments coincide, the definition of* Ψ_2 *yields, for all sufficiently large* $\top$,

$$\begin{aligned}\Psi_2(\top) &= A_2(\top)B_{12}(v)[\mu_1(\top)E_1(v) + \mu_2(\top)E_1(v)]\\ &= 4tB_{12}(v)E_1(v)\\ &= 4t[B_{12}(v) - B_{12}(2v)/16].\end{aligned}$$

The exact expression for B_{12} *gives*

$$B_{12}(v) - B_{12}(2v)/16 = v^3/6 - 15a^2v/16 + 5a^3/8.$$

Therefore,

$$\Psi_2(\top) = \top^{5/2}/12 - \left(15a^2/8\right)\top^{3/2} + \left(5a^3/2\right)\top.$$

Define

$$H(s) = s^{7/2}/42 - \left(3a^2/4\right)s^{5/2} + \left(5a^3/4\right)s^2.$$

Then, $H'(s) = \Psi_2(s)$, *and thus*

$$m_2(\top) = \int_{\sqrt{\top}/2}^{\top} \Psi_2(s)ds = H(\top) - H\left(\sqrt{\top}/2\right).$$

Since

$$H(\top)/\top^{7/2} \to 1/42 \text{ and } H\left(\sqrt{\top}/2\right)/\top^{7/2} \to 0,$$

we conclude that

$$m_2(\top)/\top^{7/2} \to 1/42,$$

and hence $m_2(\top) \to \infty$. *Accordingly,*

$$\liminf_{\top\to\infty} m_1(\top) = \liminf_{\top\to\infty} m_2(\top) = \infty.$$

Both conditions (70) and (71) are satisfied. Therefore, by Corollary 7, every solution of Equation (86), understood in the solution class specified above, is oscillatory.

Remark 5. *In Example 2,* $\sigma(\tau(\top)) = \sqrt{2\top}/2 \neq \sqrt{\top} = \tau(\sigma(\top))$. *Hence, the commutativity condition required by the criteria compared in [8,29,32,33] is not satisfied. These criteria are therefore inapplicable, whereas Corollary 7 applies without requiring commutativity. Table 1 summarizes this comparison.*

Table 1. Applicability of the compared criteria to Example 2.

Criterion	Commutativity Required	Applicable to Example 2
The criteria considered in [8,29,32,33]	Yes	No
Corollary 7	No	Yes

Example 3. *Consider the third-order neutral differential equation*

$$\left[\top^3\left(x(\top)+\frac{1}{8}x(2\top)\right)''\right]'+\sum_{i=1}^{n}\beta_i\top(x(\theta_i\top))^{1/3}=0,\quad \top\geq 1, \tag{88}$$

where $n\geq 2$, $\beta_i>0$, and $0<\theta_i<1$ for $i=1,\ldots,n$. Assume that the constants θ_i are pairwise distinct, and interpret the fractional power as the real cube root. Set

$$\begin{aligned} z(\top) &= x(\top)+\frac{1}{8}x(2\top),\\ \theta_- &= \min_{1\leq i\leq n}\theta_i,\ \ \theta_+=\max_{1\leq i\leq n}\theta_i, \end{aligned}$$

and

$$D=\sum_{i=1}^{n}\beta_i\theta_i{}^{-1/3}>0.$$

Equation (88) is a particular case of Equation (1) with

$$b_1(\top)=1, b_2(\top)=\top^3, \mu_i(\top)=\beta_i\top, d(\top)=\frac{1}{8}, \tau(\top)=2\top, \sigma_i(\top)=\theta_i\top, h(u)=u^{1/3}.$$

Assumptions (A_1)–(A_4) and (A_{5a}) are satisfied with $\alpha=1/3, k=1$ and $d_0=\frac{1}{8}$. Moreover,

$$A_2(\top)=\frac{1}{2\top^2}, A_{12}(\top)=\frac{1}{2\top}.$$

Thus, $A_{12}(1)=1/2<\infty$, and case (C_2) applies. Consequently, $\Omega(\top)=1/(2\top)$, $y(\top)=2\top z(\top)$. The canonical coefficients are

$$p_1(\top)=1/2,\ p_2(\top)=1/2.$$

Hence, $P_1(\top)=P_2(\top)=2$, and the canonical transformation yields

$$y'''(\top)+\frac{2}{\top}\sum_{i=1}^{n}\beta_i(x(\theta_i\top))^{1/3}=0.$$

Fix a base point $a=\top_*\geq 1$. Direct integration gives

$$B_1(\top)=B_2(\top)=2(\top-a), \text{ and } B_{12}(\top)=2(\top-a)^2.$$

Therefore,

$$d_0\frac{B_{12}(\tau(\top))}{B_{12}(\top)}=\frac{1}{8}[(2\top-a)/(\top-a)]^2\to\frac{1}{2}<1.$$

Condition (21) is thus satisfied eventually. Furthermore,

$$E(\top)=7/(16\top),\ E_1(\top)=[1/(2\top)]\Big(1-(1/8)[(2\top-a)/(\top-a)]^2\Big),$$

and consequently,

$$\top E_1(\top) \to \frac{1}{4}.$$

In particular, E_1 is eventually positive. More explicitly, for $\top \geq 3a$,

$$E_1(\top) \geq 7/(64\top) > 0.$$

Choose a constant r such that $1 < r < 1/\sqrt{\theta_+}$, and let $\rho(\top) = r\top$. Such a choice is possible because $\theta_+ < 1$. Since $\sigma_-(\top) = \theta_-\top$, $\sigma_+(\top) = \theta_+\top$, we obtain

$$\eta(\top) = \sigma_+(\rho(\rho(\top))) = \theta_+ r^2 \top < \top.$$

Thus, condition (62) holds.
To evaluate Ψ_1, observe that

$$\begin{aligned} & A_2(u) \sum_{i=1}^{n} \mu_i(u)(E(\sigma_i(u)))^{1/3} \\ = \; & (D/2)(7/16)^{1/3} u^{-4/3}. \end{aligned}$$

Using $P_1 = P_2 = 2$, we find

$$\begin{aligned} \Psi_1(\top) &= 2D(7/16)^{1/3} \int_{\top}^{r\top} \left[\int_s^{rs} u^{-4/3} \mathrm{d}u \right] \mathrm{d}s \\ &= 2D(7/16)^{1/3}(9/2)\left(1 - r^{-1/3}\right)\left(r^{2/3} - 1\right)\top^{2/3} \\ &= C_r \top^{2/3}, \end{aligned}$$

where $C_r = 9D(7/16)^{1/3}\left(1 - r^{-1/3}\right)\left(r^{2/3} - 1\right) > 0$. It follows that

$$\int_{\top}^{\infty} \Psi_1(s)\mathrm{d}s = \infty.$$

For the second comparison function, its definition gives

$$\Psi_2(\top) = [B_{12}(\theta_-\top)]^{1/3}/(2\top) \sum_{i=1}^{n} \beta_i (E_1(\theta_i\top))^{1/3}.$$

For all sufficiently large $\top$, every term in this expression is defined and positive. Indeed, $\top \geq 3a/\theta_-$ ensures that $\theta_i\top \geq 3a$ for every i. Since $B_{12}(\theta_-\top)/\top^2 \to 2\theta_-^2$, and $\top E_1(\theta_i\top) \to 1/(4\theta_i)$, $i = 1, \ldots, n$, we obtain

$$\top^{2/3}\Psi_2(\top) \to \theta_-^{2/3} D/2^{4/3} > 0.$$

Because $\int_{\top}^{\infty} s^{-2/3}\mathrm{d}s = \infty$, limit comparison yields

$$\int_{\top}^{\infty} \Psi_2(s)\mathrm{d}s = \infty.$$

Here $\top$ is chosen sufficiently large so that all the required domain and positivity conditions hold. Both divergence conditions in (72) are therefore satisfied. Since $\alpha = 1/3 \in (0,1)$, part (i) of Corollary 8 implies that every solution of Equation (88), in the solution class specified above, is oscillatory.

5. Conclusions

This paper establishes sufficient conditions for the oscillation of solutions of Equation (1) under the semi-canonical configuration (8) and the neutral coefficient condition (3), treating advanced and delayed neutral arguments as alternative settings. Canonical reduction provides a common analytical framework in which the corresponding eventually positive canonical functions belong to one of two classes, N_0 and N_2. The oscillation criteria are then obtained by excluding these classes.

For an advanced neutral argument, comparison, and integral criteria are established for multiple delayed terms without requiring commutativity. For a delayed neutral argument, class-specific estimates are derived under commutativity and the stated argument assumptions. While retaining the assumptions on the original neutral coefficient, this analysis allows the transformed coefficient to satisfy $0 \leq a(\top) \leq a_0 < \infty$, without requiring $a_0 < 1$. When $\alpha = 1$ and $b_1 \equiv 1$, criteria excluding N_0 are combined with a Riccati criterion excluding N_2 to establish oscillation of all solutions in the prescribed solution class.

Three examples illustrate the applicability of the criteria, including explicit sufficient parameter conditions for proportional arguments and applications of both canonical transformations. The conditions obtained are sufficient; their necessity and optimality are not established. Future work may investigate these questions and the relaxation of the commutativity assumption in the delayed neutral setting.

Author Contributions: Conceptualization, A.A.-J. and B.Q.; methodology, A.A.-J. and B.Q.; validation, A.A.-J. and B.Q.; investigation, A.A.-J. and B.Q.; resources, B.Q., F.A., and A.A.-J.; data curation, A.A.-J., I.F.B.S., F.A., and B.Q.; writing—original draft preparation, A.A.-J., F.A., I.F.B.S., and B.Q.; writing—review and editing, B.Q., A.A.-J., F.A., and I.F.B.S.; visualization, A.A.-J., I.F.B.S., and B.Q.; supervision, A.A.-J., I.F.B.S., and B.Q.; project administration, B.Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Researchers Supporting Project number (PNURSP2026R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: The original contributions presented in this paper are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: Authors would like to thank Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Győri, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations: With Applications*; Oxford Mathematical Monographs; Clarendon Press: Oxford, UK; Oxford University Press: Oxford, UK, 1991.
2. Erbe, L.H.; Kong, Q.; Zhang, B.G. *Oscillation Theory for Functional Differential Equations*; Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA, 1995; Volume 190.
3. Gopalsamy, K. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*; Mathematics and Its Applications; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1992; Volume 74.
4. Al-Jaser, A.; Serra-Capizzano, S.; Alluqmani, E.; Qaraad, B. Third-order nonlinear neutral delay differential equations with several deviating arguments: Improved oscillation criteria. *Axioms* **2025**, *14*, 850. https://doi.org/10.3390/axioms14110850.
5. Li, J.; Qu, J. High precision barycentric interpolation collocation method to solve EFK equation. *Int. J. Comput. Math.* **2025**, *102*, 1880–1898. https://doi.org/10.1080/00207160.2025.2522993.
6. Baculíková, B.; Džurina, J. Oscillation of third-order neutral differential equations. *Math. Comput. Model.* **2010**, *52*, 215–226. https://doi.org/10.1016/j.mcm.2010.02.011.
7. Masood, F.; Almutairi, A.; Iambor, L.F.; Al-Ghafri, K.S.; Bazighifan, O. Novel oscillation criteria for general third-order nonlinear neutral differential equations. *AIMS Math.* **2025**, *10*, 29607–29626. https://doi.org/10.3934/math.20251301.

8. Feng, L.; Han, Z. Oscillation of a class of third-order neutral differential equations with noncanonical operators. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 2519–2530. https://doi.org/10.1007/s40840-021-01079-x.
9. Thandapani, E.; Göktürk, B.; Özdemir, O.; Tunç, E. Oscillatory behavior of semi-canonical nonlinear neutral differential equations of third order via comparison principles. *Qual. Theory Dyn. Syst.* **2023**, *22*, 30. https://doi.org/10.1007/s12346-022-00731-6.
10. Hale, J.K.; Verduyn Lunel, S.M. *Introduction to Functional Differential Equations*; Applied Mathematical Sciences; Springer: New York, NY, USA, 1993; Volume 99. https://doi.org/10.1007/978-1-4612-4342-7.
11. Hupkes, H.J.; Augeraud-Véron, E. Well-posedness of initial value problems for functional differential and algebraic equations of mixed type. *Discret. Contin. Dyn. Syst.* **2011**, *30*, 737–765. https://doi.org/10.3934/dcds.2011.30.737.
12. Kumar, M.S.; Elayaraja, R.; Ganesan, V.; Bazighifan, O.; Al-Shaqsi, K.; Nonlaopon, K. Qualitative behavior of unbounded solutions of neutral differential equations of third-order. *Fractal Fract.* **2021**, *5*, 95. https://doi.org/10.3390/fractalfract5030095.
13. Tripathy, A.K.; Santra, S.S. Necessary and sufficient conditions for oscillation of second-order differential equations with nonpositive neutral coefficients. *Math. Bohem.* **2021**, *146*, 185–197. https://doi.org/10.21136/MB.2020.0063-19.
14. Baculíková, B.; Džurina, J. Property (A) of third-order advanced differential equations. *Math. Slovaca* **2014**, *64*, 339–346. https://doi.org/10.2478/s12175-014-0208-8.
15. Baculíková, B.; Džurina, J. Remarks on properties of Kneser solutions for third-order neutral differential equations. *Appl. Math. Lett.* **2017**, *63*, 1–5. https://doi.org/10.1016/j.aml.2016.07.005.
16. Chandrasekaran, E.; Chatzarakis, G.E.; Sakthivel, R.; Thandapani, E. Third-order nonlinear semi-canonical functional differential equations: Oscillation via new canonical transform. *Mathematics* **2024**, *12*, 3113. https://doi.org/10.3390/math12193113.
17. Baculíková, B.; Rani, B.; Selvarangam, S.; Thandapani, E. Properties of Kneser's solution for half-linear third order neutral differential equations. *Acta Math. Hungar.* **2017**, *152*, 525–533. https://doi.org/10.1007/s10474-017-0721-7.
18. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I.; Li, T.; Tunç, E. Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. *Complexity* **2019**, *2019*, 5691758. https://doi.org/10.1155/2019/5691758.
19. Ganesan, V.; Sathishkumar, M. On the oscillation of a third-order nonlinear differential equations with neutral type. *Ural Math. J.* **2017**, *3*, 122–129. http://dx.doi.org/10.15826/umj.2017.2.013.
20. Džurina, J.; Grace, S.R.; Jadlovská, I. On nonexistence of Kneser solutions of third-order neutral delay differential equations. *Appl. Math. Lett.* **2019**, *88*, 193–200. https://doi.org/10.1016/j.aml.2018.08.016.
21. Saranya, K.; Piramanantham, V.; Thandapani, E.; Tunç, E. Oscillation criteria for third-order semi-canonical differential equations with unbounded neutral coefficients. *Stud. Univ. Babeş-Bolyai Math.* **2024**, *69*, 115–125. https://doi.org/10.24193/subbmath.2024.1.08.
22. Chatzarakis, G.E.; Džurina, J.; Jadlovská, I. Oscillation properties of third-order neutral delay differential equations with noncanonical operators. *Mathematics* **2019**, *7*, 1177. https://doi.org/10.3390/math7121177.
23. Vidhyaa, K.S.; Thandapani, E.; Tunç, E. Novel oscillation criteria for third-order semi-canonical differential equations with an advanced neutral term. *Opusc. Math.* **2025**, *45*, 543–558. https://doi.org/10.7494/OpMath.2025.45.4.543.
24. Saranya, K.; Piramanantham, V.; Thandapani, E. Oscillation results for third-order semi-canonical quasi-linear delay differential equations. *Nonauton. Dyn. Syst.* **2021**, *8*, 228–238. https://doi.org/10.1515/msds-2020-0135.
25. Graef, J.R.; Jadlovská, I. Canonical representation of third-order delay dynamic equations on time scales. *Differ. Equ. Appl.* **2024**, *16*, 1–18. https://doi.org/10.7153/dea-2024-16-01.
26. Kiguradze, I.T.; Chanturia, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993; Volume 89.
27. Kiguradze, I.T. On the oscillatory character of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n \operatorname{sign} u = 0$. *Mat. Sb.* **1964**, *65*, 172–187.
28. Tang, X.H. Oscillation for first order superlinear delay differential equations. *J. Lond. Math. Soc.* **2002**, *65*, 115–122. https://doi.org/10.1112/S0024610701002678.
29. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math.* **2011**, *47*, 181–199.
30. Philos, C.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. https://doi.org/10.1007/BF01223686.
31. Koplatadze, R.G.; Chanturiya, T.A. Oscillating and monotone solutions of first-order differential equations with deviating argument. *Differ. Uravn.* **1982**, *18*, 1463–1465.
32. Jiang, Y.; Jiang, C.; Li, T. Oscillatory behavior of third-order nonlinear neutral delay differential equations. *Adv. Differ. Equ.* **2016**, *2016*, 171. https://doi.org/10.1186/s13662-016-0902-7 .
33. Sallam, R.A.; Salem, S.; El-Sheikh, M.M.A. Oscillation of solutions of third-order nonlinear neutral differential equations. *Adv. Differ. Equ.* **2020**, *2020*, 314. https://doi.org/10.1186/s13662-020-02777-9.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.