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On Meromorphic Solutions to the Generalized Liénard Equation

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Abstract

The generalized Liénard equation, an important subclass of polynomial planar vector fields closely related to the limit-cycle problem in the second part of Hilbert's 16th problem, has been extensively studied for real limit cycles, whose number depends sensitively on the degrees of the involved polynomials. This model also finds wide applications in electrical engineering, mathematical biology, population dynamics, nonlinear chemistry, seismology and economic fluctuation research. In this work, we systematically investigate meromorphic solutions of the generalized Liénard system. Using Nevanlinna theory together with the Kowalevski–Gambier method, we carry out a rigorous full classification of all admissible meromorphic solutions and prove that every non-constant meromorphic solution must be an elliptic function or one of its degenerate forms. This study extends the classical real-domain qualitative theory of Liénard systems to the complex plane and provides new complex-analytic perspectives for exploring the global structure of nonlinear oscillations.

Keywords: generalized Liénard equation; meromorphic function; Nevanlinna theory; elliptic functions

MSC: 30D35; 34M04; 34M05

1. Introduction

At the 1900 International Congress of Mathematics in Paris, Hilbert put forward a collection of 23 influential mathematical problems. The second part of Hilbert's 16th problem asks for an upper bound on the number of limit cycles of planar polynomial vector fields of given degree p. Its solution is an essential step toward the topological classification of phase portraits for generic dense families of such vector fields, namely Morse–Smale vector fields. A prominent and well-studied subclass is the generalized Liénard equation

$$u' = v - M(u), \quad v' = -Q(u), \tag{1}$$

or equivalently

$$u'' + P(u)u' + Q(u) = 0, \tag{2}$$

where $u = u(t)$ denotes the state variable, $P(u)u'$ represents a nonlinear damping term, and $Q(u)$ corresponds to an elastic restoring force derived from a potential field. Here M is a polynomial of degree $p+1$, Q is a polynomial of degree q and $M' = P$. The limit-cycle count for this system depends sensitively on the degrees of the polynomials M and Q; some low-degree cases are settled, while most combinations remain open. Originally introduced to describe a series electrical circuit with a capacitor, inductor and resistor [1,2],

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this equation also arises in diverse biological, population, chemical, seismological and economic models [3–9].

As a special case of Equation (2) where $Q(u) = u$, the classical Liénard equation admits the conjecture by Lins et al. [10] that the number of limit cycles is at most $\lfloor p/2 \rfloor$. This conjecture holds for $p \leq 4$ (see [10,11]) and fails for $p \geq 6$ (see [12,13]). On the other hand, the best known general lower bound is provided by Maesschalck and Huzak [14]. For the generalized Equation (2), results on upper bounds are only available for a small set of low-degree pairs $(\deg M, \deg Q)$. Let $L(p,q)$ denote the maximum number of limit cycles. Known values include $L(1,2) = 1$ (see [15]), $L(1,3) = 1$ (see [16,17]), $L(2,2) = 1$ (see [18]), $L(2,3) = 3$ (see [19]), $L(1,4) = 3$ (see [20]). Additional recent progress can be found in [21–23]. Nevertheless, rigorous upper bounds are still lacking for most pairs (p,q), and the general problem remains open. Classical research further addresses existence [24], uniqueness [25], stability [26,27], numerical solutions [28,29], periodic solutions [30–32], and bifurcation phenomena [33,34].

Numerical treatment for nonlinear differential equations has long relied on well-constructed finite-difference discretization schemes that preserve structural properties such as conservation and convergence; for instance, Dong et al. [35] constructed a second-order accurate two-level implicit nonlinear difference scheme for the Rosenau–Burgers equation, verifying its conservation, unique solvability and second-order convergence both theoretically and numerically. Such finite-difference techniques represent one mainstream paradigm for approximate PDE solving, and are distinct from the complex-analytic classification pursued in the present work.

It is indispensable to investigate periodic solutions of Equation (2) in the study of limit cycles. In parallel to this extensive real-domain literature, the complex-analytic investigation of meromorphic solutions offers a complementary perspective. Rooted in Nevanlinna theory, which provides a powerful, systematic framework for characterizing the growth, zero distribution, pole structure, and deficiency properties of meromorphic functions [36–39], this line of research has yielded profound results for both linear and nonlinear complex differential equations. Beyond revealing the global analytic structures and implicit symmetry properties of differential equations, meromorphic solutions also facilitate the construction of exact traveling-wave solutions for nonlinear partial differential equations, with significant theoretical and applied value. Alongside classical finite-difference methods, machine-learning PDE solvers provide an alternative route to study solitons; Zhou et al. [40] constructed an enhanced PINN for $\mathcal{PT}$-symmetric solitons, whose data-driven numerical approach complements our analytic classification of meromorphic solutions. Nasiruzzaman et al. [41] supply relevant approximation-theoretic background for operator analysis. He et al. [42] introduced the ROFT model for accelerated neural operator tuning, and its optimization strategy may guide future parameter studies of Liénard-type equations.

This paper aims to fill this gap by systematically investigating meromorphic solutions of the generalized Liénard equation. Using Nevanlinna theory, the Wiman–Valiron method and Clunie-type lemmas, we derive necessary and sufficient conditions for the existence of rational and transcendental meromorphic solutions, and obtain implicit solution forms for several subclasses. These results are formulated in our main theorems, which classify all admissible meromorphic solutions and show that every such solution is either an elliptic function or one of its degenerate forms. Here the degenerate forms refer to functions of the form $R(e^{az})$ or rational functions, where R is a rational function and a is a constant. The contributions are twofold: theoretically, this work extends the real-domain qualitative theory of generalized Liénard equations to the complex plane via this full classification, yielding new insights into the global analytic structure of Liénard-type nonlinear oscillations; practically,

it provides explicit analytic representations of these meromorphic solutions for applications in nonlinear circuits, mathematical physics and related areas.

The paper is organized as follows. Section 2 introduces the fundamental theoretical tools. Section 3 lists essential lemmas required for subsequent proofs. Section 4 presents our main results and detailed proofs. Section 5 discusses open questions related to the generalized Liénard equation. Section 6 concludes the paper.

2. Preliminaries

Nevanlinna theory provides a powerful tool for investigating entire and meromorphic solutions of complex differential equations [43]. In recent decades, this theory has been widely adopted to analyze such solutions [44–48]. In this section, we recall several fundamental results from Nevanlinna theory without proofs [43,49].

Definition 1. *For every real number $\alpha \geq 0$, we define* $\ln^+ \alpha = \max\{0, \ln \alpha\}$.

Definition 2. *Let f be a meromorphic function, and $n(r, f)$ be the number of poles of f counted with multiplicities in $|z| < r$, we define*

Counting function: $N(r, f) = \int_0^r \frac{n(r,f)-n(0,f)}{t} dt + n(0, f) \log r$,
Proximity function: $m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta$,
Characteristic function: $T(r, f) = m(r, f) + N(r, f)$.

Definition 3. *The order of a meromorphic function f is defined by*

$$\rho(f) = \limsup_{r\to\infty} \frac{\ln^+ T(r, f)}{\ln r}.$$

Definition 4. *Let $G(r)$ be a non-negative function and f be a meromorphic function such that $G(r) = o(T(r, f))$ as $r \to \infty$ outside a set of finite linear measure. We denote $G(r) = S(r, f)$. A meromorphic function g is called small function respect to f if $T(r, g) = S(r, f)$.*

Theorem 1 (The First Main Theorem)**.** *Suppose f is meromorphic in $|z| < R(\leq \infty)$. Then for all $a \in \mathbb{C}$ and $0 < r < R$, we have*

$$T\left(r, \frac{1}{f - a}\right) = T(r, f) + O(1). \tag{3}$$

Theorem 2. *Let $f, f_1, \ldots, f_n$ be meromorphic function, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta \neq \beta\gamma$ and $f \neq -\delta/\gamma$. Then*

(a) $T(r, f_1, \ldots, f_n) \leq \sum_{i=1}^n T(r, f_i)$ *for* $r \geq 1$,
(b) $T(r, f^n) = nT(r, f)$, $n \in \mathbb{N}$,
(c) $T(r, \prod_{i=1}^n f_i) \leq \sum_{i=1}^n T(r, f_i) + \log n$ *for* $r \geq 1$,
(d) $T\left(r, \frac{\alpha f+\beta}{\gamma f+\delta}\right) = T(r, f) + O(1)$.

Theorem 3. *Let f be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f)$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is $\leq n$, then

$$m(r, P(z, f)) = S(r, f).$$

Theorem 4. *Let f be a transcendental meromorphic function. Then $m(r, f'/f) = S(r, f)$. Further, if $\rho(f) < +\infty$, then $m(r, f'/f) = O(\ln r)$ as $r \to \infty$.*

Theorem 5. *A meromorphic function f is rational if and only if $T(r, f) = O(\log r)$.*

Definition 5. *Let f be an entire function whose Taylor expansion is $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we define*
The maximum term: $\mu(r) = \mu(r, f) = \max_{n \geq 0} |a_n| r^n$,
The centre index: $\nu(r) = \nu(r, f) = \max\{m : |a_m| r^m = \mu(r, f)\}$.

Theorem 6. *If f is an entire function of order $\rho(r, f)$, then*

$$\rho(r, f) = \limsup \frac{\ln^+ \nu(r, f)}{\ln r} = \limsup \frac{\ln^+ \ln^+ \mu(r, f)}{\ln r}.$$

Lemma 1. *Let g be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z be such that $|z| = r$ and that*

$$|g(z)| > M(r, g)\nu(r, g)^{-\frac{1}{4}+\delta}$$

holds. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure, i.e., $\int_F dt/t < +\infty$, such that

$$g^{(m)}(z) = \left(\frac{\nu(r, g)}{z}\right)^m (1 + o(1))g(z)$$

holds for all $m \geq 0$ and all $r \notin F$.

As a local singularity analysis tool, the Kowalevski–Gambier method [50] characterizes all admissible local series expansions of solutions near movable singular points. We briefly recap its core idea. Consider an ordinary differential equation of the form

$$E = E(f, f', \ldots, f^{(n)}) \equiv 0, \tag{4}$$

which admits a meromorphic solution f. Assume a Laurent series ansatz near a movable pole z_0

$$f(z) = \sum_{j=0}^{+\infty} u_j (z - z_0)^{j+p}, \quad u_0 \neq 0, \ \ p < 0 \tag{5}$$

where $-p$ is the pole order at z_0. Substituting (5) into (4) yields

$$E = \sum_{j=0}^{+\infty} E_j (z - z_0)^{j+q} = 0, \tag{6}$$

with q the minimal exponent among all leading powers. The terms contributing to the pole of order $-q$ at z_0 form the dominant part, denoted by $\hat{E}$. We then define the indicial equation as follows.

Definition 6. *The indicial equation of the system $E = 0$ is*

$$p(u_0, j) = \lim_{z \to z_0} (z - z_0)^{-j-q} \hat{E}'(z, u_0(z - z_0)^p)(z - z_0)^{j+p} = 0,$$

and its zeros in j are called the Fuchs indices.

In the recursive computation, each equation $E_j = 0$ is linear

$$E_j \equiv p(u_0, j)u_j + Q_j(\{u_l | l < j\}), j = 1, 2, \ldots$$

When j is not a root of the polynomial $p(u_0, j)$, the coefficient u_j is uniquely determined. Consequently, the local structure of any meromorphic solution can be explicitly described from the leading coefficient u_0.

3. Lemmas

This section presents essential lemmas required for proving our main results.

Lemma 2 ([51]). *For an ODE: $G = G(u, u, ..., u(n)) = 0$ with constant coefficients, suppose there only exist l meromorphic solution with a movable pole z_k of multiplicity k, and that the number of the k-multiple poles in one of these solutions u is more than l, if $m(r, u) = S(r, u)$, then $u(z) = R(e^{az})$, where R is a rational function with at most l pole of multiplicity k in $\mathbb{C}/\{0\}$, or u is an elliptic function.*

Lemma 3 ([43]). *Let*

$$P(z, u, u', u^{(n)}) = 0 \tag{7}$$

be an algebraic differential equation, i.e., $P(z, x_0, x_1, \ldots, x_n)$ is a polynomial in all of its arguments, and let u be a transcendental meromorphic solution of (7). If a constant $c \in \mathbb{C}$ does not solve (7), then

$$m\left(r, \frac{1}{u - c}\right) = S(r, u).$$

Lemma 4 ([52]). *If $f_j (j = 1, 2, \ldots, n + 1)(n \geq 2)$, $g_j(z)(j = 1, 2, \ldots, n)(n \geq 1)$ are entire functions satisfying the following two conditions:*

- $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}(z)$;
- *When $1 \leq j \leq n + 1, 1 \leq k \leq n$, the order of $f_j(z)$ is less than that of $e^{g_k(z)}$. When $n \geq 2$ and $1 \leq j \leq n + 1, 1 \leq h < k \leq n$, the the order of $f_j(z)$ are also less than that of $e^{g_k(z) - g_h(z)}$.*

Then $f_j(z) \equiv 0 (j = 1, 2, \ldots, n)(n \geq 1)$.

Lemma 2 is applied to derive the periodicity of solutions. Lemma 3 provides the condition $m(r, 1/(u - c)) = S(r, u)$ required for the application of Lemma 2, and Lemma 4 is used to establish algebraic relations among the coefficients of Equation (2) and the parameters in the resulting solutions. To facilitate the subsequent arguments, we state the following lemmas together with their proofs.

Lemma 5. *Let u be a transcendental meromorphic solution of the equation*

$$u'' + (a_1 u + a_0)u' + b_3 u^3 + b_2 u^2 + b_1 u + b_0 = 0, \tag{8}$$

where $a_i (i = 0, 1)$ and $b_j (j = 0, 1, 2, 3)$ are constants. Then u' must have zeros.

Proof. We may assume $b_0 \neq 0$. If not, replace u with $u - a$ for some constant a such that $b_3 a^3 + b_2 a^2 + b_1 a + b_0 \neq 0$. Applying Lemma 7, we immediately get $m(r, 1/u) = S(r, u)$. It is also straightforward to verify that all poles of u are simple. Now suppose u' has no zeros, then there exists a transcendental entire function v with only simple zeros such that $u' = 1/v^2$. Combining this with Theorems 1 and 2, we further derive that $N(r, 1/v) = T(r, v) + S(r, v)$. We now analyze five cases as follows:

Case I. $a_1 = 0$. Equation (8) simplifies to

$$u'' + a_0 u' + b_3 u^3 + b_2 u^2 + b_1 u + b_0 = 0. \tag{9}$$

Differentiate Equation (9) and divide the resulting identity by u'. Differentiate the new equation once more and divide by u' again, we then eliminate u by differentiating the second derived relation to arrive at

$$a_0u'^2u^{(4)} - 4a_0u'''u'u'' + 3a_0u''^3 + 6b_3u'^5 - 3u^{(4)}u'u'' + u^{(5)}u'^2 - u'''^2u' + 3u'''u''^2 = 0. \tag{10}$$

Recall that $u' = 1/v^2$, so Equation (10) transforms into

$$v^2v'\left(a_0v'' + 6v'''\right) - v^3\left(a_0v''' + v^{(4)}\right) + 3b_3 - 18vv'^2v'' + 6v'^4 + 7v^2v''^2 = 0. \tag{11}$$

Differentiating (11) gives

$$\begin{aligned} &a_0v\left(2v'^2v'' - 2v(z)v'''v' + v\left(v''^2 - vv^{(4)}\right)\right) + 6v'^3v'' \\ &+ vv'\left(3vv^{(4)} - 22v''^2\right) - 6vv'''v'^2 + v^2\left(20v'''v'' - vv^{(5)}\right) = 0. \end{aligned} \tag{12}$$

Substituting $v(z_0) = 0$ into (12), we get $[v'(z_0)]^3v''(z_0) = 0$. Since all zeros of v are simple, $v'(z_0) \neq 0$, and consequently $v''(z_0) = 0$. Thus every zero of v is also a zero of v''. As v is entire, the function $\phi_1 = v''/v$ has no poles and is therefore entire. Applying Lemma 4, we have $T(r, \phi_1) = m(r, \phi_1) = S(r, v)$. Substitute $v'' = \phi_1 v$ into Equation (12) to get

$$4a_0v'\phi_1' + v(a_0\phi_1'' - 16\phi_1\phi_1' + \phi_1''') = 0. \tag{13}$$

One may readily verify that all zeros of v are zeros of $a_0\phi_1'$. Combining $N(r, 1/v) = T(r, v) + S(r, v)$ and $N(r, 1/\phi_1') \leq T(r, \phi_1') = S(r, v)$, we deduce that $a_0\phi_1' \equiv 0$ and further gives $a_0\phi_1'' - 16\phi_1\phi_1' + \phi_1''' = 0$.

We claim that $\phi_1' = 0$. Suppose on the contrary that $\phi_1' \neq 0$. Then $a_0 = 0$ and $16\phi_1\phi_1' = \phi_1'''$. If ϕ_1 is a transcendental entire function, Lemma 3 yields $T(r, \phi_1) = m(r, \phi_1) = S(r, \phi_1)$, which is a contradiction. If ϕ_1 is a nonconstant polynomial, a degree count shows that the degrees on the two sides of $16\phi_1\phi_1' = \phi_1'''$ cannot match. As a result, $\phi_1' = 0$, so ϕ_1 is constant.

Now, the equation $v'' = \phi_1 v$ can be solved directly as $v(z) = C_1e^{\sqrt{\phi_1}z} + C_2e^{-\sqrt{\phi_1}z}$, where C_1 and C_2 are constants. Accordingly,

$$u(z) = -\frac{1}{2C_1\sqrt{\phi_1}\left(C_1e^{2\sqrt{\phi_1}z} + C_2\right)},\ C_1 \neq 0 \tag{14}$$

$$u(z) = -\frac{C2}{\sqrt{\phi_1}}e^{-\sqrt{\phi_1}z},\ C_1 = 0. \tag{15}$$

Substitute the case $C_1 = 0$ of (14) into (8). This produces

$$\frac{C_2}{\sqrt{\phi_1}}\left(a_0\sqrt{\phi_1} - b_2 - \phi_1\right)e^{-\sqrt{\phi_1}z} - \frac{b_3C_2^3}{\phi_1^{3/2}}e^{-3\sqrt{\phi_1}z} + \frac{b_2C_2^2}{\phi_1}e^{-2\sqrt{\phi_1}z} + b_0 = 0. \tag{16}$$

By Lemma 4, we get $b_0 = 0$, which is a contradiction.

Next substitute the case $C_1 \neq 0$ of (14) into (8), yielding

$$\begin{aligned} &2C_1^2\sqrt{\phi_1}\left(4C_1C_2\phi_1\left(a_0 + 3b_0C_1C_2 + 2\sqrt{\phi_1}\right) + b_2\left(1 - 4C_1C_2\sqrt{\phi_1}\right)\right)e^{2\sqrt{\phi_1}z} \\ &+ 4C_1^4\phi_1\left(2\sqrt{\phi_1}\left(a_0 + 3b_0C_1C_2 - 2\sqrt{\phi_1}\right) - b_2\right)e^{4\sqrt{\phi_1}z} \\ &+ 8b_0C_1^6\phi_1^{\frac{3}{2}}e^{6z\sqrt{\phi_1}} + 8b_0C_1^3C_2^3\phi_1^{\frac{3}{2}} - 4b_2C_1^2C_2^2\phi_1 + 2b_2C_1C_2\sqrt{\phi_1} - b_3 = 0. \end{aligned} \tag{17}$$

Applying Lemma 4, we obtain $8b_0C_1^6\phi_1^{\frac{3}{2}} = 0$, again a contradiction.

Case II. $b_3 = 0$. Equation (8) reduces to

$$u'' + (a_1u + a_0)u' + b_2u^2 + b_1u + b_0 = 0. \tag{18}$$

We proceed via the same argument as in Case I. Differentiate Equation (18), divide the resulting identity by $a_1u'' + 2b_2u'$, and differentiate the derived relation once again. Eliminating u via a further differentiation of the latter equation and substituting $u' = 1/v^2$ into (18), we arrive at

$$\begin{aligned} &v(-12a_1b_2v' - 12a_1v'^2v'' + 2a_1^2v'' - 24b_2v'^3) \\ &+v^2\left(4a_0b_2v'^2 - 2a_1b_1v'^2 + 4a_1v'''v' - 4a_1v''^2 + 28b_2v'v'' + 4b_2^2\right) \\ &+v^3\left(2a_1b_1v'' - 4a_0b_2v'' - 4b_2v'''\right) + 6a_1(2v'^2 + a_1)v'^2 = 0. \end{aligned} \tag{19}$$

Let z_0 denote an arbitrary zero of v. It follows from (19) that z_0 is also a zero of $2(v')^2 + a_1$, so that $v'(z_0) = \pm\sqrt{-a_1/2}$. Differentiating (19) yields

$$v'\left(a_1v''\left(7a_1 + 18v'^2\right) - 6b_2v'\left(a_1 + 2v'^2\right)\right) + P_1(v)v = 0, \tag{20}$$

where $P_1(v)$ is a polynomial in v and its derivatives with constant coefficients. From Equation (20), we deduce $a_1^2v''(z_0) = 0$. This ensures that every zero of v is a zero of v'', which guarantees the entireity of the function $\phi_2 = v''/v$. As a consequence, $T(r,\phi_2) = m(r,\phi_2) = S(r,v)$. Substitute $v'' = \phi_2 v$ into the linear combination $2a_1 \cdot (20) + 4b_2 \cdot (19)$, followed by differentiation of the resulting equation. This procedure gives

$$8v'\left(a_1^2\left((b_1 - 8\phi_2)v'^2 - 4a_1\phi_2\right) - 2a_0a_1b_2v'^2 + 4b_2^2(a_1 + 3v'^2)\right) + P_2(v,\phi_2)v = 0, \tag{21}$$

where $P_2(v,\phi_2)$ stands for a polynomial in v, ϕ_2, and their derivatives with constant coefficients. Combining the foregoing relation $v'(z_0) = \pm\sqrt{-a_1/2}$, we verify that z_0 annihilates the expression $a_1^2(b_1 - 4\phi_2) - 2a_0a_1b_2 + 8b_2^2$. The arbitrariness of the zero z_0 and $N(r,1/v) = T(r,v) + S(r,v)$ forces $a_1^2(b_1 - 4\phi_2) - 2a_0a_1b_2 + 8b_2^2 = 0$, and then ϕ_2 is a constant. Following the same line of reasoning as Case I, we finally arrive at a contradiction.

Case III. $b_3 = b_2 = 0$. Equation (8) simplifies to

$$u'' + (a_1u + a_0)u' + b_1u + b_0 = 0. \tag{22}$$

Following the same line of reasoning as in Case I, Equation (18) can be algebraically transformed into

$$2a_1(b_0v^3v' + b_1v^2 + v'^2 - vv'') + a_1^2 + b_1v^2(b_1v^2 + 6v'^2 - 2vv'') = 0. \tag{23}$$

Let z_0 be a zero of v. From Equation (23), we immediately have $v'(z_0) = \pm\sqrt{-a_1/2}$. Differentiating Equation (23) yields

$$\begin{aligned} &2v^3\left(a_1b_0v'' + 2b_1^2v' - b_1v'''\right) + 2v^2\left(3a_1b_0v'^2 + 3b_1v'v''\right) \\ &+ 2v\left(2a_1b_1v' - a_1 + 6b_1v'^3\right) + 2a_1v'v'' = 0. \end{aligned} \tag{24}$$

Hence, all zeros of v are also zeros of v'', implying that $\phi_3 = v''/v$ is entire. Consequently, $T(r,\phi_3) = m(r,\phi_3) = S(r,v)$. Substituting $v'' = \phi_3 v$ into Equation (25) yields

$$2b_1v'(a_1 + 3v'^2) + P_3(v,\phi_3)v = 0, \tag{25}$$

where $P_3(v, \phi_3)$ is a polynomial in v, ϕ_3 and their derivatives with constant coefficients. This further leads to $v'(z_0) = \pm\sqrt{-a_1/3}$, which clearly contradicts the prior conclusion $v'(z_0) = \pm\sqrt{-a_1/2}$.

Case IV. $b_3 = b_2 = b_1 = 0$. Equation (8) reduces to

$$u'' + (a_1 u + a_0)u' + b_0 = 0. \tag{26}$$

Analogous to Case I, Equation (26) transforms into

$$a_1 - 2vv'' + 2v'^2 + 2b_0 v^3 v' = 0. \tag{27}$$

Applying Lemma 3, we obtain $m(r, v') = S(r, v)$. This immediately gives $T(r, v') = m(r, v') = S(r, v)$, which yields a contradiction.

Case V. $a_1 \neq 0$ and $b_3 \neq 0$. Similar to Case I, Equation (8) can be transformed into

$$\begin{aligned} &+ a_1 v^4 (v^{(4)} v'' - v'''^2) + 12a_1 b_3 v'^2 + 6a_1 v'^6 + 3a_1^2 v'^4 + 18b_3 v'^4 + 9b_3^2 \\ &+ v^2 \Big((3a_0 b_3 - a_1 b_2) v' v'' + 2(9b_3 - a_1^2) v' v''' + 18a_1 v'^2 v''^2 \Big) \\ &- v(12a_1 b_3 v'' + 18a_1 v'^4 v'' + 4a_1^2 v'^2 v'' + 54b_3 v'^2 v'') + 3(a_1^2 + 7b_3) v^2 v''^2 \\ &+ v^3 \Big(a_1 (b_2 v''' + 2v''' v' v'' - v^{(4)} v'^2 - 7v''^3) - 3b_3 (a_0 v''' + v^{(4)}) \Big) = 0. \end{aligned} \tag{28}$$

Let z_0 be a zero of v. From Equation (28), we obtain

$$2a_1 v'(z_0)^6 + (a_1^2 + 6b_3) v'(z_0)^4 + 4a_1 b_3 v'(z_0)^2 + 3b_3^2 = 0. \tag{29}$$

Differentiating (28) yields

$$8v'v'' \Big(a_1 (6b_3 + 9v'^4) + 4a_1^2 v'^2 + 9b_3 v'^2 \Big) + P_4(v) v = 0, \tag{30}$$

where $P_4(v)$ denotes a polynomial in v and its derivatives with constant coefficients. This implies that z_0 is a zero of either v'' or $a_1(6b_3 + 9v'^4) + 4a_1^2 v'^2 + 9b_3 v'^2$. If

$$a_1 (6b_3 + 9v(z_0)'^4) + 4a_1^2 v(z_0)'^2 + 9b_3 v(z_0)'^2 = 0, \tag{31}$$

then Equations (29) and (31) indicate that the polynomials $2a_1 z^6 + (a_1^2 + 6b_3) z^4 + 4a_1 b_3 z^2 + 3b_3^2$ and $a_1(6b_3 + 9z'^4) + 4a_1^2 z'^2 + 9b_3 z'^2$ share a common root $v'(z_0)$. As a consequence, $a_1^2 - 9b_3 = 0$ and the common factor is identified as $a_1 + 3z^2$, i.e., $a_1 + 3v'(z_0)^2 = 0$. Without loss of generality, we assume $\sqrt{a_1} - \sqrt{3} v'(z_0) = 0$. Define $\phi_4 = (\sqrt{3} v' - \sqrt{a_1})/v$, so that z_0 is not a pole of ϕ_4. Substituting $v' = \frac{1}{\sqrt{3}} v \phi_4 + \sqrt{-\frac{a_1}{3}}$ and $a_1^2 - 9b_3 = 0$ into (28) gives

$$48a_1^3 + P_5(v, \phi_4) v = 0, \tag{32}$$

where $P_5(v, \phi_4)$ is a polynomial in v, ϕ_4 and their derivatives with constant coefficients. Equation (32) forces z_0 to be a zero of a_1^3, which is impossible. Therefore, all zeros of v must be zeros of v'', so the quotient $\phi_5 = v''/v$ defines an entire function satisfying $T(r, \phi_5) = m(r, \phi_5) = S(r, v)$. Inserting $v'' = v\phi_5$ into Equation (28) gives

$$\begin{aligned} &3(a_1 v'^2 + 3b_3)(2v'^4 + a_1 v'^2 + b_3) - 6\phi_5 (a_1 + 3v'^2)(a_1 v'^2 + 2b_3) v^2 \\ &- 2v'\phi_5' \Big(a_1 (a_1 + v'^2) - 6b_3 \Big) v^3 \\ &+ (a_1 \Big(b_2 \phi_5' + v'^2 \Big(18\phi_5^2 - \phi_5' \Big) \Big) - 3b_3 \Big(a_0 \phi_5' + \phi_5'' - 6\phi_5^2 \Big) + 3a_1^2 \phi_5^2) v^4 = 0. \end{aligned} \tag{33}$$

Differentiating Equation (33), we derive

$$\begin{aligned} &2v'\Big(2(a_1b_2-3a_0b_3)\phi_5'-a_1\phi_5''(a_1+3v'^2)\Big)v \\ &-12a_1\phi_5'(2v'^4+a_1v'^2+b_3)+P_6(v,\phi_5)v^2=0 \end{aligned} \tag{34}$$

where $P_6(v,\phi_5)$ is a polynomial in v, ϕ_5 and their derivatives with constant coefficients. Hence z_0 vanishes $2v'^4+a_1v'^2+b_3$. Noting that

$$(2v'^4+a_1v'^2+b_3)'=2(4v'^2+a_1)v'v''=2(4v'^2+a_1)\phi_5vv' \tag{35}$$

has z_0 as a zero. Then we derive that z_0 is a multiple zero of $2v'^4+a_1v'^2+b_3$. Combining this with Equation (34), we further deduce that z_0 is a zero of $2(a_1b_2-3a_0b_3)\phi_5'-a_1\phi_5''(a_1+3v'^2)$. In summary, the polynomials $2z^4+a_1z^2+b_3$ and $2(a_1b_2-3a_0b_3)\phi_5'(z_0)-a_1\phi_5''(z_0)(a_1+3z^2)$ admit a common zero $v'(z_0)$, which leads to

$$8(a_1b_2-3a_0b_3)^2\phi_5'(z_0)^2-2a_1^2(a_1b_2-3a_0b_3)\phi_5'(z_0)\phi_5''(z_0)-a_1^2(a_1^2-9b_3)\phi_5''(z_0)^2=0. \tag{36}$$

As the arbitrariness of the zero z_0 and $N(r,1/v)=T(r,v)+S(r,v)$, we conclude

$$8(a_1b_2-3a_0b_3)^2\phi_5'^2-2a_1^2(a_1b_2-3a_0b_3)\phi_5'\phi_5''-a_1^2(a_1^2-9b_3)\phi_5''^2=0. \tag{37}$$

If $a_1^2-9b_3=0$, we arrive at Equation (32), and the contradiction follows by an analogous argument. Next, Equation (37) indicates that either $\phi_5'=0$ or ϕ_5' has no zeros. Suppose ϕ_5' vanishes nowhere. Then we may write $\phi_5'=e^{\alpha}$, for some entire function α. Substituting this expression into (37), we arrive at

$$8(a_1b_2-3a_0b_3)^2-2a_1^2(a_1b_2-3a_0b_3)\alpha'-a_1^2(a_1^2-9b_3)\alpha'^2=0. \tag{38}$$

Solving Equation (38) yields $\alpha(z)=az+b$, so $\phi_5'=e^{az+b}$. Recall that each zero z_0 annihilates $2(a_1b_2-3a_0b_3)\phi_5'-a_1\phi_5''(a_1+3v'^2)$, which immediately implies z_0 is a zero of $2(a_1b_2-3a_0b_3)-a_1a(a_1+3v'^2)$. Since $(2(a_1b_2-3a_0b_3)-a_1a(a_1+3v'^2))'=-6a_1av'v''$, z_0 is a multiple zero of $2(a_1b_2-3a_0b_3)-a_1a(a_1+3v'^2)$. As a result, $\psi=(2(a_1b_2-3a_0b_3)-aa_1(a_1+3v'^2))/(-3aa_1v^2)$ defines an entire function satisfying $T(r,\psi)=m(r,\psi)=S(r,v)$. Substituting $v'^2=\psi v^2+d$ into Equation (33), where $d=(2(a_1b_2-3a_0b_3)-a_1^2a)/(3aa_1v^2)$, we derive

$$\begin{aligned} &3(d(2d+a_1)+b_3)(da_1+3b_3)-6(3d+a_1)(da_1+2b_3)(\phi_5-\psi)v^2 \\ &-2(a_1(d+a_1)-6b_3)v'\phi_5'v^3+2a_1v'(\phi_5-\psi)\phi_5'v^5 \\ &+\Big(3(6da_1+6b_3+a_1^2)(\phi_5-\psi)^2-(da_1+3b_3)\phi_5''+(a_1b_2-3b_3a_0)\phi_5'\Big)v^4 \\ &-a_1\Big((\psi-\phi_5)\phi_5''+\phi_5'^2+6(\phi_5-\psi)^3\Big)v^6=0. \end{aligned} \tag{39}$$

Consequently, we obtain

$$(\psi-\phi_5)\phi_5''+\phi_5'^2+6(\phi_5-\psi)^3=0 \ \text{ and } \ (\phi_5-\psi)\phi_5'=0. \tag{40}$$

Since $\phi_5'=e^{az+b}\neq 0$, the second equation in Equation (40) forces $\phi_5=\psi$, which further leads to $\phi_5'=0$ via the first relation. By a similar reasoning as in Case I, we get a contradiction. □

In the end, we discuss the case where the zeros of u are small relative to $T(r,u)$, i.e., $N(r,u)=S(r,u)$. Here we show that the corresponding solutions are entire.

Lemma 6. *If the generalized Liénard Equation (2) admits a transcendental meromorphic solution u satisfying $N(r,u) = S(r,u)$, then $q = p+1$. Furthermore, the explicit form of such a solution can be classified as follows:*

(i) For $p = 0$, u takes the form

$$\begin{aligned} u(z) &= C_1 e^{\frac{-a_0+\sqrt{a_0^2-4b_0}}{2}z} + C_2 e^{\frac{-a_0-\sqrt{a_0^2-4b_0}}{2}z} \ \text{ with } \ a_0^2 \neq 4b_0, \\ u(z) &= (C_3 + C_4 z) e^{-\frac{a_0}{2}z} \ \text{ with } \ a_0^2 = 4b_0; \end{aligned} \tag{41}$$

(ii) For $p \geq 1$, u has the form of

$$u(z) = C_5 + C_6 e^{-\frac{b_{p+1}}{a_p}z}, \tag{42}$$

where $C_i (i = 1,2,3,4,5,6)$ are constants algebraic expressions in the coefficients of Equation (2).

Proof. If $q \geq p+2$, i.e., $p+1 \leq q-1$, applying Lemma 3 to Equation (2) leads to $T(r,u) = m(r,u) + N(r,u) = S(r,u)$, a contradiction. If $q \leq p$, by applying Lemma 3 and Lemma 3 to Equation (2) gives

$$T(r,u') = m(r,\frac{u'}{u}u) + N(r,u') \leq m(r,\frac{u'}{u}) + m(r,u) + S(r,u) = S(r,u),$$

another contradiction. As a result, we must have $q = p+1$.

When $p = 0$, the solution (41) can be directly obtained by solving the linear Equation (2).

We now turn to the case $p \geq 1$. Let $P(z) = a_p z^p + \cdots + a_1 z + a_0$ and $Q(z) = b_{p+1}z^p + \cdots + b_1 z + b_0$, where $a_j (j = 0,1,\ldots,p)$, $b_j (j = 0,1,\ldots,p+1)$ are constants, $a_p \neq 0$ and $b_{p+1} \neq 0$. An application of Lemma 3 to Equation (2) gives $m(r, a_p u' + b_{p+1}u) = S(r,u)$. Define

$$\tau_1 = a_p u' + b_{p+1} u, \tag{43}$$

then τ_1 is a meromorphic function satisfying $T(r,\tau_1) = m(r,\tau_1) + N(r,\tau_1) = S(r,u)$.

For $p = 1$, substitute $u' = -b_2/a_1 u + 1/a_1\tau_1$ into Equation (2) with $p = 1$ and $q = 2$ to get

$$(a_0 a_2 - b_2)u' + (b_1 + \tau_1)u + a_1 b_0 + \tau_1' = 0. \tag{44}$$

Combining the linear Equations (43)-(44), eliminating u' to get

$$[b_2(a_0 a_2 - b_2) - a_1(b_1 + \tau_1)]u = \tau_1(a_0 a_2 - b_2) - (\tau_1' - a_1 b_0)a_1.$$

Noticing that $T(r,\tau_1) = S(r,u)$, then we have $b_2(a_0 a_2 - b_2) - a_1(b_1 + \tau_1) = 0$, which yields that τ_1 is constant. Solving (43) then produces the solution (41). For $p \geq 2$, substituting $u' = -b_{p+1}/a_p u + 1/a_p\tau_1$ into (2) produces

$$u'' + P_1(u)u' + Q_1(u) + \tau_1 u^p = 0, \tag{45}$$

where $P_1(u) = P(u) - a_p u^p$ and $Q_1(u) = Q(u) - b_{p+1}u^{p+1}$. Lemma 3 applied to (45) yields $m(r, a_{p-1}u' + b_p u) = S(r,u)$. Define

$$\tau_2 = a_{p-1}u' + (\tau_1 + b_p)u, \tag{46}$$

then τ_2 is a meromorphic function satisfying $T(r,\tau_2) = m(r,\tau_1) + N(r,\tau_1) = S(r,u)$. From the linear Equations (43) and (46) to eliminate u', we have

$$[a_{p-1}b_{p+1} - a_p(\tau_1 + b_p)]u = a_{p-1}\tau_1 - a_p\tau_2.$$

Noticing that $T(r, \tau_1) = S(r, u)$, then we have $a_{p-1}b_{p+1} - a_p(\tau_1 + b_p) = 0$, which yields that τ_1 is constant. consequently, we obtain the solution (41). □

4. Main Results and the Proofs

Theorem 7. *The generalized Liénard Equation (2) admits non-zero rational solutions if and only if $p \leq 1$ and $q \leq 3$. Moreover, any rational solution u satisfies $\lim_{z\to\infty} u = C$ for some constant C.*

Proof. Let u be a rational function solution of the Liénard Equation (2). We may assume that all zeros of u are simple. Otherwise, we replace u with $u - D$, for a suitable constant D, such that $u - D$ possesses only simple zeros.

Suppose first that u is a polynomial of degree k. Substituting u into Equation (2), one can verify that every zero of u is a zero of $u'' + a_0u' + b_0$. However, $\deg(u'' + a_0u' + b_0) \leq k - 1$, which yields a contradiction.

Next, consider the case where u admits at least one pole. We decompose u as $u(z) = r(z) + s_1(z)/t(z)$, where $r(z)$, $s_1(z)$ and $t(z)$ are polynomials with $\gcd(s_1, t) = 1$ and $\deg s_1 < \deg t$. Let z_0 be a zero of $t(z)$ with multiplicity k. Comparing the zero multiplicities of both sides of Equation (2) at z_0, we conclude that $k = 1$ and $q = p + 2$. Denote $\alpha = \mathrm{Res}[u, z_0]$. By comparing the leading singular terms at z_0, we obtain $-a_p + b_{p+2}\alpha = 0$. Substituting the decomposition $u(z) = r(z) + s_1(z)/t(z)$ back into Equation (2) and taking the limit $z \to \infty$, we deduce that $r(z)$ must be a constant. Consequently, u can be rewritten as $u(z) = r + s_1(z)/t(z) = s(z)/t(z)$, where $\deg s = \deg t$. Define $w(z) = 1/u(z) = t(z)/s(z)$, then w satisfies

$$2w'^2 - ww'' - \frac{1}{w^{p-1}}P_1(w)w' + \frac{1}{w^{p-1}}Q_1(w) = 0, \tag{47}$$

where $P_1(w)/w^p = P(1/w)$ and $Q_1(w)/w^{p+1} = Q(1/w)$. If $p \geq 2$, then z_0 annihilates $-a_pw' + b_{p+2}$, and thus annihilates $b_{p+2}s - a_pt'$. Therefore, there exists a constant β, such that $b_{p+2}s - a_pt' = \beta t$. This further gives

$$-a_pw' + b_{p+2} = \frac{a_pts' - a_pst' + b_{p+2}s^2}{s^2} = a_pw\frac{s'}{s} + \beta w. \tag{48}$$

It then follows that

$$-a_p\frac{w'}{w} + b_{p+2}u = a_p\frac{s'}{s} + \beta. \tag{49}$$

Combining the relation $-a_p + b_{p+2}\alpha = 0$ with the residue analysis of (49) at z_0, we derive $a_p = 0$, which is a contradiction. This completes the proof of the desired conclusion. □

Theorem 8. *The transcendental meromorphic solutions take the forms (41), (42), or the following expressions.*
When $q = p + 2 \geq 3$, we have

$$u(z) = \frac{s_0}{e^{az} - Y_0} + d_0; \tag{50}$$

while for $q \leq 1$ and $p \leq 3$, the solutions read

$$u(z) = \frac{s_1}{Z - Y_1} + \frac{s_2}{Z - Y_2} + d_1; \tag{51}$$

$$u(z) = \frac{s_3}{\wp(z - z_0; Y_3, Y_4) + B} + d_2. \tag{52}$$

Here s_j $(j = 0,1,2,3)$, a, Y_j $(j = 0,1,2,3,4)$, *and* d_j $(j = 0,1,2)$ *are constants, expressed algebraically in* $(a_1, a_0, b_3, b_2, b_1, b_0)$.

Proof. We may assume that the meromorphic solution $u^{[1]}(z)$ of Equation (2) admits a zero at $z = 0$. Otherwise, we can take the shifted solution $u^{[2]}(z) = u^{[1]}(z - \eta)$ for some suitable constant η. Now let u be a transcendental meromorphic solution of Equation (2) with a pole at z_1. Write the Laurent expansion of u around z_1 as

$$u(z) = \sum_{j=0}^{\infty} u_j (z - z_1)^{j-k}, \tag{53}$$

where $u_j (j = 0,1,\dots)$ are constants with $u_0 \neq 0$. We now divide the discussion into three cases:

Case I. $p \geq 2$ or $q \geq 4$. Substitute the Laurent expansion (53) into the Liénard Equation (2). One finds that z_1 is a simple pole, namely $k = 1$, and consequently $q = p + 2$. The leading terms are given by $\hat{E} = a_p u^p u' + b_q u^q$, which yield the condition $-a_p + b_q u_0 = 0$. Hence, u_0 assumes a unique value. Since $q = p + 2 > 1$, applying Lemma 3 to Equation (2) gives $m(r,u) = S(r,u)$. Hence, u has infinitely many simple poles. By Definition 6, the indicial equation

$$p(u_0, j) = a_p(j + 1), \tag{54}$$

admits no positive integer solutions. Therefore, the coefficients $u_j (j = 1,2,\dots)$ are uniquely determined by u_0. Consequently, there exists exactly one Laurent series around z_0 solving the Liénard Equation (2). It then follows from Lemma 2 that $u(z) = R(e^{az})$, where R is a rational function with only one pole, or u is an elliptic function possessing one simple pole in each period parallelogram. However, the sum of the residues within any period parallelogram must vanish for an elliptic function, which rules out the elliptic function case for u. We therefore write

$$u(z) = R(Z) = \frac{r_n}{Z^n} + \cdots + \frac{r_1}{Z} + \frac{s_0}{Z - Y_0} + Q(Z), \tag{55}$$

where Q is a polynomial in $Z = e^{az}$, $r_j (j = 1,\dots,n)$, s_0 and Y_0 are constants with $s_0 \neq 0$. Inserting Equation (55) into Equation (2) and taking $Z \to 0$ gives $r_n = \cdots = r_1 = 0$; letting $Z \to \infty$ shows Q to be a constant d_0. We thus arrive at

$$u(z) = \frac{s_0}{e^{az} - Y_0} + d_0. \tag{56}$$

Case II. $p \leq 1$ and $q = 3$. Equation (2) reduces to

$$u'' + (a_1 u + a_0)u' + b_3 u^3 + b_2 u^2 + b_1 u + b_0 = 0 \tag{57}$$

where $b_3 \neq 0$. By Lemma 3, we have $m(r,u) = S(r,u)$, which means that u has infinitely many poles. Substituting the Laurent expansion (53) into (57), we observe that z_0 is a simple pole. The leading terms are given by $\hat{E} = u'' + a_1 u u' + b_3 u^3$, yield the condition $b_3 u_0^2 - a_1 u_0 + 2 = 0$. Hence, u_0 takes at most two values, denoted by $u_{0,1}$ and $u_{0,2}$.

On the other hand, it follows from Lemma 5 that $u(z) = \sum_{j=0}^{\infty} u_{j,1} z^{j-1}$ has a zero ζ_1. Setting $w(z) = u(z) - u(\zeta_1)$, then w has ζ_1 as its multiple zero and satisfies Equation (57) with modified coefficients. Without loss of generality, we still denote this equation by (57). If $b_0 = 0$, then Equation (57) reduces to

$$u'' + (a_1 u + a_0)u' + b_3 u^3 + b_2 u^2 + b_1 u = 0. \tag{58}$$

Since ζ_1 is a multiple zero of w, it follows from Equation (58) that ζ_1 is also a zero of w''. Differentiating (58) further yields $w'''(\zeta_1)$. By mathematical induction, we conclude that $w^{(j)} = 0 (j = 0, 1, 2, \dots)$. The Taylor expansion theorem then forces $w \equiv 0$. Therefore, we focus on the case $b_0 \neq 0$. From Lemma 3, we have $m(r, 1/w) = S(r, w)$. Let $v = 1/w$, then ζ_1 is a multiple pole of v and Equation (57) transforms into

$$b_0 v^3 + b_1 v^2 + b_2 v + b_3 - a_1 v' - a_0 v v' + 2v'^2 - v v'' = 0. \tag{59}$$

Comparing the pole order at ζ_1 on both sides confirms that ζ_1 is a double pole of v. We thus expand v in the Laurent series

$$v(z) = \sum_{j=0}^{\infty} v_j (z - \zeta_1)^{j-2}, \tag{60}$$

where $v_j (j = 0, 1, \dots)$ are constants. Examining the singular dominant behavior at ζ_1 in (59), we obtain the dominant terms

$$\hat{E}(v) = b_0 v^3 + 2v'^2 - v v'', \tag{61}$$

which gives the algebraic relation $b_0 v_0 + 2 = 0$. This uniquely determines the value of v_0. Then, by Definition 6, we have

$$p(v_0, r) = -v_0^2 (r + 1)(r + 2) = 0. \tag{62}$$

Therefore, the Fuchs indices of Equation (59) are $j = -1, -2$, with no positive integer indices present. Consequently, the leading coefficient v_0 uniquely fixes all remaining coefficients in the Laurent expansion (60). It follows that there exists exactly one Laurent series with a pole at $z = \zeta_1$ solving Equation (59). By the uniqueness theorem for analytic functions, there can be at most one meromorphic solution of (57) possessing the pole z_1 and the zero $z = \zeta_1$. An analogous argument applies to $u(z) = \sum_{j=0}^{\infty} u_{j,2} z^{j-k}$ with its zero ζ_2. Hence, there can be at most two meromorphic solutions with the pole z_1 meeting (57). By Lemma 2, either $u = R(e^{az})$ for some rational function R with two simple poles in $\mathbb{C} \backslash \{0\}$, or u is an elliptic function with at most two simple poles inside each fundamental period parallelogram.

If $u = R(e^{az})$, then proceeding similarly to the derivation of solution (56), we arrive at

$$u(z) = \frac{s_1}{Z - Y_1} + \frac{s_2}{Z - Y_2} + d_1. \tag{63}$$

If u is an elliptic function, then for any rational function of u and its derivatives, the sum of the residues within a period parallelogram must vanish. Thus,

$$\forall k \in \mathbb{N},\ n \in \mathbb{N},\ \sum_{i=1}^{2} \operatorname{res} (u^{(k)})^n = 0. \tag{64}$$

The conditions can be partially expressed as

$$\begin{cases} k = 0,\ n = 1 : a_1 = 0 \\ k = 0,\ n = 2 : a_0 = 0 \end{cases}$$

As a results, Equation (57) reduces to

$$u'' + b_3 u^3 + b_2 u^2 + b_1 u + b_0 = 0. \tag{65}$$

Multiply both sides of (65) by u' and integrate the resulting identity to get

$$u'^2 + \frac{1}{4}b_3u^4 + \frac{1}{3}b_2u^3 + \frac{1}{2}b_1u^2 + b_0u + B_0 = 0, \tag{66}$$

where B_0 is a constant. Equation (66) is one of the five first-order binomial equations of Briot and Bouquet [53](p.105), from which the solution (52) follows.

Case III. $p \leq 1$ and $q \leq 2$. Equation (2) reduces to

$$u'' + (a_1u + a_0)u' + b_2u^2 + b_1u + b_0 = 0. \tag{67}$$

Substituting the Laurent expansion (53) into (67), we deduce that $a_1 \neq 0$ and z_1 is a simple pole. The leading terms are given by $\hat{E} = u'' + a_1uu'$, which yield the condition $2 - a_1u_0 = 0$. It follows from Lemma 5 that for $u(z) = \sum_{j=0}^{\infty} u_j z^{j-1}$, either u takes the form (50), or u' admits a zero ζ, where $u_0 = 2/a_1$. We only consider the latter case in what follows. Let $v = 1/(u - u(\zeta))$, then the equation becomes

$$(b_2d^2 + b_1d + b_0)v^3 + (2b_2d + b_1)v^2 - v''v + 2v'^2 - (a_1d + a_0)v'v + b_2v - a_1v' = 0, \tag{68}$$

where $d = u(\zeta)$. Let

$$v(z) = \sum_{j=0}^{\infty} v_j(z - \zeta)^{j-k} \tag{69}$$

be the Laurent series of v at the multiple pole ζ. Substituting (69) into (68), we find $k = 2$ and $b_2d^2 + b_1d + b_0 \neq 0$. Applying Lemma 3 to Equation (68), we obtain $m(r, v) = S(r, v)$. Hence, v has infinitely many poles. By reasoning similar to Case II, there exists exactly one Laurent series with a double pole at ζ. The uniqueness theorem then implies that there can be at most one meromorphic solution of (67) with a movable pole at z_0.

We now prove that u is periodic. Let z_j $(j = 1, 2, \dots)$ denote all poles of u, and define $w_j(z) = u(z + z_j - z_0)$. Each w_j is a meromorphic solution of (67) with a pole at z_0. Since the equation admits at most one solution with a movable pole at z_0, distinct w_j must coincide. This forces u to be periodic, which further implies that v has infinitely many double zeros. By Lemma 2, we deduce that $v = R(e^{az})$ or v is an elliptic function, where R is a rational function. Next, if v has simple poles, the uniqueness theorem again shows that there can be at most one meromorphic solution of (68) with a movable simple pole. Periodicity guarantees infinitely many simple poles for v, so Lemma 2 implies that R has only one simple pole. Finally, we show that $u = 1/R(e^{az}) + u(\zeta)$ and R has only one simple pole. Within a fundamental period, suppose u has at least two simple poles η_j $(j = 1, 2, \dots)$. Note that there is only one meromorphic solution of (68) with a pole at η_1. Since $u(z + \eta_2 - \eta_1)$ is also a meromorphic solution of (68) with a pole at η_1, we get $u(z) = u(z + \eta_2 - \eta_1)$, confirming periodicity of u in the fundamental period. As a consequence, either $u = 1/R(e^{az}) + u(\zeta)$ with R having only one simple pole, or u is an elliptic function possessing only one simple pole in its fundamental period. The second possibility cannot occur, because the sum of residues of an elliptic function over its fundamental parallelogram vanishes.

To conclude this part, for $u = 1/R(e^{az})$ with R having only one simple pole, arguing analogously to the derivation of (56), we obtain

$$u(z) = \frac{s_1}{e^{az} - \gamma_1} + \gamma_1, \tag{70}$$

where a, s_1, Y_1 and γ_1 are constants. For $v = R(e^{az})$ with R having only one simple pole, by a similar argument as for (56), we have

$$v(z) = \frac{s_2}{(e^{az} - Y_2)^2(e^{az} - Y_3)^\sigma} + \gamma_2, \tag{71}$$

where s_2, Y_2, Y_3, γ_2 are constants and $\sigma = 0,$ or 1. Combining (70) and (71) via the relation $u = 1/v + u(\zeta)$, we arrive at a contradiction. □

5. Some Applications

In order to investigate Hayman's conjecture, Ng and Wu [54] determined all explicit meromorphic solutions of the equation $P(z, f, f', f'') = 0$. This demonstrates that Hayman's conjecture holds for these second-order ODEs. However, their work leaves one open case unsolved, namely Equation (57).

In a related study, Liao and Ye [55] studied a second-order differential equation generalizing the Malmquist–Yosida type equation. They showed that, under the assumption that the order of the solution exceeds 2, the equation they considered reduces to nine distinct equations. Equation (57) constitutes the ninth case.

Building on the above works, the present study fully resolves the open case left by Ng and Wu and completes the proof of Hayman's conjecture for the corresponding class of second-order ODEs. Furthermore, our conclusions generalize and enrich the relevant results established by Liao and Ye.

6. Conclusions

This paper systematically studies meromorphic solutions of the generalized Liénard equation, a canonical nonlinear ODE closely related to Hilbert's 16th problem and widely used in physical, engineering, biological and economic models. Unlike most prior work focusing on qualitative properties of real solutions, we investigate the meromorphic solutions under the parameter restrictions in Theorems 7 and 8.

Combining Nevanlinna value distribution theory and the Kowalevski–Gambier method, we classify all admissible meromorphic solutions for the polynomial cases treated herein. Our rigorous derivations show that, subject to the conditions of Theorems 7 and 8, any meromorphic solution is either an elliptic function or one of its degenerate forms, i.e., rational or exponential solutions. These results extend the classical qualitative theory of generalized Liénard equations from the real line to the complex plane and provide exact analytic representations for models such as nonlinear circuits, population dynamics, chemical reactions and seismology. The method developed here offers a useful framework for studying meromorphic solutions of related nonlinear differential equations.

We emphasize that our classification holds only under the polynomial assumptions and parameter constraints of Theorems 7 and 8. Some open problems remain for future study. One direction is to classify meromorphic solutions when P and Q are general entire or meromorphic functions rather than polynomials. Another is to explore connections between the zero-pole distribution of solutions and the geometric structure of the associated dynamical system.

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