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Nonlocal Linearization and Conformal Lax Representations for Non-Autonomous Coupled Second-Order Systems

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Abstract

We study non-autonomous coupled second-order systems $x'' = \Pi_1(t,x,y,x',y')$, $y'' = \Pi_2(t,x,y,x',y')$ and the question of when such a system is carried by a generalized Sundman transformation (GST) $u = \varphi(t,x,y)$, $v = \psi(t,x,y)$, $\mathrm{d}s = \eta(t,x,y)\,\mathrm{d}t$ onto a linear coupled system with constant coefficients, including damping and restoring terms. We state the determining system for (φ,ψ,η) and an explicit constancy test for its linear and zeroth-order part. For the part of the transformed equations that is quadratic in the velocities, we prove a geometric characterization: it is realized by a GST if and only if a torsion-free affine connection, depending on the gauge function $G = \log|\eta|$, is flat. This condition is necessary but not sufficient for linearizability of the complete system. For the linear target, we describe all quadratic invariants through a Lyapunov-type matrix equation and give the correct count of independent ones, with the assumptions needed. A rank-one matrix $\mathcal{L} = ZZ^TS$ built from the transformed phase-space vector Z satisfies a conformal Lax equation $\dot{\mathcal{L}} = [\mathcal{L},\mathcal{W}] + \kappa\eta\mathcal{L}$; we prove the converse implication under an explicit nondegeneracy condition. All trace invariants of this matrix are powers of a single quadratic form, so the Lax formalism reorganizes, rather than enlarges, the set of invariants. For a damped cyclotron, we obtain two exponentially weighted first integrals, two exactly decaying quadratic quantities, and one further first integral free of exponential weight. The numerical checks are documented with convergence tables, and a script reproducing all figures is supplied.

Keywords: Lax representation; generalized Sundman transformation; coupled nonlinear systems; first integrals; non-autonomous ordinary differential equations

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1. Introduction

Non-autonomous coupled nonlinear oscillators, with or without dissipation, arise naturally in applied physics and engineering. In the presence of dissipation the classical Hamiltonian toolbox [1] is not available, and analytical methods based on integrability remain indispensable. Among them, the Lax representation occupies a distinguished place:

originally introduced for integrable partial differential equations [2,3], it has since been found for broad classes of nonlinear ordinary differential equations.

For scalar second-order equations $y'' + f(z,y)y' + g(z,y) = 0$, Sinelshchikov, Gaiur, and Kudryashov [4] proved that the existence of a quadratic first integral, the existence of a 2×2 Lax pair, and linearizability by a GST are equivalent. This was extended to cubic autonomous and non-autonomous oscillators [5,6].

For coupled systems the theory is less developed, though not untouched. Moyo and Meleshko [7] applied the GST to systems of two second-order equations and obtained necessary and sufficient linearization conditions; the targets considered there are, to our understanding, systems $u'' = k_{11}u' + k_{12}v'$, $v'' = k_{21}u' + k_{22}v'$ with constant k_{ij}. Chandrasekar, Senthilvelan, and Lakshmanan, in parts IV and V of their series [8,9], constructed independent first integrals and linearizing nonlocal transformations for coupled second-order systems by the modified Prelle–Singer method; Pradeep et al. [10] studied the converse construction, generating coupled nonlinear systems from linear ones by nonlocal transformations. λ-symmetries [11] give a further route to first integrals of dissipative second-order equations and are closely related to the Sundman factor η.

The target used here is the linear coupled system with constant coefficients

$$u'' + a_{11}u' + a_{12}v' + b_{11}u + b_{12}v = 0, \qquad v'' + a_{21}u' + a_{22}v' + b_{21}u + b_{22}v = 0, \tag{1}$$

including the damped case $a_{11} = a_{22} = \gamma > 0$. Relative to the velocity-only targets of [7], it contains the restoring terms b_{ij}; this is what produces nontrivial quadratic invariants and makes a Lax-type description meaningful.

A natural first attempt at a Lax matrix for (1) is a block 4×4 matrix built only from first derivatives of (φ, ψ). Such entries do not involve the coefficients b_{ij}, whereas a quadratic invariant of (1) must contain them. We therefore use a rank-one matrix $\mathcal{L} = ZZ^T S$ built from the full transformed phase-space vector $Z = (\varphi, \psi, \mu_1, \mu_2)^T$ (Section 4).

Contributions and scope.

1. A determining system of twelve PDEs for (φ, ψ, η) with data and unknowns made explicit (Theorem 2), and an explicit constancy test for its linear and zeroth-order part.
2. A geometric characterization of those quadratic-in-velocity terms that are realized by some GST: a connection ∇^G is flat for a suitable G. This concerns *only* Equations (8)–(10); linearizability of the complete system also requires (11)–(13). Eliminating G to obtain conditions on the given coefficients alone is not carried out here.
3. The quadratic invariants of the target with the correct count and assumptions.
4. A rank-one conformal Lax equation, with the converse implication under an explicit nondegeneracy condition. All trace invariants are powers of one quadratic form (Remark 4): the Lax formalism reorganizes the invariants of Theorem 4 but yields no further ones.
5. Exact invariants of a damped cyclotron and documented numerics with convergence tables.

Section 2 recalls the GST framework. Section 3 gives the determining system and the geometric criterion. Section 4 develops the Lax formalism. Section 5 treats three examples and Section 6 documents the numerics. Section 7 concludes.

2. Preliminaries

2.1. The GST Framework

We consider

$$x'' = \Pi_1(t, x, y, x', y'), \qquad y'' = \Pi_2(t, x, y, x', y'), \tag{2}$$

where primes denote $\mathrm{d}/\mathrm{d}t$. A *generalized Sundman transformation* (GST) is

$$u = \varphi(t,x,y), \quad v = \psi(t,x,y), \quad \mathrm{d}s = \eta(t,x,y)\,\mathrm{d}t, \tag{3}$$

with $\Delta := \varphi_x\psi_y - \varphi_y\psi_x \neq 0$ and $\eta \neq 0$. Write $D_t = \partial_t + x'\partial_x + y'\partial_y$. System (2) is *GST-linearizable* if (3) carries it into (1).

Theorem 1 (Necessity of the quadratic form)**.** *If* (2) *is GST-linearizable into* (1) *via* (3)*, then* Π_1, Π_2 *are quadratic polynomials in* (x', y')*:*

$$x'' = A_1(x')^2 + B_1x'y' + C_1(y')^2 + D_1x' + E_1y' + F_1, \tag{4}$$
$$y'' = A_2(x')^2 + B_2x'y' + C_2(y')^2 + D_2x' + E_2y' + F_2, \tag{5}$$

with $A_i, \ldots, F_i$ *functions of* (t,x,y) *determined by* φ, ψ, η *and* a_{ij}, b_{ij}*, as in Theorem 2.*

This direction is proved together with Theorem 2. The converse is false: three functions (φ, ψ, η) together with eight constants determine twelve coefficient functions, and a generic system of the form (4) and (5) is not GST-linearizable.

2.2. Lax Representations

Definition 1 (Lax representation)**.** *Let* $\mathcal{D}$ *be an open set of* (t,x,y,x',y')*-space. A pair* $(\mathcal{L}, \mathcal{W})$ *of matrix-valued smooth functions on* $\mathcal{D}$ *is a* Lax representation of (2) on $\mathcal{D}$ *if, for every* C^2 *curve* $t \mapsto (x(t), y(t))$ *whose lift* (t,x,y,x',y') *lies in* $\mathcal{D}$*,*

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = [\mathcal{L}, \mathcal{W}] := \mathcal{L}\mathcal{W} - \mathcal{W}\mathcal{L} \tag{6}$$

holds along the curve if and only if (2) *holds along it.*

The "if and only if" and the domain $\mathcal{D}$ are both essential: without them, (6) is satisfied by $\mathcal{L} = \mathcal{W} = 0$ for any equation. Below, the implication "system $\Rightarrow$ Lax equation" is proved in Theorems 5 and 6, and the converse in Proposition 3, on the set $\mathcal{D}_S$ described there. When (6) holds, $\mathrm{tr}(\mathcal{L}^k)$ is conserved for every k, since $\mathrm{d}(\mathcal{L}^k)/\mathrm{d}t = [\mathcal{L}^k, \mathcal{W}]$ (Lemma 1 below); normalizing $\mathrm{tr}(\mathcal{L}) = 0$ is a convenience, not a requirement.

For the scalar equation $y'' + f(z,y)y' + g(z,y) = 0$, Sinelshchikov et al. [4] exhibit a 2×2 pair

$$\mathcal{L} = \begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & -\ell_1 \end{pmatrix}, \tag{7}$$

for which the Lax equation is equivalent to the scalar equation and $\ell_1^2 + \ell_2\ell_3$ is a conserved quadratic. The potential term g enters through the off-diagonal entries ℓ_2, ℓ_3.

3. The Determining System and the Geometric Criterion

3.1. The Determining System

Theorem 2 (Determining system)**.** *Suppose* (2) *is carried by the GST* (3) *into* (1)*. Then,* Π_1, Π_2 *take the quadratic form* (4) *and* (5)*, and for* $\chi = \varphi$ *(row* $i = 1$*) and* $\chi = \psi$ *(row* $i = 2$*), the functions* (φ, ψ, η) *satisfy the twelve equations*

$$\eta\chi_{xx} + \eta(A_1\chi_x + A_2\chi_y) - \eta_x\chi_x = 0, \tag{8}$$
$$2\eta\chi_{xy} + \eta(B_1\chi_x + B_2\chi_y) - \eta_x\chi_y - \eta_y\chi_x = 0, \tag{9}$$
$$\eta\chi_{yy} + \eta(C_1\chi_x + C_2\chi_y) - \eta_y\chi_y = 0, \tag{10}$$

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$$2\eta\chi_{tx} + \eta(D_1\chi_x + D_2\chi_y) - \eta_t\chi_x - \eta_x\chi_t = -\eta^2(a_{i1}\varphi_x + a_{i2}\psi_x), \tag{11}$$
$$2\eta\chi_{ty} + \eta(E_1\chi_x + E_2\chi_y) - \eta_t\chi_y - \eta_y\chi_t = -\eta^2(a_{i1}\varphi_y + a_{i2}\psi_y), \tag{12}$$
$$\eta\chi_{tt} + \eta(F_1\chi_x + F_2\chi_y) - \eta_t\chi_t = -\eta^2(a_{i1}\varphi_t + a_{i2}\psi_t) - \eta^3(b_{i1}\varphi + b_{i2}\psi). \tag{13}$$

Conversely, if (φ, ψ, η) *(with* $\Delta \neq 0$, $\eta \neq 0$*) and constants* a_{ij}, b_{ij} *satisfy* (8)–(13)*, the GST* (3) *carries* (4) *and* (5) *into* (1)*.*

Proof. Since $\mathrm{d}s = \eta\,\mathrm{d}t$, $\mathrm{d}u/\mathrm{d}s = D_t\varphi/\eta$ and $\mathrm{d}^2u/\mathrm{d}s^2 = \eta^{-1}D_t(D_t\varphi/\eta) = \left(\eta D_t^2\varphi - (D_t\eta)(D_t\varphi)\right)/\eta^3$, and likewise for v with ψ. Substituting into row i of (1) and multiplying by η^3 gives, for $\chi = \varphi$ $(i = 1)$ and $\chi = \psi$ $(i = 2)$,

$$\eta D_t^2\chi - (D_t\eta)(D_t\chi) + \eta^2(a_{i1}D_t\varphi + a_{i2}D_t\psi) + \eta^3(b_{i1}\varphi + b_{i2}\psi) = 0. \tag{14}$$

Expand $D_t^2\chi = \chi_{tt} + 2x'\chi_{tx} + 2y'\chi_{ty} + (x')^2\chi_{xx} + 2x'y'\chi_{xy} + (y')^2\chi_{yy} + x''\chi_x + y''\chi_y$. Taking $\chi = \varphi, \psi$ in (14) gives two linear equations for (x'', y'') with coefficient matrix $\left(\begin{smallmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{smallmatrix}\right)$ of determinant $\Delta \neq 0$; solving expresses x'', y'' as polynomials of degree at most two in (x', y'), which is (4) and (5). Substituting back into (14) and collecting the coefficients of the six linearly independent monomials $(x')^2, x'y', (y')^2, x', y', 1$ yields exactly (8)–(13) (monomial-by-monomial derivation in Appendix A). The converse is the same computation read backwards. □

Remark 1. *No term* $D_t\eta$ *appears in* (8)–(13)*: each equation carries exactly the single partial derivative of* η *paired with the monomial selected by the coefficient extraction. Given* (φ, ψ, η) *and the constants, Equations* (8)–(10) determine $A_1, A_2, B_1, B_2, C_1, C_2$ *and* (11)–(13) *determine* D_i, E_i, F_i*; read in that direction, they are identities. They become constraints on* (φ, ψ, η) *only when* $A_i, \ldots, F_i$ *are prescribed data coming from a given system* (2)*.*

Proposition 1 (Constancy test for (11)–(13))**.** *Let* (φ, ψ, η) *with* $\Delta \neq 0$, $\eta \neq 0$ *satisfy* (8)–(10) *for given* A_i, B_i, C_i*, and let* D_i, E_i, F_i *be given. For* $i = 1, 2$ *(*$\chi = \varphi$ *for* $i = 1$, $\chi = \psi$ *for* $i = 2$*), define*

$$\rho_i^x = -\eta^{-2}\left[2\eta\chi_{tx} + \eta(D_1\chi_x + D_2\chi_y) - \eta_t\chi_x - \eta_x\chi_t\right],$$
$$\rho_i^y = -\eta^{-2}\left[2\eta\chi_{ty} + \eta(E_1\chi_x + E_2\chi_y) - \eta_t\chi_y - \eta_y\chi_t\right],$$

and

$$a_{i1} := \frac{\rho_i^x\psi_y - \rho_i^y\psi_x}{\Delta}, \qquad a_{i2} := \frac{\varphi_x\rho_i^y - \varphi_y\rho_i^x}{\Delta},$$
$$\rho_i^0 := -\eta^{-3}\left[\eta\chi_{tt} + \eta(F_1\chi_x + F_2\chi_y) - \eta_t\chi_t\right] - \eta^{-1}(a_{i1}\varphi_t + a_{i2}\psi_t).$$

Then, the GST carries (4) *and* (5) *into a system of the form* (1) *with* constant *coefficients if and only if the four functions* a_{ij} *are constant and there are constants* b_{i1}, b_{i2} *with* $\rho_i^0 = b_{i1}\varphi + b_{i2}\psi$ *identically (*$i = 1, 2$*).*

Proof. Equations (11) and (12) read $a_{i1}\varphi_x + a_{i2}\psi_x = \rho_i^x$, $a_{i1}\varphi_y + a_{i2}\psi_y = \rho_i^y$, a linear system with determinant $\Delta \neq 0$ whose solution is the displayed a_{ij}; Equation (13) reads $b_{i1}\varphi + b_{i2}\psi = \rho_i^0$. Apply the converse part of Theorem 2. □

3.2. Geometric Characterization of the Quadratic Part

Theorem 1 is only a necessary condition, and the criterion below concerns only the Equations (8)–(10), i.e., the coefficients A_i, B_i, C_i of the quadratic velocity terms. It is the analog, for coupled systems, of the Liouville–Tresse theory of point invariants of scalar second-order equations.

Theorem 3 (Geometric characterization of the quadratic part)**.** *Let $G := \log|\eta|$ and define the connection coefficients pulled back from the flat (u, v)-plane,*

$$\Gamma^x_{xx} = \frac{\varphi_{xx}\psi_y - \psi_{xx}\varphi_y}{\Delta}, \qquad \Gamma^y_{xx} = \frac{\psi_{xx}\varphi_x - \varphi_{xx}\psi_x}{\Delta}, \tag{15}$$

and analogously $\Gamma^x_{xy}, \Gamma^y_{xy}, \Gamma^x_{yy}, \Gamma^y_{yy}$ with φ_{xy}, ψ_{xy} and φ_{yy}, ψ_{yy}. If (φ, ψ, η) satisfy (8)–(10), then

$$\begin{aligned} A_1 &= -\Gamma^x_{xx} + G_x, & A_2 &= -\Gamma^y_{xx}, \\ B_1 &= -2\Gamma^x_{xy} + G_y, & B_2 &= -2\Gamma^y_{xy} + G_x, \\ C_1 &= -\Gamma^x_{yy}, & C_2 &= -\Gamma^y_{yy} + G_y. \end{aligned} \tag{16}$$

Conversely, given functions A_i, B_i, C_i of (t, x, y) (t acting as a parameter) and a function G, define a torsion-free connection ∇^G on the (x, y)-plane by

$$\begin{aligned} &\Gamma^x_{xx} := G_x - A_1, && \Gamma^y_{xx} := -A_2, && \Gamma^x_{xy} := \tfrac{1}{2}(G_y - B_1), \\ &\Gamma^y_{xy} := \tfrac{1}{2}(G_x - B_2), && \Gamma^x_{yy} := -C_1, && \Gamma^y_{yy} := G_y - C_2. \end{aligned} \tag{17}$$

Then, locally, there exist (φ, ψ) with $\Delta \neq 0$ such that $(\varphi, \psi, \eta = e^G)$ satisfy (8)–(10) if and only if ∇^G is flat, i.e., its curvature tensor vanishes identically; in dimension two, this is a system of four scalar PDEs for G. Consequently, the quadratic part of (4) and (5) is realized by some GST if and only if G can be chosen so that ∇^G is flat.

Proof. Direct part: by (15), φ and ψ simultaneously satisfy $\chi_{ij} = \Gamma^k_{ij}\chi_k$ (the unique solution of the 2×2 linear system, since $\Delta \neq 0$). Substituting into (8)–(10) and using $\eta_x/\eta = G_x$, $\eta_y/\eta = G_y$ gives, for both $\chi = \varphi, \psi$, relations of the form $(\Gamma^x_{xx} + A_1 - G_x)\chi_x + (\Gamma^y_{xx} + A_2)\chi_y = 0$, etc.; since $\Delta \neq 0$ the coefficients vanish, which is (16).

Converse: (17) is equivalent to $\nabla^G \mathrm{d}\chi = 0$, i.e.,

$$\chi_{xx} = \Gamma^x_{xx}\chi_x + \Gamma^y_{xx}\chi_y, \quad \chi_{xy} = \Gamma^x_{xy}\chi_x + \Gamma^y_{xy}\chi_y, \quad \chi_{yy} = \Gamma^x_{yy}\chi_x + \Gamma^y_{yy}\chi_y. \tag{18}$$

With $p = \chi_x$, $q = \chi_y$ this is a total (Pfaffian) system for (χ, p, q) in x, y. By Frobenius' theorem it is completely integrable, i.e., it has a three-dimensional solution space, if and only if $\partial_y\chi_{xx} = \partial_x\chi_{xy}$ and $\partial_y\chi_{xy} = \partial_x\chi_{yy}$ hold identically in (χ, p, q); substituting (18) and equating coefficients of p, q gives exactly the vanishing of the four independent curvature components of ∇^G. The solution space is spanned by the constant function and two further solutions φ, ψ (take initial (p, q) linearly independent, so $\Delta \neq 0$ near the base point); these are affine coordinates of the flat connection and reproduce A_i, B_i, C_i through (16). Conversely, if such (φ, ψ) exist, ∇^G is the pullback of the flat connection of the (u, v)-plane and is flat. □

Remark 2. *Equations (4) and (5) without the D_i, E_i, F_i terms are the pre-geodesic equations of ∇^G; one checks that (17) is a projective change of connection by the exact 1-form $\mathrm{d}G/2$, so $\eta = e^G$ is the projective gauge freedom. The statement is local, and φ, ψ are determined by ∇^G up to $(u, v) \mapsto \Lambda(u, v)^T + c$ with $\Lambda \in GL_2$.*

Corollary 1 (Complete linearizability)**.** *A system of the form (4) and (5) is (locally) GST-linearizable into (1) if and only if (i) there is G such that ∇^G in (17) is flat, and (ii) for the corresponding coordinates $(\varphi, \psi) = \Lambda(t)(\varphi_0, \psi_0)^T + c(t)$, the conditions of Proposition 1 hold for some choice of $\Lambda(t) \in GL_2$, $c(t)$ and G. Condition (i) alone is necessary but not sufficient.*

Remark 3 (How to test a given system). *(1) Verify the quadratic form* (4) *and* (5) *and read off* $A_i, \dots, F_i$. *(2) Find* $G(t,x,y)$, *making* ∇^G *flat. (3) Integrate* (18) *to get* (φ_0, ψ_0) *and set* $\eta = e^G$. *(4) Apply Proposition 1: the functions* a_{ij} *must be constant and* ρ_i^0 *must be linear in* (φ, ψ) *with constant coefficients; this is a linear-algebra/ODE problem for* $\Lambda(t), c(t)$. *Steps (2) and (4) jointly are a determined-or-overdetermined system, and eliminating* G, Λ, c *to obtain conditions on* $A_i, \dots, F_i$ *alone is* not *done in this paper. Example 1 carries out steps (1)–(4) in closed form.*

4. Lax Formalism

4.1. First-Order Form

Write the target as $U'' + \mathbf{a}U' + \mathbf{b}U = 0$ with $U = (u, v)^T$, $\mathbf{a} = (a_{ij})$, $\mathbf{b} = (b_{ij})$, and set $\mu_1 = D_t\varphi/\eta$, $\mu_2 = D_t\psi/\eta$, so that $\mu = (\mu_1, \mu_2)^T = U_s$ along trajectories. Put

$$Z = \begin{pmatrix} \varphi \\ \psi \\ \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^4, \qquad M = \begin{pmatrix} 0 & I_2 \\ -\mathbf{b} & -\mathbf{a} \end{pmatrix} \in \mathbb{R}^{4\times 4}. \tag{19}$$

In the variable s the target is $Z_s = MZ$, and along trajectories of (2),

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \eta MZ. \tag{20}$$

Let $\lambda_1, \dots, \lambda_4$ be the eigenvalues of M.

Lemma 1 (Telescoping). *If* $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{W}] + \kappa(t)\mathcal{L}$, *then for every* $k \geq 1$, $\dfrac{\mathrm{d}(\mathcal{L}^k)}{\mathrm{d}t} = [\mathcal{L}^k, \mathcal{W}] + k\kappa(t)\mathcal{L}^k$.

Proof. By the product rule, $\mathrm{d}(\mathcal{L}^k)/\mathrm{d}t = \sum_{j=0}^{k-1} \mathcal{L}^j \dot{\mathcal{L}}\, \mathcal{L}^{k-1-j}$. Insert $\dot{\mathcal{L}} = \mathcal{L}\mathcal{W} - \mathcal{W}\mathcal{L} + \kappa\mathcal{L}$. The commutator part is $\sum_{j=0}^{k-1}(\mathcal{L}^{j+1}\mathcal{W}\mathcal{L}^{k-1-j} - \mathcal{L}^j\mathcal{W}\mathcal{L}^{k-j})$, which telescopes to $\mathcal{L}^k\mathcal{W} - \mathcal{W}\mathcal{L}^k$, and the remaining part is $k\kappa\mathcal{L}^k$. □

Hence, $\mathrm{d}\,\mathrm{tr}(\mathcal{L}^k)/\mathrm{d}t = k\kappa\ \mathrm{tr}(\mathcal{L}^k)$; for $\kappa \equiv 0$, one recovers $\mathrm{d}\,\mathrm{tr}(\mathcal{L}^k)/\mathrm{d}t = 0$.

4.2. Quadratic Invariants of the Target

Theorem 4 (Quadratic invariants). *Let* S *be a complex symmetric* 4×4 *matrix,* $\mathcal{Q}(Z) := Z^T S Z$, *and* $\Lambda_M(S) := M^T S + SM$. *Then:*

(i) $\mathrm{d}\mathcal{Q}/\mathrm{d}s = \kappa\mathcal{Q}$ *along solutions of* $Z_s = MZ$ *for all initial data if and only if* $\Lambda_M(S) = \kappa S$.

(ii) *The operator* Λ_M *acting on the* 10*-dimensional space of symmetric matrices has spectrum* $\{\lambda_i + \lambda_j : 1 \leq i \leq j \leq 4\}$ *counted with algebraic multiplicity. Consequently, the space of conserved quadratic forms* ($\kappa = 0$) *has dimension* $\dim\ker\Lambda_M \leq m_0 := \#\{(i \leq j) : \lambda_i + \lambda_j = 0\}$, *with equality if* M *is diagonalizable. In general, the inequality can be strict.*

(iii) *Along every trajectory of* (2), $\mathrm{d}\mathcal{Q}(Z(t))/\mathrm{d}t = \kappa\,\eta\,\mathcal{Q}(Z(t))$. *Hence, the* normalized *quantity*

$$\mathcal{Q}(Z(t)) \exp\Big(-\kappa \int_{t_0}^{t} \eta(\sigma)\,\mathrm{d}\sigma\Big) \tag{21}$$

is an exact first integral of (2), *whereas the* unnormalized $\mathcal{Q}(Z(t)) = \mathcal{Q}(Z(t_0))\exp(\kappa\int_{t_0}^{t}\eta)$ *evolves with the known rate. If* S *is real positive semidefinite and* $\kappa < 0$, $\mathcal{Q}$ *is a non-negative quantity that decays at exactly this rate; it is not* (21), *which is constant.*

Proof. (i) $\mathrm{d}\mathcal{Q}/\mathrm{d}s = Z_s^T S Z + Z^T S Z_s = Z^T(M^T S + SM)Z$; this equals $\kappa Z^T S Z$ for all Z iff the two symmetric matrices agree.

(ii) If $M = P\,\mathrm{diag}(\lambda_i)P^{-1}$, let w_i be the rows of P^{-1} and $S_{ij} = \frac{1}{2}(w_i^T w_j + w_j^T w_i)$, $i \le j$. These ten matrices are a basis of the symmetric matrices and $\Lambda_M(S_{ij}) = (\lambda_i + \lambda_j)S_{ij}$. For general M, the coefficients of the characteristic polynomial of Λ_M are polynomials in the entries of M; the identity $\det(x - \Lambda_M) = \prod_{i\le j}(x - \lambda_i - \lambda_j)$ holds for the dense set of diagonalizable M and hence for all M. The kernel of Λ_M has dimension of, at most, the algebraic multiplicity of the eigenvalue 0, with equality when Λ_M is diagonalizable there. If M is diagonalizable, so is Λ_M by the basis above.

(iii) By (20), $\mathrm{d}\mathcal{Q}/\mathrm{d}t = \eta Z^T \Lambda_M(S) Z = \kappa\eta\mathcal{Q}$; integrate. □

Proposition 2 (The count can be strict)**.** *For* $\mathbf{a} = \mathbf{b} = 0$ *all ten sums* $\lambda_i + \lambda_j$ *vanish, but only four linearly independent quadratic forms are conserved, namely* μ_1^2, $\mu_1\mu_2$, μ_2^2 *and* $\varphi\mu_2 - \psi\mu_1$.

Proof. Write $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{smallmatrix}\right)$. Here, $M = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$ and $\Lambda_M(S) = \left(\begin{smallmatrix} 0 & S_{11} \\ S_{11} & S_{12}+S_{12}^T \end{smallmatrix}\right)$. So, $\Lambda_M(S) = 0$ iff $S_{11} = 0$ and S_{12} is antisymmetric ($S_{12} = \sigma\varepsilon$), with S_{22} arbitrary symmetric: $1 + 3 = 4$ parameters. The forms are $Z^T S Z = 2\sigma(\varphi\mu_2 - \psi\mu_1) + \mu^T S_{22}\mu$. □

Corollary 2 (Hamiltonian case)**.** *If* $\mathbf{a}$ *is antisymmetric and* $\mathbf{b}$ *is symmetric, then* $S = \mathrm{diag}(\mathbf{b}, I_2)$ *solves* $\Lambda_M(S) = 0$, *giving the energy*

$$\mathscr{E} = \tfrac{1}{2}(\mu_1^2 + \mu_2^2) + \tfrac{1}{2}(b_{11}\varphi^2 + 2b_{12}\varphi\psi + b_{22}\psi^2). \tag{22}$$

If, moreover, $\mathbf{a} = \Omega\varepsilon$ *with* $\Omega \neq 0$ *and* $\mathbf{b} = b_0 I_2$ *with* $b_0 > 0$, *the solution space of* $\Lambda_M(S) = 0$ *is exactly two-dimensional, spanned by* $\mathscr{E}$ *and the shifted angular momentum*

$$\mathscr{M} = \varphi\mu_2 - \psi\mu_1 - \tfrac{\Omega}{2}(\varphi^2 + \psi^2). \tag{23}$$

Proof. Expand $M^T S + S M = 0$ in blocks, with $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{smallmatrix}\right)$ symmetric:

$$\begin{aligned} &(1,1):\ S_{12}\mathbf{b} + (S_{12}\mathbf{b})^T = 0, \quad (1,2):\ S_{11} = \mathbf{b}^T S_{22} + S_{12}\mathbf{a}, \\ &(2,1):\ S_{11} = \mathbf{a}^T S_{12}^T + S_{22}\mathbf{b}, \quad (2,2):\ S_{12} + S_{12}^T = \mathbf{a}^T S_{22} + S_{22}\mathbf{a}. \end{aligned}$$

With $S_{12} = 0$, $S_{22} = I_2$ these reduce to $S_{11} = \mathbf{b} = \mathbf{b}^T$ and $0 = \mathbf{a}^T + \mathbf{a}$, true by hypothesis; then, $Z^T S Z = U^T \mathbf{b} U + |\mu|^2 = 2\mathscr{E}$. For the second claim, $(1,1)$ gives $b_0(S_{12} + S_{12}^T) = 0$, so $S_{12} = \sigma\varepsilon$; $(2,2)$ gives $\Omega(S_{22}\varepsilon - \varepsilon S_{22}) = 0$, so the symmetric S_{22} commutes with ε and $S_{22} = \alpha I_2$; $(1,2)$ gives $S_{11} = (b_0\alpha - \Omega\sigma)I_2$, and $(2,1)$ gives the same. Hence, $Z^T S Z = \alpha(|\mu|^2 + b_0|U|^2) + \sigma\big(2(\varphi\mu_2 - \psi\mu_1) - \Omega|U|^2\big) = 2\alpha\mathscr{E} + 2\sigma\mathscr{M}$. This is consistent with Theorem 4(ii): the eigenvalues of M are $\pm i\nu_1, \pm i\nu_2$, $\nu_{1,2} = \frac{1}{2}(\Omega \pm \sqrt{\Omega^2 + 4b_0})$, with $\nu_1\nu_2 = -b_0 \neq 0$, $\nu_1 \pm \nu_2 \neq 0$, so M is diagonalizable and exactly two of the ten sums vanish. □

4.3. The Conformal Lax Equation and Its Converse

Theorem 5 (Conformal Lax pair, forward implication)**.** *Let* S *be complex symmetric with* $\Lambda_M(S) = \kappa S$, $\kappa \in \mathbb{C}$, *and set*

$$\mathcal{L} := Z Z^T S, \qquad \mathcal{W} := -\eta M. \tag{24}$$

Then, along every trajectory of (2),

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = [\mathcal{L}, \mathcal{W}] + \kappa\eta\mathcal{L}, \tag{25}$$

and consequently, for every $k \geq 1$, $\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}(\mathcal{L}^k) = k\kappa\eta\ \operatorname{tr}(\mathcal{L}^k)$, *so that* $\operatorname{tr}(\mathcal{L}^k)\exp(-k\kappa\int_{t_0}^{t}\eta\,\mathrm{d}\sigma)$ *is constant. If* $\kappa = 0$, (25) *is a genuine isospectral Lax equation, and* $\mathcal{L}_0 := \mathcal{L} - \frac{1}{4}\operatorname{tr}(\mathcal{L})I_4$ *satisfies* $\dot{\mathcal{L}}_0 = [\mathcal{L}_0, \mathcal{W}]$ *with* $\operatorname{tr}(\mathcal{L}_0^2) = \frac{3}{4}\mathcal{Q}^2$.

Proof. By (20) and $\Lambda_M(S) = \kappa S$, $\dot{\mathcal{L}} = \eta MZZ^TS + \eta ZZ^TM^TS = \eta MZZ^TS + \eta ZZ^T(\kappa S - SM) = \eta(M\mathcal{L} - \mathcal{L}M) + \kappa\eta\mathcal{L} = [\mathcal{L}, -\eta M] + \kappa\eta\mathcal{L}$. The trace statement follows from Lemma 1. For the last claim, $\mathcal{L}_0$ differs from $\mathcal{L}$ by a multiple of the identity, whose trace is constant when $\kappa = 0$. Since $\mathcal{L}$ has rank one, $\mathcal{L}^2 = (Z^TSZ)\mathcal{L} = \mathcal{Q}\mathcal{L}$, so $\operatorname{tr}(\mathcal{L}) = \mathcal{Q}$, $\operatorname{tr}(\mathcal{L}^2) = \mathcal{Q}^2$, and $\operatorname{tr}(\mathcal{L}_0^2) = \mathcal{Q}^2 - \frac{1}{2}\mathcal{Q}^2 + \frac{1}{4}\mathcal{Q}^2 = \frac{3}{4}\mathcal{Q}^2$. □

Proposition 3 (Converse implication and nondegeneracy)**.** *Let* $\Lambda_M(S) = \kappa S$, *and let* $t \mapsto (x(t), y(t))$ *be a* C^2 *curve on an interval* I *along which* $\Delta \neq 0$, $\eta \neq 0$, *and* (8)–(13) *hold, so that* $Z(t) = (\varphi, \psi, D_t\varphi/\eta, D_t\psi/\eta)^T$ *is defined. If* $\mathcal{L} = ZZ^TS$ *satisfies* (25) *on* I *and, for all* t *in a dense subset of* I, $Z(t) \neq 0$ *and either* $SZ(t) \neq 0$ *or the lower-right block* S_{22} *of* $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{smallmatrix}\right)$ *is invertible, then the curve solves* (2) *on* I. *Equivalently,* $(\mathcal{L}, \mathcal{W})$ *is a Lax representation of* (2) *in the sense of Definition 1 on* $\mathcal{D}_S = \{Z \neq 0\} \cap \{SZ \neq 0 \text{ or } \det(S_{22}) \neq 0\}$. *If* S_{22} *is invertible, then* $\mathcal{D}_S = \{Z \neq 0\}$. *If* S_{22} *is singular, at points with* $SZ = 0$ *the Lax equation only gives* $S\Xi = 0$, *which may leave part of* Ξ *undetermined.*

Proof. Put $\Xi := \dot{Z} - \eta MZ$. Since $\dot{\varphi} = \eta\mu_1$ and $\dot{\psi} = \eta\mu_2$ hold for every curve, $\Xi = (0, 0, \xi)^T$ with $\xi = \dot{\mu} + \eta(\mathbf{a}\mu + \mathbf{b}U) = \eta\,(U_{ss} + \mathbf{a}U_s + \mathbf{b}U)$. Write $Y := SZ$ so $\mathcal{L} = ZY^T$. The computation in the proof of Theorem 5, applied to a general curve, gives

$$\dot{\mathcal{L}} - [\mathcal{L}, \mathcal{W}] - \kappa\eta\mathcal{L} = \Xi Y^T + Z(S\Xi)^T.$$

Suppose this vanishes at a time t with $Z \neq 0$. If $Y \neq 0$, choose v with $Y^Tv \neq 0$. Multiplying on the right by v gives $\Xi\,(Y^Tv) = -Z\,(S\Xi)^Tv$, so $\Xi = \theta Z$ for a scalar θ. Then, $\theta ZY^T + Z(\theta SZ)^T = 2\theta ZY^T = 0$, hence $\theta = 0$ and $\Xi = 0$ at t. If instead $Y = 0$, the identity reduces to $Z(S\Xi)^T = 0$, so $S\Xi = 0$; since $\Xi = (0, \xi)^T$, we have $S\Xi = (S_{12}\xi, S_{22}\xi)^T$, and invertibility of S_{22} gives $\xi = 0$. Since Ξ is continuous and vanishes on a dense set, $\Xi \equiv 0$ on I. Finally, $\xi_i = \eta^{-2}\times$(left side of (14) for $\chi = \varphi, \psi$), so $\Xi = 0$ means that (14) holds along the curve; by Theorem 2 and $\Delta \neq 0$ this is equivalent to $(x'', y'') = (\Pi_1, \Pi_2)$. □

Remark 4 (Trace invariants)**.** *Since* $\mathcal{L}$ *has rank one,* $\mathcal{L}^k = \mathcal{Q}^{k-1}\mathcal{L}$ *and* $\operatorname{tr}(\mathcal{L}^k) = \mathcal{Q}^k$ *for all* k. *Thus, all higher traces are powers of the single quadratic form* $\mathcal{Q}$ *and yield* no additional independent first integrals*: the independent invariants are exactly those of Theorem 4. Moreover,* $(\mathcal{L}, \mathcal{W})$ *is built* from *the linearizing GST, so it is a reformulation of the quadratic invariants of the target rather than an independent integrability test. Its value is structural: it shows that the dissipative case is described by a conformal, rather than isospectral, Lax equation. The notion of a "block Lax pair that decouples" is not available here: for* $\mathbf{a} = 0$, $\mathbf{b}$ *diagonal,* $\mathcal{L}$ *is a single rank-one matrix and is not a direct sum of two* 2×2 *matrices.*

4.4. A Block Lax Pair for the Hamiltonian Case and the Scalar Specialization

Theorem 6 (Block Lax pair)**.** *Assume* $\mathbf{a}$ *antisymmetric,* $\mathbf{b}$ *symmetric, and take* $S = \operatorname{diag}(\mathbf{b}, I_2)$, *as in Corollary 2. Writing* $U = (\varphi, \psi)^T$, $\mu = (\mu_1, \mu_2)^T$, *the matrix* $\mathcal{L} = ZZ^TS$ *has the block form*

$$\mathcal{L} = \begin{pmatrix} UU^T\mathbf{b} & U\mu^T \\ \mu U^T\mathbf{b} & \mu\mu^T \end{pmatrix}, \qquad \mathcal{W} = -\eta M = \eta\begin{pmatrix} 0 & -I_2 \\ \mathbf{b} & \mathbf{a} \end{pmatrix}, \tag{26}$$

and satisfies $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{W}]$ *identically along trajectories of* (2), *with* $\operatorname{tr}(\mathcal{L}) = U^T\mathbf{b}U + |\mu|^2 = 2\mathscr{E}$ *(a direct block-by-block verification is in Appendix B).*

Proof. Theorem 5 with $\kappa = 0$ (applicable by Corollary 2) and $ZZ^T = \left(\begin{smallmatrix} UU^T & U\mu^T \\ \mu U^T & \mu\mu^T \end{smallmatrix}\right)$. □

Remark 5. *Conversely, since $S_{22} = I_2$ is invertible, Proposition 3 shows that on $\{Z \neq 0\}$, the Lax equation $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{W}]$ implies (2), for every symmetric* **b** *(invertible or not). Hence, $(\mathcal{L}, \mathcal{W})$ is a Lax representation in the sense of Definition 1 on $\{Z \neq 0\}$.*

Corollary 3 (Specialization to one equation)**.** *For a single equation with target $u_{ss} + bu = 0$, $b \neq 0$, $M = \left(\begin{smallmatrix} 0 & 1 \\ -b & 0 \end{smallmatrix}\right)$ and $\Lambda_M(S) = 0$ forces $S = \text{const} \cdot \text{diag}(b, 1)$. Taking the constant 1, the traceless part of $\mathcal{L} = ZZ^T S$, $Z = (u, u_s)^T$, is*

$$\mathcal{L}_0 = \begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & -\ell_1 \end{pmatrix}, \quad \ell_1 = \tfrac{1}{2}(bu^2 - u_s^2),\ \ell_2 = uu_s,\ \ell_3 = buu_s, \tag{27}$$

which has the form (7), and $\ell_1^2 + \ell_2\ell_3 = \left(\frac{1}{2}(bu^2 + u_s^2)\right)^2 = E^2$.

Proof. For $S = \left(\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right)$, $\Lambda_M(S) = \left(\begin{smallmatrix} -2bs_2 & s_1 - bs_3 \\ s_1 - bs_3 & 2s_2 \end{smallmatrix}\right) = 0$ iff $s_2 = 0$, $s_1 = bs_3$. Then, $\mathcal{L} = \left(\begin{smallmatrix} bu^2 & uu_s \\ buu_s & u_s^2 \end{smallmatrix}\right)$, $\text{tr}(\mathcal{L}) = bu^2 + u_s^2 = 2E$, and subtracting $\frac{1}{2}\text{tr}(\mathcal{L})I_2$ gives (27); finally, $\ell_1^2 + \ell_2\ell_3 = \frac{1}{4}(bu^2 - u_s^2)^2 + bu^2u_s^2 = \frac{1}{4}(bu^2 + u_s^2)^2$. □

Remark 6. *This is the same recipe $\mathcal{L} = ZZ^T S$ applied with $n = 1$; it recovers the shape (7) in which the potential coefficient enters through the off-diagonal entries. It is not a decoupling limit of the 4×4 matrix.*

5. Illustrative Examples

All numerical statements below refer to the methodology and error tables of Section 6.

Example 1 (Coupled system with an explicit closed-form solution)**.** *Consider*

$$x'' + \frac{1}{x}(x')^2 + \frac{1}{y}x'y' + q(t)x' = 0, \qquad y'' + \frac{1}{y}(y')^2 + \frac{1}{x}x'y' + q(t)y' = 0. \tag{28}$$

Proposition 4. *System (28) is GST-linearizable onto $u_{ss} = v_{ss} = 0$ (the case* **a** $=$ **b** $= 0$ *of (1)) by*

$$\varphi = x, \quad \psi = y, \quad \eta(t, x, y) = \frac{e^{-\int q\,\mathrm{d}t}}{xy}, \quad \Delta = 1. \tag{29}$$

Among transformations with $\varphi = x, \psi = y$ and this target, η is unique up to the choice of the antiderivative. (The choice $\eta = xy$ obtained by dropping the exponential factor does not *linearize (28): it gives $u_{ss} = x'\left(-xyq - 2xy' - 2x'y\right)/(x^3y^3) \neq 0$.) The transformed velocities*

$$\mu_1 = xy\,x'e^{\int q\,\mathrm{d}t}, \qquad \mu_2 = xy\,y'e^{\int q\,\mathrm{d}t} \tag{30}$$

are separately conserved, and hence so are all quadratic forms in μ_1, μ_2. In particular, $I := \mu_1^2 + \mu_2^2 = (xy)^2[(x')^2 + (y')^2]e^{2\int q\,\mathrm{d}t}$ is conserved. With $u = x = c_1 s + c_2$, $v = y = c_3 s + c_4$ and $\Theta(t) := \int e^{-\int q\,\mathrm{d}t}\mathrm{d}t$, the general solution is given implicitly by

$$\Theta(t) = c_1c_3\frac{s^3}{3} + \frac{c_1c_4 + c_2c_3}{2}s^2 + c_2c_4 s + c_5. \tag{31}$$

Proof. Apply Theorem 2 with $\chi = \varphi = x$ ($\chi_x = 1$, other derivatives zero) and $A_1 = -1/x$, $B_1 = -1/y$: Equations (8) and (9) give $(\log\eta)_x = -1/x$, $(\log\eta)_y = -1/y$, so $\eta = h(t)/(xy)$; (11) with $D_1 = -q$, $a_{1j} = 0$ gives $h'/h = -q$. The case $\chi = \psi = y$ reproduces the same η and forces $a_{2j} = b_{ij} = 0$. Since $u_{ss} = 0$, $\mu_1 = u_s$ is constant

and equals $D_t\varphi/\eta = x'/\eta$, which is (30); directly, $\frac{\mathrm{d}}{\mathrm{d}t}\left(xyx'e^{\int q}\right) = e^{\int q}\left[y(x')^2 + xx'y' + xyx'' + qxyx'\right] = 0$ on substituting $x'' = -x^{-1}(x')^2 - y^{-1}x'y' - qx'$. The conservation of $\varphi\mu_2 - \psi\mu_1$ follows from μ constant and $u_s = \mu_1$, $v_s = \mu_2$. For (31), $\mathrm{d}s = \eta\,\mathrm{d}t$ reads $e^{-\int q}\mathrm{d}t = xy\,\mathrm{d}s = (c_1 s + c_2)(c_3 s + c_4)\,\mathrm{d}s$; integrate. □

Since this is the case $\mathbf{a} = \mathbf{b} = 0$, *Proposition 2 predicts four independent quadratic first integrals:* $\mu_1^2, \mu_1\mu_2, \mu_2^2$ *and* $\varphi\mu_2 - \psi\mu_1 = xy\,e^{\int q}(xy' - yx')$, *although all ten sums* $\lambda_i + \lambda_j$ *vanish. Numerically, for* $q(t) = 0.3\sin t$, $x(1) = 1$, $y(1) = 2$, $x'(1) = 0.4$, $y'(1) = -0.3$, *the quantities* μ_1, μ_2, *and* $\varphi\mu_2 - \psi\mu_1$ *are constant to within* 4×10^{-13} *over* $t \in [1, 3]$.

Remark 7. *In the language of Remark 3, for Example 1 one finds* $G = -\log(xy) - \int q\,\mathrm{d}t$ *with* ∇^G *flat,* $(\varphi_0, \psi_0) = (x, y)$, $\Lambda = I_2$, $c = 0$, *and the constancy test of Proposition 1 returns* $a_{ij} = b_{ij} = 0$.

Example 2 (Undamped cyclotron: a charged particle in a time-varying magnetic field)**.** *We consider the model system*

$$x'' + \frac{1}{x}(x')^2 - \frac{1}{t}x' + \frac{\Omega t y}{x}y' + \frac{\omega^2 t^2}{2}x = 0, \tag{32}$$

$$y'' + \frac{1}{y}(y')^2 - \frac{1}{t}y' - \frac{\Omega t x}{y}x' + \frac{\omega^2 t^2}{2}y = 0. \tag{33}$$

It is a position-dependent-mass-type model problem (cf. [12] for a quantum-mechanical setting) designed so that the GST below linearizes it; it is not taken from [12].

Proposition 5. *Under* $\varphi = x^2$, $\psi = y^2$, $\eta = t$ *(so* $s = \frac{1}{2}(t^2 - t_0^2)$*), system (32) and (33) is carried into the constant-coefficient target*

$$u_{ss} + \Omega v_s + \omega^2 u = 0, \qquad v_{ss} - \Omega u_s + \omega^2 v = 0, \tag{34}$$

i.e., $a_{12} = -a_{21} = \Omega$, $b_{11} = b_{22} = \omega^2$, *all other coefficients zero. By Corollary 2, it possesses exactly two independent quadratic first integrals,*

$$\mathscr{E} = \frac{2\left[x^2(x')^2 + y^2(y')^2\right]}{t^2} + \frac{\omega^2}{2}(x^4 + y^4), \tag{35}$$

$$\mathscr{M} = \frac{2(x^2yy' - y^2xx')}{t} - \frac{\Omega}{2}(x^4 + y^4). \tag{36}$$

In the variable $w := u + iv$, (34) *becomes* $w_{ss} - i\Omega w_s + \omega^2 w = 0$, *an equation with constant coefficients, with roots* $\lambda_{1,2} = \frac{i}{2}(\Omega \pm \sqrt{\Omega^2 + 4\omega^2})$.

Proof. With $u = x^2$: $u_{tt} = 2(x')^2 + 2xx''$, and from (32), $xx'' = -(x')^2 + xx'/t - \Omega tyy' - \omega^2 t^2 x^2/2$, so $u_{tt} = 2xx'/t - 2\Omega tyy' - \omega^2 t^2 u$. With $\mathrm{d}s = t\,\mathrm{d}t$: $u_{ss} = u_{tt}/t^2 - u_t/t^3 = -2\Omega yy'/t - \omega^2 u = -\Omega v_s - \omega^2 u$, since $v_s = 2yy'/t$. The second equation is identical in form. The rest is Corollary 2 with $b_0 = \omega^2$, $\mu_1 = 2xx'/t$, $\mu_2 = 2yy'/t$ substituted into (22), (23). The complex form follows by adding i times the second equation to the first. □

For $\Omega = 1$, $\omega = 0.5$, $x(1) = y(1) = 1$, $x'(1) = y'(1) = 0$: $\lambda_1 = 1.207107\,i$, $\lambda_2 = -0.207107\,i$, *and* $w(s) = (1 + i)\left[0.146447\,e^{\lambda_1 s} + 0.853553\,e^{\lambda_2 s}\right]$ *solves (34) with* $w(0) = 1 + i$, $w_s(0) = 0$. *Then,* $x = \sqrt{\mathrm{Re}(w)}$, $y = \sqrt{\mathrm{Im}(w)}$ *up to the real domain of existence* $t < 2.64536$ *(where* $y \to 0$*). Note* $x \not\equiv y$: *the system is not symmetric under* $x \leftrightarrow y$ *(that exchange sends* $\Omega \mapsto -\Omega$*). The closed form agrees with a high-accuracy integration of (32) and (33) to* 1.5×10^{-13} *on* $[1, 2.6]$ *(Section 6).*

Example 3 (Damped cyclotron)**.**

$$x'' + \frac{1}{x}(x')^2 + \Big(\gamma t - \frac{1}{t}\Big)x' + \frac{\Omega t y}{x}y' + \frac{\omega^2 t^2}{2}x = 0, \tag{37}$$

$$y'' + \frac{1}{y}(y')^2 - \frac{\Omega t x}{y}x' + \Big(\gamma t - \frac{1}{t}\Big)y' + \frac{\omega^2 t^2}{2}y = 0, \tag{38}$$

with $\gamma > 0$*. Put* $w := x^2 + iy^2$, $\beta := \gamma - i\Omega$, $w_s = 2(xx' + iyy')/t$, $s = \frac{1}{2}(t^2 - t_0^2)$*, and let* λ_1, λ_2 *be the roots of* $\lambda^2 + \beta\lambda + \omega^2 = 0$*.*

Theorem 7 (Damped cyclotron: exact invariants and decaying quantities)**.** *Under* $\varphi = x^2$, $\psi = y^2$, $\eta = t$*, system* (37) *and* (38) *is carried into* $u_{ss} + \gamma u_s + \Omega v_s + \omega^2 u = 0$, $v_{ss} + \gamma v_s - \Omega u_s + \omega^2 v = 0$*; equivalently,* $w_{ss} + \beta w_s + \omega^2 w = 0$*. Consequently:*

(i) Exponentially weighted first integrals. *With* $\mathcal{Q} := w_s^2 + \beta w w_s + \omega^2 w^2 = (w_s - \lambda_1 w)(w_s - \lambda_2 w)$,

$$\mathcal{Q}\, e^{\beta s} = const, \tag{39}$$

which gives two real first integrals: writing $\mathcal{Q} = \mathcal{Q}_R + i\mathcal{Q}_I$,

$$K_1 = e^{\gamma s}\big(\mathcal{Q}_R \cos\Omega s + \mathcal{Q}_I \sin\Omega s\big), \qquad K_2 = e^{\gamma s}\big(\mathcal{Q}_I \cos\Omega s - \mathcal{Q}_R \sin\Omega s\big). \tag{40}$$

(ii) Two exactly decaying quantities. *The non-negative quantities* $\mathcal{Q}_1 := |w_s - \lambda_2 w|^2$, $\mathcal{Q}_2 := |w_s - \lambda_1 w|^2$ *satisfy exactly* $\mathrm{d}\mathcal{Q}_j/\mathrm{d}t = \kappa_j t\, \mathcal{Q}_j$ *with* $\kappa_1 = 2\,\mathrm{Re}(\lambda_1) < 0$, $\kappa_2 = 2\,\mathrm{Re}(\lambda_2) < 0$*; hence,* $\mathcal{Q}_j(t) = \mathcal{Q}_j(t_0) e^{\kappa_j s}$*.*

(iii) A first integral without exponential weight. *If* $\kappa_1 \neq \kappa_2$ *and* $\mathcal{Q}_1(t_0), \mathcal{Q}_2(t_0) > 0$*, then* $K := \mathcal{Q}_1^{\kappa_2}\mathcal{Q}_2^{-\kappa_1}$ *is a first integral of* (37) *and* (38) *that carries no factor* $e^{\gamma s}$ *or* $e^{\beta s}$*. It is* not *independent of* t *as a function on* (t, x, y, x', y')*-space:* K *depends on* t *through* $w_s = 2(xx' + iyy')/t$*.*

Proof. The target identification is direct substitution, as in Proposition 5, now the term $\gamma t x'$ in (37) contributes $-2\gamma t x x' = -\gamma t u_t$ to u_{tt}, hence $-\gamma u_s$ to u_{ss} (similarly for v); setting $w = u + iv$ gives $w_{ss} + (\gamma - i\Omega)w_s + \omega^2 w = 0$. (i) Using $\lambda_1 + \lambda_2 = -\beta$, $\lambda_1\lambda_2 = \omega^2$, one has directly, for $j = 1, 2$, $\dfrac{\mathrm{d}}{\mathrm{d}s}(w_s - \lambda_j w) = w_{ss} - \lambda_j w_s = \lambda_{3-j} w_s - \lambda_1\lambda_2 w = \lambda_{3-j}(w_s - \lambda_j w)$. Hence, $\mathrm{d}\mathcal{Q}/\mathrm{d}s = (\lambda_1 + \lambda_2)\mathcal{Q} = -\beta\mathcal{Q}$, which is (39). Separating real and imaginary parts with $e^{\beta s} = e^{\gamma s}(\cos\Omega s - i\sin\Omega s)$ gives (40). (ii) From the relation above, $\mathrm{d}(w_s - \lambda_2 w)/\mathrm{d}s = \lambda_1(w_s - \lambda_2 w)$, so $\mathcal{Q}_1 = |w_s - \lambda_2 w|^2$ evolves as $e^{2\,\mathrm{Re}\,\lambda_1 s}$; likewise $\mathcal{Q}_2$. Both real parts are negative. Since $\mathrm{d}s = t\,\mathrm{d}t$ this is $\mathrm{d}\mathcal{Q}_j/\mathrm{d}t = \kappa_j t \mathcal{Q}_j$. (iii) $\mathrm{d}\ln K/\mathrm{d}s = \kappa_2\kappa_1 - \kappa_1\kappa_2 = 0$. □

For $\gamma = 0.5$, $\Omega = 1$, $\omega = 0.5$, $x(1) = y(1) = 1$, $x'(1) = y'(1) = 0$ (so $w(t_0) = 1 + i$, $w_s(t_0) = 0$ and $\mathcal{Q}_1(t_0), \mathcal{Q}_2(t_0) > 0$), the roots are

$$\lambda_1 = -0.067804 - 0.186073\,i, \qquad \lambda_2 = -0.432196 + 1.186073\,i, \tag{41}$$

so $\kappa_1 = -0.135607$ and $\kappa_2 = -0.864393$; both real parts are negative for all s. The local minimum of $x(t)$ near $t \approx 2.197$, and the inflections near $t \approx 1.683$ and $t \approx 2.617$ are the effect of the two different decay rates combined with $x = \sqrt{\mathrm{Re}(w)}$; they are not associated with $t = 1/\sqrt{\gamma}$. The eigenvalues of the Lax partner $\mathcal{W} = -\eta M$ are not Lyapunov exponents of the flow and carry a reparameterization factor from $\mathrm{d}s = \eta\,\mathrm{d}t$.

Remark 8 (On Theorem 7)**.** *(a) For the damped target, the eigenvalues of* M *are* $\lambda_1, \lambda_2, \bar\lambda_1, \bar\lambda_2$ *and every pairwise sum has a negative real part when* $\gamma > 0$*. By Theorem 4(ii), there is therefore* no quadratic first integral without exponential weight *(*$\kappa = 0$*). This does not mean that the*

system has no first integral: K_1, K_2, and K are first integrals, with weights $e^{\beta s}$ (complex $\kappa = -\beta$) in the first two cases and built from the two real decay rates κ_1, κ_2 in the third; it is precisely why the conformal formulation (Theorem 5) is needed. The number of real decaying forms (two) matches the two real sums $\lambda_j + \bar{\lambda}_j = \kappa_j$.

(b) $\mathcal{Q}_1$, $\mathcal{Q}_2$ are positive semidefinite in Z (each vanishes on a two-dimensional real subspace), and trajectories of (37) and (38) leave the domain $x, y > 0$ in finite time. We therefore call them "Lyapunov-type" quantities with exactly known decay, not Lyapunov functions for an equilibrium of the original system.

Proof that $\mathrm{Re}(\lambda_j) < 0$**.** No root lies on the imaginary axis: for $\lambda = iy$, $y \in \mathbb{R}$, the equation $\lambda^2 + (\gamma - i\Omega)\lambda + \omega^2 = 0$ reads $(\omega^2 - y^2 + \Omega y) + i\gamma y = 0$, so $\gamma y = 0$; hence, $y = 0$ and $\omega^2 = 0$, a contradiction. For fixed $\gamma > 0, \omega > 0$ the roots depend continuously on Ω, so they cannot cross the imaginary axis as Ω varies; at $\Omega = 0$ they are the roots of $\lambda^2 + \gamma\lambda + \omega^2$, which have negative real parts. Hence, $\mathrm{Re}(\lambda_j) < 0$ for all Ω. □

6. Numerical Methodology and Verification

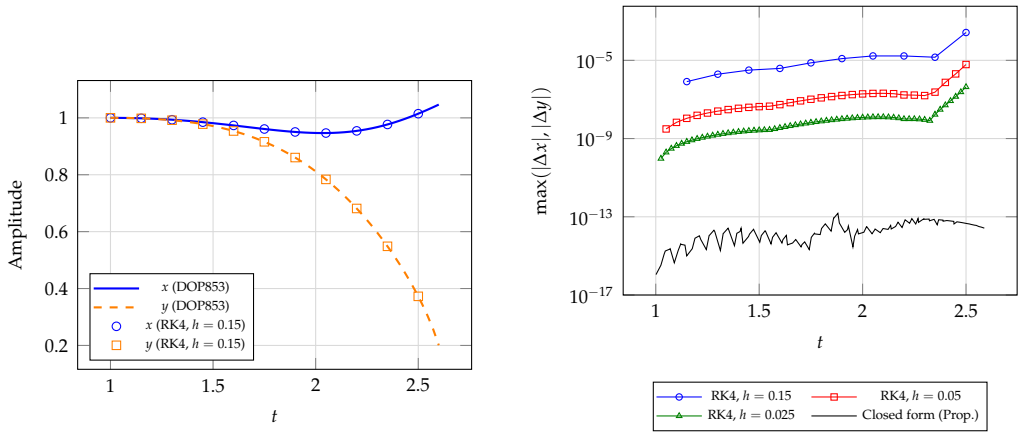
6.1. Solvers, Sampling, and Error Measures

The results for Example 2 are shown in Figures 1 and 2, and those for Example 3 in Figures 3 and 4.

- **Reference solution.** The original nonlinear system (32) and (33) or (37) and (38) is integrated in the variables (x, y, x', y') with the adaptive Dormand–Prince method of order 8, relative tolerance 10^{-13}, absolute tolerance 10^{-14}, and sampled by dense output on a uniform grid of 4001 points in $[1, 2.6]$ (Example 2) and $[1, 3.19]$ (Example 3). The solution ceases to exist when $y \to 0$, at $t = 2.64536$ (Example 2), and $t = 3.19693$ (Example 3); both intervals end before it.
- **Closed forms** (Proposition 5, Theorem 7) are used only as an independent cross-check; they agree with the reference to 1.5×10^{-13} (Example 2) and 1.3×10^{-13} (Example 3), maximum of $|\Delta x|, |\Delta y|$ over the grid.
- **Fixed-step RK4.** The classical fourth-order Runge–Kutta scheme with constant step h on the same variables, evaluated at its own nodes $t_k = 1 + kh$. Errors in Figures 1b and 3b and Table 1 are $\max(|\Delta x|, |\Delta y|)$ against the reference dense output at those nodes. Panels (a) of Figures 1 and 3 use $h = 0.15$.
- **Invariants.** The invariants in Figures 2 and 4 and Table 2 are evaluated along the reference (DOP853) solution of the nonlinear system, not along the closed form. RK4 curves are shown for contrast. Since $K_1(t_0) = 0$ for these data, K_1, K_2 are measured as $|K_j - K_j(t_0)|/|\mathcal{Q}(t_0)|$ with $|\mathcal{Q}(t_0)| = 1/2$; all other quantities as relative errors.

Table 1. Error of fixed-step RK4 against the DOP853 reference: $\max(|\Delta x|, |\Delta y|)$ at $t = 2.5$ (Example 2) and $t = 3.0$ (Example 3); observed order $= \log_2(e_{2h}/e_h)$. The step $h = 0.15$ does not divide $[1, 3]$ and is therefore not listed for Example 3.

h	Ex. 2 Error	Order	Ex. 3 Error	Order
0.15	2.65×10^{-4}	–	–	–
0.1	7.22×10^{-5}	–	8.50×10^{-6}	–
0.05	6.11×10^{-6}	3.56	6.49×10^{-7}	3.71
0.025	4.35×10^{-7}	3.81	4.32×10^{-8}	3.91
0.0125	2.87×10^{-8}	3.92	2.74×10^{-9}	3.98
0.00625	1.84×10^{-9}	3.96	1.72×10^{-10}	4.00



(a) Trajectories (b) Error vs. DOP853 reference

Figure 1. Example 2. (**a**) Reference solution and fixed-step RK4 with $h = 0.15$; $x \not\equiv y$. (**b**) Error histories against the reference: RK4 with $h = 0.15, 0.05, 0.025$ at the RK4 nodes, and the closed form of Proposition 5 .

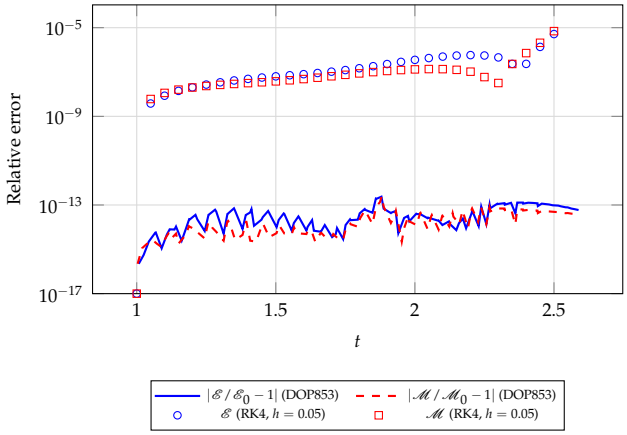
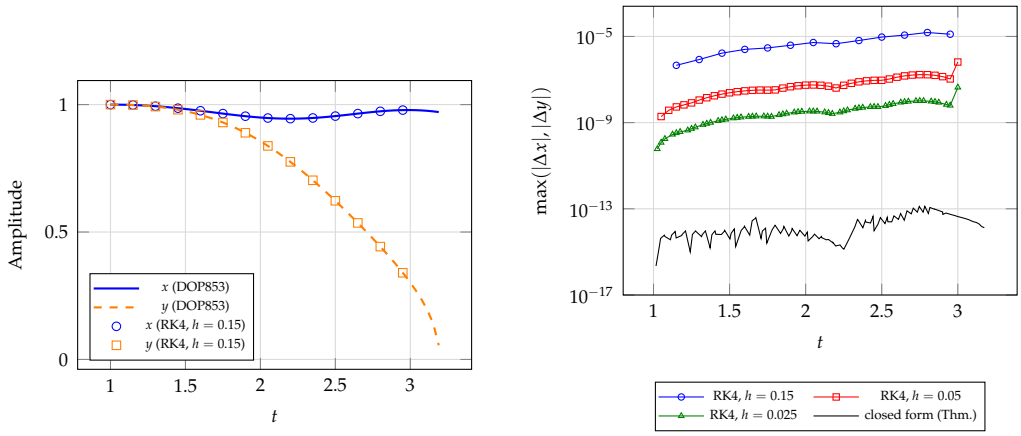


Figure 2. Example 2: relative errors of the two invariants $\mathscr{E}$, $\mathscr{M}$ of (35) and (36) along the DOP853 solution of the nonlinear system, and along fixed-step RK4 with $h = 0.05$.

Table 2. Maximum drift of the invariants over the sampling grid. Example 2: $\max\left(|\mathscr{E}/\mathscr{E}_0 - 1|, |\mathscr{M}/\mathscr{M}_0 - 1|\right)$ on $[1, 2.6]$ (DOP853) and on the RK4 nodes up to $t = 2.5$. Example 3: $|\mathcal{Q}e^{\beta s} - (\mathcal{Q}e^{\beta s})_{t_0}|/|\mathcal{Q}(t_0)|$ on $[1, 3.19]$ (DOP853) and on the RK4 nodes up to $t = 3.0$. For DOP853, atol = 0.1·rtol.

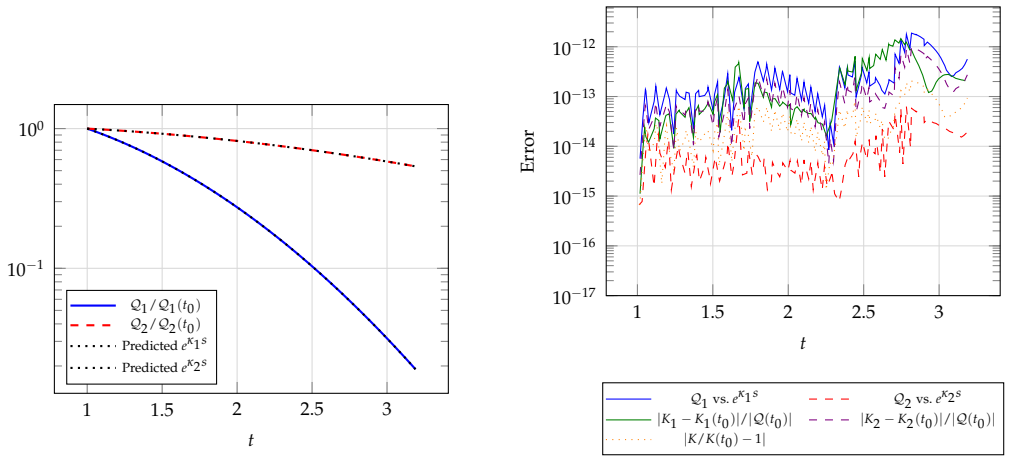
Integrator	Example 2	Example 3
DOP853, rtol 10^{-8}	5.2×10^{-9}	4.1×10^{-8}
DOP853, rtol 10^{-10}	1.1×10^{-10}	1.2×10^{-9}
DOP853, rtol 10^{-12}	1.6×10^{-12}	1.7×10^{-11}
DOP853, rtol 10^{-13}	2.5×10^{-13}	1.6×10^{-12}
RK4, $h = 0.15$	2.4×10^{-4}	1.1×10^{-3}
RK4, $h = 0.05$	7.0×10^{-6}	3.2×10^{-5}

https://doi.org/10.3390/math14214092



(a) Trajectories (b) Error vs. DOP853 reference

Figure 3. Example 3 ($\gamma = 0.5$, $\Omega = 1$, $\omega = 0.5$, same data). (**a**) DOP853 reference and RK4 with $h = 0.15$; the trajectory ends at $t \approx 3.197$, where $y \to 0$. (**b**) Error histories against the reference: RK4 with $h = 0.15, 0.05, 0.025$ at the RK4 nodes, and the closed form of Theorem 7.



(a) Decay of $\mathcal{Q}_1, \mathcal{Q}_2$ (b) Errors (DOP853, nonlinear system)

Figure 4. Example 3. (**a**) $\mathcal{Q}_1, \mathcal{Q}_2$ normalized by their initial values, with the predicted rates $e^{K_j s}$. (**b**) Errors of the decay law and of the invariants K_1, K_2, K, along the DOP853 solution of the nonlinear system. Each curve is the maximum over each of 120 consecutive intervals of the 4001-point grid; K_1, K_2 are normalized by $|\mathcal{Q}(t_0)|$.

6.2. Convergence Tables

For the reference run (rtol 10^{-13}), the individual errors in Figure 4 are 1.9×10^{-12}, 6.2×10^{-14} ($\mathcal{Q}_2$), 1.5×10^{-12} (K_1), 9.7×10^{-13} (K_2), and 2.0×10^{-13} (K). The tables show two things: the reference solver is converged to about 10^{-13}, and the fixed-step RK4 curves of Figures 1a and 3a ($h = 0.15$) agree with it only to about 10^{-4}, with fourth-order convergence as expected. Statements of agreement at the 10^{-10}–10^{-12} level refer to the adaptive solver and the closed forms, not to RK4.

7. Conclusions

We have studied GST-linearization of non-autonomous coupled second-order systems onto linear constant-coefficient targets and the associated Lax-type structures. The main results are:

1. A determining system of twelve PDEs (Theorem 2) with the roles of data and unknowns explicit, and a constancy test for its linear and zeroth-order part (Proposition 1).

2. A geometric criterion for the quadratic velocity terms (Theorem 3): they are realized by a GST iff ∇^G is flat for some G. This is necessary but not sufficient for linearizability of the complete system, which requires in addition the conditions of Proposition 1 (Corollary 1). The elimination of G is open.
3. The quadratic invariants of the target (Theorem 4); their number equals the multiplicity of 0 among the sums $\lambda_i + \lambda_j$ when M is diagonalizable and can be strictly smaller otherwise (Proposition 2).
4. A rank-one conformal Lax equation, with converse implication under the nondegeneracy condition of Proposition 3 (Theorem 5, Proposition 3). Its trace invariants are powers of one quadratic form, so it reorganizes the invariants of item 3 without adding new ones (Remark 4).
5. For the damped cyclotron: two exponentially weighted first integrals, two exactly decaying quadratic quantities, and one first integral without exponential weight (Theorem 7), with numerical checks documented in Section 6.

Open directions: eliminating the gauge G in Theorem 3 to obtain testable conditions on the given coefficients (compare the Liouville–Tresse conditions); the relation of $\eta = e^G$ to λ-symmetries [11]; the extension to $n \geq 3$ equations with $2n \times 2n$ matrices M, Z, where the number of independent quadratic invariants is governed by Theorem 4(ii) over the $\binom{2n+1}{2}$ pairwise sums; and a systematic comparison with the coupled-system results of [8,9].

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Notation

To avoid the clashes noted by the referees, the symbols below are used with a single meaning throughout.

Symbol	Meaning	Symbol	Meaning
Δ	Jacobian $\varphi_x\psi_y - \varphi_y\psi_x$	$\mathscr{E}$	Energy (22)
ε	Matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	$\mathscr{M}$	Shifted angular momentum (23)
$\mathcal{L}, \mathcal{W}$	Lax matrix and its partner	$\mathcal{Q}, \mathcal{Q}_j$	Quadratic form $Z^T S Z$; decaying forms
Λ_M	Operator $S \mapsto M^T S + SM$	κ, κ_j	Conformal constant; decay rates
μ_1, μ_2	Transformed velocities u_s, v_s	$c_1, \ldots, c_5$	Integration constants (Example 1)
$A_i, \ldots, F_i$	Quadratic coefficients in (4) and (5)	ℓ_1, ℓ_2, ℓ_3	Entries in (7)

Appendix A. Derivation of the Determining System

Fix $\chi \in \{\varphi, \psi\}$ (row $i = 1$ for φ, $i = 2$ for ψ) and recall (14).

Step 1. With $D_t\chi = \chi_t + x'\chi_x + y'\chi_y$ and the quadratic form (4) and (5),

$$D_t^2\chi = \chi_{tt} + 2x'\chi_{tx} + 2y'\chi_{ty} + (x')^2\left[\chi_{xx} + A_1\chi_x + A_2\chi_y\right] + x'y'\left[2\chi_{xy} + B_1\chi_x + B_2\chi_y\right] + (y')^2\left[\chi_{yy} + C_1\chi_x + C_2\chi_y\right] + x'\left[D_1\chi_x + D_2\chi_y\right] + y'\left[E_1\chi_x + E_2\chi_y\right] + \left[F_1\chi_x + F_2\chi_y\right].$$

Step 2. With $D_t\eta = \eta_t + x'\eta_x + y'\eta_y$, $(D_t\eta)(D_t\chi) = \eta_t\chi_t + x'(\eta_t\chi_x + \eta_x\chi_t) + y'(\eta_t\chi_y + \eta_y\chi_t) + (x')^2\eta_x\chi_x + x'y'(\eta_x\chi_y + \eta_y\chi_x) + (y')^2\eta_y\chi_y$.

Step 3. $\eta^2(a_{i1}D_t\varphi + a_{i2}D_t\psi) = \eta^2(a_{i1}\varphi_t + a_{i2}\psi_t) + x'\eta^2(a_{i1}\varphi_x + a_{i2}\psi_x) + y'\eta^2(a_{i1}\varphi_y + a_{i2}\psi_y)$, and $\eta^3(b_{i1}\varphi + b_{i2}\psi)$ is a pure constant term.

Step 4. Substituting in (14) and equating to zero the coefficient of each of the six monomials $(x')^2, x'y', (y')^2, x', y', 1$ (independent because x', y' are free at fixed (t, x, y)) gives

$$(x')^2: \ \eta[\chi_{xx} + A_1\chi_x + A_2\chi_y] - \eta_x\chi_x = 0, \tag{8}$$

$$x'y': \ \eta[2\chi_{xy} + B_1\chi_x + B_2\chi_y] - (\eta_x\chi_y + \eta_y\chi_x) = 0, \tag{9}$$

$$(y')^2: \ \eta[\chi_{yy} + C_1\chi_x + C_2\chi_y] - \eta_y\chi_y = 0, \tag{10}$$

$$x': \ 2\eta\chi_{tx} + \eta[D_1\chi_x + D_2\chi_y] - \eta_t\chi_x - \eta_x\chi_t + \eta^2(a_{i1}\varphi_x + a_{i2}\psi_x) = 0, \tag{11}$$

$$y': \ 2\eta\chi_{ty} + \eta[E_1\chi_x + E_2\chi_y] - \eta_t\chi_y - \eta_y\chi_t + \eta^2(a_{i1}\varphi_y + a_{i2}\psi_y) = 0, \tag{12}$$

$$1: \ \eta[\chi_{tt} + F_1\chi_x + F_2\chi_y] - \eta_t\chi_t + \eta^2(a_{i1}\varphi_t + a_{i2}\psi_t) + \eta^3(b_{i1}\varphi + b_{i2}\psi) = 0. \tag{13}$$

Taking $\chi = \varphi$ and $\chi = \psi$ gives twelve equations. No $D_t\eta$ survives, because each term of $(D_t\eta)(D_t\chi)$ has already been assigned to a single monomial.

Appendix B. Block-by-Block Verification of Theorem 6

Write $\mathcal{L} = \left(\begin{smallmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ \mathcal{L}_{21} & \mathcal{L}_{22}\end{smallmatrix}\right)$, $\mathcal{L}_{11} = UU^T\mathbf{b}$, $\mathcal{L}_{12} = U\mu^T$, $\mathcal{L}_{21} = \mu U^T\mathbf{b}$, $\mathcal{L}_{22} = \mu\mu^T$, and $\mathcal{W} = \left(\begin{smallmatrix}0 & -\eta I\\ \eta\mathbf{b} & \eta\mathbf{a}\end{smallmatrix}\right)$. Along trajectories $\dot{U} = \eta\mu$, $\dot{\mu} = -\eta\mathbf{b}U - \eta\mathbf{a}\mu$; recall $\mathbf{b}^T = \mathbf{b}$, $\mathbf{a}^T = -\mathbf{a}$.

Block (1,1). $\dot{\mathcal{L}}_{11} = \eta\mu U^T\mathbf{b} + \eta U\mu^T\mathbf{b}$ and $[\mathcal{L}, \mathcal{W}]_{11} = \mathcal{L}_{12}(\eta\mathbf{b}) - (-\eta I)\mathcal{L}_{21} = \eta U\mu^T\mathbf{b} + \eta\mu U^T\mathbf{b}$.

Block (1,2). $\dot{\mathcal{L}}_{12} = \eta\mu\mu^T + U(-\eta\mathbf{b}U - \eta\mathbf{a}\mu)^T = \eta\mu\mu^T - \eta UU^T\mathbf{b} + \eta U\mu^T\mathbf{a}$, and $[\mathcal{L}, \mathcal{W}]_{12} = \mathcal{L}_{11}(-\eta I) + \mathcal{L}_{12}(\eta\mathbf{a}) - (-\eta I)\mathcal{L}_{22} = -\eta UU^T\mathbf{b} + \eta U\mu^T\mathbf{a} + \eta\mu\mu^T$.

Block (2,1). $\dot{\mathcal{L}}_{21} = (-\eta\mathbf{b}U - \eta\mathbf{a}\mu)U^T\mathbf{b} + \eta\mu\mu^T\mathbf{b}$, and $[\mathcal{L}, \mathcal{W}]_{21} = \mathcal{L}_{22}(\eta\mathbf{b}) - \eta\mathbf{b}\mathcal{L}_{11} - \eta\mathbf{a}\mathcal{L}_{21} = \eta\mu\mu^T\mathbf{b} - \eta\mathbf{b}UU^T\mathbf{b} - \eta\mathbf{a}\mu U^T\mathbf{b}$.

Block (2,2). $\dot{\mathcal{L}}_{22} = (-\eta\mathbf{b}U - \eta\mathbf{a}\mu)\mu^T + \mu(-\eta\mathbf{b}U - \eta\mathbf{a}\mu)^T = -\eta\mathbf{b}U\mu^T - \eta\mathbf{a}\mu\mu^T - \eta\mu U^T\mathbf{b} + \eta\mu\mu^T\mathbf{a}$, and $[\mathcal{L}, \mathcal{W}]_{22} = \mathcal{L}_{21}(-\eta I) + \mathcal{L}_{22}(\eta\mathbf{a}) - \eta\mathbf{b}\mathcal{L}_{12} - \eta\mathbf{a}\mathcal{L}_{22}$, which is the same.

All four blocks of $\dot{\mathcal{L}} - [\mathcal{L}, \mathcal{W}]$ vanish identically.

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