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Differentiability of the Functional in the Problem of Reconstructing the Density of a Gravitational Anomaly

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Abstract

This article addresses the problem of reconstructing the density distribution of a gravitational anomaly from measurements of the vertical component of the gravitational field gradient. The study is motivated by the growing demand for reliable mathematical methods in gravity exploration and inverse modeling. The direct problem is formulated using the Poisson equation with homogeneous boundary conditions on an extended computational domain, while the inverse problem is posed as the minimization of a smooth quadratic misfit functional. A sensitivity analysis is performed to derive the functional gradient, and the Gateaux derivative is rigorously established through the solution of an adjoint boundary value problem. The analysis reveals a fundamental feature of the adjoint formulation: the right-hand side of the adjoint equation contains the derivative of the Dirac delta function. This singularity makes the direct application of classical gradient-based optimization methods numerically impractical, despite the formal differentiability of the objective functional. The obtained results provide a theoretical justification for abandoning conventional gradient optimization techniques in this class of inverse gravimetric problems and motivate the use of derivative-free optimization methods. The proposed framework establishes a mathematical foundation for the development of robust numerical algorithms for gravitational density reconstruction and outlines promising directions for future research.

Keywords: inverse gravimetric problem; gravitational anomaly; density reconstruction; Poisson equation; adjoint equation; Gateaux derivative; Dirac delta function; derivative-free optimization; ill-posed inverse problems.

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1. Introduction

Gravity inversion plays a fundamental role in modern geophysical exploration because it enables the reconstruction of subsurface density distributions from gravity measurements. Accurate density reconstruction is essential for mineral prospecting, oil and gas exploration, geological mapping, and the investigation of the internal structure of the Earth. Mathematical models based on the Poisson equation provide the theoretical foundation for describing the relationship between the density distribution and the observed gravitational field. Since density cannot be measured directly, its determination is formulated as an inverse gravimetric problem, which belongs to the class of ill-posed inverse problems and requires specialized mathematical and computational techniques to obtain stable solutions [1–6].

Various approaches have been developed to solve inverse gravimetric problems. Classical methods include regularization techniques, iterative inversion algorithms, and optimization-based approaches, where the unknown density distribution is recovered by minimizing the functional mismatch between the observed and calculated gravity data [7–13]. Gradient-based optimization methods are among the most widely used because of their computational efficiency and well-established theoretical properties. However, their implementation requires the existence and numerical evaluation of the functional derivative, which is commonly obtained through sensitivity analysis and adjoint-state methods [14–19].

Recent studies have focused on regularized inversion techniques, Bayesian inversion, joint inversion of gravity and magnetic data, and machine learning approaches for density reconstruction. Li and Oldenburg [9] proposed a three-dimensional gravity inversion framework that remains one of the most influential methods in geophysical exploration. Martyshko et al. [12] developed efficient parallel algorithms for large-scale gravimetric inversion. Recent optimization-based approaches increasingly rely on adjoint-state methods [15–18]; however, their successful application assumes that the computed gradient can be accurately evaluated. The mathematical consequences of singular source terms in the adjoint equation have received considerably less attention.

Although adjoint methods provide an efficient framework for gradient computation, their applicability depends on the mathematical properties of the corresponding adjoint problem. In many inverse problems, the differentiability of the objective functional is assumed to guarantee the successful application of gradient optimization algorithms. However, the existence of the Gateaux derivative alone does not necessarily ensure the numerical applicability of these methods. In particular, singular source terms appearing in the adjoint equation can significantly complicate or even prevent the practical computation of the gradient [20–25].

The present study investigates this issue for the inverse gravimetric problem of reconstructing the density distribution from measurements of the vertical component of the gravity field gradient. The direct problem is formulated by the Poisson equation with homogeneous boundary conditions on an extended computational domain, whereas the inverse problem is posed as the minimization of a quadratic misfit functional. A rigorous sensitivity analysis is performed to derive the Gateaux derivative of the functional through an adjoint boundary value problem.

The main contribution of this work is the theoretical demonstration that the adjoint equation contains the derivative of the Dirac delta function in its right-hand side. Although the objective functional is formally differentiable, this singularity makes the direct application of conventional gradient-based optimization methods numerically impractical. Consequently, the obtained results provide a mathematical justification for employing derivative-free optimization methods in this class of inverse gravimetric problems [20–24]. The proposed analysis establishes a theoretical basis for the development of robust numerical algorithms for density reconstruction and may be extended to other inverse problems governed by elliptic partial differential equations.

2. Problem Statement

The problem arose in geophysics, where mathematical modeling serves as a tool for its solution. Its roots lie in the practical necessity of solving inverse problems in gravimetric exploration [26–30].

The problem arises in geophysics, where mathematical modeling provides an important tool for the interpretation of gravimetric observations. The direct and inverse problems considered in this study are based on the Poisson equation describing the gravitational potential generated by a density distribution.

Let Ω' denote the extended computational domain. The gravitational potential $\eta(x,y)$ generated by the density distribution $\psi(x,y)$ is determined from the direct problem

$$\Delta\eta(x,y) = -4\pi G\psi(x,y), \qquad (x,y) \in \Omega', \tag{1}$$

subject to the homogeneous boundary condition

$$\eta(x,y)|_{\partial\Omega'} = 0. \tag{2}$$

Here, $G = 6.6743 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the gravitational constant, $\psi(x,y)$ represents the density distribution of the anomaly, and $\eta(x,y)$ denotes the corresponding gravitational potential.

The computational domain is extended in all directions so that the gravitational field generated by the anomaly can be approximated by zero on the artificial boundary $\partial\Omega'$. This leads to the homogeneous boundary condition (2).

The quantities measured in the physical experiment are the values of the vertical component of the gravitational field gradient along the observation profile NK. These data are denoted by

$$\left.\frac{\partial\eta(x,y)}{\partial y}\right|_{NK} = \eta_2(x), \qquad x \in [N,K]. \tag{3}$$

Equation (3) is not an additional condition imposed on the state equation and does not form part of the direct problem. Instead, it describes the observational data obtained from gravimetric measurements. In other words, the direct problem consists of solving (1) subject to (2) for a prescribed density distribution, while (3) specifies the quantity of the state that is observed and subsequently used in the formulation of the inverse problem.

For simplicity, we assume that the anomaly occupies a known domain $\Omega_0 \subset \Omega'$ and that its density contrast with respect to the surrounding medium is spatially constant. Thus,

$$\psi(x,y) = \begin{cases} \psi_0, & (x,y) \in \Omega_0, \\ 0, & (x,y) \notin \Omega_0, \end{cases} \tag{4}$$

where ψ_0 is the unknown density contrast.

Under this assumption, the inverse problem consists of determining the parameter ψ_0, and consequently reconstructing the density distribution $\psi(x,y)$ within the class of piecewise constant functions defined by (4), from the observed gravity-gradient data $\eta_2(x)$ on the profile NK.

The direct problem is therefore formulated as follows: for a prescribed value of ψ_0, determine the gravitational potential from (1) and (2) and calculate the corresponding gravity-gradient values along the observation profile. The inverse problem is to determine the value of ψ_0 for which the calculated gravity-gradient data agree the best with the measured data $\eta_2(x)$.

To solve the inverse problem, we introduce the quadratic misfit functional

$$I(\psi_0) = \int_N^K \left(\left.\frac{\partial\eta(x,y;\psi_0)}{\partial y}\right|_{y=0} - \eta_2(x) \right)^2 dx \to \min. \tag{5}$$

Although the present formulation contains a single unknown parameter ψ_0, the dependence of the objective function on this parameter is determined through the solution of the boundary-value problem (1) and (2). The variational formulation used below makes it possible to analyze the sensitivity of the state with respect to the density parameter and to derive the corresponding adjoint problem. In addition, the same derivation naturally extends to the case of a spatially varying density parameter $\psi(x,y)$, for which the objective becomes a functional in the conventional infinite-dimensional sense.

The formulation of the direct problem is as follows: *determine the value of the gravitational field gradient $\eta_2(x)$ on the NK segment, given the density distribution of the anomaly $\psi(x,y)$ in the extended domain Ω'*

In practice, however, the density distribution $\psi(x,y)$ of the anomaly is unknown. Instead, the values of the gravitational field gradients along the NK segment are known, measured by gravimeters within the framework of geodynamic monitoring of an oil field.

The formulation of the inverse problem is therefore as follows: *determine the density distribution of the anomaly $\psi(x,y)$ in the extended domain Ω', given the values of the gravitational field gradient $\eta_2(x)$ on the NK segment (see Figure 1).*

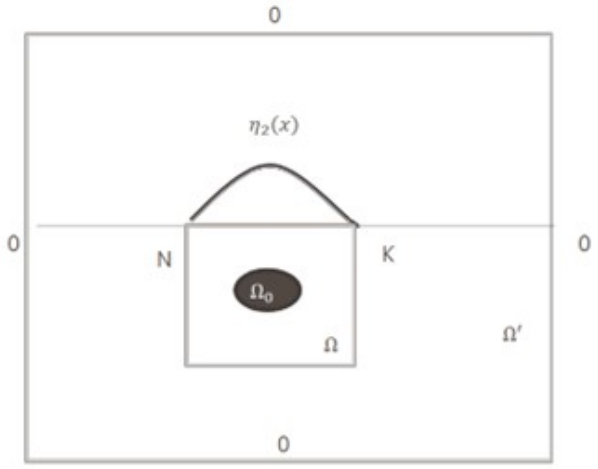


Figure 1. Studying area of the problem statement.

To solve the inverse problem, it is necessary to reformulate it as an optimization problem. For this purpose, we introduce a functional of the form (6), which represents the quadratic deviation of the vertical components of the gravitational field gradient from the values measured by gravimeters.

$$I(\psi_0) = \int_N^K \Big(\frac{\partial \eta(x,y)}{\partial y}|_{NK} - \eta_2(x) \Big)^2 dx \to min. \tag{6}$$

Thus, the optimization problem is formulated as follows: *it is required to determine the value ψ_0 from the condition of minimizing functional (6), where ψ_0 is defined by expression (4).*

Classically, optimization problems are solved using gradient-based methods. The application of such methods requires the computation of the gradient of the functional.

3. Results: Differentiation of the Functional

Our goal is to minimize functional (6). Problems of this nature are usually solved using various gradient-based methods. The practical application of these methods requires computing the derivative of the functional.

Theorem 1. *The derivative of the functional I at the point ψ_0 is given by*

$$I'(\psi_0) = 4\pi G \iint_\Omega p(x,y)\, dx\, dy \tag{7}$$

where $p(x,y)$ is the solution of the adjoint system, which has the following form:

$$\Delta p(x,y) = \alpha(x) \frac{\partial}{\partial y}\Big(2\Big(\frac{\partial \eta(x,y)}{\partial y} - \eta_2(x) \Big) \delta(y) \Big) \tag{8}$$

with boundary conditions:

$$p(0;y) = 0 \tag{9}$$

$$p(2P + L; y) = 0 \tag{10}$$

$$p(x; -M - P) = 0 \tag{11}$$

$$p(x; P) = 0 \tag{12}$$

We now proceed to find the derivative of functional (6).

Proof of Theorem 1. We use the definition of the Gateaux derivative:

$$(I'(\psi_0), h) = \lim_{\sigma \to 0} \Big(\frac{I(\psi_0 + \sigma h) - I(\psi_0)}{\sigma} \Big).$$

Since the functional is related to the density ψ_0 through the field potential state Equation (1), we introduce additional notation.

Let the density ψ_0 generate the field potential (1) denoted by $y(x, y)$, and let the density $\psi_0 + \sigma h$ generate the field potential (1) denoted by $q(x, y)$, that is

$$\psi_0 \to v(x, y)$$

$$\psi_0 + \sigma h \to q(x, y).$$

Then

$$I_*(\psi_0) = \int_N^K \Big(\frac{\partial v(x, 0)}{\partial y} - \eta_2(x) \Big)^2 dx,$$

$$I_*(\psi_0 + \sigma h) = \int_N^K \Big(\frac{\partial q(x, 0)}{\partial y} - \eta_2(x) \Big)^2 dx.$$

We denote the increment of the functional by:

$$\Delta I_*(\psi_0) = I_*(\psi_0 + \sigma h) - I_*(\psi_0) = \int_N^K \left[\Big(\frac{\partial q(x, 0)}{\partial y} - \eta_2(x) \Big)^2 - \Big(\frac{\partial v(x, 0)}{\partial y} - \eta_2(x) \Big)^2 \right] dx =$$

$$= \int_N^K \Big(\frac{\partial q(x, 0)}{\partial y} + \frac{\partial v(x, 0)}{\partial y} - 2\eta_2(x) \Big) \Big(\frac{\partial q(x, 0)}{\partial y} + \frac{\partial v(x, 0)}{\partial y} - 2\eta_2(x) \Big) dx.$$

Let $q(x, 0) - v(x, 0) = \eta(x, 0)$. It follows that $q(x, 0) = v(x, 0) + \eta(x, 0)$, therefore

$$\frac{\partial \eta(x, 0)}{\partial y} = \frac{\partial q(x, 0)}{\partial y} - \frac{\partial v(x, 0)}{\partial y} \; and \; \frac{\partial q(x, 0)}{\partial y} = \frac{\partial v(x, 0)}{\partial y} + \frac{\partial \eta(x, 0)}{\partial y}.$$

Therefore, the increment of the functional takes the form:

$$\Delta I_*(\psi_0) = \int_N^K 2 \Big(\frac{\partial v(x, 0)}{\partial y} - \eta_2(x) \Big) \frac{\partial \eta(x, 0)}{\partial y} dx + \int_N^K \Big(\frac{\partial \eta(x, 0)}{\partial y} \Big)^2 dx.$$

Ultimately, we have:

$$I_*(\psi_0 + \sigma h) - I_*(\psi_0) = \int_N^K 2 \Big(\frac{\partial v(x, 0)}{\partial y} - \eta_2(x) \Big) \frac{\partial \eta(x, 0)}{\partial y} dx + \int_N^K \Big(\frac{\partial \eta(x, 0)}{\partial y} \Big)^2 dx. \tag{13}$$

Next, we need to calculate

$$(I'_*(\psi_0), h) = \lim_{\sigma \to 0} \frac{I_*(\psi_0 + \sigma h) - I_*(\psi_0)}{\sigma}$$

but Equation (13) does not explicitly depend on σ. The parameter σ is related to Equation (13) through the system state Equation (1) in our problem, this is the gravitational field potential.

$$\psi_0 \to v(x,y),$$

$$\psi_0 + \sigma h \to q(x,y)$$

This means that the density ψ_0 corresponds to the gravitational field potential described by the function $v(x,y)$. Similarly, the density $\psi_0 + \sigma h$ corresponds to the field potential $q(x,y)$.

We express the problem formulation (1)–(6) using the two functions $q(x,y)$ and $v(x,y)$.

$$\Delta v(x,y) = -4\pi\psi(\psi_0)G, \qquad \Delta q(x,y) = -4\pi\psi(\psi_0 + \sigma h)G,$$

$$v(x,y)|_{\partial\Omega'} = 0, \qquad q(x,y)|_{\partial\Omega'} = 0,$$

$$\psi(\psi_0) = \begin{cases} 0, & (x,y) \notin \Omega_0 \\ \psi_0, & (x,y) \in \Omega_0 \end{cases}, \qquad \psi(\psi_0 + \sigma h) = \begin{cases} 0, & (x,y) \notin \Omega_0 \\ \psi_0 + \sigma h, & (x,y) \in \Omega_0 \end{cases}.$$

We now subtract these two equations and introduce the notation $\eta(x,y) = q(x,y) - v(x,y)$. The resulting equation can then be written as:

$$\Delta\eta(x,y) = -4\pi G[\psi(\psi_0) + \sigma h) - \psi(\psi_0)].$$

We denote $[\psi(\psi_0) + \sigma h) - \psi(\psi_0)] = \hat{\psi}$, then

$$\Delta\eta(x,y) = -4\pi G\hat{\psi}, \tag{14}$$

$$\eta(x,y)|_{\partial\Omega'} = 0$$

where $\partial\Omega'$ is the boundary of the whole extended domain

$$\hat{\psi} = \begin{cases} 0, & (x,y) \notin \Omega_0 \\ \sigma h, & (x,y) \in \Omega_0 \end{cases}.$$

Equation (14) is artificially multiplied by $\lambda(x,y)$ (where $\lambda(x,y)$ is arbitrary), and then integrated over the entire domain under consideration.

$$\int_0^{2P+L}\int_{-M-P}^{P} \lambda(x,y)\frac{\partial^2\eta(x,y)}{\partial x^2}\,dx\,dy + \int_0^{2P+L}\int_{-M-P}^{P} \lambda(x,y)\frac{\partial^2\eta(x,y)}{\partial y^2}\,dx\,dy =$$

$$= -4\pi G\int_0^{2P+L}\int_{-M-P}^{P} \lambda(x,y)\hat{\psi}\,dx\,dy = -4\pi G\sigma h\int_0^{2P+L}\int_{-M-P}^{P} \lambda(x,y)\,dx\,dy. \tag{15}$$

Let the first double integral in Equation (15) be denoted by I_{1*}. We then apply integration by parts twice.

$$I_{1*} = \int_0^{2P+L}\int_{-M-P}^{P} \lambda(x,y)\frac{\partial^2\eta(x,y)}{\partial x^2}\,dx\,dy =$$

$$= \int_{-M-P}^{P}\left(\lambda(x,y)\frac{\partial\eta(x,y)}{\partial x}\Big|_0^{2P+L} - \int_0^{2P+L}\frac{\partial\eta(x,y)}{\partial x}\frac{\partial\lambda(x,y)}{\partial x}\,dx\right)dy =$$

$$= \int_{-M-P}^{P}\left(\lambda(x,y)\frac{\partial\eta(x,y)}{\partial x}\Big|_0^{2P+L} - \frac{\partial\lambda(x,y)}{\partial x}\eta(x,y)\Big|_0^{2P+L} + \int_0^{2P+L}\eta(x,y)\frac{\partial^2\lambda(x,y)}{\partial x^2}\,dx\right)dy.$$

The term

$$\frac{\partial\lambda(x,y)}{\partial x}\,\eta(x,y)\Big|_0^{2P+L}$$

vanishes because $\eta(x,y) = 0$ on the boundary of the domain. Hence,

$$I_{1^*} = \int_{-M-P}^{P} \left(\lambda(x,y) \frac{\partial \eta(x,y)}{\partial x} \Big|_0^{2P+L} + \int_0^{2P+L} \eta(x,y) \frac{\partial^2 \lambda(x,y)}{\partial x^2} \, dx \right) dy.$$

Similarly, we denote the second double integral in Equation (15) by I_{2^*} and integrate it by parts twice.

$$\begin{aligned} I_{2^*} &= \int_0^{2P+L} \int_{-M-P}^{P} \lambda(x,y) \frac{\partial^2 \eta(x,y)}{\partial y^2} \, dy \, dx = \\ &= \int_0^{2P+L} \left(\lambda(x,y) \frac{\partial \eta(x,y)}{\partial y} \Big|_{-M-P}^{P} - \int_{-M-P}^{P} \frac{\partial \eta(x,y)}{\partial y} \frac{\partial \lambda(x,y)}{\partial y} \, dy \right) dx = \\ &= \int_0^{2P+L} \left(\lambda(x,y) \frac{\partial \eta(x,y)}{\partial y} \Big|_{-M-P}^{P} - \frac{\partial \lambda(x,y)}{\partial y} \eta(x,y) \Big|_{-M-P}^{P} + \int_{-M-P}^{P} \eta(x,y) \frac{\partial^2 \lambda(x,y)}{\partial y^2} \, dy \right) dx. \end{aligned}$$

The term

$$\frac{\partial \lambda(x,y)}{\partial y} \eta(x,y) \Big|_{-M-P}^{P}$$

vanishes since $\eta(x,y) = 0$ on the boundary of the domain. Hence,

$$I_{2^*} = \int_0^{2P+L} \left(\lambda(x,y) \frac{\partial \eta(x,y)}{\partial y} \Big|_{-M-P}^{P} + \int_{-M-P}^{P} \eta(x,y) \frac{\partial^2 \lambda(x,y)}{\partial y^2} \, dy \right) dx.$$

Substituting the obtained expressions for I_{1^*} and I_{2^*} into Equation (15), we obtain

$$\begin{aligned} &\int_0^{2P+L} \int_{-M-P}^{P} \eta(x,y) \, \Delta\lambda(x,y) \, dx \, dy + \\ &\quad + \int_0^{2P+L} \lambda(x,y) \frac{\partial \eta(x,y)}{\partial y} \Big|_{y=-M-P} dx - \int_0^{2P+L} \lambda(x,y) \frac{\partial \eta(x,y)}{\partial y} \Big|_{y=0} dx + \\ &\quad + \int_{-M-P}^{P} \lambda(x,y) \frac{\partial \eta(x,y)}{\partial x} \Big|_0^{2P+L} dy = \\ &= -4\pi G \sigma h \int_{\Omega'} \lambda(x,y) \, dx \, dy, \qquad \forall \lambda(x,y). \end{aligned} \tag{16}$$

Our goal was to explicitly identify σ from Equation (13) in order to subsequently separate it and take the limit, thus obtaining the Gateaux derivative. In Equation (16), σ is explicitly expressed. Therefore, it is necessary to establish a connection between Equations (16) and (13).

Since λ is arbitrary, we can choose it so that Equation (16) coincides with Equation (13). Because the values of the gravitational field gradient measured by the gravimeter are located inside the investigated domain rather than on the boundary, this will produce a response in the adjoint problem within the interior region. Consequently, it can be assumed that the boundary values in the adjoint problem are also zero, analogous to the original problem formulation.

Hence, we obtain zero boundary conditions for the adjoint problem. Let us now denote $\lambda(x,y) = p(x,y)$.

$$\begin{aligned} p(0,y) &= 0, \\ p(2P+L,y) &= 0, \\ p(x,-M-P) &= 0, \\ p(x,P) &= 0. \end{aligned}$$

We now derive the adjoint equation. We start from Equation (13). Recall that it has the following form. The second term is neglected since it contains σ^2; after dividing by σ and passing to the limit as $\sigma \to 0$, this term vanishes:

$$I_*(\psi_0 + \sigma h) - I_*(\psi_0) = \int_N^K 2 \left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x) \right) \frac{\partial \eta(x,0)}{\partial y} \, dx + \int_N^K \left(\frac{\partial \eta(x,0)}{\partial y} \right)^2 dx. \tag{12}$$

Our goal is to transform the first term in (13) into a double integral in order to establish an analogy with the double integral term in (16). To this end, we rewrite (13) in a two-dimensional form.

Since Equation (13) does not depend on the variable y, we artificially multiply its first term by the Dirac delta function $\delta(y)$. We then integrate with respect to y over the entire domain under consideration. Thus, we obtain

$$\int_N^K 2\left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x)\right)\frac{\partial \eta(x,0)}{\partial y}\,dx = \int_{-M-P}^{P}\int_N^K 2\left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x)\right)\frac{\partial \eta(x,y)}{\partial y}\,\delta(y)\,dx\,dy.$$

Next, we extend the interval $[N,K]$ to the boundary of the domain under consideration (see Figure 1). Introduce the function

$$\alpha(x) = \begin{cases} 0, & x \notin [N,K], \\ 1, & x \in [N,K]. \end{cases}$$

Then the corresponding double integral over the entire domain takes the form

$$\int_{-M-P}^{P}\int_0^{2P+L} 2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x)\right)\frac{\partial \eta(x,y)}{\partial y}\,\delta(y)\,\alpha(x)\,dx\,dy.$$

Since in Equation (16) we need $\eta(x,y)$ in its "pure" form (i.e., without derivatives), we integrate by parts in y.

$$\int_{-M-P}^{P}\int_0^{2P+L} 2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x)\right)\frac{\partial \eta(x,y)}{\partial y}\,\delta(y)\,\alpha(x)\,dx\,dy =$$

$$= \int_0^{2P+L}\alpha(x)\int_{-M-P}^{P} 2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x)\right)\delta(y)\frac{\partial \eta(x,y)}{\partial y}\,dy\,dx.$$

The first term from the integration by parts vanishes because $\eta(x,y) = 0$ on the boundaries. The second term then becomes

$$-\int_{-M-P}^{P}\int_0^{2P+L}\alpha(x)\,\frac{\partial}{\partial y}\left[2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x)\right)\delta(y)\right]\eta(x,y)\,dx\,dy.$$

Comparing the obtained expression with Equation (16), and recalling that $\lambda(x,y)$ was arbitrary, we now choose a specific function $p(x,y)$, which is a solution of the adjoint system:

$$\Delta p(x,y) = \alpha(x)\,\frac{\partial}{\partial y}\left[2\left(\frac{\partial \eta(x,y)}{\partial y} - \eta_2(x)\right)\delta(y)\right], \tag{17}$$

with homogeneous Dirichlet boundary conditions

$$\begin{aligned} p(0,y) &= 0, \\ p(2P+L,y) &= 0, \\ p(x,-M-P) &= 0, \\ p(x,P) &= 0. \end{aligned}$$

□

4. Rigorous Analysis of the Adjoint Equation

The formal derivation above yields the adjoint equation in the form (16). However, from a mathematical standpoint, Equation (16) must be understood in the sense of distributions (Schwartz distributions), since its right-hand side contains the first derivative of the Dirac delta function, $\delta'(y)$. The presence of this singular term necessitates a careful analysis of the existence and uniqueness of a weak solution for the adjoint problem.

Definition 1. *(Weak formulation). We seek a function $p(x,y) \in H_0^1(\Omega')$ such that for all test functions $\varphi(x,y) \in H_0^1(\Omega')$, the following variational equality holds:*

$$\int_{\Omega'} \nabla p(x,y) \cdot \nabla \varphi(x,y)\, dxdy = -\int_0^{2P+L} \alpha(x) \left[2\left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x) \right) \frac{\partial \varphi(x,0)}{\partial y} \right] dx. \tag{18}$$

Proof. ([**Justification of** (18)]). Let us define the singular source term as a functional $F \in H^{-1}(\Omega')$ acting on a test function φ:

$$\langle F, \varphi \rangle = \left\langle \alpha(x) \frac{\partial}{\partial y} \left(2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x) \right) \delta(y) \right), \varphi(x,y) \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega')$ and $H_0^1(\Omega')$. Using the definition of the distributional derivative, we have:

$$\langle F, \varphi \rangle = -\int_{\Omega'} \alpha(x) \left(2\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x) \right) \delta(y) \right) \frac{\partial \varphi(x,y)}{\partial y}\, dxdy.$$

By the sifting property of the Dirac delta function, this reduces to:

$$\langle F, \varphi \rangle = -\int_0^{2P+L} \alpha(x) \left[2\left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x) \right) \frac{\partial \varphi(x,0)}{\partial y} \right] dx.$$

Thus, the weak formulation (18) is obtained. □

Theorem 2. *(Existence and uniqueness of a weak solution). Let the misfit term $\left(\frac{\partial v(x,y)}{\partial y} - \eta_2(x) \right) \in L^2(\Omega')$. Then the variational problem (18) admits a unique weak solution $p(x,y) \in H_0^1(\Omega')$.*

Proof. We rewrite (18) in the standard bilinear form:

$$B(p,\varphi) = \int_{\Omega'} \nabla p \cdot \nabla \varphi\, dxdy = L(\varphi),$$

where

$$L(\varphi) = -\int_0^{2P+L} \alpha(x) g(x) \frac{\partial \varphi(x,0)}{\partial y} dx,$$

with $g(x) = 2\left(\frac{\partial v(x,0)}{\partial y} - \eta_2(x) \right)$.

Clearly, the bilinear form $B(p,\varphi)$ is coercive (by the Poincaré inequality) and continuous on $H_0^1(\Omega') \times H_0^1(\Omega')$. It remains to show that the linear functional $L(\varphi)$ is bounded (continuous) on $H_0^1(\Omega')$. Using the trace theorem, there exists a constant $C_{\text{tr}} > 0$ such that:

$$\left| \frac{\partial \varphi(x,0)}{\partial y} \right|_{L^2(N,K)} \leq C_{\text{tr}} \|\varphi\|_{H^1(\Omega')}.$$

Therefore,

$$|L(\varphi)| \leq 2 \left\| \frac{\partial v(x,0)}{\partial y} - \eta_2(x) \right\|_{L^2(N,K)} \cdot \left\| \frac{\partial \varphi(x,0)}{\partial y} \right\|_{L^2(N,K)} \leq C_{\text{tr}} \|g\|_{L^2} \|\varphi\|_{H^1(\Omega')}.$$

Hence, $L(\varphi)$ is a continuous linear functional. By the Lax–Milgram theorem, there exists a unique $p \in H_0^1(\Omega')$ satisfying (18). Thus, the adjoint Equation (16) has a well-defined weak solution in the Sobolev space $H_0^1(\Omega')$.

This analysis demonstrates that while the strong form (16) contains the singular term $\delta'(y)$, its weak formulation is well-posed and involves only the evaluation of the trace of the test function on the measurement profile. This distinction is crucial: although the problem is rigorously defined mathematically, the presence of the distributional derivative in the strong form implies that any numerical scheme attempting to discretize (16) directly must handle the singular source with special care, such as through regularization or the construction of

appropriate discrete test functions that satisfy the variational form (18). Conversely, this justifies the use of derivative-free methods, as discussed below, which bypass the need for a computable gradient in the strong sense.

Thus, we have obtained the adjoint problem (a Poisson equation with four homogeneous Dirichlet boundary conditions). Next, we consider the difference

$$I_*(\psi_0 + \sigma h) - I_*(\psi_0) = 4\pi G\,\sigma h \iint_{\Omega'} p(x,y)\,dx\,dy + \int_0^L \left(\frac{\partial \eta(x,0)}{\partial y}\right)^2 dx.$$

Dividing by σ and passing to the limit as $\sigma \to 0$, we obtain the Gateaux derivative:

$$(I_*'(\psi_0), h) = \lim_{\sigma \to 0} \frac{I_*(\psi_0 + \sigma h) - I_*(\psi_0)}{\sigma} = 4\pi G \iint_{\Omega'} p(x,y)\,dx\,dy\; h.$$

Therefore, the Gateaux derivative is

$$I_*'(\psi_0) = 4\pi G \iint_{\Omega'} p(x,y)\,dx\,dy,$$

which was to be proven. □

We have found the derivative, which, in principle, allows us to provide a complete description of gradient-type algorithms.

However, the term

$$\int_0^L \left(\frac{\partial \eta(x,0)}{\partial y}\right)^2 dx$$

vanishes when dividing by σ and taking the limit as $\sigma \to 0$, since the integral contains a second-order term in σ. After division by σ, only one factor of σ remains, and as σ tends to zero, the entire expression tends to zero.

The difficulty of this problem formulation lies in the adjoint equation. Namely, the right-hand side of Equation (17) contains the derivative of the Dirac delta function. It is unclear how to handle the derivative of the delta function and how to compute it numerically.

5. Discussion

The present study addresses the inverse gravimetry problem from the standpoint of optimization theory. Unlike many existing studies that focus primarily on the numerical implementation of inversion algorithms, this work investigates the mathematical properties of the optimization functional underlying the reconstruction of density distributions from gravimetric observations.

The obtained results demonstrate that the quadratic misfit functional is Gateaux differentiable, and its derivative can be represented through the solution of an adjoint boundary-value problem. This conclusion is consistent with the general theory of PDE-constrained optimization, where adjoint equations provide an efficient tool for gradient computation. However, the inverse gravimetry problem considered here possesses an important distinguishing feature. Owing to the fact that the observational data are available only along the measurement profile, the adjoint equation contains the derivative of the Dirac delta function. This singular term fundamentally complicates the numerical evaluation of the functional gradient.

The obtained result indicates that the existence of a mathematical derivative does not necessarily imply its practical applicability. Although the optimization functional is smooth in the Gateaux sense, the corresponding gradient cannot be efficiently computed using conventional finite-difference, finite-element, or projection techniques without introducing additional regularization or approximation procedures. Consequently, the direct application of classical gradient-based optimization methods, including the steepest descent and conjugate gradient methods, becomes computationally challenging.

Compared with traditional inverse gravimetry approaches based on Tikhonov regularization, iterative gradient methods, or adjoint-state optimization, the present analysis reveals an additional theoretical limitation associated with the natural formulation of the problem. The difficulty does not originate from numerical discretization but from the mathematical structure of the adjoint equation itself. This observation provides a deeper understanding of why conventional optimization techniques may exhibit poor robustness when applied to practical gravimetric inversion problems.

The presented theoretical analysis also identifies promising directions for future numerical developments. Since the optimization functional remains well defined while its gradient is difficult to evaluate, derivative-free optimization algorithms appear to be particularly suitable. Methods such as Nelder–Mead, Pattern Search, Differential Evolution, Particle Swarm Optimization, and Bayesian optimization based on surrogate models can potentially provide efficient alternatives for solving the proposed inverse problem.

The current study has several limitations. First, the analysis is restricted to a two-dimensional mathematical model with homogeneous Dirichlet boundary conditions. Second, the work is focused on the theoretical derivation of the functional derivative and does not include numerical experiments or comparisons between optimization algorithms. These aspects will be addressed in future studies involving realistic geological models and field gravimetric observations from oil and gas reservoirs.

6. Conclusions

This paper investigated the mathematical properties of the optimization functional arising in the inverse gravimetry problem for density reconstruction. The inverse problem was formulated as the minimization of a quadratic misfit functional subject to the Poisson equation describing the gravitational potential.

The main scientific contribution of this work is the derivation of the Gateaux derivative of the objective functional through an adjoint boundary-value problem. The obtained analytical representation establishes the theoretical foundation for optimization-based approaches to inverse gravimetry.

A fundamental result of the study is the identification of a singular term involving the derivative of the Dirac delta function in the adjoint equation. Although the functional is differentiable, this singularity makes the direct numerical implementation of classical gradient-based optimization algorithms extremely difficult. Therefore, the existence of the analytical gradient does not automatically guarantee its computational applicability.

These findings provide a mathematical explanation for the limitations of conventional gradient methods in practical gravimetric inversion and justify the consideration of derivative-free optimization techniques as a more appropriate computational framework for this class of inverse problems.

The proposed theoretical framework forms the basis for further research aimed at developing efficient numerical algorithms for inverse gravimetry. Future work will focus on the implementation and comparison of derivative-free optimization methods, extension of the model to three-dimensional problems, incorporation of real gravimetric measurements from oil and gas fields, and integration of physics-informed neural networks to improve the stability and accuracy of density reconstruction.

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