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A Unified Quadratic Algebraic Approach to Hadamard–Fibonacci Transformations

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Abstract

In this paper, we introduce and study a Hadamard-type transformation associated with quadratic Fibonacci numbers over the parametrized quadratic algebra $K_p = \mathbb{R}[h]/(h^2 - p)$ for $p \in \mathbb{R}$. Rather than generalizing the scalar recurrence relation itself, we change the algebra in which the sequence takes its values and define the quadratic Fibonacci elements by $\mathcal{F}_n^{(p)} := F_n + hF_{n+1}$, inspired by the relation $\alpha^n = F_n\alpha + F_{n-1}$ for the golden ratio α and the matrix representations of Fibonacci sequences. To proceed, we employ generating functions, spectral operator theory, and the geometry of K_p based on the parameter p. By using Binet-type representations, we obtain a closed formula for the transform of an arbitrary generating function and describe the corresponding operator in a spectral way. In particular, we show that the monomials form an eigenbasis of the Hadamard–Fibonacci operator and that the eigenvalues are exactly the quadratic Fibonacci elements. Furthermore, we examine the preservation of rationality and algebraicity, the scaling of singularities, and the asymptotic growth rate under iteration, accompanied by explicit applications to harmonic numbers, Catalan numbers, and Stirling numbers of the second kind.

Keywords: Hadamard product; generating functions; Fibonacci numbers; quadratic algebra; analytic combinatorics

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1. Introduction

Generating functions and Hadamard products are basic tools in combinatorics, number theory, and algebraic analysis [1–3]. For two formal power series $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, the classical Hadamard product is defined by the coefficientwise product $(A * B)(x) = \sum_{n\geq 0} a_n b_n x^n$. This product is used often in analytic combinatorics, in coefficient-extraction problems and in asymptotic analysis [1,4].

Hypercomplex number systems, which are extensions of the real numbers to real algebras of higher dimension, have a long history in algebraic geometry and in mathematical physics [5]. In classical algebra, it is known that there are essentially three ways to extend the real numbers to a real algebra of dimension two, that is, the complex numbers $z = x + iy$ where we have $i^2 = -1$, the hyperbolic numbers $z' = x' + jy'$ where we have $j^2 = 1$, and the dual numbers $z'' = x'' + ky''$ where we have $k^2 = 0$. The interested reader is encouraged to see [5] for more information regarding the extended complex numbers and

the corresponding algebras induced. But as an introductory statement, we recall that the algebras induced by the set of extended complex numbers are commutative and can be presented geometrically by setting $h^2 = p$ for $h \in \{i, j, k\}$ and $p \in \mathbb{R}$ in general. There are also noncommutative algebras of higher dimension, such as the quaternions [6], the octonions [7] and the sedenions [8]. They are obtained from the real numbers by the doubling procedure called the Cayley–Dickson process, and this process can be continued to obtain the general 2^n-ions [9–11].

The hyperbolic numbers are also called Lorentz, split-complex, double, duplex, or perplex numbers in the literature. Cockle already considered them with complex coefficients in 1849 [12], and they are useful for measuring distances in the Lorentz space-time plane [13]. In a similar way, Cheng and Thompson [14] introduced dual numbers with complex coefficients, and Akar et al. [15] studied the dual hyperbolic numbers.

For number sequences, one often studies structural properties by putting classical recurrences into such algebras. But in the usual approach, one generalizes the scalar recurrence relation itself, as in the classical Horadam sequences studied in [16]. Recently, Soykan [17] introduced and studied in detail generalized Fibonacci and Lucas components in the two-dimensional Clifford algebra of the hyperbolic numbers. Similar extensions were given for the hyperbolic Jacobsthal numbers [18] and for hyperbolic Fibonacci sequences [19]. In all these constructions, a single hypercomplex algebra is fixed in advance.

In this paper, we follow a different point of view. We do not change the Fibonacci recurrence, and we do not fix one particular hypercomplex algebra. Instead, we extend the algebra where the sequence takes its values and introduce a single operator-theoretic transformation by using the parametrized quadratic algebra $K_p = \mathbb{R}[h]/(h^2 - p)$ for $p \in \mathbb{R}$. Then we consider two consecutive Fibonacci numbers, put them together, and define the quadratic Fibonacci numbers by $\mathcal{F}_n^{(p)} := F_n + hF_{n+1}$, inspired by the relation $\alpha^n = F_n\alpha + F_{n-1}$ for the golden ratio α and $Q^n = F_nQ + F_{n-1}I$ for the Q-matrix Q studied in [20]. Our definition has also been partially motivated by the extended complex numbers, such as hyperbolic and dual numbers parametrized in K_p.

The main aim of this paper is to define and study a Hadamard-type transformation associated with the quadratic Fibonacci numbers $\mathcal{F}_n^{(p)}$. To proceed, we employ generating functions, spectral operator theory, and the geometry of K_p based on the value of the parameter p. We show that this transformation splits into separate branches, that it preserves rationality and algebraicity, and that its behavior depends on the sign of p, which could be positive, negative, or zero, as we observed above. Depending on the sign of p, we will get hyperbolic geometry [21–24], complex geometry, and the dual geometry, respectively. For more information regarding the geometry induced by these three cases, the interested reader is referred to [25].

The paper is organized as follows. The Notations section establishes the specific conventional notations used throughout the paper. In Section 2, we give the algebraic, topological, and asymptotic properties of K_p that are relevant to our constructive results, and we recall the basic properties of the Fibonacci numbers. In Section 3, we prove the Binet-type representation of the quadratic elements and define the Hadamard–Fibonacci transformation. In Section 4, we study the operator and its spectral properties, including a criterion for coefficientwise invertibility. In Section 5, we describe the geometric decomposition in the different cases. In Sections 6 and 7, we prove the preservation of rationality and give explicit applications to combinatorial numbers (harmonic, Catalan and Stirling numbers). Section 8 is the conclusion.

Notations

Throughout the paper, we will use $\mathbb{R}$ and $\mathbb{C}$ as our ground fields interchangeably, unless otherwise stated. p will be an arbitrary element of $\mathbb{R}$, unless otherwise stated, and h, x, y and a will be indeterminates or functions depending on context. n, k and m will be in $\mathbb{N}_0$, and the capital letters A, B, P, S and Q will refer to polynomial functions or formal power series in x or z. N will represent the algebraic norm function on K_p and α, β will be assumed to be the golden ratios.

2. Preliminaries

This section mostly concerns the algebraic, topological, and asymptotic facts that are relevant to our constructive results and the corresponding proofs.

Let $p \in \mathbb{R}$ be a fixed parameter, h an indeterminate, and $\mathbb{R}[h]$ the usual polynomial ring. Consider the quadratic algebra $K_p = \mathbb{R}[h]/I$, where $I = \langle h^2 - p \rangle$ is the principal ideal generated by $h^2 - p$.

Definition 1. *Let $p \in \mathbb{R}$. Consider two consecutive Fibonacci numbers F_n, F_{n+1} for any $n \in \mathbb{N}$. Then the two-parameter unital associative algebra ${}^{np}\mathscr{F}$ of order n is defined as a subalgebra of the quadratic algebra $K_p = \mathbb{R}[h]/\langle h^2 - p \rangle$, such that it consists of elements of the form $\mathcal{F}_n^{(p)} := F_n + hF_{n+1}$, for $h^2 = p$. We call ${}^{np}\mathscr{F}$ the algebra of quadratic Fibonacci elements, and $\mathcal{F}_n^{(p)}$ will be called the quadratic Fibonacci numbers or elements of K_p, interchangeably.*

Remark 1.

(i) Note that, by construction, as ${}^{np}\mathscr{F}$ is defined as a subalgebra of K_p, our definition will be automatically well-defined using the algebra structures of K_p and number-theoretic properties of the Fibonacci numbers.

(ii) Note that in the above Definition, by "order n" we do not mean the usual meaning of an order in an algebraic structure, but we simply refer to n in F_n.

(iii) For any $\mathcal{F}_n^{(p)} \in {}^{np}\mathscr{F}$, its algebraic conjugate is naturally defined as $\overline{\mathcal{F}_n^{(p)}} = F_n - hF_{n+1}$, and hence one may naturally equip ${}^{np}\mathscr{F}$ with an algebraic norm given by $N(\mathcal{F}_n^{(p)}) = (F_n^{(p)})^2 - p(F_{n+1}^{(p)})^2$. Later on, we will consider this algebraic operator property in more detail.

(iv) Note that, in part, our definition will include some specific parts of the complex, the hyperbolic, and the dual numbers (and the associated algebraic-geometrical properties) for an appropriate value of p as discussed before.

(v) Note that by the nature of our definition, one should not consider it as a one-parameter (or any other sort) deformation, as it fails to satisfy the necessary and sufficient conditions.

Note that as ${}^{np}\mathscr{F}$ is defined as a subalgebra of the quadratic algebra K_p, any element $z \in K_p$ (and consequently the elements $\mathcal{F}_n^{(p)} \in {}^{np}\mathscr{F}$) inherits the algebraic structure of the extended complex numbers, i.e., the algebra of hyperbolic and dual numbers for $p = 1$ and $p = 0$, respectively. Thus, the multiplicative structure and operations on ${}^{np}\mathscr{F}$ are entirely governed by the relation $h^2 = p$ in K_p. We recall the following fundamental algebraic properties of K_p that naturally apply to ${}^{np}\mathscr{F}$.

Lemma 1. *Let $z = x + hy \in K_p$. Then the following hold:*

(i) The norm $N(z) = x^2 - py^2$ is multiplicative, that is, $N(z_1 z_2) = N(z_1)N(z_2)$ for all $z_1, z_2 \in K_p$.

(ii) An element $z \in K_p$ has a multiplicative inverse z^{-1} if and only if $N(z) \neq 0$. In this case, $z^{-1} = \frac{\bar{z}}{N(z)} = \frac{x - hy}{x^2 - py^2}$.

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(iii) If $p > 0$, K_p contains nontrivial zero divisors satisfying $N(z) = 0$ for $z \neq 0$. If $p < 0$, K_p is a field isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra via the explicit mapping $x + hy \mapsto x + i\sqrt{|p|}y$.

The next result is of immense importance as it links analytic number theory (specifically the quadratic Fibonacci numbers) and functional complex analysis by studying the matrix representation theory of K_p and equipping it with the usual matrix sup operator norm. We will use this result when we study the power series, their convergence radius, and the growth rate of the multipliers.

Lemma 2. *For $z = x + hy$ as in Lemma 1, an arbitrary element in K_p, let $L_z : K_p \to End(K_p) : w \mapsto L_z(w) := zw$ be the left regular representation of K_p. Note that one can naturally assign the following matrix representation to L_z:*

$$M_z := \begin{pmatrix} x & py \\ y & x \end{pmatrix}. \tag{1}$$

Also one may naturally equip K_p with the following induced sup *operator norm:*

$$\|z\| := \|M_z\|_{op} := \max_{\|v\|_2=1} \|M_z v\|_2 \tag{2}$$

for $\|\cdot\|_2$ the Euclidean norm. Then the following statements hold:

(i) The $\mathbb{R}$-algebra homomorphism $\mathcal{D} : K_p \to M_2(\mathbb{R}) : z \mapsto \mathcal{D}(z) := M_z$ is a unital injective homomorphism. Moreover, $(K_p, \|\cdot\|)$ is a two-dimensional real unital Banach algebra.

(ii) The operator norm $\|\cdot\|$ is submultiplicative:

$$\|z_1 z_2\| \leq \|z_1\|\|z_2\| \quad \text{for all } z_1, z_2 \in K_p. \tag{3}$$

(iii) $\|\cdot\|$ is equivalent to the Euclidean norm $|z|_2 = \sqrt{x^2 + y^2}$; it follows that the coefficient growth rate is independent of the choice of norm, and $\lim_{n\to\infty} \|\mathcal{F}_n^{(p)}\|^{1/n} = \alpha$.

Proof. Writing $z_1 z_2 = (x_1 x_2 + p y_1 y_2) + h(x_1 y_2 + x_2 y_1)$ and comparing entries gives $M_{z_1 z_2} = M_{z_1} M_{z_2}$, while $M_1 = I_2$ and $z \mapsto M_z$ is clearly $\mathbb{R}$-linear; so it is a unital $\mathbb{R}$-algebra homomorphism. It is injective because $M_z(1,0)^\top = (x,y)^\top$ recovers both coordinates of z. Expanding the determinant yields $\det M_z = x^2 - py^2 = N(z)$. As $M_2(\mathbb{R})$ with the operator norm is a Banach algebra and $z \mapsto M_z$ is an isometry onto the closed subalgebra $\{M_z : z \in K_p\}$, $(K_p, \|\cdot\|)$ inherits the Banach-algebra structure, proving (i). The submultiplicativity (ii) follows directly from the standard operator norm bounds $\|AB\|_{\text{op}} \leq \|A\|_{\text{op}}\|B\|_{\text{op}}$. Finally, all norms on $K_p \cong \mathbb{R}^2$ are equivalent, giving $\|z\| \asymp |z|_2$, which along with $\mathcal{F}_n^{(p)} = F_n + hF_{n+1}$ and $F_n \sim \alpha^n/\sqrt{5}$ establishes $\lim_{n\to\infty} \|\mathcal{F}_n^{(p)}\|^{1/n} = \alpha$, proving (iii). □

Considering the extended complex number algebra structure of K_p that we pointed out earlier in the introduction and the geometrical structures imposed by $h^2 - p$ for different values of the parameter p can acquire, note that for $p = 1$ the algebra K_1 will be isomorphic to the hyperbolic (or the split-complex) numbers as the algebraic model of the Minkowski space-time plane possessing zero divisors with nontrivial idempotents, and the sequences restricted to this hyperbolic case are studied in [13,17]. Consequently, the elements $\mathcal{F}_n^{(1)}$ could be identified as the hyperbolic Fibonacci numbers studied in [19].

Recall that the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, with $F_0 = 0$ and $F_1 = 1$. The roots of its characteristic polynomial $\lambda^2 - \lambda - 1 = 0$ are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, which are called the golden ratios and satisfy the classical Binet representation and relations $\alpha^n = F_n\alpha + F_{n-1}$ and $\beta^n = F_n\beta + F_{n-1}$. We now recall the following result that will be used later throughout the paper.

Lemma 3 (See [20]). *Let α and β be the characteristic roots defined above. Then*

(i) For all $n \geq 0$, the classical Binet formula holds:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \tag{4}$$

(ii) Since $|\beta| < 1 < \alpha$, the term with α dominates as $n \to \infty$. In particular, the ratio of consecutive terms satisfies: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$.

(iii) The Cauchy–Hadamard coefficient limit for the sequence is $\lim_{n\to\infty} |F_n|^{1/n} = \alpha$.

3. Binet-Type Representation and the Hadamard–Fibonacci Transformation

This section could be considered the core part of the paper when it comes to the algebraic structures, such as the Binet-type formula for the quadratic Fibonacci numbers, and the closed form that it yields for the Hadamard-type transformation. The closed form will provide us with a powerful tool to study algebraicity and the singularities affected by the Hadamard–Fibonacci transformation.

Before we study the transformation, we need a closed-form expression for the elements $\mathcal{F}_n^{(p)}$, the multipliers in the transformation.

Lemma 4 (Binet-type Representation). *For every integer $n \geq 0$, the quadratic Fibonacci element $\mathcal{F}_n^{(p)} \in K_p$ satisfies the Binet-type representation:*

$$\mathcal{F}_n^{(p)} = \frac{(1 + h\alpha)\alpha^n - (1 + h\beta)\beta^n}{\sqrt{5}}. \tag{5}$$

Proof. Substituting the classical Binet formula (4) into $\mathcal{F}_n^{(p)} = F_n + hF_{n+1}$ gives $\mathcal{F}_n^{(p)} = \frac{\alpha^n - \beta^n}{\sqrt{5}} + h\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}}\right) = \frac{(\alpha^n + h\alpha^{n+1}) - (\beta^n + h\beta^{n+1})}{\sqrt{5}}$, and collecting the α^n and β^n terms yields the stated formula. □

Let $\mathbb{R}[[x]]$ be the ring of formal power series over the reals. For a generating function $A(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{R}[[x]]$, we define its Hadamard–Fibonacci transform $\mathcal{H}_{F,A}^{(p)}(x) \in K_p[[x]]$ coefficientwise:

$$\mathcal{H}_{F,A}^{(p)}(x) := \sum_{n\geq 0} \mathcal{F}_n^{(p)} a_n x^n.$$

Theorem 1 (Transformation Formula). *Let $A(x) = \sum_{n\geq 0} a_n x^n$ be a formal power series. Then the transform satisfies the closed-form identity:*

$$\mathcal{H}_{F,A}^{(p)}(x) = \frac{1}{\sqrt{5}}\Big[(1 + h\alpha)A(\alpha x) - (1 + h\beta)A(\beta x)\Big]. \tag{6}$$

Proof. Substituting the Binet representation of Lemma 4 into the definition of the transform gives

$$\mathcal{H}_{F,A}^{(p)}(x) = \sum_{n\geq 0}\left[\frac{(1 + h\alpha)\alpha^n - (1 + h\beta)\beta^n}{\sqrt{5}}\right]a_n x^n = \frac{1}{\sqrt{5}}\sum_{n\geq 0}[(1 + h\alpha)a_n(\alpha x)^n - (1 + h\beta)a_n(\beta x)^n].$$

Note that to obtain (6), we used Lemma 4, and as every $A(x) \in \mathbb{R}[[x]]$ admits a unique expansion $\sum_{n\geq 0} a_n x^n$, i.e., the monomials form a basis of $\mathbb{R}[[x]]$, the diagonal action follows. □

Remark 2. *While Hadamard-type transformations and generalized linear recurrent sequences over scalar rings continue to be actively investigated in recent literature (e.g., Koshy [20], Kazenas [26], and Alecci [27]), our quadratic K_p-algebraic framework introduces several distinct structural features:*

(i) ***Classical Scalar Hadamard Shifts vs. K_p-Extension:*** *Recent studies on the applications of the Hadamard product on recursive sequences over scalar fields $\mathbb{R}$ or $\mathbb{C}$ [26,27] show that, by contrast, embedding the transformation directly into $K_p[[x]]$ replaces scalar multipliers with nontrivial algebraic elements $\mathcal{F}_n^{(p)} = \alpha^n u - \beta^n \omega$, and shifts to a structural quadratic ring setting by generalizing the scalar argument.*

(ii) ***Algebraic Parameter p and Zero-Divisor Dynamics:*** *In contrast with the standard Fibonacci identities and linear recurrence properties studied in [20,27], the presence of the deformation parameter p in our generalized definition of the quadratic hyperbolic numbers $\mathcal{F}_n^{(p)} = F_n + hF_{n+1}$ induces a nontrivial operator-theoretic phenomenon; specifically, this happens when we deal with the case where $p > 0$. In this case, K_p contains zero divisors isomorphic to the split-complex numbers. This fact leads to nontrivial kernel structures and certain spectral properties that naturally do not exist in classical (field-based) Hadamard operators.*

(iii) ***Analytical Convergence and Domain Expansion:*** *Despite the fact that the classical recurrence-sequence-based studies mostly focus on coefficientwise product relations [26], our method of employing the Binet-type algebraic factors $(1 + h\alpha)$ and $(1 + h\beta)$, explicitly established in Theorem 1, scales the radius of convergence by α^{-1} and provides a direct bridge between the quadratic ring operators and the analytical domain transformations.*

4. Diagonalization on Monomials

For every $c \in \mathbb{R}$, let $(S_c A)(x) := A(cx)$ denote the canonical scaling operator on formal power series $A(x)$. Then we have the following definition.

Definition 2 (Formal Operator Action and Topological Setting)**.** *The Hadamard–Fibonacci transformation $T_F^{(p)} : K_p[[x]] \to K_p[[x]]$ is defined as the K_p-linear endomorphism*

$$T_F^{(p)} := \frac{1}{\sqrt{5}}\Big[(1 + h\alpha)S_\alpha - (1 + h\beta)S_\beta\Big]. \tag{7}$$

Its action on any formal power series $A(x) = \sum_{n\geq 0} a_n x^n \in K_p[[x]]$ is given by termwise K_p-linear extension:

$$T_F^{(p)}\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} \mathcal{F}_n^{(p)} a_n x^n. \tag{8}$$

When acting on the Fréchet algebra of formal power series $K_p[[x]]$ endowed with the standard x-adic topology, $T_F^{(p)}$ will be a continuous K_p-linear endomorphism.

Proposition 1 (Diagonal Action and Spectrum)**.** *The monomials $\{x^n\}_{n\geq 0}$ form a topological basis for $K_p[[x]]$ with respect to the x-adic topology, and $T_F^{(p)}$ acts diagonally on this basis. Specifically, for every $n \geq 0$, x^n is an eigenfunction of $T_F^{(p)}$ with corresponding eigenvalue $\mathcal{F}_n^{(p)} \in K_p$, that is,*

$$T_F^{(p)}(x^n) = \mathcal{F}_n^{(p)} x^n \tag{9}$$

holds. Consequently, the point spectrum of the operator on $K_p[[x]]$ will be explicitly given by $\sigma_p(T_F^{(p)}) = \left\{\mathcal{F}_n^{(p)} : n \geq 0\right\}$.

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Proof. Consider $T_F^{(p)}$ from Equation (7). Then we have

$$\begin{aligned} T_F^{(p)}(x^n) &= \frac{1}{\sqrt{5}}\left[(1+h\alpha)S_\alpha(x^n) - (1+h\beta)S_\beta(x^n)\right] \\ &= \frac{1}{\sqrt{5}}\left[(1+h\alpha)(\alpha^n x^n) - (1+h\beta)(\beta^n x^n)\right] \\ &= \frac{1}{\sqrt{5}}\left[(1+h\alpha)\alpha^n - (1+h\beta)\beta^n\right]x^n \\ &= \mathcal{F}_n^{(p)} x^n. \end{aligned} \tag{10}$$

Since every $A(x) \in K_p[[x]]$ expands uniquely as $\sum_{n\geq 0} a_n x^n$, diagonal action follows directly. □

Remark 3. *Note that following Proposition 1 and Equation (9), one may naturally have the iterated formula as follows using properties of the quadratic Fibonacci numbers* $\mathcal{F}_n^{(p)}$*:*

$$\begin{aligned} (T_F^{(p)})^m(x^n) &= (T_F^{(p)})^{m-1}(T_F^{(p)}(x^n)) \\ &= (T_F^{(p)})^{m-1}(\mathcal{F}_n^{(p)} x^n) \\ &= \mathcal{F}_n^{(p)}((T_F^{(p)})^{m-1}(x^n)) \\ &= \mathcal{F}_n^{(p)}((T_F^{(p)})^{m-2}(T_F^{(p)}(x^n))) \\ &= \mathcal{F}_n^{(p)}((T_F^{(p)})^{m-2}(\mathcal{F}_n^{(p)} x^n)) \\ &= (\mathcal{F}_n^{(p)})^2((T_F^{(p)})^{m-2}(x^n)) \\ &= \dots \\ &= (\mathcal{F}_n^{(p)})^m x^n. \end{aligned} \tag{11}$$

Theorem 2. *Let $p \in \mathbb{R}$ be fixed. The Hadamard–Fibonacci transformation endomorphism $T_F^{(p)} : K_p[[x]] \longrightarrow K_p[[x]]$ defined by*

$$T_F^{(p)}(A)(x) := \sum_{n\geq 0} \mathcal{F}_n^{(p)} a_n x^n \tag{12}$$

is coefficientwise invertible (i.e., possesses a two-sided inverse endomorphism $(T_F^{(p)})^{-1} : K_p[[x]] \to K_p[[x]]$ satisfying $(T_F^{(p)})^{-1} \circ T_F^{(p)} = id_{K_p[[x]]}$) if and only if $p \neq \left(\frac{F_n}{F_{n+1}}\right)^2$ for all $n \geq 0$, and the reverse direction also holds. In this case, for any series $B(x) = \sum_{n\geq 0} b_n x^n \in K_p[[x]]$,

$$(T_F^{(p)})^{-1}\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} \frac{F_n - hF_{n+1}}{F_n^2 - pF_{n+1}^2} b_n x^n.$$

Moreover, these forbidden values accumulate at $p = \alpha^{-2} = \frac{3-\sqrt{5}}{2}$.

Proof. Following Equation (9), it could be easily seen that $T_F^{(p)}$ is invertible whenever all its eigenvalues, which are $\mathcal{F}_n^{(p)} = F_n + hF_{n+1}$ for $n \geq 0$, are invertible and vice versa. Then using Lemma 1, it could easily be verified that $N(\mathcal{F}_n^{(p)}) = F_n^2 - pF_{n+1}^2$ holds, and hence as a result we have that $\mathcal{F}_n^{(p)}$ is invertible if and only if

$$F_n^2 - pF_{n+1}^2 \neq 0, \tag{13}$$

holds. Then, dividing Equation (13) by F_{n+1}^2 will yield $p \neq \left(\frac{F_n}{F_{n+1}}\right)^2$. Now the claim is that

$$g : K_p[[x]] \to K_p[[x]] : \sum_{n\geq 0} b_n x^n \mapsto g\left(\sum_{n\geq 0} b_n x^n\right)(x) := \sum_{n\geq 0} \frac{F_n - hF_{n+1}}{F_n^2 - pF_{n+1}^2} b_n x^n \tag{14}$$

provides a two-sided inverse for $T_F^{(p)}$, as we have

$$\begin{aligned} g \circ T_F^{(p)}\left(\sum_{n\geq 0} a_n x^n\right) &= g\left(\sum_{n\geq 0}(F_n + hF_{n+1})a_n x^n\right) \\ &= \sum_{n\geq 0}\left(\frac{F_n - hF_{n+1}}{F_n^2 - pF_{n+1}^2}\right)(F_n + hF_{n+1})a_n x^n \\ &= \sum_{n\geq 0}\left(\frac{F_n^2 - pF_{n+1}^2}{F_n^2 - pF_{n+1}^2}\right)a_n x^n \\ &= \sum_{n\geq 0} a_n x^n = \mathrm{id}_{K_p[[x]]}. \end{aligned} \tag{15}$$

The other direction, i.e., $T_F^{(p)} \circ g\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} a_n x^n$, proceeds in an almost similar way. □

Remark 4. *Following the number-theoretic features of the quadratic Fibonacci elements, the following statements could be easily verified (note that as here we use the algebraic origin of $\mathcal{F}_n^{(p)}$, we refer to them as elements of K_p):*

(a) For $p = -1$, the operator $T_F^{(-1)}$ is coefficientwise invertible.
(b) For $p = 1$, the operator $T_F^{(1)}$ is not coefficientwise invertible.
(c) Following Remark 3 and Equations (11) and (12), for a fixed integer $m \geq 1$ and $A(x) = \sum_{n\geq 0} a_n x^n \in K_p[[x]]$, the iterated transformation will yield

$$\left(T_F^{(p)}\right)^m (A)(x) = \sum_{n\geq 0}\left(\mathcal{F}_n^{(p)}\right)^m a_n x^n. \tag{16}$$

To continue, define $C_R := \{A(x) = \sum_{n\geq 0} a_n x^n \in K_p[[x]] : \limsup_{n\to\infty} \|a_n\|^{1/n} = \frac{1}{R} < \infty\}$. Then we may propose the following proposition.

Proposition 2 (Spectral growth under iteration)**.** *Let $A(x) = \sum_{n\geq 0} a_n x^n \in C_R$ be a convergent power series with coefficients in K_p having radius of convergence $R > 0$. Fix an integer $m \geq 1$, and consider the iterated transform $(T_F^{(p)})^m(A)(x) = \sum_{n\geq 0}(\mathcal{F}_n^{(p)})^m a_n x^n$. Then $(T_F^{(p)})^m(A)(x)$ belongs to C_{R_m}, where the transformed radius of convergence R_m satisfies:*

$$R_m = \frac{R}{\alpha^m}.$$

Proof. By the Cauchy–Hadamard formula, the radius of convergence R of the series $A(x) = \sum_{n\geq 0} a_n x^n$ satisfies $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$. By Lemma 2 (iii) and Lemma 4, the limit of the growth rate for the quadratic Fibonacci elements exists and is given by $\lim_{n\to\infty} \|\mathcal{F}_n^{(p)}\|^{1/n} = \alpha > 0$.

For a fixed integer $m \geq 1$, consider the coefficients $b_n = (\mathcal{F}_n^{(p)})^m a_n$ of the iterated transform $(T_F^{(p)})^m(A)(x) = \sum_{n\geq 0} b_n x^n$. Applying the Cauchy–Hadamard formula yields

$$\frac{1}{R_m} = \limsup_{n\to\infty} \|b_n\|^{1/n} = \limsup_{n\to\infty}\left(\|\mathcal{F}_n^{(p)}\|^{m/n}\|a_n\|^{1/n}\right).$$

Since the ordinary limit $\lim_{n\to\infty} \|\mathcal{F}_n^{(p)}\|^{m/n} = \alpha^m$ exists and is strictly positive, we use the standard limit property $\limsup(c_n x_n) = (\lim c_n)(\limsup x_n)$ for sequences with a convergent positive factor c_n. This yields the exact equality in both directions:

$$\frac{1}{R_m} = \left(\lim_{n\to\infty} \|\mathcal{F}_n^{(p)}\|^{1/n}\right)^m \limsup_{n\to\infty} \|a_n\|^{1/n} = \alpha^m \cdot \frac{1}{R}.$$

Taking the reciprocal gives $R_m = \frac{R}{\alpha^m}$, which completes the proof. □

5. Splitting, Rotation and Shift

Let $e_1 := \frac{1+h}{2}$, and $e_2 := \frac{1-h}{2}$ be idempotents that satisfy the following relations:

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0, \quad e_1 + e_2 = 1. \tag{17}$$

If one decomposes $z = x + hy$ as $z = (x+y)e_1 + (x-y)e_2$ using the newly defined idempotents e_1 and e_2, then the Hadamard–Fibonacci transformation would split into two independent parts. So we may propose the following result.

Proposition 3 (Hyperbolic decomposition)**.** *For $p = 1$, the Hadamard–Fibonacci operator decomposes into two independent components:*

$$T_F^{(1)} = T_+ \oplus T_-. \tag{18}$$

More precisely, with respect to the idempotent basis $e_1 = \frac{1+h}{2}$ and $e_2 = \frac{1-h}{2}$, the quadratic Fibonacci eigenvalues decompose as

$$\mathcal{F}_n^{(1)} = F_n + hF_{n+1} = (F_n + F_{n+1})e_1 + (F_n - F_{n+1})e_2 = F_{n+2}e_1 - F_{n-1}e_2,$$

with $F_{-1} = 1$.

Proof. Every element of K_1 could be uniquely decomposed as $z = z_1 e_1 + z_2 e_2$ using relations (17), and as $h = e_1 - e_2$, we obtain $\mathcal{F}_n^{(1)} = (F_n + F_{n+1})e_1 + (F_n - F_{n+1})e_2 = F_{n+2}e_1 - F_{n-1}e_2$. The two scalar channels act independently, establishing $T_F^{(1)} = T_+ \oplus T_-$. □

Note that for $p = -1$, the algebra K_{-1} is isomorphic to $\mathbb{C}$, so the scaling coefficients $1 + h\alpha$ and $1 + h\beta$ (where h plays the role of $i = \sqrt{-1}$) can be expressed in exact polar form as $R_\alpha e^{i\theta_\alpha}$ and $R_\beta e^{i\theta_\beta}$, where $R_\alpha = \sqrt{\frac{5+\sqrt{5}}{2}}$ and $R_\beta = \sqrt{\frac{5-\sqrt{5}}{2}}$.

When $p = 0$, one obtains the dual algebra K_0 where $h = \varepsilon$ with $\varepsilon^2 = 0$. Here h is a non-zero nilpotent generator representing an infinitesimal unit rather than an idempotent.

In this case, separating real and dual cases gives

$$\mathcal{H}_{F,A}^{(0)}(x) = \frac{1}{\sqrt{5}}[A(\alpha x) - A(\beta x)] + \varepsilon \frac{1}{\sqrt{5}}[\alpha A(\alpha x) - \beta A(\beta x)].$$

The nilpotent part of the dual case produces an index-shifted Hadamard–Fibonacci series $\sum_{n\geq 0} F_{n+1} a_n x^n$.

6. Algebraicity and Rescaled Singularities

The singularity analysis used in the next results is a consequence of the standard framework of analytic combinatorics studied in [1].

Theorem 3 (Preservation of algebraicity)**.** *Let $A(x) \in \mathbb{C}[[x]]$ be algebraic over the rational field $\mathbb{C}(x)$, and write the transformed series as*

$$\mathcal{H}_{F,A}^{(p)}(x) = u(x) + hv(x),$$

for $u(x) = \frac{A(\alpha x) - A(\beta x)}{\sqrt{5}}$ and $v(x) = \frac{\alpha A(\alpha x) - \beta A(\beta x)}{\sqrt{5}}$ algebraic over $\mathbb{C}(x)$. Then both component series $u(x)$ and $v(x)$ are algebraic over $\mathbb{C}(x)$.

Proof. Since $A(x)$ is algebraic over $\mathbb{C}(x)$, $A(\alpha x)$ and $A(\beta x)$ are also algebraic over $\mathbb{C}(x)$, and then the linear combinations $u(x)$ and $v(x)$ will be algebraic over $\mathbb{C}(x)$ as well. □

Proposition 4. *If* $A(x) = \frac{P(x)}{Q(x)} \in \mathbb{C}(x)$ *is rational, then* $\mathcal{H}^{(p)}_{F,A}(x)$ *is also rational, that is,* $\mathcal{H}^{(p)}_{F,A}(x) \in K_p \otimes \mathbb{C}(x)$.

If $A = P/Q$ is in lowest terms, with $Q(x) = \prod_{j=1}^{m}(1-\rho_j x)^{m_j}$, then the singularities are transported to $x = \frac{1}{\alpha\rho_j}$ and $x = \frac{1}{\beta\rho_j}$.

If $A(x)$ admits a partial fraction decomposition $A(x) = \sum_{j=1}^{m}\sum_{\ell=1}^{m_j}\frac{c_{j,\ell}}{(1-\rho_j x)^{\ell}} + P(x)$, where $c_{j,\ell} \in \mathbb{C}$ are partial fraction coefficients, ρ_j^{-1} are the distinct poles of $A(x)$, and $P(x)$ is a polynomial, then $A(\alpha x)$ and $A(\beta x)$ shift poles explicitly, more explicitly multiplying pole locations by α^{-1} and β^{-1}.

7. Applications

Recall the generating function of the harmonic numbers $H_n = \sum_{k=1}^{n}\frac{1}{k}$, which is $S(x) = \sum_{n\geq 1} H_n x^n = -\frac{\log(1-x)}{1-x}$. Now using $S(x)$ one can easily see that $\sum_{n\geq 1}\mathcal{F}_n^{(p)}H_n x^n = \frac{1}{\sqrt{5}}[(1+h\alpha)S(\alpha x) - (1+h\beta)S(\beta x)]$. Hence, the only singularity of $S(x)$ is at $x = 1$, and therefore the transform will have singularities at $x = \frac{1}{\alpha}$ and $x = \frac{1}{\beta}$; since $|\frac{1}{\alpha}| < |\frac{1}{\beta}|$ holds, the radius of convergence will drop from 1 to $\frac{1}{\alpha}$ by Proposition 2 for $m = 1$.

For Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, the generating function is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. The branch point at $x = 1/4$ shifts to $x = 1/(4\alpha)$ and $x = 1/(4\beta)$.

For Stirling numbers of the second kind $S(n,k)$, representing the number of ways to partition n elements into k non-empty sets, $A_k(x) = \sum_{n\geq k} S(n,k)x^n = \frac{x^k}{\prod_{j=1}^{k}(1-jx)}$. Applying $T_F^{(p)}$ shifts poles from $x = 1/j$ to $x = 1/(j\alpha)$ and $x = 1/(j\beta)$.

8. Conclusions

We defined a Hadamard–Fibonacci transformation over the quadratic algebra $K_p = \mathbb{R}[h]/(h^2 - p)$ and provided a unified algebraic and operator-theoretic framework. The closed dilation formula reduces analysis to quadratic Fibonacci elements $\mathcal{F}_n^{(p)}$. The parameter p governs invertibility, singularity movement, and structural decomposition across hyperbolic ($p > 0$), complex ($p < 0$), and dual ($p = 0$) geometries.

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