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Tsallis Entropy Measures for Concomitants of *k*-Record Values: Characterization and Nonparametric Estimation with Applications to Breaking Strength Data

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Abstract

First, we examine the finiteness of a commonly used formulation of cumulative Tsallis entropy and show that, for distributions with unbounded support, the defining integral is generally divergent. This motivates an alternative formulation based on the cumulative distribution function (CDF) that is finite under appropriate moment conditions and admits a natural interpretation in terms of the mean of a generalized maximum. Second, we develop a comprehensive framework for quantifying uncertainty in concomitants of *k*-record values $Z_{[n,k]}$ arising from the Sarmanov (SAR(ζ)) family of bivariate distributions, using cumulative Tsallis entropy (CTE) as the primary information measure. We derive explicit closed-form expressions for the CTE, its past-lifetime dynamic variant, and the associated Rényi entropy for the concomitants $Z_{[n,k]}$ under the SAR(ζ) dependence structure. A quantile-based reformulation is employed throughout, enabling the results to hold for arbitrary marginal distributions without requiring closed-form CDFs. Furthermore, we establish several fundamental theoretical properties, including uniqueness of the quantile density characterization, relationships to the mean inactivity time, and monotonicity classes that provide nonparametric criteria for model discrimination. To facilitate practical implementation, we introduce a nonparametric kernel-based estimator for the Tsallis entropy, with comprehensive comparisons of bandwidth selection methods. Finally, a real-world dataset is used to illustrate the proposed methodology.

Keywords: cumulative Tsallis entropy; Rényi entropy; Sarmanov family; concomitants of *k*-record values; quantile-based entropy; nonparametric estimation

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1. Introduction

Information theory provides fundamental tools for quantifying uncertainty and randomness in probability distributions. Boltzmann [1] was the first in physics to talk about the concept of entropy. So, he came up with a clear concept of entropy, which is a way to measure how disorganized particles are in a closed system. One of the most important and widely used measures in this field is Shannon entropy, introduced by Shannon [2], which plays an important role in measuring the average uncertainty of a random variable (RV).

For a continuous RV Y with probability density function (PDF) $f_Y(y)$, Shannon entropy is defined as

$$H(Y) = -\int_0^\infty f_Y(y) \ln f_Y(y) dy.$$

1.1. Rényi and Tsallis Entropies

There are times when the Shannon entropy might not be the best choice because of important extended versions. There are a lot of different generalized entropy measures out there. These measures are more useful in real life because they are smooth and have a large dynamic range with respect to certain conditions that make them applicable in practice.

A variety of extensions of the classical Shannon entropy have been proposed in the information theory literature to better capture uncertainty. Among the most prominent are the generalized entropy measure introduced by Tsallis [3] and Rényi [4], both of which incorporate the parameter δ, which allows the measure to adapt to different characteristics of probability distributions. Let Y be an absolutely continuous RV with a PDF $f_Y(y)$, then Rényi's entropy of order δ is defined as

$$\mathcal{R}_\delta(Y) = \frac{1}{1-\delta} \ln\left(\int_0^\infty f_Y^\delta(y) dy\right), \ 0 < \delta \neq 1.$$

It can be readily verified that $\mathcal{R}_\delta(Y)$ converges to $H(Y)$ as $\delta \to 1$. The presence of the parameter δ enhances flexibility while maintaining the additive nature of Shannon entropy. Another notable generalization is the Tsallis entropy introduced by Tsallis [3], which represents a fundamentally non-additive extension of Shannon entropy. This measure has gained significant importance in areas such as statistical mechanics and thermodynamics. For a comprehensive overview and applications, see Cartwright [5]. Tsallis entropy is yielded by

$$H_\delta(Y) = \frac{1}{\delta-1}\left(1 - \int_0^\infty f_Y^\delta(y) dy\right), \ 0 < \delta \neq 1. \tag{1}$$

Similar to Rényi entropy, Tsallis entropy reduces to Shannon entropy in the limit as $\delta \to 1$. Moreover, these two generalized measures are related through

$$\mathcal{R}_\delta(Y) = \frac{1}{\delta-1} \ln\Big(1 - (\delta-1) H_\delta(Y)\Big).$$

Beck [6] highlighted the advantages of Tsallis entropy over Rényi entropy within the framework of generalized statistical mechanics, emphasizing that Tsallis entropy satisfies important properties such as concavity and Lesche stability (cf. Lesche, [7]), which are not generally exhibited by Rényi entropy. Due to its non-additive (non extensive) nature, Tsallis entropy provides a powerful tool for modeling complex systems characterized by long-range dependence, memory effects, and multifractal structures (see Tsallis et al., [8]). Further discussions on entropy and its generalizations can be found in Tsallis [3]. Additionally, Wilk and Wlodarczyk [9] demonstrated scenarios in which Tsallis entropy yields meaningful results, whereas Shannon entropy may fail to adequately describe the underlying information.

Recently, Rao et al. [10] introduced an alternative measure to the differential entropy known as the cumulative residual entropy (CREN), based on the survival function (SF) $\bar{F}_Y(y) = P(Y > y)$ instead of the PDF $f_Y(y)$. CREN is considered to be more stable, as SF is

more regular than the PDF and possesses more mathematical properties. Further, the CDF exists even if the PDF does not exist. Rao et al. [10] characterized CREN as

$$\mathcal{CRE}(Y) = -\int_0^\infty \bar{F}_Y(y) \ln \bar{F}_Y(y) dy. \tag{2}$$

Some applications of (2) are presented in Asadi and Zohrevand [11]. An information measure similar to (2) is the cumulative entropy, defined as (see Di Crescenzo and Longobardi [12]).

$$\mathcal{CE}(Y) = -\int_0^\infty F_Y(y) \ln F_Y(y) dy, \tag{3}$$

where $F(.)$ is the CDF of a RV Y. However, since the argument of the logarithm is a probability, we have $0 \leq \mathcal{CE}(Y) \leq \infty$, whereas $H(Y)$ may be negative in the absolutely continuous case. Moreover, $\mathcal{CE}(Y) = 0$ if and only if Y is a constant. From (2) and (3), it follows that the cumulative entropy and the CREN are linked by the following relation (see, Di Crescenzo and Longobardi [13])

$$\mathcal{CRE}(Y) + \mathcal{CE}(Y) = \int_0^\infty h_Y(y) dy,$$

where $h_Y(y) = -(F_Y(y) \ln F_Y(y) + \bar{F}_Y(y) \ln \bar{F}_Y(y))$ is the partition entropy of Y evaluated at y (see [14]). We point out that if $X = aY + b$, with $a \in \mathbb{R}^+$, and $b \in \mathbb{R}$, then $\mathcal{CE}(X) = a\mathcal{CE}(Y)$.

The measure of uncertainty in a past lifetime plays an important role in different areas such as information theory, reliability theory, survival analysis, economics, business, forensic science, and other related fields. However, in many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance, if at time t, a system which is observed only at certain preassigned inspection times is found to be down, then the uncertainty of the system life relies on the past, i.e., on which instant in $(0, t)$ it has failed. A wide variety of research is available on entropy measures and their applications in past lifetimes. For details, one can refer to Di Crescenzo and Longobardi [12], Sachlas and Papaioannou [15], and the references therein. Motivated by the classical cumulative entropy and Tsallis-type entropy, Cali et al. [16] introduced the CTE. For a non-negative absolutely continuous RV Y, the CDF-based CTE of order δ is defined by

$$CTE_{\delta,\mathrm{CDF}}(Y) = \frac{1}{\delta - 1} \int_0^\infty \left[F_Y(y) - F_Y^\delta(y) \right] dy, \qquad \delta > 0, \quad \delta \neq 1. \tag{4}$$

This definition is particularly important because it reduces to the classical cumulative entropy as δ tends to one. Indeed, under the appropriate regularity conditions,

$$\lim_{\delta \to 1} CTE_{\delta,\mathrm{CDF}}(Y) = -\int_0^\infty F_Y(y) \ln F_Y(y) dy = \mathcal{CE}(Y).$$

Thus, (4) provides a Tsallis-type generalization of the classical cumulative entropy defined in (3). Moreover, for the convergence of the measure $CTE_{\delta,\mathrm{CDF}}(Y)$, we need to

consider $F_Y(y) - F_Y^{\delta}(y) \to 0$, as $y \to \infty$. Using the Taylor expansion for small $\bar{F}_Y(y) = 1 - F_Y(y)$, we obtain

$$\begin{aligned} F_Y(y) - F_Y^{\delta}(y) &= (1 - \bar{F}_Y(y)) - (1 - \bar{F}_Y(y))^{\delta} \\ &= (1 - \bar{F}_Y(y)) - \left(1 - \delta\bar{F}_Y(y) + \frac{\delta(\delta-1)}{2}\bar{F}_Y^2(y) + O(\bar{F}_Y^3(y))\right) \\ &= (\delta - 1)\bar{F}_Y(y) - \frac{\delta(\delta-1)}{2}\bar{F}_Y^2(y) - O(\bar{F}_Y^3(y)). \end{aligned}$$

Therefore, $F_Y(y) - F_Y^{\delta}(y) \sim \delta\bar{F}_Y(y)$ as $y \to \infty$. Thus, the measure $CTE_{\delta,\mathrm{CDF}}(Y)$ converges if and only if $E(Y) = \int_0^{\infty} \bar{F}_Y(y)\, dy < \infty$.

The CTE has several important applications. In reliability engineering, it can be used to quantify uncertainty associated with component and system lifetimes. In survival analysis, it provides measures of uncertainty concerning failure-time distributions and past lifetimes. It can also be employed in the study of aging properties, stochastic orderings, characterization problems, and entropy-based statistical inference. More generally, the parameter δ provides additional flexibility for emphasizing different features of the underlying distribution.

Further details on classical and modern entropy measures and their applications in probability, statistics, and data analysis, with particular emphasis on recent advances in dynamic cumulative entropy and entropy formulations under dependence structures, including copula-based and multivariate settings, can be found in the comprehensive survey by Kumar et al. [17], see also Asadi and Zohrevand [11], Kharazmi and Balakrishnan [18], and Sunoj et al. [19].

1.2. Concomitants of k−Record Values

Let (Y_i, Z_i), $i = 1, 2, ...$, denote a sequence of independent and identically distributed bivariate random vectors having the continuous joint CDF $F_{Y,Z}(y, z) = P(Y \le y, Z \le z)$. Suppose that the analysis is concerned primarily with the $k-$record values generated by the first component Y. The observation of the second component Z corresponding to a particular $k-$record of Y is referred to as the concomitant of that record. Concomitants of $k-$record values have attracted considerable attention because of their applications in a variety of statistical and reliability settings; see, among others, Bdair and Raqab [20], Chacko and Mary [21], and Thomas et al. [22]. In particular, if $Z_{[n,k]}$ denotes the concomitant associated with the nth upper $k-$record value of Y, its density can be expressed as

$$f_{[n,k]}(z) = \int_0^{\infty} f_{Z|Y}(z|y) g_{n,k}(y) dy,$$

where $f_{Z|Y}(z|y)$ represents the conditional density of Z given $Y = y$, and $g_{n,k}(y)$ is the density of the corresponding nth upper $k-$record value.

When only the marginal distributions of two variables are known, constructing a suitable joint distribution with these prescribed marginals becomes particularly important for modeling their dependence. One useful class for this purpose is the Farlie–Gumbel–Morgenstern (FGM) family. Its joint CDF and corresponding PDF are provided by

$$F_{Y,Z}(y, z) = F_Y(y)F_Z(z)[1 + \lambda\bar{F}_Y(y)\bar{F}_Z(z)]$$

and

$$f_{Y,Z}(y, z) = f_Y(y)f_Z(z)[1 + \lambda(2F_Y(y) - 1)(2F_Z(z) - 1)],\ -1 \le \lambda \le 1,$$

where $f_Y(y)$, $f_Z(z)$, and $F_Y(y)$, $F_Z(z)$ denote the marginal PDFs and CDFs of the RVs Y and Z, respectively, while $\bar{F}_Y$ and $\bar{F}_Z$ are the associated SFs. Despite its convenient structure and wide applicability, the classical FGM family is restricted to relatively weak dependence, with Spearman's rank correlation lying approximately in the interval $(-0.33, 0.33)$. This limitation has motivated the development of a number of generalized FGM-type models capable of representing stronger dependence; examples include the extensions proposed by Bairamov and Kotz [23], Bekrizadeh et al. [24], Cambanis [25], Huang and Kotz ([26,27]), and Kimeldorf and Sampson [28]. These generalized families have subsequently been investigated in several areas, including the theory of concomitants and information measures by Abd Elgawad et al. [29], Alawady et al. [30], and Husseiny et al. [31], as well as reliability and bivariate dependence modeling by Mansour et al. [32].

More recently, the Sarmanov family, denoted by SAR(ζ), has emerged as a competitive alternative to the existing FGM extensions; see Balakrishnan and Lai [33]. Its properties and applications have been studied by Alawady et al. [34], Barakat et al.[35], and references therein. The corresponding joint CDF and PDF are, respectively,

$$F_{Y,Z}(y,z) = F_Y(y)F_Z(z)\Big[1 + 3\zeta\bar{F}_Y(y)\bar{F}_Z(z) + 5\zeta^2\Big(2F_Y(y)-1\Big)\Big(2F_Z(z)-1\Big)\bar{F}_Y(y)\bar{F}_Z(z)\Big]$$

and

$$\begin{aligned} f_{Y,Z}(y,z) &= f_Y(y)f_Z(z)\Big[1 + 3\zeta\Big(2F_Y(y)-1\Big)\Big(2F_Z(z)-1\Big) \\ &\quad + \frac{5}{4}\,\zeta^2\Big(3\Big(2F_Y(y)-1\Big)^2 - 1\Big)\Big(3\Big(2F_Z(z)-1\Big)^2 - 1\Big)\Big], \ |\zeta| \le \frac{\sqrt{7}}{5}. \end{aligned} \tag{5}$$

It is important to note that the bound $|\zeta| \le \frac{\sqrt{7}}{5}$ is a necessary condition for the bivariate density (5) to be non-negative for all y and z; this bound does not depend on the marginal distributions. The correlation coefficient ρ between Y and Z, however, depends on the marginals. For uniform marginals, we have $\rho = \zeta$, providing the range ± 0.529 (cf. Balakrishnan and Lai, [33]; page 74).

The joint distributional structure of the SAR family and its use in studying concomitants of k-record values have been investigated in earlier work, particularly by Alawady et al. [34], Mansour et al. [32], and related studies. Accordingly, the joint CDF and PDF of the SAR model displayed above are recalled as established distributional components rather than claimed as new results of the present paper. The contribution of this work lies rather in the development and analysis of CTE and related information measures for these concomitants. In particular, we investigate the finiteness of alternative CTE formulations and develop a CDF-based formulation suitable for distributions with unbounded support under appropriate moment conditions. We also derive quantile-based representations, establish theoretical properties, and propose a nonparametric estimation procedure with an application to breaking-strength data.

Building on the established SAR distributional framework, this paper develops new results on cumulative Tsallis entropy and related information measures for concomitants of k-record values. Its contributions concern the formulation, analytical characterization, theoretical properties, and nonparametric estimation of these measures, rather than the introduction of new joint CDF or PDF expressions for the SAR family. First, we begin (Section 2) by investigating the finiteness of a widely employed formulation of CTE. We show that, for distributions supported on an unbounded domain, the defining integral is generally divergent. This limitation motivates the introduction of an alternative formulation based on the CDF, which is finite under suitable moment conditions and admits a natural interpretation in terms of the mean of a generalized maximum. Second, we

derive explicit analytical expressions for the CTE, its past-lifetime dynamic counterpart, and the corresponding Rényi entropy for the concomitants of k-record values drawn from the SAR(ζ) family. These derivations, presented in Section 3, are general and hold for arbitrary marginal distributions. Moreover, we provide closed-form examples for uniform and Weibull marginals, demonstrating the sensitivity of these measures to both the dependence parameter (ζ) and the record indices n and k. Third, in Section 4, we introduce a nonparametric estimation procedure based on kernel density estimation (KDE), providing practitioners with a practical tool for entropy estimation from real data. Also, we establish several theoretical properties, including uniqueness and monotonicity characterizations, which are essential for model selection and interpretation. Fourth, in Section 5, we illustrate the proposed methodology using a well-known dataset of breaking strengths of jute fibers measured at two gauge lengths, demonstrating its practical applicability in a reliability setting. Finally, Section 6 concludes the paper with a summary of our findings and directions for future research.

2. Preliminary Results on Cumulative Tsallis Entropy

In the present section, we consider a cumulative Tsallis-type functional based directly on the power $[F_Y(y)]^\delta$. This functional has been considered in alternative formulations of CTE; see, for example, Mohamed [36] and Kumar [37].

2.1. Cumulative Tsallis Entropy for Distributions with Finite Support

Suppose that Y is a non-negative RV with support contained in $(0, b)$, where $0 < b < \infty$. For every $\delta > 0$,

$$0 \leq [F_Y(y)]^\delta \leq 1,$$

we have

$$0 \leq \int_0^b [F_Y(y)]^\delta \, dy \leq b.$$

This motivates the following bounded-support (BS) cumulative Tsallis-type functional:

$$CTE_{\delta,\mathrm{BS}}(Y) = \frac{1}{\delta - 1}\left[b - \int_0^b [F_Y(y)]^\delta dy\right], \qquad \delta > 0, \quad \delta \neq 1. \tag{6}$$

Both b and $\int_0^b [F_Y(y)]^\delta dy$ have the same physical dimensions. Hence, the subtraction in (6) is dimensionally consistent.

The functional in (6) is a BS cumulative Tsallis-type functional adopted in the present study. It should be distinguished from the CDF-based CTE in (4). However, when the support of Y is contained in $(0, b)$, the CDF-based CTE in (4) becomes

$$CTE_{\delta,\mathrm{CDF}}(Y) = \frac{1}{\delta - 1}\int_0^b \left[F_Y(y) - [F_Y(y)]^\delta\right] dy.$$

We can now establish the relationship between the two functionals.

Lemma 1. *Let Y be a non-negative, absolutely continuous RV with support contained in $(0, b)$, and let $\delta > 0$, $\delta \neq 1$. Then,*

$$CTE_{\delta,CDF}(Y) = CTE_{\delta,\mathrm{BS}}(Y) - \frac{E(Y)}{\delta - 1}. \tag{7}$$

Proof. Since

$$(\delta - 1)CTE_{\delta,\mathrm{BS}}(Y) = \int_0^b \left[1 - [F_Y(y)]^\delta\right] dy,$$

we obtain

$$CTE_{\delta,\mathrm{BS}}(Y) - CTE_{\delta,\mathrm{CDF}}(Y) = \frac{1}{\delta-1}\int_0^b \Big[1 - [F_Y(y)]^\delta - F_Y(y) + [F_Y(y)]^\delta\Big]\,dy$$
$$= \frac{1}{\delta-1}\int_0^b [1 - F_Y(y)]\,dy = \frac{E(Y)}{\delta-1},$$

where the last equality follows from the well-known mean-survival identity for a non-negative RV with finite mean. This proves (7). □

Lemma 1 shows that, for BS distributions, the difference between the two functionals is completely determined by the mean of Y. In particular, if $E(Y) > 0$, then

$$CTE_{\delta,\mathrm{BS}}(Y) > CTE_{\delta,\mathrm{CDF}}(Y), \qquad \delta > 1,$$

whereas

$$CTE_{\delta,\mathrm{BS}}(Y) < CTE_{\delta,\mathrm{CDF}}(Y), \qquad 0 < \delta < 1.$$

Moreover, the value of the BS functional at $\delta = 1$ can be defined by continuity. By L'Hôpital's rule,

$$CTE_{1,\mathrm{BS}}(Y) = \lim_{\delta\to 1} CTE_{\delta,\mathrm{BS}}(Y) \quad = -\int_0^b F_Y(y)\ln F_Y(y)\,dy$$
$$= \mathcal{CE}(Y),$$

provided that the integral is finite. Thus, the BS functional and the CDF-based CTE have the same continuous limit, namely the classical cumulative entropy, as $\delta \to 1$. The two cumulative Tsallis-type functionals examined in this study serve different purposes:

1. The CDF-based CTE $CTE_{\delta,\mathrm{CDF}}$ is defined directly through $F_Y - F_Y^\delta$ and provides a continuous extension of the classical cumulative entropy.
2. The BS functional $CTE_{\delta,\mathrm{BS}}$ is particularly useful when the support is known to be contained in a finite interval $(0, b)$. The constant b ensures dimensional consistency and naturally arises from integrating the constant function over the bounded support.

The relationship established in Lemma 1 shows that the two functionals differ by a term involving the mean of Y, divided by $\delta - 1$. This difference can be important in applications because the two measures may produce different quantitative rankings of distributions, particularly when the distributions being compared have different means or when δ is close to unity.

Thus, the two quantities are mathematically related but conceptually distinct: (4) is the CDF-based CTE, whereas (6) is the BS cumulative Tsallis-type functional considered in the present study.

2.2. A Critical Remark on the Definition of Cumulative Tsallis Entropy

Let Y be a non-negative RV with CDF F_Y. A form of CTE that has appeared in a portion of the literature (e.g., Kumar, [37], and Mohamed, [36]) is

$$CTE_{\delta,\mathrm{O}}(Y) = \frac{1}{\delta-1}\left[1 - \int_0^\infty F_Y^\delta(y)\,dy\right], \qquad \delta > 0, \quad \delta \neq 1. \tag{8}$$

Although (8) has been referred to as a CTE, caution is required when it is applied to RVs with unbounded support. Indeed, if

$$\sup\{y : F_Y(y) < 1\} = \infty,$$

then

$$\lim_{y\to\infty} F_Y^{\delta}(y) = 1$$

for every $\delta > 0$. Consequently,

$$\int_0^{\infty} F_Y^{\delta}(y)\, dy = \infty,$$

and hence, the quantity in (8) is not finite. More precisely,

$$CTE_{\delta,\mathrm{O}}(Y) = \begin{cases} +\infty, & 0 < \delta < 1, \\ -\infty, & \delta > 1 \end{cases}$$

whenever Y has an infinite right endpoint. This issue is important because it is not resolved merely by assuming that δ is an integer. In the ordinary order-statistic setting, F_Y^{δ} is the CDF of the maximum of a sample of size δ when δ is a positive integer. In the generalized order-statistic framework of Kamps [38], however, order-statistic constructions can involve real-value parameters. Thus, powers of the form F_Y^{δ} with $\delta > 0$ not necessarily an integer arise naturally in generalized-order-statistic settings and can admit a corresponding generalized-maximum interpretation.

A more suitable formulation for distributions with unbounded support is the CDF-based CTE $CTE_{\delta,\mathrm{CDF}}$ defined in (4). In contrast to (8), its integrand tends toward zero as $y \to \infty$. Moreover, under appropriate moment conditions, the resulting integral is finite. Indeed,

$$CTE_{\delta,\mathrm{CDF}}(Y) = \frac{1}{\delta - 1}\left\{ \int_0^{\infty} \left[1 - F_Y^{\delta}(y)\right] dy - E(Y) \right\},$$

where we have used the mean-survival identity

$$E(Y) = \int_0^{\infty} [1 - F_Y(y)]\, dy.$$

Whenever F_Y^{δ} can be interpreted as the CDF of a corresponding generalized maximum, denoted by $Y^*_{\delta:\delta}$, and the latter has a finite mean, we have

$$\int_0^{\infty} \left[1 - F_Y^{\delta}(y)\right] dy = E(Y^*_{\delta:\delta}). \tag{9}$$

Consequently,

$$CTE_{\delta,\mathrm{CDF}}(Y) = \frac{E\left(Y^*_{\delta:\delta}\right) - E(Y)}{\delta - 1}. \tag{10}$$

For an ordinary sample of integer size $m \geq 2$, the generalized maximum reduces to the ordinary sample maximum $Y_{m:m}$, and (10) becomes

$$CTE_{m,\mathrm{CDF}}(Y) = \frac{E(Y_{m:m}) - E(Y)}{m - 1}. \tag{11}$$

Thus, for integer $m \geq 2$, the CDF-based CTE can be interpreted as the increase in the mean induced by replacing the parent RV with its sample maximum, normalized by the sample-size parameter minus one. The generalized-order-statistic representation in (10) extends this interpretation to positive non-integer values of δ whenever the corresponding generalized maximum is well defined.

The preceding observations also suggest an alternative unbounded-support extension of the BS functional (6), namely

$$CTE_{\delta,\mathrm{U}}(Y) = \frac{1}{\delta - 1} \int_0^{\infty} \left[1 - F_Y^{\delta}(y)\right] dy, \qquad \delta > 0, \quad \delta \neq 1. \tag{12}$$

The notation $CTE_{\delta,\mathrm{U}}$ emphasizes that this is the unbounded-support extension of the BS functional introduced in (6). When

$$\sup\{y : F_Y(y) < 1\} = b < \infty,$$

we have

$$CTE_{\delta,\mathrm{U}}(Y) = \frac{1}{\delta - 1}\int_0^b \left[1 - F_Y^{\delta}(y)\right] dy = CTE_{\delta,\mathrm{BS}}(Y),$$

so the two formulations are consistent. Furthermore, whenever $E(Y^*_{\delta:\delta}) < \infty$, (9) shows that

$$CTE_{\delta,\mathrm{U}}(Y) = \frac{E(Y^*_{\delta:\delta})}{\delta - 1}.$$

Combining this identity with (10) yields

$$CTE_{\delta,\mathrm{CDF}}(Y) = CTE_{\delta,\mathrm{U}}(Y) - \frac{E(Y)}{\delta - 1}$$

whenever the quantities involved are finite.

It follows that the two functionals should not be regarded as identical. Rather, they are closely related through the mean of the parent RV. The CDF-based CTE, $CTE_{\delta,\mathrm{CDF}}$, has the advantage of reducing to the classical cumulative entropy as $\delta \to 1$, whereas $CTE_{\delta,\mathrm{U}}$ provides a direct extension of the BS functional based on the integral of $1 - F_Y^{\delta}$.

Recommendation 1. *For applications involving non-negative RVs with unbounded support, researchers should exercise caution when using the formulation* (8)*. Although this form occurs in the literature, it is generally infinite for distributions with an infinite right endpoint and therefore does not provide a finite entropy measure in the usual unbounded-support setting. The alternative formulation* (12) *removes this particular divergence whenever* $E(Y^*_{\delta:\delta}) < \infty$*, while also admitting a natural generalized-order-statistic interpretation. Accordingly,* (12) *is a more suitable formulation when a finite Tsallis-type cumulative measure is required for unbounded-support distributions.*

2.3. Finiteness of the Modified Cumulative Tsallis Entropy

Unlike the expression involving $\int_0^{\infty} F_Y^{\delta}(y)dy$, the integral in the modified CTE (12) involves the tail quantity $1 - F_Y^{\delta}(y)$, which tends toward zero as $y \to \infty$. This modification is essential for obtaining a finite measure for distributions with unbounded support. Indeed, if $Y^*_{\delta:\delta}$ denotes the corresponding generalized maximum whose CDF is $F_{Y^*_{\delta:\delta}}(y) = F_Y^{\delta}(y)$, then, for a non-negative RV, we have (9). Consequently, $\mathrm{CTE}_{\delta,\mathrm{U}}(Y) = \frac{E(Y^*_{\delta:\delta})}{\delta-1}$; hence, the proposed measure is finite whenever $E\left(Y^*_{\delta:\delta}\right) < \infty$.

The finiteness of (12) can be illustrated by several standard unbounded-support distributions. For an exponential RV, $F_Y(y) = 1 - e^{-\lambda y}$, $\lambda, y > 0$, and

$$\int_0^{\infty} \left[1 - (1 - e^{-\lambda y})^{\delta}\right] dy = \frac{\psi(\delta + 1) + \gamma}{\lambda} < \infty,$$

where $\psi(\cdot)$ denotes the digamma function and γ is Euler's constant. In particular, for an integer $\delta = m \geq 2$,

$$\mathrm{CTE}_{m,\mathrm{U}}(Y) = \frac{H_m}{\lambda(m-1)} < \infty,$$

where H_m is the mth harmonic number.

Similarly, if Y has a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, then

$$1 - F_Y^{\delta}(y) = 1 - [1 - \exp\{-(y/\beta)^{\alpha}\}]^{\delta} \sim \delta \exp\{-(y/\beta)^{\alpha}\}, \qquad y \to \infty,$$

which establishes the finiteness of (12) for every $\delta > 0$, $\delta \neq 1$. The same tail argument applies to the normal distribution, since

$$1 - F_Y^{\delta}(y) \sim \delta\{1 - F_Y(y)\}, \qquad y \to \infty,$$

and the normal SF is integrable on $[0, \infty)$. More generally, since

$$1 - (1 - u)^{\delta} \sim \delta u, \qquad u \downarrow 0,$$

we have

$$1 - F_Y^{\delta}(y) \sim \delta\{1 - F_Y(y)\}, \qquad y \to \infty.$$

Thus, for a non-negative RV Y,

$$\int_0^{\infty} [1 - F_Y^{\delta}(y)]\, dy < \infty \quad \text{if and only if} \quad E(Y) < \infty$$

under the usual tail-integrability conditions. Equivalently, the modified measure is finite whenever the associated generalized maximum $Y^*_{\delta:\delta}$ has a finite mean. This demonstrates that the modified definition is well suited to unbounded-support models commonly encountered in reliability and survival analysis.

3. Quantile-Based Characterization of Cumulative Tsallis Entropy

This section develops the principal entropy representations for concomitants of k-record values under the SAR dependence structure. The underlying SAR distributional formulas are established results, as discussed in Section 1; here, they serve as the starting point for the entropy analysis. By expressing the CDF of $Z_{[n,k]}$ as a polynomial in the marginal CDF F_Z, we obtain quantile-based representations of the CDF-based CTE that apply to a broad class of marginal distributions, including cases in which the marginal CDF is not available in closed form. We then investigate the associated dynamic entropy measure and other theoretical properties developed in this paper.

3.1. Quantile Representation of the CTE and Illustrative Examples

Theorem 1. *Let $Z_{[n,k]}$ denote the concomitant associated with the nth upper k-record in the SAR(ζ) model. Suppose that Z is a non-negative absolutely continuous RV with CDF F_Z, and let*

$$Q_Z(p) = F_Z^{-1}(p), \quad 0 < p <$$

denote its quantile function. Assume that Q_Z is absolutely continuous on $(0,1)$, with quantile density $q(p) = Q_Z'(p)$ almost everywhere. At points where $f_Z(Q_Z(p)) > 0$, this quantile density can equivalently be written as

$$q(p) = \frac{1}{f_Z(Q_Z(p))}.$$

Provided the defining integral is finite (equivalently, the integral on the right-hand side below is finite), the CDF-based CTE admits the following quantile representation:

$$CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) = \frac{1}{\delta - 1} \int_0^1 \left\{ G_{[n,k]}(p) - G_{[n,k]}^{\delta}(p) \right\} q(p)\, dp, \tag{13}$$

where

$$G_{[n,k]}(p) = p\Big(A_{n,k} + B_{n,k}p + C_{n,k}p^2\Big), \quad 0 \le p \le 1,$$

$$A_{n,k} = 1 - 3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)},$$

$$B_{n,k} = 3\Omega_{n,k}^{(1)} - \frac{15}{2}\Omega_{n,k}^{(2)},$$

$$C_{n,k} = 5\Omega_{n,k}^{(2)},$$

$$\Omega_{n,k}^{(1)} = \zeta\left[1 - 2\left(\frac{k}{k+1}\right)^n\right],$$

and

$$\Omega_{n,k}^{(2)} = \zeta^2\left[12\left(\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right) + 2\right].$$

On the other hand, let the marginal RV Z be absolutely continuous with bounded support $(0, b)$, where $0 < b < \infty$, and δ is an integer. Then,

$$CTE_{\delta,\mathrm{BS}}(Z_{[n,k]}) = \frac{1}{\delta - 1}\left(b - \sum_{i=0}^{\delta}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}A_{n,k}^{\delta-i}B_{n,k}^{i-j}C_{n,k}^{j}\int_0^1 p^{\delta+i+j}q(p)dp\right).$$

Finally, let the marginal RV Z be absolutely continuous with bounded support $(0, b)$, where $0 < b < \infty$, and δ is a non-integer. Then,

$$CTE_{\delta,\mathrm{BS}}\left(Z_{[n,k]}\right) = \frac{1}{\delta - 1}\left[b - \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{\delta}{i}\binom{i}{j}\left(3\Omega_{n,k}^{(1)}\right)^{i-j} \times \left(\frac{5}{2}\Omega_{n,k}^{(2)}\right)^{j}\int_0^1 p^{\delta}(p-1)^i(2p-1)^j q(p)\,dp\right], \tag{14}$$

provided that

$$3\left|\Omega_{n,k}^{(1)}\right| + \frac{5}{2}\left|\Omega_{n,k}^{(2)}\right| < 1. \tag{15}$$

Proof. The CDF of $Z_{[n,k]}$ that is part of the SAR (ζ) described in Alawady et al. [34] is presented as

$$F_{[n,k]}(z) = F_Z(z)\left[1 + 3\Omega_{n,k}^{(1)}(F_Z(z) - 1) + \frac{5}{2}\Omega_{n,k}^{(2)}(2F_Z^2(z) - 3F_Z(z) + 1)\right]. \tag{16}$$

Collecting the constant, linear, and quadratic terms in $F_Z(z)$, yields

$$F_{[n,k]}(z) = F_Z(z)\left[A_{n,k} + B_{n,k}F_Z(z) + C_{n,k}F_Z^2(z)\right] = G_{[n,k]}(F_Z(z)).$$

Since $z = Q_Z(p)$, we have $dz = q(p)\,dp$. Moreover, as z ranges from 0 to ∞, $p = F_Z(z)$ ranges from 0 to 1. Hence, by the change of variables $p = F_Z(z)$,

$$\begin{aligned} CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) &= \frac{1}{\delta - 1}\int_0^{\infty}\left[F_{[n,k]}(z) - F_{[n,k]}^{\delta}(z)\right]dz \\ &= \frac{1}{\delta - 1}\int_0^1\left[G_{[n,k]}(p) - G_{[n,k]}^{\delta}(p)\right]q(p)\,dp. \end{aligned}$$

This completes the proof of the first part of the theorem (for arbitrary support of $F_Z(z)$). We now proceed to prove Part 2. In this case, $F_Z(z)$ has finite support $(0, b)$, and δ

is assumed to be an integer. Since δ is a positive integer, the multinomial expansion is finite. First, we write

$$\left[A_{n,k} + B_{n,k}F_Z(z) + C_{n,k}F_Z^2(z)\right]^{\delta} = \sum_{i=0}^{\delta} \binom{\delta}{i} A_{n,k}^{\delta-i}\left[B_{n,k}F_Z(z) + C_{n,k}F_Z^2(z)\right]^{i}.$$

Applying the binomial theorem once more yields

$$\begin{aligned}\left[B_{n,k}F_Z(z) + C_{n,k}F_Z^2(z)\right]^{i} &= \sum_{j=0}^{i} \binom{i}{j} B_{n,k}^{i-j} C_{n,k}^{j} F_Z^{i-j}(z) F_Z^{2j}(z) \\ &= \sum_{j=0}^{i} \binom{i}{j} B_{n,k}^{i-j} C_{n,k}^{j} F_Z^{i+j}(z).\end{aligned}$$

Therefore,

$$F_{[n,k]}^{\delta}(z) = \sum_{i=0}^{\delta}\sum_{j=0}^{i} \binom{\delta}{i}\binom{i}{j} A_{n,k}^{\delta-i} B_{n,k}^{i-j} C_{n,k}^{j} F_Z^{\delta+i+j}(z).$$

Integrating over the support of Z, we obtain

$$\int_0^b F_{[n,k]}^{\delta}(z)\,dz = \sum_{i=0}^{\delta}\sum_{j=0}^{i} \binom{\delta}{i}\binom{i}{j} A_{n,k}^{\delta-i} B_{n,k}^{i-j} C_{n,k}^{j} \int_0^b F_Z^{\delta+i+j}(z)\,dz.$$

Now, apply the probability integral transformation $p = F_Z(z)$, $z = Q_Z(p)$, and $dz = q(p)\,dp$. Since $F_Z(0) = 0$ and $F_Z(b) = 1$, it follows that $\int_0^b F_Z^{\delta+i+j}(z)\,dz = \int_0^1 p^{\delta+i+j} q(p)\,dp$. Hence,

$$\int_0^b F_{[n,k]}^{\delta}(z)\,dz = \sum_{i=0}^{\delta}\sum_{j=0}^{i} \binom{\delta}{i}\binom{i}{j} A_{n,k}^{\delta-i} B_{n,k}^{i-j} C_{n,k}^{j} \int_0^1 p^{\delta+i+j} q(p)\,dp.$$

Finally, substituting this expression into

$$CTE_{\delta,\mathrm{BS}} = \frac{1}{\delta-1}\left[b - \int_0^b F_{[n,k]}^{\delta}(z)\,dz\right]$$

completes the proof of this part. We now turn to proving Part 3 when δ is a non-integer. Since $2F_Z^2(z) - 3F_Z(z) + 1 = \big(F_Z(z) - 1\big)\big(2F_Z(z) - 1\big)$, we can rewrite (16) as

$$F_{[n,k]}(z) = F_Z(z)\left[1 + \big(F_Z(z) - 1\big)\left\{3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)}\big(2F_Z(z) - 1\big)\right\}\right].$$

Consequently,

$$F_{[n,k]}^{\delta}(z) = F_Z^{\delta}(z)\left[1 + \big(F_Z(z) - 1\big)\left\{3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)}\big(2F_Z(z) - 1\big)\right\}\right]^{\delta}.$$

Put

$$x(z) = \big(F_Z(z) - 1\big)\left\{3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)}\big(2F_Z(z) - 1\big)\right\}.$$

Since $0 \le F_Z(z) \le 1$, we have $|F_Z(z) - 1| \le 1$ and $|2F_Z(z) - 1| \le 1$. Therefore,

$$|x(z)| \le |F_Z(z) - 1| \left[3\left|\Omega_{n,k}^{(1)}\right| + \frac{5}{2}\left|\Omega_{n,k}^{(2)}\right| |2F_Z(z) - 1| \right] \le 3\left|\Omega_{n,k}^{(1)}\right| + \frac{5}{2}\left|\Omega_{n,k}^{(2)}\right| < 1. \tag{17}$$

Thus, the generalized binomial expansion $(1 + x(z))^\delta = \sum_{i=0}^{\infty} \binom{\delta}{i} x^i(z)$ is valid throughout the support. Thus,

$$F_{[n,k]}^{\delta}(z) = F_Z^{\delta}(z) \sum_{i=0}^{\infty} \binom{\delta}{i} \left(F_Z(z) - 1\right)^i \left[3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)} \left(2F_Z(z) - 1\right) \right]^i. \tag{18}$$

Applying the ordinary finite binomial theorem to the term inside the brackets provides

$$\left[3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)} \left(2F_Z(z) - 1\right) \right]^i = \sum_{j=0}^{i} \binom{i}{j} \left(3\Omega_{n,k}^{(1)}\right)^{i-j} \left(\frac{5}{2}\Omega_{n,k}^{(2)}\right)^j \left(2F_Z(z) - 1\right)^j. \tag{19}$$

Substituting (19) into (18), we obtain

$$F_{[n,k]}^{\delta}(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\delta}{i} \binom{i}{j} \left(3\Omega_{n,k}^{(1)}\right)^{i-j} \left(\frac{5}{2}\Omega_{n,k}^{(2)}\right)^j F_Z^{\delta}(z) \left(F_Z(z) - 1\right)^i \left(2F_Z(z) - 1\right)^j.$$

Under condition (17), the generalized binomial series is uniformly convergent on the support. Moreover, the factors involving $F_Z(z)$ are uniformly bounded, while the quantile density is integrable, since

$$\int_0^1 q(p)\, dp = Q_Z(1) - Q_Z(0) = b < \infty.$$

Consequently, the uniformly convergent series may be integrated term by term against the integrable quantile density. Thus,

$$\begin{aligned} \int_0^b F_{[n,k]}^{\delta}(z)\, dz = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{\delta}{i} \binom{i}{j} \left(3\Omega_{n,k}^{(1)}\right)^{i-j} \left(\frac{5}{2}\Omega_{n,k}^{(2)}\right)^j \\ \times \int_0^b F_Z^{\delta}(z) \left(F_Z(z) - 1\right)^i \left(2F_Z(z) - 1\right)^j dz. \end{aligned} \tag{20}$$

Now, prepare the quantile transformation $p = F_Z(z)$, $z = Q_Z(p)$, $dz = q(p)\, dp$, where $q(p) = Q_Z'(p)$ is the quantile density of Z. Since the support of Z is contained in $[0, b]$, the limits of integration become $p = 0$ and $p = 1$. Therefore,

$$\int_0^b F_Z^{\delta}(z) \left(F_Z(z) - 1\right)^i \left(2F_Z(z) - 1\right)^j dz = \int_0^1 p^{\delta} (p-1)^i (2p-1)^j q(p)\, dp. \tag{21}$$

Substitute (21) into (20) and then use (6), yields (14). This is what was required to prove. The proof of the theorem is now complete. □

Remark 1. *The condition* (15) *can be expressed explicitly in terms of the SAR dependence parameter* ζ*, the record order* k*, and the record index* n*. Indeed, define*

$$A_{n,k}^* = 1 - 2\left(\frac{k}{k+1}\right)^n \quad \text{and} \quad B_{n,k}^* = 2 + 12\left[\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right].$$

Since

$$\Omega_{n,k}^{(1)} = \zeta A_{n,k}^* \quad \text{and} \quad \Omega_{n,k}^{(2)} = \zeta^2 B_{n,k}^*,$$

the above condition is equivalent to

$$3|A^*_{n,k}|\,|\zeta| + \frac{5}{2}|B^*_{n,k}|\,\zeta^2 < 1.$$

Consequently, when $B^*_{n,k} \neq 0$*, a sufficient and explicit admissible range for the dependence parameter is*

$$|\zeta| < \frac{\sqrt{9A^{*2}_{n,k} + 10|B^*_{n,k}|} - 3|A^*_{n,k}|}{5|B^*_{n,k}|}.$$

When $B^*_{n,k} = 0$*, the restriction reduces to*

$$|\zeta| < \frac{1}{3|A^*_{n,k}|}, \qquad A^*_{n,k} \neq 0.$$

For fixed k as $n \to \infty$*, and also for fixed n as* $k \to \infty$*, the admissible bound converges to*

$$|\zeta| < \frac{\sqrt{29} - 3}{10} \approx 0.2385.$$

Thus, the condition provides a transparent restriction on the dependence parameter under which the generalized binomial expansion used in Theorem 1 is uniformly valid over the support.

Example 1. *In this example, we evaluate the quantile-based* $CTE_{\delta,\mathrm{BS}}(Z_{[n,k]})$ *and* $CTE_{\delta,CDF}(Z_{[n,k]})$ *for the standard uniform and Weibull distributions, respectively.*

(i) Uniform distribution: *We consider the case where* $\delta = 4$*,* $n = \{2, 3, 5, 8, 10\}$*,* $k = 2$*, and* $\zeta = \{-0.4, -0.2, 0.2, 0.4\}$*. For* $F_Z(z) = z$*,* $0 < z < 1$*, the quantile density is* $q(p) = 1$*, and*

$$I_{\delta+i+j} = \int_0^1 p^{\delta+i+j} q(p)\, dp = \frac{1}{\delta + i + j + 1}.$$

Table 1 presents the behavior of $CTE_{4,\mathrm{BS}}(Z_{[n,k]})$ *for the standard uniform distribution for selected values of n when* $k = 2$*. It is observed that the measure increases with n for* $\zeta > 0$*, reflecting the enhanced effect of positive dependence induced by the* $SAR(\zeta)$ *structure, whereas for* $\zeta < 0$*, a decreasing trend is observed. This pattern is also observed in the other numerical examples considered below. Moreover, the magnitude of variation becomes more pronounced as n increases, highlighting the sensitivity of the proposed measure to both the dependence parameter* ζ *and the record index n.*

Take $\delta = 1.5$ *and apply the third part of Theorem 1. The quantile density is* $q(p) = 1$*, and*

$$I^{\delta}_{i,j} = \int_0^1 p^{\delta}(p-1)^i (2p-1)^j dp.$$

Expand $(2p-1)^j$ *using the binomial theorem*

$$(2p-1)^j = \sum_{r=0}^{j} \binom{j}{r} (2p)^r (-1)^{j-r}.$$

Then,

$$I^{\delta}_{i,j} = \sum_{r=0}^{j} \binom{j}{r} (2)^r (-1)^{j-r} \int_0^1 p^{\delta+r}(p-1)^i dp = (-1)^i \sum_{r=0}^{j} \binom{j}{r} (2)^r (-1)^{j-r} \beta(\delta + r + 1, i + 1).$$

Tables 1 and 2 show that the choice of δ *substantially affects the magnitude and monotonicity of the CTE. For* $\delta = 1.5$ *(Table 2), the values are substantially larger (roughly 1.0–1.4) than those*

for $\delta = 2$ (Table 1, roughly 0.21–0.27), reflecting the heavier weight assigned to the CDF term when δ is closer to unity. Moreover, in Table 2 the CTE exhibits a pronounced non-monotonic pattern in n and k, with a clear inversion at $k = 2$ where the trend reverses, whereas Table 1 shows a smoother, more gradual variation. This suggests that smaller δ amplifies the sensitivity of the entropy measure to both the dependence parameter ζ and the record indices, making δ a critical tuning parameter for uncertainty quantification in SAR concomitants.

(ii) Weibull distribution: *For the Weibull model $F_Z(z) = 1 - \exp\left\{-\left(\frac{z}{\beta}\right)^{\alpha}\right\}$, $z \geq 0$, $\alpha, \beta > 0$, we have $Q_Z(p) = \beta[-\ln(1-p)]^{1/\alpha}$, and therefore, $q(p) = \frac{\beta}{\alpha}\frac{[-\ln(1-p)]^{1/\alpha-1}}{1-p}$. Thus, we obtain*

$$CTE_{\delta,CDF}(Z_{[n,k]}) = \frac{\beta}{\alpha(\delta-1)}\int_0^1 \left[G_{[n,k]}(p) - G^{\delta}_{[n,k]}(p)\right]\frac{[-\ln(1-p)]^{1/\alpha-1}}{1-p}\,dp. \tag{22}$$

Although the representation in (22) is already explicit, numerical evaluation can be facilitated by introducing the transformation $t = -\ln(1-p)$. Then, $p = 1 - e^{-t}$, $dp = e^{-t}\,dt$, and $1 - p = e^{-t}$. Consequently, $\frac{dp}{1-p} = dt$ and $[-\ln(1-p)]^{1/\alpha-1} = t^{1/\alpha-1}$. Therefore, (22) becomes

$$CTE_{\delta,CDF}(Z_{[n,k]}) = \frac{\beta}{\alpha(\delta-1)}\int_0^{\infty} \left[G_{[n,k]}(1-e^{-t}) - G^{\delta}_{[n,k]}(1-e^{-t})\right]t^{1/\alpha-1}\,dt.$$

We take $\alpha = 2$, $\beta = 1$, and $\delta = 2$. For a numerical illustration, we consider $\zeta \in \{\pm 0.2, \pm 0.5\}$. The resulting values of $CTE_{2,CDF}(Z_{[n,k]})$, obtained by numerical integration using Mathematica Version:13.2.1.0, are reported in Table 3.

Table 1. $CTE_{4,\mathrm{BS}}(Z_{[n,2]})$ for Uniform$(0,1)$ distribution.

n	$\zeta = -0.4$	$\zeta = -0.2$	$\zeta = 0.2$	$\zeta = 0.4$
2	0.2576	0.2630	0.2689	0.2698
3	0.2393	0.2545	0.2764	0.2840
5	0.2175	0.2447	0.2842	0.2976
8	0.2048	0.2392	0.2883	0.3041
10	0.2017	0.2379	0.2893	0.3055

Table 2. $CTE_{1.5,\mathrm{BS}}(Z_{[n,k]})$ for uniform $(0,1)$ distribution.

k	n	$\zeta = -0.5$	$\zeta = -0.4$	$\zeta = -0.2$	$\zeta = 0.2$	$\zeta = 0.4$	$\zeta = 0.5$
1	2	0.92953	0.985756	1.095	1.30067	1.39678	1.443
2	2	1.13385	1.14855	1.17571	1.22139	1.23971	1.24759
2	3	0.97574	1.02286	1.11376	1.28126	1.35711	1.39284
3	2	1.25183	1.24363	1.22383	1.17245	1.14117	1.12406
3	3	1.10633	1.12725	1.16575	1.22997	1.25526	1.26591
3	4	0.994731	1.03833	1.12176	1.27268	1.33924	1.36996

Table 3. $CTE_{2,\mathrm{CDF}}(Z_{[n,k]})$ for Weibull distribution $(\beta = 1, \alpha = 2)$.

k	n	$\zeta = 0.2$	$\zeta = -0.2$	$\zeta = 0.5$	$\zeta = -0.5$
1	2	0.266657	0.249024	0.269735	0.227177
2	2	0.258879	0.254909	0.247675	0.237411
2	3	0.263624	0.249145	0.257854	0.221451
3	3	0.25847	0.252872	0.241929	0.227261
4	4	0.258221	0.251734	0.238615	0.221497
4	5	0.261136	0.248817	0.246476	0.214569

3.2. Quantile–Based Formulation of CTE in Past Lifetime

Let $T_t = (t - Z_{[n,k]}) \mid \{Z_{[n,k]} < t\}$ denote the past lifetime (or inactivity time) of the concomitant $Z_{[n,k]}$ at age t. That is, T_t is the RV with the conditional distribution of $t - Z_{[n,k]}$ given $Z_{[n,k]} < t$. The reversed hazard rate and the mean inactivity time (MIT) of $Z_{[n,k]}$ are provided, respectively, by

$$\mathrm{H}_{[n,k]}(t) = \frac{f_{[n,k]}(t)}{F_{[n,k]}(t)}$$

and

$$\widetilde{M}_{[n,k]}(t) = E\Big(t - Z_{[n,k]} \mid Z_{[n,k]} < t\Big) = \frac{1}{F_{[n,k]}(t)} \int_0^t F_{[n,k]}(z)dz.$$

Let $Q_{[n,k]}(u) = F^{-1}_{[n,k]}(u)$, $0 < u < 1$ denote the quantile function of $Z_{[n,k]}$, and let $q_{[n,k]}(u) = Q'_{[n,k]}(u)$ be its quantile density. The corresponding quantile-based reversed hazard rate is

$$\gamma_{[n,k]}(u) = \mathrm{H}_{[n,k]}\Big(Q_{[n,k]}(u)\Big) = \frac{1}{uq_{[n,k]}(u)}.$$

Furthermore, the quantile version of the MIT is

$$\Lambda_{[n,k]}(u) = \widetilde{M}_{[n,k]}\Big(Q_{[n,k]}(u)\Big) = \frac{1}{u}\int_0^u pq_{[n,k]}(p)dp.$$

Consequently,

$$q_{[n,k]}(u) = \Lambda'_{[n,k]}(u) + \frac{\Lambda_{[n,k]}(u)}{u}, \tag{23}$$

and hence,

$$Q_{[n,k]}(u) = \Lambda_{[n,k]}(u) + \int_0^u \frac{\Lambda_{[n,k]}(p)}{p}dp.$$

We now introduce the quantile formulation of the dynamic CTE in a past lifetime. For $\delta > 0$, $\delta \neq 1$, define

$$CTE_\delta(Z_{[n,k]}; t) = \frac{1}{\delta - 1}\int_0^t \left[\frac{F_{[n,k]}(z)}{F_{[n,k]}(t)} - \left\{\frac{F_{[n,k]}(z)}{F_{[n,k]}(t)}\right\}^\delta\right]dz.$$

Equivalently,

$$CTE_\delta(Z_{[n,k]}; t) = \frac{1}{\delta - 1}\left[\widetilde{M}_{[n,k]}(t) - \frac{1}{F^\delta_{[n,k]}(t)}\int_0^t F^\delta_{[n,k]}(z)dz\right].$$

Setting $t = Q_{[n,k]}(u)$ and using $F_{[n,k]}\Big(Q_{[n,k]}(p)\Big) = p$ and $dQ_{[n,k]}(p) = q_{[n,k]}(p)dp$, we obtain the following quantile representation:

$$CTE_\delta\Big(Z_{[n,k]}; Q_{[n,k]}(u)\Big) = \frac{1}{\delta - 1}\left[\Lambda_{[n,k]}(u) - \frac{1}{u^\delta}\int_0^u p^\delta q_{[n,k]}(p)dp\right], \qquad 0 < u < 1. \tag{24}$$

Thus, the dynamic CTE in a past lifetime can be expressed entirely in terms of the quantile density of the SAR concomitant. For convenience, define

$$C_{\delta,[n,k]}(u) = CTE_\delta\Big(Z_{[n,k]}; Q_{[n,k]}(u)\Big), \qquad 0 < u < 1.$$

The following theorem establishes a characterization of the SAR concomitant through this quantile-based CTE.

Theorem 2 (Characterization of the SAR concomitant via quantile CTE). *Let $Z_{[n,k]}$ be a non-negative absolutely continuous SAR concomitant whose support has lower endpoint of zero. Assume that $F_{[n,k]}$ is strictly increasing and that its quantile function $Q_{[n,k]}$ is twice differentiable on $(0,1)$. For $\delta > 0$ and $\delta \neq 1$, define*

$$C_{\delta,[n,k]}(u) = CTE_{\delta}\Big(Z_{[n,k]}; Q_{[n,k]}(u)\Big), \quad 0 < u < 1.$$

If $C_{\delta,[n,k]}$ is twice differentiable on $(0,1)$, then the quantile density is uniquely determined by

$$q_{[n,k]}(u) = \frac{\delta}{u} C_{\delta,[n,k]}(u) + (\delta+2) C'_{\delta,[n,k]}(u) + u C''_{\delta,[n,k]}(u), \quad 0 < u < 1. \tag{25}$$

Consequently,

$$Q_{[n,k]}(u) = \int_0^u \left[\frac{\delta}{p} C_{\delta,[n,k]}(p) + (\delta+2) C'_{\delta,[n,k]}(p) + p C''_{\delta,[n,k]}(p)\right] dp. \tag{26}$$

Hence, the distribution of $Z_{[n,k]}$ is uniquely determined by its quantile-based dynamic CTE.

Proof. From (24),

$$(\delta-1) C_{\delta,[n,k]}(u) = \Lambda_{[n,k]}(u) - \frac{1}{u^{\delta}} \int_0^u p^{\delta} q_{[n,k]}(p)\, dp.$$

Differentiating both sides with respect to u provides

$$(\delta-1) C'_{\delta,[n,k]}(u) = \Lambda'_{[n,k]}(u) - q_{[n,k]}(u) + \frac{\delta}{u}\Big[\Lambda_{[n,k]}(u) - (\delta-1) C_{\delta,[n,k]}(u)\Big]. \tag{27}$$

By (23), we obtain

$$q_{[n,k]}(u) = \Lambda'_{[n,k]}(u) + \frac{\Lambda_{[n,k]}(u)}{u}.$$

Substitution into (27) yields

$$C'_{\delta,[n,k]}(u) = \frac{\Lambda_{[n,k]}(u) - \delta C_{\delta,[n,k]}(u)}{u},$$

and therefore,

$$\Lambda_{[n,k]}(u) = \delta C_{\delta,[n,k]}(u) + u C'_{\delta,[n,k]}(u). \tag{28}$$

Differentiating (28), we obtain

$$\Lambda'_{[n,k]}(u) = (\delta+1) C'_{\delta,[n,k]}(u) + u C''_{\delta,[n,k]}(u).$$

Using again (23),

$$q_{[n,k]}(u) = \Lambda'_{[n,k]}(u) + \frac{\Lambda_{[n,k]}(u)}{u}.$$

Hence,

$$\begin{aligned} q_{[n,k]}(u) &= (\delta+1) C'_{\delta,[n,k]}(u) + u C''_{\delta,[n,k]}(u) + \frac{\delta}{u} C_{\delta,[n,k]}(u) + C'_{\delta,[n,k]}(u) \\ &= \frac{\delta}{u} C_{\delta,[n,k]}(u) + (\delta+2) C'_{\delta,[n,k]}(u) + u C''_{\delta,[n,k]}(u), \end{aligned}$$

which proves (25). Since the lower endpoint of the support is zero, $Q_{[n,k]}(0+) = 0$, and $Q'_{[n,k]}(u) = q_{[n,k]}(u)$ and the integration of (25) from 0 to u yields (26). Therefore, the quan-

tile function, and hence, the CDF $F_{[n,k]}$, is uniquely determined by $C_{\delta,[n,k]}$. This completes the proof. □

The characterization can also be expressed directly in terms of the quantile mean inactivity function. From (28), we have

$$\Lambda_{[n,k]}(u) = \delta C_{\delta,[n,k]}(u) + uC'_{\delta,[n,k]}(u).$$

Thus, the quantile-based CTE in a past lifetime determines the quantile mean inactivity function and, through (23), determines the quantile density $q_{[n,k]}$. This provides an alternative route to the characterization of the SAR concomitant.

3.3. Properties of the Cumulative Tsallis Entropy for $Z_{[n,k]}$

Throughout this subsection, let Z be a non-negative absolutely continuous RV with CDF F_Z, and let $Z_{[n,k]}$ denote the concomitant associated with the nth upper k-record in the SAR(ζ) model. Using the quantile representation (13) provided in Theorem 1, for integer $\delta \geq 2$, the power $G^{\delta}_{[n,k]}(p)$ can, if desired, be written explicitly by the ordinary multinomial expansion. In particular,

$$G^{\delta}_{[n,k]}(p) = \left(A_{n,k}p + B_{n,k}p^2 + C_{n,k}p^3\right)^{\delta} = \sum_{\substack{r,s,t\geq 0\\ r+s+t=\delta}} \frac{\delta!}{r!s!t!} A^r_{n,k} B^s_{n,k} C^t_{n,k} p^{r+2s+3t}.$$

Consequently,

$$CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) = \frac{1}{\delta-1}\int_0^1 \left[G_{[n,k]}(p) - \sum_{\substack{r,s,t\geq 0\\ r+s+t=\delta}} \frac{\delta!}{r!s!t!} A^r_{n,k} B^s_{n,k} C^t_{n,k} p^{r+2s+3t}\right] q(p)\, dp.$$

Lemma 2 (MIT representation of the CDF-based CTE)**.** *Let $Z_{[n,k]}$ be a non-negative absolutely continuous RV with CDF $F_{[n,k]}$, and let*

$$\widetilde{M}_{[n,k]}(z) = \frac{1}{F_{[n,k]}(z)}\int_0^z F_{[n,k]}(x)\, dx, \qquad F_{[n,k]}(z) > 0$$

denote its MIT. If $\delta > 0$, $\delta \neq 1$, and $CTE_{\delta,CDF}(Z_{[n,k]})$ is finite, then

$$CTE_{\delta,CDF}(Z_{[n,k]}) = E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})F^{\delta-1}_{[n,k]}(Z_{[n,k]})\right]. \tag{29}$$

Proof. For $b > 0$, put $H(b) = \int_0^b F_{[n,k]}(x)\, dx$. By the definition of the MIT,

$$\widetilde{M}_{[n,k]}(z)F_{[n,k]}(z) = \int_0^z F_{[n,k]}(x)\, dx.$$

Hence, for every $b > 0$,

$$E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})F^{\delta-1}_{[n,k]}(Z_{[n,k]})\mathbf{1}_{\{Z_{[n,k]}\leq b\}}\right] = \int_0^b H(z)F^{\delta-2}_{[n,k]}(z)f_{[n,k]}(z)\, dz = \frac{1}{\delta-1}\int_0^b H(z)\, dF^{\delta-1}_{[n,k]}(z).$$

Integration with parts on the finite interval $[0, b]$ yields

$$E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})F^{\delta-1}_{[n,k]}(Z_{[n,k]})\mathbf{1}_{\{Z_{[n,k]}\leq b\}}\right] = \frac{H(b)F^{\delta-1}_{[n,k]}(b)}{\delta-1} - \frac{1}{\delta-1}\int_0^b F^{\delta}_{[n,k]}(z)\, dz.$$

Since

$$H(b) = \int_0^b F_{[n,k]}(z)\, dz = b - \int_0^b \left[1 - F_{[n,k]}(z)\right] dz,$$

we obtain

$$E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})F_{[n,k]}^{\delta-1}(Z_{[n,k]})\mathbf{1}_{\{Z_{[n,k]}\le b\}}\right] = \frac{1}{\delta-1}\int_0^b \left[F_{[n,k]}(z) - F_{[n,k]}^{\delta}(z)\right] dz + \frac{1}{\delta-1}\left[H(b)F_{[n,k]}^{\delta-1}(b) - \int_0^b F_{[n,k]}(z)\,dz\right].$$

The last bracket converges to zero as $b \to \infty$ because $F_{[n,k]}(b) \to 1$ and, under the assumed finiteness of the CDF-based CTE, the truncated identity has a finite limit. Consequently, letting $b \to \infty$ yields

$$E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})F_{[n,k]}^{\delta-1}(Z_{[n,k]})\right] = \frac{1}{\delta-1}\int_0^\infty \left[F_{[n,k]}(z) - F_{[n,k]}^{\delta}(z)\right] dz.$$

This is precisely (29). □

It follows immediately from (29) that

$$CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) \le E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})\right], \qquad \delta > 1,$$

whereas

$$CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) \ge E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})\right], \quad 0 < \delta < 1.$$

Theorem 3. *Suppose that $Z_{[n,k]}$ is a non-negative absolutely continuous RV. Then, for $\delta > 0$, $\delta \neq 1$,*

$$CTE_{\delta,CDF}(Z_{[n,k]}) \ge 0.$$

Moreover, if $Z_{[n,k]}$ is nondegenerate and the CTE is finite, then

$$CTE_{\delta,CDF}(Z_{[n,k]}) > 0.$$

Proof. For $0 < \delta < 1$ and $0 \le F_{[n,k]}(z) \le 1$,

$$F_{[n,k]}^{\delta}(z) \ge F_{[n,k]}(z),$$

while $\delta - 1 < 0$. Hence, the integrand in (13), including its prefactor, is non-negative. Similarly, if $\delta > 1$, then

$$F_{[n,k]}^{\delta}(z) \le F_{[n,k]}(z),$$

and $\delta - 1 > 0$, so the CTE is again non-negative. If $Z_{[n,k]}$ is nondegenerate and absolutely continuous, then $0 < F_{[n,k]}(z) < 1$ on a set of positive Lebesgue measure. Therefore, $F_{[n,k]}(z) - F_{[n,k]}^{\delta}(z)$ has a strict sign on a set of positive measure, and consequently, the finite CTE is strictly positive.

Conversely, if the CTE is zero, then the non-negative integrand in (13) must vanish almost everywhere. Hence, $F_{[n,k]}(z)$ can take only the values 0 and 1 almost everywhere, which corresponds to a degenerate distribution. □

Lemma 3. *Let $Y_{[n,k]} = \phi(Z_{[n,k]})$, where ϕ is a strictly increasing differentiable function. Then,*

$$CTE_{\delta,CDF}(Z_{[n,k]}) = \frac{1}{\delta-1}\int_{\max\{0,\phi^{-1}(0)\}}^\infty \left[F_{[n,k]}(z) - F_{[n,k]}^{\delta}(z)\right]\phi'(z)\,dz, \tag{30}$$

whenever the integral is finite.

Proof. Since ϕ is strictly increasing,

$$F_{Y_{[n,k]}}(y) = F_{[n,k]}(\phi^{-1}(y)).$$

Applying the definition of the CTE and making the change of variable $y = \phi(z)$ yields (30). □

In particular, if

$$\phi(u) = au + b, \qquad a > 0, \quad b \geq 0,$$

then

$$CTE_{\delta,\mathrm{CDF}}(aZ_{[n,k]} + b) = aCTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}).$$

Thus, the CTE is translation-invariant for non-negative translations and homogeneous of degree one with respect to positive scale transformations.

Theorem 4. *Let $Z_{[n,k]}$ be a non-negative absolutely continuous RV with the finite cumulative Shannon entropy*

$$\mathcal{CE}(Z_{[n,k]}) = -\int_0^\infty F_{[n,k]}(z) \ln F_{[n,k]}(z)\, dz.$$

Then,

$$CTE_{\delta,CDF}(Z_{[n,k]}) \leq \mathcal{CE}(Z_{[n,k]}), \qquad \delta > 1, \tag{31}$$

and

$$CTE_{\delta,CDF}(Z_{[n,k]}) \geq \mathcal{CE}(Z_{[n,k]}), \qquad 0 < \delta < 1. \tag{32}$$

Proof. From (13),

$$CTE_{\delta,\mathrm{CDF}}(Z_{[n,k]}) = \frac{1}{\delta - 1} \int_0^\infty F_{[n,k]}(z) \left[1 - F_{[n,k]}^{\delta-1}(z)\right] dz.$$

For $u > 0$, $1 - u \leq -\ln u$. If $\delta > 1$, then

$$\frac{1 - F_{[n,k]}^{\delta-1}(z)}{\delta - 1} \leq -\ln F_{[n,k]}(z),$$

which yields (31). If $0 < \delta < 1$, division by the negative quantity $\delta - 1$ reverses the inequality, yielding (32). □

It is also consistent with the identity

$$\mathcal{CE}(Z_{[n,k]}) = E\left[\widetilde{M}_{[n,k]}(Z_{[n,k]})\right],$$

which provides an alternative interpretation of the comparison in terms of the MIT.

Example 2. *Let Z be uniformly distributed on $(0, c)$, where $c > 0$. Then,*

$$\mathcal{CE}(Z) = \frac{c}{4},$$

and

$$CTE_{\delta,CDF}(Z) = \frac{c}{2(\delta + 1)}.$$

Consequently,

$$CTE_{\delta,CDF}(Z) \leq \mathcal{CE}(Z), \qquad \delta > 1,$$

whereas

$$CTE_{\delta,CDF}(Z) \geq \mathcal{CE}(Z), \qquad 0 < \delta < 1,$$

which confirms the inequalities in the preceding theorem.

3.4. Past Rényi Entropy

Theorem 5. *Let $Z_{[n,k]}$ be the concomitant of the nth k-record value arising from the SAR(ζ) family. For $\delta > 0, \delta \neq 1$, the past Rényi entropy of order δ, as defined by Gupta and Nanda [39], is provided by*

$$\mathcal{R}_{\delta}^{(\zeta)}(Z_{[n,k]};t) = \mathcal{R}_{\delta}^{(\zeta)}(Z) + \frac{1}{1-\delta}\ln\left[\frac{\mathcal{J}(t)}{F_{[n,k]}^{\delta}(t)}\right], \tag{33}$$

where

$$\mathcal{J}(t) = \int_0^u q(p)^{1-\delta}\left[1 + 3\Omega_{n,k}^{(1)}(2p-1) + \frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2p-1)^2 - 1\right\}\right]^{\delta} dp, \qquad u = F_Z(t), \tag{34}$$

and

$$\mathcal{R}_{\delta}^{(\zeta)}(Z) = \frac{1}{1-\delta}\ln\int_0^{\infty} f_Z^{\delta}(z)\,dz \tag{35}$$

is the corresponding Rényi entropy of Z, under the convention adopted in (35).

Proof. Starting from the definition of the past Rényi entropy for the concomitant, we have

$$\mathcal{R}_{\delta}^{(\zeta)}(Z_{[n,k]};t) = \frac{1}{1-\delta}\ln\int_0^t \left(\frac{f_{[n,k]}(z)}{F_{[n,k]}(t)}\right)^{\delta} dz. \tag{36}$$

From the SAR family representation provided by Alawady et al. [34], the PDF of $Z_{[n,k]}$ is

$$\begin{aligned} f_{[n,k]}(z) = &\left(1 - 3\Omega_{n,k}^{(1)} + \frac{5}{2}\Omega_{n,k}^{(2)}\right) f_Z(z) \\ &+ 3\left(\Omega_{n,k}^{(1)} - \frac{5}{2}\Omega_{n,k}^{(2)}\right) f_{V_1}(z) + 5\Omega_{n,k}^{(2)} f_{V_2}(z). \end{aligned} \tag{37}$$

Using $f_{V_1}(z) = 2F_Z(z)f_Z(z)$ and $f_{V_2}(z) = 3F_Z^2(z)f_Z(z)$, Equation (37) can be written as

$$f_{[n,k]}(z) = f_Z(z)\left[1 + 3\Omega_{n,k}^{(1)}(2F_Z(z)-1) + \frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2F_Z(z)-1)^2 - 1\right\}\right]. \tag{38}$$

Substituting (38) into (36), we obtain

$$\begin{aligned} \mathcal{R}_{\delta}^{(\zeta)}(Z_{[n,k]};t) = \frac{1}{1-\delta}\ln\Bigg\{ &\frac{1}{F_{[n,k]}^{\delta}(t)}\int_0^t f_Z^{\delta}(z)\Big[1 + 3\Omega_{n,k}^{(1)}(2F_Z(z)-1) \\ &+ \frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2F_Z(z)-1)^2 - 1\right\}\Big]^{\delta} dz\Bigg\}. \end{aligned}$$

Now, make the probability integral transformation $p = F_Z(z)$, $dz = q(p)\,dp$, $f_Z(z) = \frac{1}{q(p)}$, and let $u = F_Z(t)$. Then,

$$\begin{aligned} &\int_0^t f_Z^{\delta}(z)\left[1 + 3\Omega_{n,k}^{(1)}(2F_Z(z)-1) + \frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2F_Z(z)-1)^2 - 1\right\}\right]^{\delta} dz \\ &= \int_0^u q(p)^{1-\delta}\left[1 + 3\Omega_{n,k}^{(1)}(2p-1) + \frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2p-1)^2 - 1\right\}\right]^{\delta} dp. \end{aligned}$$

Furthermore, the same transformation over the whole support provides

$$\int_0^{\infty} f_Z^{\delta}(z)\,dz = \int_0^1 q(p)^{1-\delta}\,dp.$$

Therefore, using (35), we obtain

$$\begin{aligned}\mathcal{R}_{\delta}^{(\zeta)}(Z_{[n,k]};t) &= \frac{1}{1-\delta}\ln\int_0^{\infty} f_Z^{\delta}(z)\,dz \\ &\quad + \frac{1}{1-\delta}\ln\left[\frac{\displaystyle\int_0^u q(p)^{1-\delta}\left[1+3\Omega_{n,k}^{(1)}(2p-1)+\frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2p-1)^2-1\right\}\right]^{\delta}dp}{F_{[n,k]}^{\delta}(t)}\right] \\ &= \mathcal{R}_{\delta}^{(\zeta)}(Z)+\frac{1}{1-\delta}\ln\left[\frac{\displaystyle\int_0^u q(p)^{1-\delta}\left[1+3\Omega_{n,k}^{(1)}(2p-1)+\frac{5}{4}\Omega_{n,k}^{(2)}\left\{3(2p-1)^2-1\right\}\right]^{\delta}dp}{F_{[n,k]}^{\delta}(t)}\right]. \qquad (39)\end{aligned}$$

Defining $\mathcal{J}(t)$ as in (34), Equation (39) yields the result in (33). □

For integer $\delta \geq 1$, the power appearing in (34) can be expanded by the binomial theorem twice. In this case,

$$\begin{aligned}\mathcal{J}(t) &= \sum_{j=0}^{\delta}\sum_{v=0}^{j}\binom{\delta}{j}\binom{j}{v}\left(3\Omega_{n,k}^{(1)}\right)^{j-v}\left(\frac{5}{4}\Omega_{n,k}^{(2)}\right)^{v} \\ &\quad\times\int_0^u q(p)^{1-\delta}(2p-1)^{j-v}\left\{3(2p-1)^2-1\right\}^{v}dp, \qquad u=F_Z(t).\end{aligned}$$

For noninteger δ, the corresponding generalized binomial expansion is an infinite series and therefore requires a convergence condition. A sufficient condition ensuring absolute convergence uniformly for $0 \leq p \leq 1$ is (15). Indeed,

$$|2p-1| \leq 1, \quad \left|3(2p-1)^2-1\right| \leq 2, \quad 0 \leq p \leq 1.$$

Hence, under condition (15), the generalized binomial expansion can be used in (34). Without such a condition, the exact integral representation (34) should be retained without expanding the power δ.

3.5. Past Tsallis Entropy

Theorem 6. *Let Z be a continuous RV representing the life length of a system, and let $Z_{[n,k]}$ denote the concomitant of the nth k-record value generated from Z. For $\delta \in \{2,3,\ldots\}$, define*

$$u = F_Z(t), \qquad 0 < u < 1,$$

and let

$$q(p) = \frac{d}{dp}F_Z^{-1}(p) = \frac{1}{f_Z(F_Z^{-1}(p))}, \qquad 0 < p < 1.$$

Then, the past Tsallis entropy of $Z_{[n,k]}$ at time t is provided by

$$\mathcal{T}_{\delta}^{(\zeta)}\left(Z_{[n,k]};t\right) = \frac{1}{\delta-1}\left[1-\frac{1}{F_{[n,k]}^{\delta}(t)}\sum_{j=0}^{\delta}\sum_{v=0}^{j}\binom{\delta}{j}\binom{j}{v}\left(3\Omega_{n,k}^{(1)}\right)^{j-v}\left(\frac{5}{4}\Omega_{n,k}^{(2)}\right)^{v} \times\int_{0}^{u}q(p)^{\,1-\delta}(2p-1)^{j-v}\left\{3(2p-1)^{2}-1\right\}^{v}dp\right]. \tag{40}$$

Proof. By definition, the past Tsallis entropy of $Z_{[n,k]}$ at time t is

$$\begin{aligned}\mathcal{T}_{\delta}^{(\zeta)}\left(Z_{[n,k]};t\right) &= \frac{1}{\delta-1}\left[1-\int_{0}^{t}\left\{\frac{f_{[n,k]}(z)}{F_{[n,k]}(t)}\right\}^{\delta}dz\right]\\ &= \frac{1}{\delta-1}\left[1-\frac{1}{F_{[n,k]}^{\delta}(t)}\int_{0}^{t}f_{[n,k]}^{\delta}(z)\,dz\right].\end{aligned}$$

Using the representation of the density of $Z_{[n,k]}$ under $SAR(\zeta)$,

$$f_{[n,k]}(z)=f_{Z}(z)\left[1+3\Omega_{n,k}^{(1)}\{2F_{Z}(z)-1\}+\frac{5}{4}\Omega_{n,k}^{(2)}\{3(2F_{Z}(z)-1)^{2}-1\}\right],$$

we obtain

$$\mathcal{T}_{\delta}^{(\zeta)}\left(Z_{[n,k]};t\right) = \frac{1}{\delta-1}\left[1-\frac{1}{F_{[n,k]}^{\delta}(t)}\int_{0}^{t}f_{Z}^{\delta}(z)\left[1+3\Omega_{n,k}^{(1)}\{2F_{Z}(z)-1\} +\frac{5}{4}\Omega_{n,k}^{(2)}\{3(2F_{Z}(z)-1)^{2}-1\}\right]^{\delta}dz\right]. \tag{41}$$

Since δ is a positive integer, the binomial expansion yields

$$\begin{aligned}&\left[1+3\Omega_{n,k}^{(1)}\{2F_{Z}(z)-1\}+\tfrac{5}{4}\Omega_{n,k}^{(2)}\{3(2F_{Z}(z)-1)^{2}-1\}\right]^{\delta}\\ &=\textstyle\sum_{j=0}^{\delta}\binom{\delta}{j}\left[3\Omega_{n,k}^{(1)}\{2F_{Z}(z)-1\}+\tfrac{5}{4}\Omega_{n,k}^{(2)}\{3(2F_{Z}(z)-1)^{2}-1\}\right]^{j}\\ &=\textstyle\sum_{j=0}^{\delta}\sum_{v=0}^{j}\binom{\delta}{j}\binom{j}{v}\left(3\Omega_{n,k}^{(1)}\right)^{j-v}\left(\tfrac{5}{4}\Omega_{n,k}^{(2)}\right)^{v}\\ &\quad\times(2F_{Z}(z)-1)^{j-v}\{3(2F_{Z}(z)-1)^{2}-1\}^{v}.\end{aligned} \tag{42}$$

Finally, make the transformation

$$p=F_{Z}(z),\qquad z=F_{Z}^{-1}(p),\qquad dz=q(p)\,dp.$$

Consequently,

$$f_{Z}^{\delta}(z)\,dz=f_{Z}^{\delta}(F_{Z}^{-1}(p))q(p)\,dp=q^{1-\delta}(p)\,dp,$$

and the limits $z=0$ and $z=t$ become $p=0$ and $p=u=F_{Z}(t)$, respectively. Substituting these expressions into (41) and using (42) yields (40). □

Remark 2. *The above finite-sum representation is derived for the positive integer δ, for which the ordinary binomial expansion terminates. If $\delta>1$ is a non-integer, the binomial expansion is*

generally an infinite series and requires a convergence condition. In particular, a sufficient condition for its uniform validity over $0 \le F_Z(z) \le 1$ *is* (15). *Indeed,*

$$\left|3\Omega_{n,k}^{(1)}(2F_Z(z)-1)+\frac{5}{4}\Omega_{n,k}^{(2)}\{3(2F_Z(z)-1)^2-1\}\right| \le 3|\Omega_{n,k}^{(1)}|+\frac{5}{2}|\Omega_{n,k}^{(2)}|<1.$$

Under this condition, the generalized binomial series converges, although it no longer reduces to the finite sum appearing in the theorem. The condition (15) *is sufficient rather than necessary.*

4. Nonparametric Estimation

Nonparametric estimation techniques are widely used for estimating entropy and related information measures when the underlying marginal distribution is unknown. Among these methods, KDE provides a flexible and effective way to estimate an unknown PDF. Motivated by this approach, we develop a kernel-based nonparametric plug-in estimator for the density-based Tsallis entropy of the concomitant $Z_{[n,k]}$.

It is important to distinguish the functional considered in this section from the CDF-based CTE developed in Section 2. Here, the objective is to estimate the ordinary density-based Tsallis entropy, which is defined in (1) (note that $\int_0^\infty f_Y(y)dy = 1$) is

$$H_\delta(Y) = \frac{1}{\delta-1}\left\{1-\int_0^\infty f_Y^\delta(y)\,dy\right\}, \qquad \delta>0, \quad \delta\neq 1 \tag{43}$$

for the concomitant $Z_{[n,k]}$.

4.1. Kernel Estimation of the Marginal Distribution

Suppose that $Z_1, Z_2, \ldots, Z_N$ constitute a random sample from the marginal distribution of Z, with unknown density f_Z and CDF F_Z. Let K be a non-negative kernel satisfying

$$K(u) \ge 0, \quad \int_{-\infty}^{\infty} K(u)\,du = 1,$$

and let $h > 0$ denote the bandwidth. The kernel density estimator of f_Z is

$$\widehat{f}_N(z) = \frac{1}{Nh}\sum_{i=1}^{N} K\left(\frac{z-Z_i}{h}\right).$$

The corresponding integrated kernel estimator of F_Z is

$$\widehat{F}_N(z) = \int_{-\infty}^{z} \widehat{f}_N(t)\,dt = \frac{1}{N}\sum_{i=1}^{N} \mathcal{K}\left(\frac{z-Z_i}{h}\right),$$

where $\mathcal{K}(u) = \int_{-\infty}^{u} K(v)\,dv$ is the CDF associated with the kernel K.

The bandwidth h controls the degree of smoothing. In the numerical study below, several standard bandwidth-selection procedures are considered, including the rule-of-thumb (RT), direct plug-in (DPI), unbiased cross-validation (UCV), and biased cross-validation (BCV) methods. For more details about PDF estimation methods, see Parzen [40] and Silverman [41].

4.2. Plug-In Estimator of the Tsallis Entropy

Recall from the SAR model that the density of the concomitant $Z_{[n,k]}$ can be written as

$$f_{[n,k]}(z) = f_Z(z)\left[1+3\Omega_{n,k}^{(1)}\{2F_Z(z)-1\}+\frac{5}{4}\Omega_{n,k}^{(2)}\left\{3\{2F_Z(z)-1\}^2-1\right\}\right].$$

Thus, replacing the unknown marginal quantities f_Z and F_Z by their kernel estimates $\widehat{f}_N$ and $\widehat{F}_N$, respectively, provides the following plug-in function:

$$\widehat{f}_{[n,k]}(z) = \widehat{f}_N(z)\left[1 + 3\Omega^{(1)}_{n,k}\{2\widehat{F}_N(z) - 1\} + \frac{5}{4}\Omega^{(2)}_{n,k}\left\{3\{2\widehat{F}_N(z) - 1\}^2 - 1\right\}\right]. \tag{44}$$

For convenience, define

$$\mathcal{L}_{n,k}(z) = 1 + 3\Omega^{(1)}_{n,k}\{2\widehat{F}_N(z) - 1\} + \frac{5}{4}\Omega^{(2)}_{n,k}\left\{3\{2\widehat{F}_N(z) - 1\}^2 - 1\right\}.$$

Then,

$$\widehat{f}_{[n,k]}(z) = \widehat{f}_N(z)\mathcal{L}_{n,k}(z). \tag{45}$$

Consequently, a natural kernel plug-in estimator of the density-based Tsallis entropy of $Z_{[n,k]}$ is

$$\widehat{H}^{(\zeta)}_{\delta}(Z_{[n,k]}) = \frac{1}{\delta - 1}\int_0^{\infty}\left[\widehat{f}_{[n,k]}(z) - \widehat{f}^{\delta}_{[n,k]}(z)\right]dz, \qquad \delta > 0, \quad \delta \neq 1. \tag{46}$$

Using (45), this can equivalently be written as

$$\widehat{H}^{(\zeta)}_{\delta}(Z_{[n,k]}) = \frac{1}{\delta - 1}\int_0^{\infty}\left\{\widehat{f}_N(z)\mathcal{L}_{n,k}(z) - \left[\widehat{f}_N(z)\mathcal{L}_{n,k}(z)\right]^{\delta}\right\}dz. \tag{47}$$

Since $f_{[n,k]}$ is a PDF on the support of $Z_{[n,k]}$, its integral is equal to one. The corresponding finite-sample plug-in function $\widehat{f}_{[n,k]}$, however, need not integrate exactly to one over $[0, \infty)$. This can occur, in particular, when an unbounded-support kernel such as the Gaussian kernel is used because of boundary leakage and finite-sample estimation error. Equation (46) should therefore be understood as a DPI estimate of the target entropy functional rather than as a claim that $\widehat{f}_{[n,k]}$ is an exactly normalized density.

For non-integer values of δ, the direct numerical evaluation of (47) is preferable to expanding the term $[\mathcal{L}_{n,k}(z)]^{\delta}$ by a generalized binomial series. As discussed earlier in the paper, such an expansion becomes an infinite series when δ is non-integer and requires an additional convergence condition. No such series expansion is required in the DPI procedure.

When δ is a non-integer, we require $\mathcal{L}_{n,k}(z) \geq 0$ throughout the domain over which the numerical integral is evaluated. Together with the non-negativity of the kernel estimator $\widehat{f}_N(z)$, this guarantees that $\widehat{f}_{[n,k]}(z)$ is non-negative and that its real power $\widehat{f}^{\delta}_{[n,k]}(z)$ is well defined. For integer δ, this restriction is not needed for the algebraic definition of the power, although non-negativity remains desirable for interpreting $\widehat{f}_{[n,k]}$ as a density estimate.

4.3. Numerical Implementation

The integral in (46) is evaluated numerically. In particular, the integration should not be restricted to the intervals between successive observations. Such a restriction would be inappropriate for an unbounded-support kernel, such as the Gaussian kernel, because $\widehat{f}_N(z)$ generally remains positive outside the interval between the smallest and largest observations.

For a compactly supported kernel, the numerical integration can be restricted to the effective support of the resulting kernel estimate. For an unbounded-support kernel, such as the Gaussian kernel, the integral over $[0, \infty)$ is evaluated numerically using a sufficiently large upper truncation point together with an appropriate numerical quadrature rule. The truncation point is increased until the resulting entropy estimate is numerically stable.

The bandwidth has an important effect on the resulting entropy estimate. We therefore consider four commonly used bandwidth-selection procedures: the RT bandwidth h_{RT}, the DPI bandwidth h_{DPI}, the UCV bandwidth h_{UCV}, and the BCV bandwidth h_{BCV}. For each selected bandwidth, the corresponding $\widehat{f}_N$ and $\widehat{F}_N$ are substituted into (44), and the resulting function is used in (46).

For illustration, Figure 1 compares the Gaussian kernel estimates obtained using the different bandwidth-selection procedures for $N = 30$. The resulting RT bandwidth in this example is 112.746. The figure demonstrates the effect of the bandwidth selection procedure on the degree of smoothing and consequently on the estimated entropy. Figure 1 presents the uncorrected Gaussian-kernel density estimates as a baseline illustration of the effect of bandwidth selection. Because the Gaussian kernel has support over the entire real line, these estimates may assign positive density to negative values, even though breaking strength is non-negative. This behavior is a boundary-leakage effect of the uncorrected estimator and should not be interpreted as evidence that negative breaking strengths are physically possible. The reflection-based boundary correction used in the subsequent data analysis is discussed in Section 5.4.

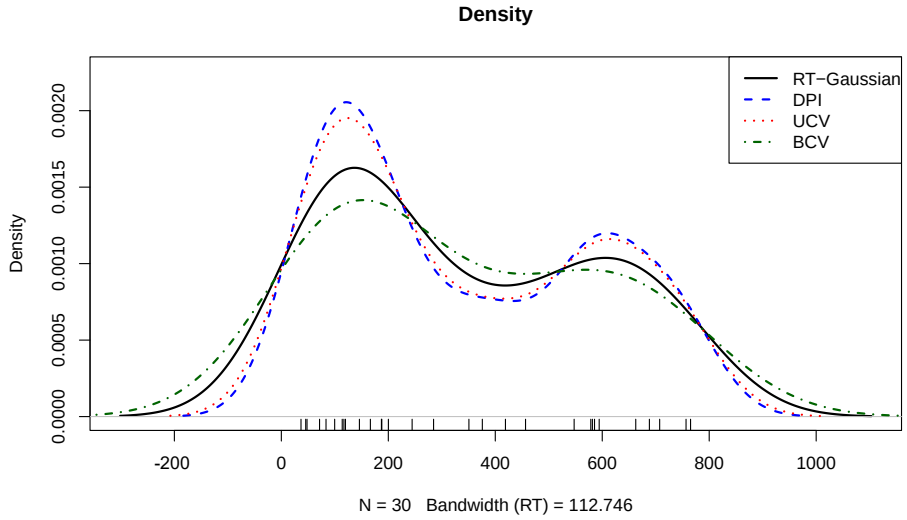


Figure 1. Comparison of the kernel density estimates obtained using different bandwidth-selection procedures for $N = 30$.

Table 4 reports the resulting estimates of $\widehat{H}_{\delta}^{(\zeta)}(Z_{[n,k]})$ for $\zeta = 0.2$ and $\zeta = 0.5$. The entries in the table are entropy estimates, not bandwidth values; hence, the bandwidth-selection procedures are identified in the column headings by the corresponding subscripts. The numerical results in Table 4 show several features of the proposed estimator. For fixed N, the estimated Tsallis entropy is smaller for $\zeta = 0.5$ than for $\zeta = 0.2$ for all four bandwidth-selection procedures considered. As N increases from 10 to 30, the entropy estimates generally decrease, whereas for larger sample sizes they exhibit moderate fluctuations rather than a uniformly monotone pattern. Thus, no general monotonic relationship between the sample size and the estimated entropy should be inferred from this numerical experiment.

The choice of bandwidth also affects the estimated entropy. In the present numerical study, the RT bandwidth produces the largest entropy estimate for each considered value of N and ζ, whereas the DPI bandwidth generally produces the smallest estimate. These findings are empirical characteristics of the present data and numerical experiment and should not be interpreted as a general theoretical ordering of the entropy estimates under the four bandwidth-selection methods.

Overall, the proposed procedure provides a direct and flexible model-assisted nonparametric approach to estimating the Tsallis entropy of the concomitant $Z_{[n,k]}$. It combines a nonparametric estimate of the unknown marginal distribution with the SAR dependence structure, while avoiding the additional convergence restrictions associated with generalized binomial expansions for non-integer values of δ.

Table 4. Nonparametric estimates of the density-based Tsallis entropy $\widehat{H}_{\delta}^{(\zeta)}(Z_{[n,k]})$ for different bandwidth selection procedures.

	$\zeta = 0.2$				$\zeta = 0.5$			
N	$\widehat{H}_{\delta,\mathrm{RT}}$	$\widehat{H}_{\delta,\mathrm{DPI}}$	$\widehat{H}_{\delta,\mathrm{UCV}}$	$\widehat{H}_{\delta,\mathrm{BCV}}$	$\widehat{H}_{\delta,\mathrm{RT}}$	$\widehat{H}_{\delta,\mathrm{DPI}}$	$\widehat{H}_{\delta,\mathrm{UCV}}$	$\widehat{H}_{\delta,\mathrm{BCV}}$
10	0.50401	0.49893	0.50187	0.50301	0.46593	0.45843	0.46266	0.46438
20	0.47118	0.45162	0.46269	0.46720	0.41494	0.38557	0.40219	0.40897
30	0.39970	0.37227	0.38721	0.39371	0.32618	0.28572	0.30781	0.31740
70	0.48041	0.45816	0.47030	0.47558	0.40981	0.37708	0.39503	0.40277
90	0.49205	0.47256	0.48328	0.48788	0.42230	0.39229	0.40882	0.41590
100	0.48758	0.47007	0.47962	0.48377	0.41911	0.39191	0.40678	0.41323
150	0.48519	0.46513	0.47595	0.48075	0.41231	0.38181	0.39830	0.40559
200	0.49434	0.47658	0.48623	0.49046	0.42169	0.39358	0.40886	0.41555

5. Application to Breaking Strengths of Jute Fiber Data

To illustrate the practical applicability of the proposed CDF-based CTE for concomitants of k-record values, we analyze a real data set on the breaking strengths of jute fibers. The data were previously considered by Xia et al. [42] and consist of paired breaking-strength measurements obtained at two different gauge lengths. In the notation used throughout this paper, Y denotes the breaking strength measured at a gauge length of 10 mm, whereas Z denotes the corresponding breaking strength measured at a gauge length of 20 mm.

5.1. Description and Exploratory Analysis

The two samples, each containing $N = 30$ observations, are

$$\begin{aligned} Y = \{&693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48,\\ &108.94, 50.16, 671.49, 183.16, 257.44, 727.23, 291.27, 101.15,\\ &376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93,\\ &590.48, 212.13, 303.90, 506.60, 530.55, 177.25\}, \end{aligned}$$

and

$$\begin{aligned} Z = \{&71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16,\\ &662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36,\\ &765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60,\\ &119.86, 48.0130, 200.16, 36.75, 244.53, 83.55\}. \end{aligned}$$

The observations are naturally treated as paired measurements because the two gauge-length measurements correspond to the same experimental units. The principal descriptive statistics are presented in Table 5.

Both samples exhibit substantial variability, with the breaking strengths covering a wide range. The means are somewhat larger than the medians, indicating moderate right-skewness, although the skewness coefficients are not large. The negative kurtosis values indicate distributions that are flatter than a normal distribution.

Because the two measurements are paired, their dependence is also relevant to the SAR model. The sample Pearson correlation coefficient is $r_{YZ} = -0.18314$, indicating

a weak negative linear association between the two measurements. The corresponding Spearman rank correlation is also negative and weak. It should be emphasized that the Pearson correlation coefficient is not itself the SAR dependence parameter ζ.

Table 5. Descriptive statistics for the breaking-strength data.

Statistic	Y (10 mm)	Z (20 mm)
Minimum	43.930	36.750
Maximum	778.170	765.140
Mean	365.7297	340.7400
Median	313.8650	264.5850
First quartile (Q_1)	160.420	116.205
Third quartile (Q_3)	602.275	582.5925
Variance	53,972.990	61,174.375
Standard deviation	232.321	247.3345
Standard error	42.4158	45.1569
Skewness	0.393	0.380
Kurtosis	−1.238	−1.410

5.2. Marginal Distributional Analysis

Before estimating the CDF-based CTE, we examine the marginal distributions of Y and Z. Several standard lifetime distributions are considered, including the Lindley, gamma, generalized exponential, exponential, Weibull, Lomax, Rayleigh, Gumbel, inverse Weibull, and Gumbel-type models. For the Y sample, the Lindley distribution has the smallest Akaike information criterion (AIC), corrected AIC (AICC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), and consistent AIC (CAIC)-type information criteria among the candidate models considered. Its corresponding the Kolmogorov–Smirnov (KS) statistic and p-value are

$$D_{\mathrm{KS}} = 0.10270, \qquad p = 0.87791.$$

The gamma and generalized exponential distributions provide very similar fits, with p-values 0.84800 and 0.84359, respectively.

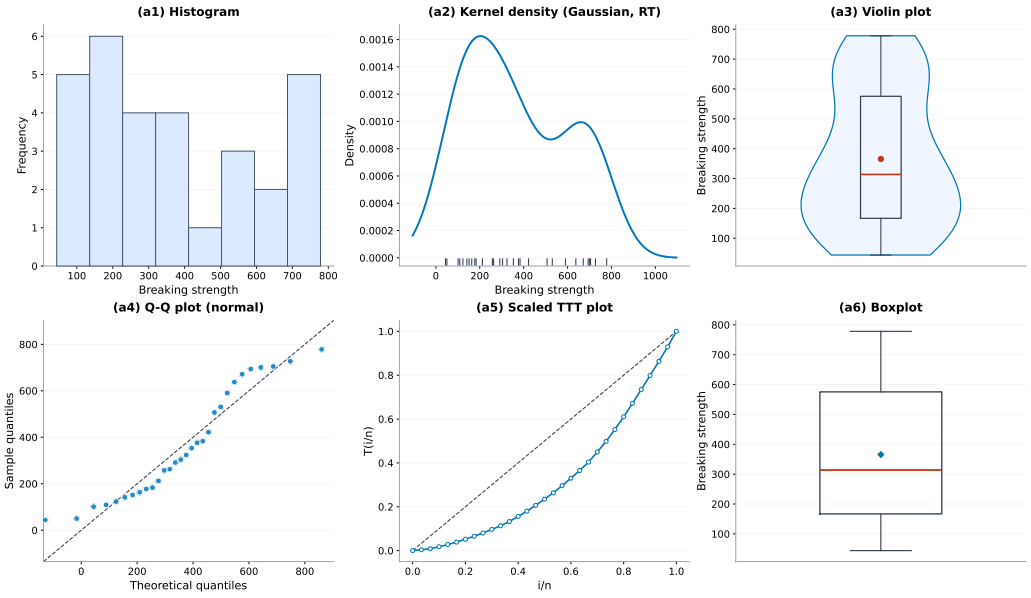
For the Z sample, the Lindley distribution again has a competitive fit according to the information criteria, while the Lomax distribution gives a relatively small KS statistic,

$$D_{\mathrm{KS}} = 0.13136, \qquad p = 0.63134.$$

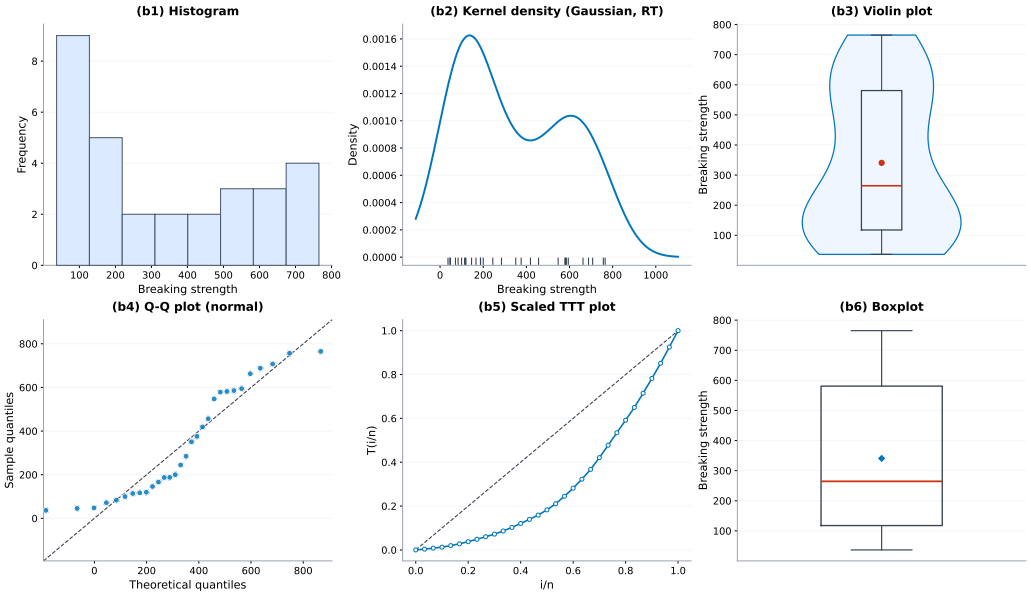
The Weibull and exponential distributions also provide acceptable fits. In contrast, the Rayleigh model provides a p-value of approximately 0.026, and hence, would be rejected at the 5% significance level if the reported KS p-value is interpreted in the usual way.

The numerical goodness-of-fit results should be regarded primarily as descriptive comparisons. In particular, because the parameters are estimated from the same data, the usual tabulated KS p-values do not automatically have their nominal finite-sample interpretation unless an appropriate parameter-estimation correction or bootstrap procedure is used.

The goodness-of-fit graphics for the two samples, Figure 2, provide complementary histogram, kernel-density, violin, Q–Q, TTT, and boxplot diagnostics.



(**a**) Dataset Y



(**b**) Dataset Z

Figure 2. Goodness-of-fit assessment for the two jute-fiber breaking-strength data sets.

5.3. Estimation of the SAR Dependence Parameter

The proposed CTE for the concomitant $Z_{[n,k]}$ depends on the dependence parameter ζ through

$$\Omega_{n,k}^{(1)} = \zeta\left[1 - 2\left(\frac{k}{k+1}\right)^n\right]$$

and

$$\Omega_{n,k}^{(2)} = \zeta^2\left[2 + 12\left\{\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right\}\right].$$

For the paired observations, we estimate ζ by the rank-based pseudo-likelihood (PL) associated with the SAR copula. Define

$$U_i = \frac{R_{Y_i} - 1/2}{N}, \qquad V_i = \frac{R_{Z_i} - 1/2}{N}, \qquad i = 1, \ldots, N,$$

where R_{Y_i} and R_{Z_i} denote the ranks of Y_i and Z_i, respectively. The pseudo-log-likelihood is

$$\ell_{\mathrm{PL}}(\zeta) = \sum_{i=1}^{N} \ln c_\zeta(U_i, V_i),$$

where $c_\zeta(u,v) = 1 + 3\zeta(2u-1)(2v-1) + \frac{5}{4}\zeta^2\left[3(2u-1)^2-1\right]\left[3(2v-1)^2-1\right]$, $-\frac{\sqrt{7}}{5} \le \zeta \le \frac{\sqrt{7}}{5}$. For these data, the resulting estimate is $\widehat{\zeta} = -0.16593$. Thus, the fitted SAR model indicates weak negative dependence, consistent in direction with the negative sample Pearson and Spearman associations. Nevertheless, $\widehat{\zeta}$ should not be interpreted as a correlation coefficient; it is the fitted dependence parameter of the SAR copula. Moreover,

$$|\widehat{\zeta}| = 0.16593 < 0.2385.$$

Hence, the fitted estimate lies within the limiting admissible range derived in Remark 1 for fixed k as $n \to \infty$. This comparison supports the applicability of the expansion-based results in the present empirical setting, subject to the convergence condition stated in Remark 1.

5.4. Nonparametric Estimation of the Marginal CDF

The CDF-based formulation of the CTE requires an estimate of the marginal CDF F_Z, rather than an estimate of the ordinary density-based Tsallis functional. We therefore construct a kernel estimator of F_Z from the observed Z values.

Since the breaking-strength variable is non-negative, a boundary correction is desirable. We use the reflection version of the Gaussian kernel estimator,

$$\widehat{f}^{R}_{Z,N}(z) = \frac{1}{Nh}\sum_{i=1}^{N}\left[\phi\left(\frac{z-Z_i}{h}\right) + \phi\left(\frac{z+Z_i}{h}\right)\right], \qquad z \ge 0,$$

where $\phi(\cdot)$ denotes the standard normal density and h is the bandwidth. Since the breaking-strength variable is supported on the non-negative half-line, the reflection method is used to reduce boundary bias near zero. Accordingly, the uncorrected estimates in Figure 1 serve only as a baseline for illustrating bandwidth effects, whereas the reflection-based estimator is used in the subsequent boundary-corrected analysis. Integrating the reflected estimator yields

$$\widehat{F}^{R}_{Z,N}(z) = \frac{1}{N}\sum_{i=1}^{N}\left[\Phi\left(\frac{z-Z_i}{h}\right) + \Phi\left(\frac{z+Z_i}{h}\right) - 1\right], \qquad z \ge 0,$$

where $\Phi(\cdot)$ is the standard normal distribution function. For the subsequent CTE calculation, we use the RT bandwidth $h = h_{\mathrm{RT}} = 112.746$. The use of the reflection correction is particularly appropriate here because both breaking-strength variables are supported on $[0, \infty)$.

5.5. Nonparametric Estimation of the CDF of the Concomitant $Z_{[n,k]}$

The principal advantage of the present formulation is that the estimated marginal CDF can be inserted directly into the exact CDF structure obtained theoretically for the SAR concomitant. Recall that (go back to (16))

$$F_{[n,k]}(z) = G_{[n,k]}\left(F_Z(z)\right),$$

where

$$G_{[n,k]}(u) = u\left[1 + 3\Omega_{n,k}^{(1)}(u-1) + \frac{5}{2}\Omega_{n,k}^{(2)}(2u^2 - 3u + 1)\right].$$

Accordingly, the nonparametric estimate of the CDF of the concomitant is defined by

$$\widehat{F}_{[n,k]}(z) = G_{[n,k]}\left(\widehat{F}_{Z,N}^{R}(z)\right),$$

where, in the present application,

$$\widehat{\Omega}_{n,k}^{(1)} = \widehat{\zeta}\left[1 - 2\left(\frac{k}{k+1}\right)^n\right],$$

and

$$\widehat{\Omega}_{n,k}^{(2)} = \widehat{\zeta}^2\left[2 + 12\left\{\left(\frac{k}{k+2}\right)^n - \left(\frac{k}{k+1}\right)^n\right\}\right].$$

5.6. Nonparametric Estimation of the CDF-Based CTE

For $\delta > 0$, $\delta \neq 1$, the nonparametric estimator of $CTE_{\delta,\mathrm{CDF}}\left(Z_{[n,k]}\right)$ is naturally defined as

$$\widehat{CTE}_{\delta,N}\left(Z_{[n,k]}\right) = \frac{1}{\delta - 1}\int_0^\infty \left[\widehat{F}_{[n,k]}(z) - \widehat{F}_{[n,k]}^{\delta}(z)\right] dz.$$

Substituting the estimated concomitant CDF yields

$$\widehat{CTE}_{\delta,N}\left(Z_{[n,k]}\right) = \frac{1}{\delta - 1}\int_0^\infty \left\{G_{[n,k]}\left(\widehat{F}_{Z,N}^{R}(z)\right) - \left[G_{[n,k]}\left(\widehat{F}_{Z,N}^{R}(z)\right)\right]^{\delta}\right\} dz.$$

The quantity $\widehat{F}_{[n,k]}^{\delta}(z)$ can be evaluated directly for both integer and non-integer values of δ.

The numerical integral is evaluated over the positive support with the upper truncation increased until the reported CTE values are stable to the required numerical precision. This is important because the breaking-strength observations are not compactly supported even though the empirical sample has a finite maximum.

For illustration, using $\widehat{\zeta} = -0.16593$, the reflected Gaussian KDE with $h = 112.746$, and direct numerical quadrature, the resulting estimates are summarized in Table 6 for several combinations of n and k.

Table 6. Nonparametric estimates of the CDF-based CTE for the concomitant $Z_{[n,k]}$.

n	k	$\widehat{CTE}_{1.5,N}$	$\widehat{CTE}_{2,N}$
2	1	171.7721	146.3582
2	2	165.3998	142.3606
3	2	168.5482	144.2983
3	3	164.3458	141.5830
4	3	166.8506	143.1638
4	4	163.7598	141.1472
5	4	165.8183	142.4619
5	5	163.3869	140.8688

The estimates are positive, as required by the general positivity property of the CDF-based CTE. They also show that the magnitude of the measure depends on both the entropy of order δ and the record indices (n, k). As expected from the definition, the values for $\delta = 1.5$ are larger than those for $\delta = 2$.

The estimates in Table 6 generally exhibit a decrease in the CTE as the record indices increase. A possible explanation is that changing the record indices alters the selection

of observations and, consequently, the distribution of the associated concomitant values. In combination with the fitted negative SAR dependence, this selection mechanism may affect the distributional weights relevant to the CTE and lead to lower estimated values for the index combinations considered here. In this context, lower CTE estimates indicate a reduction in the uncertainty measured by the chosen entropy functional. This pattern should, however, be interpreted as an empirical observation rather than a universal monotonicity property: the estimates need not decrease at every increase in the record indices, and their behavior may depend on the entropy order, the fitted dependence parameter, the marginal distribution, and the particular indices considered.

5.7. Discussion

The application demonstrates how the theoretical results for concomitants of k-records can be implemented when the marginal distribution is unknown. The procedure separates the two sources of modeling information. The dependence structure is represented by the SAR parameter ζ, while the marginal distribution of Z is estimated nonparametrically through a boundary-corrected kernel CDF.

This distinction is important for the present data. The two breaking-strength samples are paired and exhibit weak negative dependence, and the fitted SAR parameter is likewise negative. Consequently, the estimated CTE incorporates the observed direction of dependence rather than treating the two measurements as independent.

The bandwidth comparison further indicates that smoothing choices can affect nonparametric entropy estimates substantially. Hence, bandwidth selection should be reported explicitly in applications, and sensitivity analyses based on alternative bandwidth selectors can be useful when the sample size is small.

Finally, the present analysis should be viewed as an illustration of the proposed methodology rather than as evidence that the SAR model is uniquely correct for the jute-fiber data. A more comprehensive empirical analysis could compare the SAR specification with other dependence models and could use bootstrap methods to quantify the sampling uncertainty of $\widehat{\zeta}$ and of the resulting CTE estimates. Such extensions are beyond the scope of the present illustration.

6. Conclusions and Future Work

In this paper, we developed a theoretical and computational framework for studying CTE and related information measures for concomitants of k-record values arising from the SAR family of bivariate distributions. A first important aspect of the study was to examine the finiteness of a commonly used formulation of CTE. For distributions with unbounded support, the corresponding density-based integral may diverge. This observation motivates the CDF-based formulation adopted in this paper, which is finite under appropriate moment conditions and has a natural interpretation in terms of a generalized maximum. Thus, the proposed formulation provides a suitable basis for entropy analysis in a broad class of reliability and survival models.

From the theoretical standpoint, we derived explicit representations of the CDF-based CTE, its past-lifetime dynamic version, and the corresponding Rényi entropy for the concomitant $Z_{[n,k]}$ under the SAR(ζ) dependence structure. We also established several structural properties, including characterization results, relationships with the MIT, and monotonicity-based classes for model discrimination. These results provide a unified analytical framework for studying the effect of both the dependence parameter ζ and the record indices (n, k) on the uncertainty associated with concomitant of record values.

A further advantage of the proposed framework is its quantile-based formulation. By expressing the CTE in terms of the quantile density of the marginal distribution, the re-

sulting representations remain applicable when the marginal distribution does not possess a convenient closed-form CDF or when direct analytical calculations are difficult. The uniform and Weibull examples illustrate the sensitivity of the proposed measures to the dependence parameter and to the record indices. In particular, positive and negative dependence can produce different patterns as the record index changes. However, the numerical results also indicate that such behavior need not be universally monotone and may depend jointly on δ, ζ, n, and k. This highlights the role of the entropy order and record structure in determining the sensitivity of the proposed measures.

For practical estimation, we developed a nonparametric kernel-based procedure for estimating the CDF-based Tsallis entropy from observations of the concomitant variable. The numerical investigation demonstrates that bandwidth selection can have a substantial effect on the resulting entropy estimates. Consequently, bandwidth selection should be treated as an important component of the practical implementation rather than as a purely technical issue. The proposed procedure also provides a flexible alternative to fully parametric entropy estimation when the marginal distribution is not specified in advance.

The application to the breaking-strength data of jute fibers illustrates the practical use of the proposed methodology in a reliability setting. The analysis demonstrates how the CTE-based framework can be used to quantify uncertainty associated with concomitant observations and to investigate the effect of dependence and record structure on the resulting information measures. The application therefore serves as an illustration of how the theoretical results can be transferred to real reliability data.

Several directions remain open for future research. First, the present framework can be extended to other generalized information measures, including Sharma–Mittal and Havrda–Charvát entropies, and their behavior under SAR-type dependence can be investigated. Second, asymptotic properties of the proposed nonparametric estimator, including consistency, bias, variance, and data-driven bandwidth selection, deserve further study. Bootstrap procedures could also be developed to quantify the uncertainty associated with the estimated entropy measures and the dependence parameter. Third, extensions to other classes of dependence models and to higher-dimensional concomitants would broaden the applicability of the proposed approach.

A further direction for future research is to compare the SAR dependence structure with alternative bivariate dependence models using appropriate goodness-of-fit criteria and measures of dependence. Such a comparative study could clarify the relative suitability of these models for different applications and provide a more comprehensive assessment of their empirical performance.

Finally, the framework developed here has potential applications in reliability engineering, survival analysis, quality control, and environmental statistics, particularly in settings where dependence and extreme observations occur simultaneously. Further investigation of these applications may provide useful connections between generalized entropy measures, record-value theory, and statistical modeling of dependent extreme events.

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