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Cube-Difference Factor Absorbing Primary Ideals of Commutative Rings

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Abstract

Let R be a commutative ring with nonzero identity. A proper ideal I of R is *cube-difference factor absorbing primary* (cdf-primary) if, for all $a, b \in R$, $a^3 - b^3 \in I$ implies $a - b \in I$ or $a^2 + ab + b^2 \in \sqrt{I}$. This is the cubic counterpart of square-difference factor absorbing primarity, and it lies strictly between the cdf-absorbing and the cdf-semiprimary conditions. A recurring theme throughout is the *protective role* played by the prime 3, which has no counterpart in the quadratic theory and already separates the two conditions at the level of $\mathbb{Z}$: $14\mathbb{Z}$ is sdf-primary but not cdf-primary, whereas $21\mathbb{Z}$ is cdf-primary but not sdf-primary. We characterize the cdf-primary radical ideals of an arbitrary commutative ring without any finiteness or comaximality assumption: a radical ideal is cdf-primary if and only if at most one of its minimal primes has a residue domain whose fraction field contains a primitive cube root of unity and, if there is one, every other minimal prime contains 3. We also show that the radical of a cdf-primary ideal is always cdf-primary. Over $\mathbb{Z}$, for $n \geq 2$, the ideal $n\mathbb{Z}$ is cdf-primary exactly when n is a prime power, or $n = 3p^k$ with $p \neq 3$ prime, or n is squarefree with no prime factor congruent to 1 modulo 3. We also treat a weak variant and the behavior of cdf-primarity under the standard ring constructions.

Keywords: cdf-primary ideal; cdf-absorbing ideal; cdf-semiprimary ideal; cdf-zero pair; primitive cube root of unity; ZPI-ring; Dedekind domain; localization

MSC: 13A15; 11A41; 13B30; 13F05; 13F10

1. Introduction

Throughout this paper, all rings are commutative with nonzero identity, and R denotes such a ring. For a ring R and an ideal I, let $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \geq 1\}$ denote the radical of I. An element $p \in R$ is a *prime element* if it is nonzero, is not a unit, and generates a prime ideal. We write $U(R)$ for the group of units and $\mathrm{char}(R)$ for the characteristic; for a field F, we also write $F^\times = U(F)$ for its multiplicative group. The nilradical of R is $\mathrm{Nil}(R) = \sqrt{(0)}$. For a prime power q, we write $\mathbb{F}_q$ for the field with q elements so that $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime p. For a positive integer n, we write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime (p). We write $\mathrm{Spec}(R)$ for the set of prime ideals of R and $\mathrm{Frac}(D)$ for the fraction field of an integral domain D. For a prime ideal P of R, we call R/P the *residue domain* at P—the *residue field* when P is maximal—and $\mathrm{char}(R/P)$ the *residue characteristic*.

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We follow [1] for standard terminology. We also use polynomial and formal power series rings, and the localization $S^{-1}R$ of R at a multiplicatively closed set S with $1 \in S$ and $0 \notin S$, together with the extended ideal $S^{-1}I$. A nonzero integer n is *squarefree* if $p^2 \nmid n$ for every prime p; equivalently, $n\mathbb{Z} = \sqrt{n\mathbb{Z}}$.

A ring R is called a *ZPI-ring* if every proper ideal of R can be expressed as a finite product of prime ideals. This corresponds to a *general* Z.P.I.-ring in the terminology of [2], where the unqualified term is reserved for rings in which such factorizations are unique. Throughout this paper, we use ZPI-ring in this general sense. By [2] (see also [3]), this holds exactly when R is a finite direct product of Dedekind domains (fields included) and *special principal ideal rings*—that is, local principal ideal rings whose maximal ideal is nonzero and nilpotent. In a ZPI-ring, every primary ideal is a prime power (Result 1 and Corollary 1 [2]). The distinct minimal primes of a proper ideal are moreover pairwise comaximal, and every proper ideal factors as $I = P_1^{k_1} \dots P_r^{k_r}$ with $P_1, \dots, P_r$ its distinct minimal primes; both follow from the structure theorem, since a special principal ideal ring has a single prime, the minimal primes of a proper ideal of a Dedekind domain are the distinct maximal ideals occurring in its factorization—the single prime (0) when the ideal is zero—and primes coming from different direct factors are comaximal. The exponents need not be unique. They are unique for a nonzero proper ideal of a Dedekind domain but not in general: $(0) = (0)^k$ in $\mathbb{Z}$ for every $k \geq 1$, and $(0) = (2)^k$ in $\mathbb{Z}_4$ for every $k \geq 2$. We fix one such factorization throughout. Note that an arbitrary product of primes need not consist of minimal primes: in $\mathbb{Z}$, $(0) = (0) \cdot (7)$.

Prime and primary ideals have been generalized in many directions; see for example [4,5]. These conditions have recently been studied in other settings as well: for submodules [6], and relative to an expansion function δ of ideals, both for square differences [7] and for cube differences [8]. Anderson, Badawi, and Coykendall [9] recently introduced square-difference factor absorbing (sdf-absorbing) ideals, exploiting the factorization $a^2 - b^2 = (a - b)(a + b)$. A proper ideal I is sdf-absorbing if, for nonzero $a, b \in R$, $a^2 - b^2 \in I$ implies $a - b \in I$ or $a + b \in I$. Khashan, Yetkin Çelikel, and Tekir [10] then introduced square-difference factor absorbing primary (sdf-primary) ideals, where $a^2 - b^2 \in I$ implies $a - b \in I$ or $a + b \in \sqrt{I}$, so that one factor is kept in I while the other is relaxed to the radical $\sqrt{I}$.

The cubic counterpart, exploiting $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, was developed in [11,12]. The former introduces *cdf-absorbing* ideals. A proper ideal I is cdf-absorbing if $a^3 - b^3 \in I$ implies $a - b \in I$ or $a^2 + ab + b^2 \in I$. The latter introduces *cdf-semiprimary* ideals, again proper, in which each of the two alternatives is relaxed to $\sqrt{I}$, so that one asks for $a - b \in \sqrt{I}$ or $a^2 + ab + b^2 \in \sqrt{I}$. This paper treats the intermediate case, keeping $a - b$ in I and relaxing $a^2 + ab + b^2$ to $\sqrt{I}$. Thus, in the first alternative, cdf-primarity requires $a - b$ itself to lie in I, whereas cdf-semiprimality requires only $a - b \in \sqrt{I}$. The two conditions are different: $4\mathbb{Z} \times 2\mathbb{Z}$ in $\mathbb{Z} \times \mathbb{Z}$ is cdf-semiprimary but not cdf-primary (Example 2), because for $a = (2, 1)$ and $b = (0, 1)$, the factor $a - b = (2, 0)$ lies in $\sqrt{I}$ but not in I. A proper ideal I is *cdf-primary* if

$$a^3 - b^3 \in I \implies (a - b) \in I \text{ or } (a^2 + ab + b^2) \in \sqrt{I}.$$

This is the cubic analogue of the sdf-primary condition, but it behaves differently for a reason that is purely arithmetic. Viewed as a quadratic in a, the polynomial $a^2 + ab + b^2$ has discriminant $-3b^2$. Since -3 is a square modulo, a prime $p > 3$ exactly when $p \equiv 1 \pmod 3$, the residue of p modulo 3 decides, for such p, whether the polynomial can vanish modulo p without $p \mid a - b$. The primes 2 and 3 fall outside this criterion and are treated directly. Lemma 1 and Corollary 1 record the three cases. For $p \equiv 2 \pmod 3$, the divisibility must be carried by the factor $a - b$; for $p = 3$, the two factors vanish together, which is the exceptional case; and for $p \equiv 1 \pmod 3$, the factor $a^2 + ab + b^2$ can carry it alone. It is this exceptional

behavior of 3, which we call its protective role, that makes $21\mathbb{Z} = 3 \cdot 7\,\mathbb{Z}$ cdf-primary, while $14\mathbb{Z} = 2 \cdot 7\,\mathbb{Z}$, which is sdf-primary by Example 5 [10], is not. The two conditions are in fact incomparable, $21\mathbb{Z}$ being cdf-primary but not sdf-primary (Remark 25).

The three cubic classes form a strict hierarchy, cdf-absorbing $\subsetneq$ cdf-primary $\subsetneq$ cdf-semiprimary (Proposition 1, Examples 1 and 2). Every primary ideal is cdf-primary (Proposition 2), but $10\mathbb{Z}$ is cdf-primary without being primary or prime (Example 3); the condition therefore acquires content once primarity fails. We also introduce a weak variant, together with the cdf-zero pairs that witness failures of cdf-primarity, and show it to be incomparable with cdf-semiprimarity (Remark 7). In two classes of rings, every proper ideal is cdf-primary but for opposite reasons: in characteristic 3, the second alternative always holds (Proposition 3, strengthening the semiprimary statement of [12]), and in a Boolean ring, the first (Proposition 4). The class is moreover stable under passage to $(I{:}e)$ at an idempotent $e \notin I$ and under finite intersections of ideals with a common radical (Propositions 5 and 6). If $\sqrt{I}$ is cdf-primary, then I is cdf-semiprimary (Lemma 3); ordered by inclusion, the cdf-primary ideals of R have as maximal elements exactly the maximal ideals of R (Remark 5).

Cdf-primarity, that is, the property of being cdf-primary, depends only on the quotient R/I (Lemma 5), whereas the weak variant does not (Remark 8)—which is why parts (1) and (5) of Theorem 7 have no weak analogue. For an arbitrary radical ideal, the quotient description and localization yield the central result of this paper: an explicit criterion, the $\mathscr{B}$-condition, which records which of the minimal primes are of Type B—their residue domain having a fraction field that contains a primitive cube root of unity (Definition 4)—and which of the remaining ones contain 3 (Theorem 1). Moreover, if a proper ideal I of any ring is cdf-primary, then $\sqrt{I}$ is cdf-primary and even cdf-absorbing (Theorem 2, Remark 14); hence the criterion, applied to the minimal primes of I, is a necessary condition for every cdf-primary ideal. In ZPI-rings, we make it explicit when 3 is a unit and when (3) is maximal (Proposition 10, Theorem 3). The contrast with the quadratic theory persists: here, 3 plays the protective role that 2 plays there; the residue characteristics decide the types over $\mathbb{Z}$ but not over a general ZPI-ring (Remark 21); and where $2 \in U(R)$ forces every sdf-primary ideal to be primary (Proposition 2 [10]), $3 \in U(R)$ does not (Remark 15).

Over $\mathbb{Z}$, this gives a complete classification: for $n \geq 2$, the ideal $n\mathbb{Z}$ is cdf-primary precisely when n is a prime power, or $n = 3p^k$ for a prime $p \neq 3$, or n is squarefree with no prime factor congruent to 1 modulo 3 (Theorem 6, the squarefree case being Theorem 5). For Dedekind domains, we prove the cubic analogue of one implication of Theorem 4 [10] (Theorem 4); the converse fails as stated, and any genuine characterization must also constrain the non-radical ideals (Remark 22). Finally, cdf-primarity pulls back along every ring homomorphism and pushes forward along a surjection whose kernel lies inside the ideal (Proposition 12); it transfers through every construction that leaves R/I unchanged—quotients, the ideal (I, x) of $R[x]$, idealization, amalgamated algebras, and $I \times T$ in $R \times T$—in each case for the same reason (Theorem 7). Localization requires a separate argument (Theorem 8, Proposition 13), and for a product $I_1 \times I_2$ of two proper ideals, we give a partial transfer result together with the exact criterion and its extension to finite products (Theorem 9, Propositions 14 and 15).

To our knowledge, the following have no direct counterpart in the quadratic papers [9,10] or in the earlier cubic papers [11,12]: the reduction to the zero ideal of R/I (Lemma 5); the notion of Type B prime and the criterion of Theorem 1, with its consequence Theorem 2; Propositions 9 and 10; Theorems 3, 5 and 6; and the exact product criterion of Proposition 14 with its extension to finite products, Proposition 15. The incomparability with sdf-primarity (Remark 25) and the weak variant of Section 3 are likewise specific to the cubic setting.

Sections 2 and 3 develop the basic theory, Section 4 the general criterion, Section 5 the case of ZPI-rings, Dedekind domains and $\mathbb{Z}$, Section 6 transfer under ring constructions, Section 7 the questions left open, and Section 8 summarizes the results.

2. Definition and Basic Properties

Definition 1. *A proper ideal I of R is a cube-difference factor absorbing primary (cdf-primary) ideal of R if, for all $a, b \in R$,*

$$a^3 - b^3 \in I \implies (a-b) \in I \text{ or } (a^2+ab+b^2) \in \sqrt{I}.$$

We record two immediate consequences of the definition.

Remark 1.

(1) *Replacing b by $-b$ shows that I is cdf-primary if and only if $a^3 + b^3 \in I$ implies $a + b \in I$ or $a^2 - ab + b^2 \in \sqrt{I}$.*
(2) *Every prime ideal P is cdf-primary. Indeed, if $a^3 - b^3 \in P$, then $(a-b)(a^2+ab+b^2) \in P$, so $a - b \in P$ or $a^2 + ab + b^2 \in P = \sqrt{P}$. In fact, P is cdf-absorbing.*

The examples below rest on the behavior of $a^2 + ab + b^2$ modulo a prime, which falls into three cases according to the residue of that prime modulo 3.

Lemma 1. *Let p be a prime.*

(a) *If $p \equiv 2 \pmod 3$, then $p \nmid a^2 + ab + b^2$ for all $a, b \in \mathbb{Z}$ with $p \nmid b$; consequently, for such a and b, $p \mid a^3 - b^3$ if and only if $p \mid a - b$.*
(b) *If $p = 3$, then $3 \mid a^2 + ab + b^2$ if and only if $3 \mid a - b$, for all $a, b \in \mathbb{Z}$.*
(c) *If $p \equiv 1 \pmod 3$, then for every b with $p \nmid b$, there is an a with $p \mid a^2 + ab + b^2$ and $p \nmid a - b$.*

Proof. Throughout, $\mathbb{F}_p^\times$ is cyclic of order $p-1$, and any root ω of $t^2 + t + 1$ in $\mathbb{F}_p$ satisfies $\omega^3 = 1$, since $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$.

Case $p \equiv 2 \pmod 3$, including $p = 2$: Here, $3 \nmid p - 1$, so $\mathbb{F}_p^\times$ has no element of order 3. A root ω of $t^2 + t + 1$ lies in $\mathbb{F}_p^\times$, since $\omega \cdot \omega^2 = 1$, and its order divides 3; order 3 being impossible, $\omega = 1$, whence $3 = \omega^2 + \omega + 1 = 0$ in $\mathbb{F}_p$ and $p = 3$, contrary to $p \equiv 2 \pmod 3$. So, $t^2 + t + 1$ has no root in $\mathbb{F}_p$. Now, if $p \mid a^2 + ab + b^2$ with $p \nmid b$, then dividing by b^2 would make a/b such a root; hence, $p \nmid a^2 + ab + b^2$. Since $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ and p is prime, it follows that $p \mid a^3 - b^3$ if and only if $p \mid a - b$. This gives (a).

Case $p = 3$: Since $-2 \equiv 1 \pmod 3$, we have $(a-b)^2 = a^2 - 2ab + b^2 \equiv a^2 + ab + b^2 \pmod 3$. So, $3 \mid a^2 + ab + b^2 \Leftrightarrow 3 \mid (a-b)^2 \Leftrightarrow 3 \mid a - b$. This gives (b).

Case $p \equiv 1 \pmod 3$: Fix b with $p \nmid b$. Here, $3 \mid p - 1$, so $\mathbb{F}_p^\times$ contains an element of order 3, that is, a primitive cube root of unity $\omega \neq 1$ (i.e., $\omega^3 = 1$, $\omega \neq 1$, so $\omega^2 + \omega + 1 = 0$). Then, $a \equiv \omega b \pmod p$ gives $a^2 + ab + b^2 \equiv b^2(\omega^2 + \omega + 1) = 0 \pmod p$ and $a - b \equiv b(\omega - 1) \not\equiv 0 \pmod p$ (since $\omega \neq 1$ and $p \nmid b$). This gives (c). □

The equivalence in Lemma 1(a) survives the loss of the hypothesis $p \nmid b$, although the non-divisibility $p \nmid a^2 + ab + b^2$ does not. We record that unrestricted form; parts (b) and (c) repeat Lemma 1, the latter in the weaker existential form used below, so that the three residue classes can be cited together.

Corollary 1. *Let p be a prime.*

(a) *If $p \equiv 2 \pmod 3$, then for all $a, b \in \mathbb{Z}$ we have $p \mid a^3 - b^3$ if and only if $p \mid a - b$.*
(b) *If $p = 3$, then $3 \mid a - b$ if and only if $3 \mid a^2 + ab + b^2$, for all $a, b \in \mathbb{Z}$.*

(c) *If* $p \equiv 1 \pmod 3$, *then* $p \mid a^3 - b^3$ *does not imply* $p \mid a - b$: *there are* $a, b \in \mathbb{Z}$ *with* $p \mid a^3 - b^3$ *and* $p \nmid a - b$.

Proof. Part (a): If $p \nmid b$, Lemma 1(a) gives $p \nmid a^2 + ab + b^2$. Since p is prime and $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ with the second factor not divisible by p, we obtain $p \mid a^3 - b^3 \Leftrightarrow p \mid a - b$. If instead $p \mid b$, then $a^3 - b^3 \equiv a^3 \pmod p$, so $p \mid a^3 - b^3 \Leftrightarrow p \mid a \Leftrightarrow p \mid a - b$ (using $p \mid b$ in the last step). Either way, $p \mid a^3 - b^3 \Leftrightarrow p \mid a - b$.

Part (b) is Lemma 1(b). For part (c), take $b = 1$ and let a be as in Lemma 1(c); then, $p \mid a^2 + ab + b^2$, so $p \mid a^3 - b^3 = (a-b)(a^2+ab+b^2)$, while $p \nmid a - b$. □

Remark 2. *Part (b) of Corollary 1 does not extend to higher powers of 3: for* $a = 10$ *and* $b = 1$, *we have* $a - b = 9 = 3^2$, *while* $a^2 + ab + b^2 = 111 = 3 \cdot 37$, *so* 3^2 *divides* $a - b$ *but not* $a^2 + ab + b^2$.

The cdf-absorbing, cdf-primary and cdf-semiprimary conditions are related by the following implications.

Proposition 1. *Let* I *be a proper ideal of* R. *If* I *is cdf-absorbing, then* I *is cdf-primary; and if* I *is cdf-primary, then* I *is cdf-semiprimary:*

$$\textit{cdf-absorbing} \implies \textit{cdf-primary} \implies \textit{cdf-semiprimary}.$$

Proof. Both implications follow from $I \subseteq \sqrt{I}$: for the first, apply it to the second alternative; for the second, apply it to the first. □

Neither implication of Proposition 1 is reversible, as Examples 1 and 2 will show. Before turning to them, we record the main source of cdf-primary ideals on which the first of these two examples rests.

Proposition 2. *Every primary ideal is cdf-primary.*

Proof. Let I be primary and $(a-b)(a^2+ab+b^2) = a^3 - b^3 \in I$. If $a^2 + ab + b^2 \in \sqrt{I}$, the second alternative holds. Otherwise, $(a^2+ab+b^2)^k \notin I$ for all $k \geq 1$, so by the primary condition applied to $(a-b)(a^2+ab+b^2) \in I$, we obtain $a - b \in I$, and the first alternative holds. □

Example 1. *The ideal* $I = 8\mathbb{Z}$ *is cdf-primary but not cdf-absorbing.*

I is not cdf-absorbing. Taking $a = 2$ *and* $b = 0$ *gives* $a^3 - b^3 = 8 \in I$, *yet* $a - b = 2 \notin 8\mathbb{Z}$ *and* $a^2 + ab + b^2 = 4 \notin 8\mathbb{Z}$, *so neither factor is absorbed by* I.

I is cdf-primary. In the principal ideal domain $\mathbb{Z}$, *the primary ideals are* (0) *and the prime powers* (p^k) *[1], so* $8\mathbb{Z} = (2)^3$ *is primary; and every primary ideal is cdf-primary by Proposition 2.*

Example 2. *The ideal* $I = 4\mathbb{Z} \times 2\mathbb{Z}$ *in* $R = \mathbb{Z} \times \mathbb{Z}$ *(with* $\sqrt{I} = 2\mathbb{Z} \times 2\mathbb{Z}$*) is cdf-semiprimary but not cdf-primary.*

I is not cdf-primary. Take $a = (2,1)$ *and* $b = (0,1)$. *Then,* $a^3 - b^3 = (8,0) \in I$, *yet* $a - b = (2,0) \notin I$ *(since* $2 \notin 4\mathbb{Z}$*) and* $a^2 + ab + b^2 = (4,3) \notin \sqrt{I}$ *(since* $3 \notin 2\mathbb{Z}$*). Neither alternative of Definition 1 holds.*

I is cdf-semiprimary. Let (a_1, a_2) *and* (b_1, b_2) *be elements of* R *with* $(a_1^3 - b_1^3, a_2^3 - b_2^3) \in I$. *Since* $I = 4\mathbb{Z} \times 2\mathbb{Z}$, *the integer* 2 *divides each component of the cube difference. Corollary 1 (a), applied with* $p = 2$, *therefore gives* $2 \mid a_1 - b_1$ *and* $2 \mid a_2 - b_2$. *Consequently,* $(a_1 - b_1, a_2 - b_2) \in 2\mathbb{Z} \times 2\mathbb{Z} = \sqrt{I}$, *which is the first alternative in the definition of a cdf-semiprimary ideal. Thus,* I *is cdf-semiprimary.*

The converse of Proposition 2 is false, as the following example shows.

Example 3. *The ideal* $10\mathbb{Z}$ *is cdf-absorbing, hence cdf-primary, but not primary.*

By the description of the primary ideals of $\mathbb{Z}$ *given in Example 1, and since* $10 = 2 \cdot 5$ *is neither* 0 *nor a prime power,* $10\mathbb{Z}$ *is not primary. For the other assertion, note that* $2 \equiv 5 \equiv 2 \pmod 3$*, so Corollary 1(a) applies to both prime factors: if* $10 \mid a^3 - b^3$*, then* $2 \mid a - b$ *and* $5 \mid a - b$*, whence* $10 \mid a - b$*. The first alternative therefore always holds.*

The next two examples show how differently a prime $p \equiv 2 \pmod 3$ and the prime 3 behave when paired with a prime $q \equiv 1 \pmod 3$.

Example 4. *The ideal* $I = 14\mathbb{Z}$ *is not cdf-primary. Here,* $14 = 2 \cdot 7$ *and* $\sqrt{I} = 14\mathbb{Z}$*. Since* $7 \equiv 1 \pmod 3$*, Lemma 1(c) applies, the relevant primitive cube roots of unity in* $\mathbb{F}_7$ *being* 2 *and* 4*. Taking* $a = 9$ *and* $b = 1$ *(so that* $a \equiv 2 \pmod 7$*), we obtain* $a^3 - b^3 = 728 = 14 \cdot 52 \in I$*, while* $a - b = 8 \notin 14\mathbb{Z}$ *and* $a^2 + ab + b^2 = 91 = 7 \cdot 13 \notin 14\mathbb{Z}$*. Neither alternative of the cdf-primary condition holds.*

Example 5. *The ideal* $I = 21\mathbb{Z}$ *is cdf-primary. Here,* $21 = 3 \cdot 7$ *and* $\sqrt{I} = 21\mathbb{Z}$*. Suppose* $a^3 - b^3 \in I$*, i.e.,* $21 \mid (a - b)(a^2 + ab + b^2)$*. Since* 3 *is prime and* $3 \mid (a - b)(a^2 + ab + b^2)$*, it divides one of the two factors, and by Corollary 1(b), either alternative gives* $3 \mid a - b$*; consequently,* $3 \mid a^2 + ab + b^2$ *as well. For the* 7*-part: since* 7 *is prime and* $7 \mid (a - b)(a^2 + ab + b^2)$*, either* $7 \mid a - b$ *or* $7 \mid a^2 + ab + b^2$*. In the first case,* $3 \mid a - b$ *and* $7 \mid a - b$ *give* $21 \mid a - b$*, so* $a - b \in I$*. In the second case,* $3 \mid a^2 + ab + b^2$ *and* $7 \mid a^2 + ab + b^2$ *give* $21 \mid a^2 + ab + b^2$*, so* $a^2 + ab + b^2 \in \sqrt{I}$*. In all cases, one alternative of the cdf-primary condition holds.*

The contrast rests on Corollary 1(b). Since $3 \mid a - b$ *and* $3 \mid a^2 + ab + b^2$ *are equivalent, the prime* 3 *cannot separate the two alternatives; once it divides the product, it divides both factors, in particular whichever one* 7 *divides. The prime* 2 *is under no such constraint, and in* $14\mathbb{Z}$*, it does separate them. We refer to this behavior of* 3*—its inability to distinguish the two alternatives, and hence its harmlessness alongside a prime* $q \equiv 1 \pmod 3$*—as the protective role of* 3*.*

The defining conditions of the three classes are compared in Table 1; in each row, I is a proper ideal of R and $a, b \in R$.

Table 1. Comparison of the cdf-absorbing, cdf-primary, and cdf-semiprimary conditions.

Class	Conclusion When $a^3 - b^3 \in I$	Reference
cdf-absorbing	$a - b \in I$ or $a^2 + ab + b^2 \in I$	[11]
cdf-primary	$a - b \in I$ or $a^2 + ab + b^2 \in \sqrt{I}$	Definition 1
cdf-semiprimary	$a - b \in \sqrt{I}$ or $a^2 + ab + b^2 \in \sqrt{I}$	[12]

The Table 1 omits the primary ideals, which are not a link in the chain, since by Proposition 2, they lie inside the cdf-primary class but are incomparable with the cdf-absorbing ideals. The ideal $8\mathbb{Z} = (2)^3$ is primary but not cdf-absorbing (Example 1); conversely, $10\mathbb{Z}$ is cdf-absorbing but not primary (Example 3).

The Table 1 also omits one of the four conditions obtained by placing each of the two factors in I or in $\sqrt{I}$, namely the *dual* condition

$$a^3 - b^3 \in I \implies a - b \in \sqrt{I} \text{ or } a^2 + ab + b^2 \in I.$$

It lies between the cdf-absorbing and the cdf-semiprimary conditions, again because $I \subseteq \sqrt{I}$, and for radical I, it coincides with them (Lemma 2). Every primary ideal satisfies it: if I

is primary and $(a-b)(a^2+ab+b^2) \in I$, then $a^2+ab+b^2 \in I$ or $a-b \in \sqrt{I}$. Over $\mathbb{Z}$, every cdf-primary ideal $n\mathbb{Z}$ satisfies it. By Theorem 6, it suffices to check the three forms of n; write $d = a-b$ and $s = a^2+ab+b^2$. Form (i) is primary, and in form (iii), the ideal is cdf-absorbing, since each prime factor of n divides d by Corollary 1(a),(b). For $n = 3p^k$ with $p \neq 3$, if $n \mid ds$, then $3 \mid d$ and $3 \mid s$ by Corollary 1(b); if $p \mid d$, then $3p \mid d$, so $d \in \sqrt{n\mathbb{Z}}$, and otherwise $p^k \mid s$, so $n \mid s$. The dual condition does not imply cdf-primarity: $45\mathbb{Z}$ satisfies it, because $45 \mid ds$ gives $5 \mid d$ by Corollary 1(a) and $3 \mid d$ by Corollary 1(b), so that $d \in 15\mathbb{Z} = \sqrt{45\mathbb{Z}}$; yet $45\mathbb{Z}$ is not cdf-primary (Remark 12). Whether cdf-primarity implies the dual condition in an arbitrary ring is open (Section 7).

In characteristic 3, the cube difference is itself a cube, and this already forces the cdf-primary condition on every proper ideal. More generally, the same conclusion holds for any proper ideal I satisfying $3 \in \sqrt{I}$.

Proposition 3. *Let I be a proper ideal of R. If $3 \in \sqrt{I}$, then I is cdf-primary. In particular, if $\mathrm{char}(R) = 3$, then every proper ideal of R is cdf-primary.*

Proof. Suppose that $a^3 - b^3 \in I$. Since $I \subseteq \sqrt{I}$ and $3 \in \sqrt{I}$, we have $(a-b)^3 = (a^3-b^3) - 3ab(a-b) \in \sqrt{I}$.

As $\sqrt{I}$ is radical, it follows that $a-b \in \sqrt{I}$. Therefore, $a^2+ab+b^2 = (a-b)^2 + 3ab \in \sqrt{I}$.

Thus, the second alternative of Definition 1 holds, and hence I is cdf-primary. If $\mathrm{char}(R) = 3$, then $3 = 0 \in \sqrt{I}$ for every proper ideal I, so the final assertion follows. □

Remark 3. *Proposition 3 strengthens (Proposition 2.7 [12]), which establishes the cdf-semiprimary conclusion for every proper ideal of a ring of characteristic 3. Proposition 3 does not require $\mathrm{char}(R) = 3$; it is enough that $3 \in \sqrt{I}$ for the proper ideal I, and the stronger cdf-primary conclusion follows. This conclusion is also reflected in Lemma 4, which shows that when $3 \in \sqrt{I}$,*

$$a - b \in \sqrt{I} \quad \Longleftrightarrow \quad a^2 + ab + b^2 \in \sqrt{I}.$$

The proof above directly establishes the second alternative $a^2+ab+b^2 \in \sqrt{I}$ of Definition 1. The first alternative ($a-b \in I$, not merely $\sqrt{I}$) may fail. In $R = \mathbb{F}_3[x]$ with $I = (x^2)$, the pair $a = x$, $b = 0$ has $a^3 - b^3 = x^3 \in I$, while $a - b = x \notin I$.

Boolean rings form a second class of rings all of whose proper ideals are cdf-primary. Here, in contrast with the $3 \in \sqrt{I}$ case, it is the first alternative that holds.

Proposition 4. *Every proper ideal of a Boolean ring is cdf-primary.*

Proof. In a Boolean ring, $a^3 = a$ for all a. Hence, $a^3 - b^3 = a - b \in I$ whenever $a^3 - b^3 \in I$. The first alternative of Definition 1 holds. □

When I is its own radical, the cdf-absorbing, cdf-primary and cdf-semiprimary conditions have nothing left to distinguish them.

Lemma 2. *Let I be a proper ideal of R. If I is radical, that is $\sqrt{I} = I$, then the conditions cdf-absorbing, cdf-primary and cdf-semiprimary coincide for I.*

Proof. Since $\sqrt{I} = I$, all three conditions read: $a^3 - b^3 \in I$ implies $a - b \in I$ or $a^2 + ab + b^2 \in I$. □

A one-sided version of the collapse survives without the radical hypothesis.

Lemma 3. *If the radical of a proper ideal I of R is cdf-primary, then I is cdf-semiprimary.*

Proof. Let $a^3 - b^3 \in I$. Then, $a^3 - b^3 \in \sqrt{I}$, so the cdf-primarity of $\sqrt{I}$ gives $a - b \in \sqrt{I}$ or $a^2 + ab + b^2 \in \sqrt{\sqrt{I}} = \sqrt{I}$. Both alternatives place their factor in $\sqrt{I}$, which is the cdf-semiprimary condition for I. □

Read modulo $\sqrt{I}$; the two alternatives can be told apart precisely when $3 \notin \sqrt{I}$.

Lemma 4. *Let I be a proper ideal of R. The following conditions are equivalent:*

(1) *For all $a, b \in R$: $a - b \in \sqrt{I} \Leftrightarrow a^2 + ab + b^2 \in \sqrt{I}$.*
(2) *$3 \in \sqrt{I}$.*
(3) *$\mathrm{char}(R/\sqrt{I}) = 3$.*

Proof. (1) ⇒ (2): Setting $a = b = 1$ gives $a - b = 0 \in \sqrt{I}$, so (1) yields $a^2 + ab + b^2 = 3 \in \sqrt{I}$.

(2) ⇒ (1): Since $3 \in \sqrt{I}$, we have $3ab \in \sqrt{I}$ for all $a, b \in R$, and $a^2+ab+b^2 = (a-b)^2+3ab$. Thus, $a^2 + ab + b^2 \in \sqrt{I}$ if and only if $(a - b)^2 \in \sqrt{I}$. Since $\sqrt{I}$ is a radical ideal, $(a - b)^2 \in \sqrt{I}$ if and only if $a - b \in \sqrt{I}$. Hence, $a^2 + ab + b^2 \in \sqrt{I} \Leftrightarrow a - b \in \sqrt{I}$.

(2) ⇔ (3): $3 \in \sqrt{I}$ says that $3 = 0$ in $R/\sqrt{I}$, that is, $\mathrm{char}(R/\sqrt{I}) \mid 3$; and $\sqrt{I}$ is proper, so the characteristic is not 1. □

The cdf-primary condition is inherited by the ideal quotient $(I : e)$ at an idempotent e outside I.

Proposition 5. *Let I be cdf-primary and $e \in R \setminus I$ an idempotent. Then, $(I : e) = \{r \in R : re \in I\}$ is a cdf-primary ideal of R.*

Proof. The ideal $(I : e)$ is proper since $1 \notin (I : e)$ (as $e = 1 \cdot e \notin I$). We claim $\sqrt{(I : e)} = (\sqrt{I} : e)$. Since e is idempotent, $e^n = e$ for all $n \geq 1$, so $r^n e = r^n e^n = (re)^n$. Hence, $r^n e \in I$ if and only if $(re)^n \in I$, which means $r \in \sqrt{(I : e)}$ if and only if $re \in \sqrt{I}$, i.e., $r \in (\sqrt{I} : e)$. Thus, $\sqrt{(I : e)} = (\sqrt{I} : e)$.

Now, let $a^3 - b^3 \in (I : e)$. Then, $(ae)^3 - (be)^3 = (a^3 - b^3)e^3 = (a^3 - b^3)e \in I$. By the cdf-primarity of I, applied to the pair (ae, be), either $(a - b)e \in I$ or $(a^2 + ab + b^2)e^2 = (a^2 + ab + b^2)e \in \sqrt{I}$, where the last equality again uses $e^2 = e$. In the first case, $a - b \in (I : e)$. In the second, $a^2 + ab + b^2 \in (\sqrt{I} : e) = \sqrt{(I : e)}$. □

Remark 4. *Both hypotheses on e are used. That $e \notin I$ makes $(I : e)$ proper, and idempotence enters three times: in $r^n e = (re)^n$, in $e^3 = e$ and in $e^2 = e$. The first of these is the essential one, since it supplies the inclusion $(\sqrt{I} : e) \subseteq \sqrt{(I : e)}$; the opposite inclusion holds for every e, but equality can fail without idempotence. In $R = \mathbb{Z}_4$ with $I = (0)$ and $e = 2$, for instance, $\sqrt{(0 : 2)} = \{0, 2\}$ while $(\sqrt{(0)} : 2) = \mathbb{Z}_4$.*

Like the primary condition, the cdf-primary condition is stable under finite intersections of ideals sharing a common radical.

Proposition 6. *Let $m \geq 1$ and let $I_1, \ldots, I_m$ be cdf-primary ideals of R with a common radical $\sqrt{I_i} = V$ for all $i = 1, \ldots, m$. Then, $I = I_1 \cap \ldots \cap I_m$ is cdf-primary with $\sqrt{I} = V$.*

Proof. The ideal I is proper, being contained in the proper ideal I_1. Since $I \subseteq I_i$ for each i, we have $\sqrt{I} \subseteq \sqrt{I_i} = V$. Conversely, for any $z \in V$, choose n_i with $z^{n_i} \in I_i$ and set $N = \max_i n_i$,

which exists because there are finitely many I_i; then, $z^N \in I_i$ for all i, so $z^N \in I$ and $z \in \sqrt{I}$. Hence, $\sqrt{I} = V$.

Now, suppose $a^3 - b^3 \in I$ and $a - b \notin I$. Then, there exists i with $a - b \notin I_i$. Since I_i is cdf-primary and $a^3 - b^3 \in I \subseteq I_i$, we obtain $a^2 + ab + b^2 \in \sqrt{I_i} = V = \sqrt{I}$. Hence, I is cdf-primary. □

Ordered by inclusion, the cdf-primary ideals of R have an easily described set of maximal elements.

Remark 5. *Every maximal ideal is prime, hence cdf-primary by Remark 1 (2), and every cdf-primary ideal is proper and therefore contained in a maximal ideal. A maximal ideal M is therefore a maximal element, since any cdf-primary ideal containing M is proper and hence equals M. Conversely, a maximal element J of the family is proper, hence contained in some maximal ideal M; since M is itself cdf-primary, the maximality of J forces $J = M$. The maximal elements are therefore exactly the maximal ideals of R, and in particular, they always exist. The family is also closed under unions of chains: if $\{I_\alpha\}$ is a nonempty chain of cdf-primary ideals, then $L = \bigcup_\alpha I_\alpha$ is a proper ideal because $1 \notin I_\alpha$ for every α, and if $a^3 - b^3 \in L$, then $a^3 - b^3 \in I_\alpha$ for some α, so $a - b \in I_\alpha \subseteq L$ or $a^2 + ab + b^2 \in \sqrt{I_\alpha} \subseteq \sqrt{L}$, the last inclusion holding because $I_\alpha \subseteq L$ and the radical is monotone.*

Combined with Proposition 6, this gives the following structure. For a prime ideal P of R, let $C(P)$ be the set of cdf-primary ideals with radical P. It is closed under finite intersections by Proposition 6 and under unions of nonempty chains: the union L of such a chain $\{I_\alpha\}$ is cdf-primary, and $L \subseteq P$ gives $\sqrt{L} \subseteq P = \sqrt{I_\alpha} \subseteq \sqrt{L}$, so $L \in C(P)$. Its largest element is P: every member is contained in its radical P, and P itself lies in $C(P)$, being prime (Remark 1 (2)).

The quotients $\mathbb{Z}_n$ provide useful examples of cdf-primary ideals. The proper ideals of $\mathbb{Z}_n$ are of the form $d\mathbb{Z}/n\mathbb{Z}$, where $d \mid n$ and $d > 1$. As will be shown in Section 6, $d\mathbb{Z}/n\mathbb{Z}$ is cdf-primary in $\mathbb{Z}_n$ if and only if $d\mathbb{Z}$ is cdf-primary in $\mathbb{Z}$. Together with the characterization that will be established in Section 5, this determines the cdf-primary ideals of $\mathbb{Z}_n$. In particular, it will follow that every proper ideal of $\mathbb{Z}_8$ and $\mathbb{Z}_9$ is cdf-primary, since the corresponding divisors are prime powers. When $d = n$, the ideal is (0). Since $14\mathbb{Z}$ is not cdf-primary (Example 4), whereas $21\mathbb{Z}$ is cdf-primary (Example 5), the result in Section 6 will imply that (0) is cdf-primary in $\mathbb{Z}_{21}$ but not in $\mathbb{Z}_{14}$.

3. Weakly Cdf-Primary Ideals

Definition 2. *A proper ideal I of R is weakly cdf-primary if, for all $a, b \in R$,*

$$0 \neq a^3 - b^3 \in I \implies (a - b) \in I \text{ or } (a^2 + ab + b^2) \in \sqrt{I}.$$

The pairs that violate the cdf-primary condition recur throughout, so we name them.

Definition 3. *Let I be a proper ideal of R. A cdf-zero pair of I—the "zero" refers to $a^3 - b^3$ vanishing modulo I, not to a or b—is a pair (a, b) with $a, b \in R$, $a^3 - b^3 \in I$, $a - b \notin I$ and $a^2 + ab + b^2 \notin \sqrt{I}$.*

Comparing Definitions 1 and 3, we see that a proper ideal I is cdf-primary if and only if it has no cdf-zero pair.

Remark 6. *If (a, b) is a cdf-zero pair of I, then so is (b, a), and $a \neq b$. Indeed, $b^3 - a^3 = -(a^3 - b^3) \in I$, $b^2 + ba + a^2 = a^2 + ab + b^2 \notin \sqrt{I}$, and $b - a \in I$ holds if and only if $a - b \in I$, which fails by hypothesis; and $a = b$ would give $a - b = 0 \in I$.*

Every cdf-primary ideal is weakly cdf-primary. The zero ideal is vacuously weakly cdf-primary, since $0 \neq a^3 - b^3 \in (0)$ is impossible.

The weak condition can be read off from the cdf-zero pairs alone.

Proposition 7. *A proper ideal I of R is weakly cdf-primary if and only if every cdf-zero pair (a, b) of I satisfies $a^3 = b^3$ in R.*

Proof. Suppose I is weakly cdf-primary, and let (a, b) be any cdf-zero pair of I so that $a^3 - b^3 \in I$, $a - b \notin I$ and $a^2 + ab + b^2 \notin \sqrt{I}$. If $a^3 - b^3$ was nonzero, the weak condition would give $a - b \in I$ or $a^2 + ab + b^2 \in \sqrt{I}$, contradicting the choice of the pair; hence, $a^3 = b^3$.

Conversely, suppose every cdf-zero pair of I has vanishing cube difference, and let $a, b \in R$ with $0 \neq a^3 - b^3 \in I$. If neither $a - b \in I$ nor $a^2 + ab + b^2 \in \sqrt{I}$, then (a, b) is a cdf-zero pair with $a^3 \neq b^3$, which is contrary to the hypothesis. So, one of the two alternatives holds, and I is weakly cdf-primary. □

Thus, for a weakly cdf-primary ideal, cdf-primarity can fail only at pairs whose cube difference is zero in R—not merely zero modulo I. Since $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, this can be restated in terms of zero-divisors of R: for a cdf-zero pair, $a - b \notin I$ gives $a - b \neq 0$ and $a^2 + ab + b^2 \notin \sqrt{I}$ gives $a^2 + ab + b^2 \neq 0$, so $a^3 = b^3$ says that these two nonzero elements annihilate each other. Hence, I is weakly cdf-primary if and only if $a - b \in \mathrm{Ann}_R(a^2 + ab + b^2)$ for every cdf-zero pair (a, b) of I; the annihilators are taken in R and not in R/I, which is consistent with Remark 8 below. The next three examples exhibit ideals that are weakly cdf-primary but not cdf-primary. In the first two, the ideal is zero and the weak condition holds vacuously; the ideal is radical in Example 6 and not in Example 7. In the third, the ideal is nonzero, and the condition has to be checked on pairs that actually occur.

Example 6. *In $R = \mathbb{Z}_{35}$, $I = (0)$ is radical ($\sqrt{I} = I$) and weakly cdf-primary but not cdf-primary. It is vacuously weakly cdf-primary, as noted above. To see that (0) is not cdf-primary, take $a = 16$ and $b = 1$. Then, $a^3 - b^3 = 4095 = 3^2 \cdot 5 \cdot 7 \cdot 13$, and since $5 \cdot 7 = 35 \mid 4095$, we have $a^3 - b^3 = 0$ in $\mathbb{Z}_{35}$, so $a^3 - b^3 \in I$. However, $a - b = 15 \notin (0)$ and $a^2 + ab + b^2 = 256 + 16 + 1 = 273 = 3 \cdot 7 \cdot 13$; since $35 \nmid 273$, we have $a^2 + ab + b^2 \notin \sqrt{(0)} = (0)$. Hence, $(16, 1)$ is a cdf-zero pair. So, the collapse of cdf-absorbing, cdf-primary and cdf-semiprimary for radical ideals (Lemma 2) does not extend to the weak condition.*

Example 7. *In $R = \mathbb{Z}_{20}$, $I = (0)$ with $\sqrt{I} = (10)$ is weakly cdf-primary but not cdf-primary. Again, (0) is vacuously weakly cdf-primary. To see that it is not cdf-primary, take $a = 2$ and $b = 12$. Then, $a^3 - b^3 = 8 - 1728 = -1720 \equiv 0 \pmod{20}$, so $a^3 - b^3 = 0 \in I$. Yet $a - b = -10 \not\equiv 0 \pmod{20}$, so $a - b \notin I$; and $a^2 + ab + b^2 = 4 + 24 + 144 = 172 \equiv 12 \pmod{20}$, which is not in $\sqrt{I} = (10) = \{0, 10\}$. Hence, $(2, 12)$ is a cdf-zero pair.*

Example 8. *In $R = \mathbb{Z}_{40}$, let $I = (20)$, with $\sqrt{I} = (10)$. This ideal is not cdf-primary, but it is weakly cdf-primary. The pair $(2, 12)$ of Example 7, read in $\mathbb{Z}_{40}$, shows that I is not cdf-primary: $a^3 - b^3 = -1720 \equiv 0 \pmod{40}$ lies in I, while $a - b = -10 \notin I$ and $a^2 + ab + b^2 = 172 \equiv 12 \pmod{40}$ does not lie in $\sqrt{I} = \{0, 10, 20, 30\}$. For weak cdf-primarity, note that 20 is the only nonzero element of $I = \{0, 20\}$, so it suffices to show that $a^3 - b^3 \equiv 20 \pmod{40}$ forces $a - b \equiv 20 \pmod{40}$. For modulo 5, we have $5 \mid a^3 - b^3$, hence $5 \mid a - b$ by Corollary 1 (a). For modulo 8, we have $a^3 - b^3 \equiv 4$; since $c^3 \equiv 0$ for even c and $c^3 \equiv c$ for odd c modulo 8, both a and b must be odd and then $a - b \equiv a^3 - b^3 \equiv 4 \pmod{8}$. By the Chinese Remainder Theorem, $a - b \equiv 20 \pmod{40}$, that is, $a - b \in I$. The verification is not vacuous: $a = 21$ and $b = 1$ give $a^3 - b^3 = 9260 \equiv 20 \pmod{40}$ and $a - b = 20 \in I$.*

Proposition 8. *For a ring R, the following are equivalent.*

(1) Every weakly cdf-primary proper ideal of R is cdf-primary; that is, the weak and the ordinary conditions coincide for all proper ideals of R.

(2) The zero ideal of R is cdf-primary: $a^3 = b^3$ implies $a = b$ or $a^2 + ab + b^2 \in \mathrm{Nil}(R)$.

Condition (2) holds if cubing is injective on R (as on $\mathbb{Z}$) or if R is a domain; for reduced R, it holds exactly when $\mathrm{Min}(R)$ satisfies the $\mathscr{B}$-condition.

Proof. (1) ⇒ (2): The zero ideal is vacuously weakly cdf-primary, hence cdf-primary by (1). (2) ⇒ (1): let I be weakly cdf-primary and suppose (a, b) is a cdf-zero pair of I. By Proposition 7, $a^3 = b^3$; moreover, $a \neq b$ because $a - b \notin I$, and $a^2 + ab + b^2 \notin \sqrt{I} \supseteq \mathrm{Nil}(R)$. So, $a^3 - b^3 = 0$, $a - b \neq 0$ and $a^2 + ab + b^2 \notin \sqrt{(0)}$, which makes (a, b) a cdf-zero pair of the zero ideal, contradicting (2). Hence, I has no cdf-zero pair and is cdf-primary.

If cubing is injective, $a^3 = b^3$ gives $a = b$. If R is a domain, $a^3 = b^3$ gives $(a - b)(a^2 + ab + b^2) = 0$, so $a = b$ or $a^2 + ab + b^2 = 0 \in \mathrm{Nil}(R)$. If R is reduced, its zero ideal is radical and Theorem 1 applies. □

The rings in the three examples above are quotients of $\mathbb{Z}$ on which cubing is not injective, and none of them satisfies (2): the zero ideals of $\mathbb{Z}_{35}$ and $\mathbb{Z}_{20}$ are not cdf-primary (Examples 6 and 7), and neither is that of $\mathbb{Z}_{40}$ by the pair $(2, 12)$. Injectivity of cubing is sufficient for (2) but not necessary: cubing is not injective on $\mathbb{Z}_9$, where $3^3 = 0^3$ with $3 \neq 0$, yet 9 is a prime power, so every proper ideal of $\mathbb{Z}_9$ is cdf-primary (see the discussion closing Section 2), and in $\mathbb{Z}_9$, the two conditions agree.

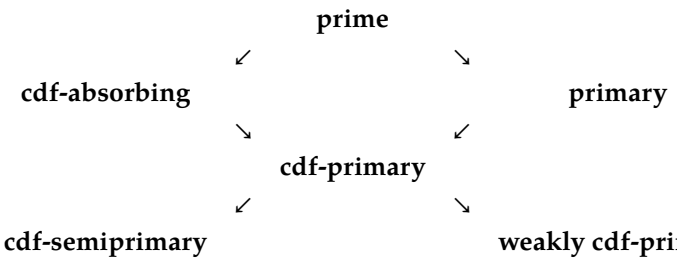
The two weakenings of the cdf-primary condition considered so far—relaxing the first alternative to the radical, and restricting attention to nonzero cube-differences—are independent of one another.

Remark 7. *The classes cdf-semiprimary and weakly cdf-primary are incomparable.*

cdf-semiprimary ⇏ weakly cdf-primary. The ideal $63\mathbb{Z}$ is cdf-semiprimary but not weakly cdf-primary. For the first assertion, $\sqrt{63\mathbb{Z}} = 21\mathbb{Z}$ is cdf-primary (Example 5), so $63\mathbb{Z}$ is cdf-semiprimary by Lemma 3. To see that $63\mathbb{Z}$ is not weakly cdf-primary, take $a = 22$ and $b = 1$. Then, $a^3 - b^3 = 10647 = 63 \cdot 169 \neq 0$, while $a - b = 21 \notin 63\mathbb{Z}$ and $a^2 + ab + b^2 = 507 = 3 \cdot 169$ with $21 \nmid 507$, so $a^2 + ab + b^2 \notin \sqrt{63\mathbb{Z}} = 21\mathbb{Z}$.

weakly cdf-primary ⇏ cdf-semiprimary. The zero ideal of $\mathbb{Z}_{35}$ serves again. It is vacuously weakly cdf-primary, and since it is radical, it is cdf-semiprimary if and only if it is cdf-primary (Lemma 2). Example 6 shows that it is not.

The six classes considered so far fit into a single diagram of implications.



Read from the top down and from left to right, the arrows are justified by Remark 1(2); the standard implication that a prime ideal is primary; Proposition 1; Proposition 2; Proposition 1 again; and, for the last arrow, the fact that Definition 2 restricts the hypothesis of Definition 1. All six are strict.

The witnesses follow: $10\mathbb{Z}$ for *prime* ⊊ *cdf-absorbing* and for *primary* ⊊ *cdf-primary* (Example 3; $10 = 2 \cdot 5$ is not prime); $4\mathbb{Z} = (2)^2$ for *prime* ⊊ *primary* (Example 1 describes the

primary ideals of $\mathbb{Z}$); $8\mathbb{Z}$ for *cdf-absorbing* $\subsetneq$ *cdf-primary* (Example 1); $4\mathbb{Z} \times 2\mathbb{Z}$ for *cdf-primary* $\subsetneq$ *cdf-semiprimary* (Example 2); and the zero ideal of $\mathbb{Z}_{35}$ for *cdf-primary* $\subsetneq$ *weakly cdf-primary* (Example 6).

Neither row of two is joined by an arrow in either direction: primary and cdf-absorbing are incomparable (Examples 1 and 3; witnesses $8\mathbb{Z}$ and $10\mathbb{Z}$), and so are cdf-semiprimary and weakly cdf-primary (Remark 7; witnesses $63\mathbb{Z}$ and the zero ideal of $\mathbb{Z}_{35}$). Outside the diagram, cdf-primary and sdf-primary ideals are incomparable as well with witnesses $21\mathbb{Z}$ and $14\mathbb{Z}$ (Remark 25).

4. Characterizations for an Arbitrary Commutative Ring

The cdf-primary condition offers two alternatives of different natures: one asks for membership in I and the other only in $\sqrt{I}$. We begin by showing that both are already visible in the quotient R/I. For a radical ideal of an arbitrary ring, that observation becomes an explicit criterion on the residue domains of the minimal primes. The consequences for the radical of an arbitrary ideal are recorded at the end of the section together with the limits of the criterion for non-radical ideals and the proof that the radical of a cdf-primary ideal is cdf-primary.

Lemma 5. *Let I be a proper ideal of R and write $\bar{R} = R/I$. The following are equivalent.*

(1) I is cdf-primary.
(2) The zero ideal of $\bar{R}$ is cdf-primary in $\bar{R}$; that is, for all $\xi, \eta \in \bar{R}$ with $\xi^3 = \eta^3$, either $\xi = \eta$ or $\xi^2 + \xi\eta + \eta^2 \in \mathrm{Nil}(\bar{R})$.
(3) For every $\bar{P} \in \mathrm{Spec}(\bar{R})$ and all $\xi, \eta \in \bar{R}$ with $\xi^3 = \eta^3$ and $\xi \neq \eta$, one has $\xi^2 + \xi\eta + \eta^2 \in \bar{P}$.
(4) There is no triple (a, b, P) with P a prime of R containing I, $a^3 - b^3 \in I$, $a - b \notin I$ and $a^2 + ab + b^2 \notin P$.

Proof. Let $\pi : R \to \bar{R}$ be the quotient map and set $\xi = \pi(a)$, $\eta = \pi(b)$; since π is surjective, every pair of elements of $\bar{R}$ arises in this way. Then, $a^3 - b^3 \in I$ if and only if $\xi^3 = \eta^3$; $a - b \in I$ if and only if $\xi = \eta$; and $a^2 + ab + b^2 \in \sqrt{I}$ if and only if $\xi^2 + \xi\eta + \eta^2 \in \sqrt{I}/I = \mathrm{Nil}(\bar{R})$. Reading Definition 1 through these three equivalences turns (1) into (2).

We next prove (1) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (3) both by contraposition. Write $\sqrt{I}$ as the intersection of the primes containing I, which $P \mapsto P/I$ identifies with the primes of $\bar{R}$. Then, $a^2 + ab + b^2 \notin \sqrt{I}$ says precisely that $a^2 + ab + b^2 \notin P$ for at least one prime $P \supseteq I$. A failure of cdf-primarity is a pair (a, b) with $a^3 - b^3 \in I$, $a - b \notin I$ and $a^2 + ab + b^2 \notin \sqrt{I}$; by the previous sentence, this is the same as a triple as in (4); and this same failure is likewise, transported to $\bar{R}$ along π and $P \mapsto P/I$, which is the same as a violation of (3). Hence, (1) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (3), which with (1) $\Leftrightarrow$ (2) above gives the lemma. □

Remark 8. *The reduction in Lemma 5(2) has no counterpart for the weak condition, because weak cdf-primarity is not determined by the quotient ring. Read in $\mathbb{Z}$, the integers $a = 16$ and $b = 1$ underlying Example 6 give $a^3 - b^3 = 4095 \neq 0$ with $4095 \in 35\mathbb{Z}$, while $a - b = 15 \notin 35\mathbb{Z}$ and $a^2 + ab + b^2 = 273 \notin 35\mathbb{Z} = \sqrt{35\mathbb{Z}}$; so $35\mathbb{Z}$ is not weakly cdf-primary in $\mathbb{Z}$. The zero ideal of $\mathbb{Z}_{35}$, by contrast, is weakly cdf-primary, vacuously. Yet $\mathbb{Z}/35\mathbb{Z}$ and $\mathbb{Z}_{35}/(0)$ are the same ring. The obstruction is that the hypothesis $a^3 - b^3 \neq 0$ refers to R and not to R/I.*

Consequently, no characterization of the weak condition can depend on the isomorphism class of R/I alone, and the weak analogues of parts (1) and (5) of Theorem 7 both fail. Part (1) allows any $K \subseteq I$ and fails already at $K = I$, where I/K is the zero ideal of R/I and hence weakly cdf-primary for every I, while I itself need not be, as $35\mathbb{Z}$ shows; and part (5) fails at $(0) \times \mathbb{Z}_2$ in $\mathbb{Z}_{35} \times \mathbb{Z}_2$, where $a = (1, 0)$ and $b = (16, 1)$ give $a^3 - b^3 = (0, 1)$, a nonzero element of the ideal, while $a - b = (20, 1)$ lies outside the ideal and $a^2 + ab + b^2 = (28, 1)$ outside its radical. What the weak condition does

admit is the intrinsic reformulation of Proposition 7, which speaks about pairs in R rather than about R/I.

By Lemma 5(4), a failure of cdf-primarity is witnessed by a single prime above I. The dichotomy that governs a single prime is whether the fraction field of its residue domain contains a primitive cube root of unity; we name it next.

Definition 4. *A prime ideal P of R is of* Type B *if $\xi^2+\xi\eta+\eta^2=0$ has a solution $\xi,\eta\in R/P$ with $\eta\neq 0$ and $\xi\neq\eta$ and of* Type A *otherwise. Equivalently, P is of Type B exactly when* $\mathrm{Frac}(R/P)$ *contains a primitive cube root of unity, i.e., when t^2+t+1 has a root in* $\mathrm{Frac}(R/P)$ *other than* 1.

The two formulations agree. Given ξ and η as in the first, dividing by η^2 makes ξ/η a root of t^2+t+1, and this root differs from 1 because $\xi\neq\eta$. Conversely, write a root $\omega\neq 1$ as α/β with $\alpha,\beta\in R/P$ and $\beta\neq 0$; then, $\xi=\alpha$, $\eta=\beta$ is a solution, since $\alpha^2+\alpha\beta+\beta^2=\beta^2(\omega^2+\omega+1)=0$, and $\alpha\neq\beta$ because $\omega\neq 1$. In the first formulation, the requirement $\eta\neq 0$ is redundant, since any solution with $\xi\neq\eta$ has it automatically: were η zero, the equation would reduce to $\xi^2=0$ and hence, R/P being a domain, force $\xi=0=\eta$, contrary to $\xi\neq\eta$.

Remark 9. *Five observations are worth recording separately.*

(1) A prime P containing 3 is of Type A, whatever its residue domain. Indeed, $3\in P$ means $\mathrm{char}(R/P)=3$ *(Lemma 4 applied to P), and in characteristic 3, the identity $\xi^2+\xi\eta+\eta^2=(\xi-\eta)^2$ holds, so a solution with $\xi\neq\eta$ would make $\xi-\eta$ a nonzero element whose square is zero, which a domain does not have.*

(2) When R/P is the prime field $\mathbb{F}_p$, P is of Type B exactly when $p\equiv 1 \pmod 3$ and of Type A exactly when $p\not\equiv 1 \pmod 3$, by Lemma 1(a), (b) and (c); item (1) covers the case $p=3$ as well.

(3) Item (2) is the case $k=1$ of the following uniform criterion. When R/P is a finite field $\mathbb{F}_{p^k}$, the group $\mathbb{F}_{p^k}^{\times}$ is cyclic of order p^k-1, so a primitive cube root of unity is present exactly when $3\mid p^k-1$, and P is of Type B precisely then. For $k>1$, the characteristic need not decide the type, since $3\mid p^k-1$ can occur while $3\nmid p-1$: a prime with residue field $\mathbb{F}_4$ or $\mathbb{F}_{25}$ is of Type B, whereas one with residue field $\mathbb{F}_2$ or $\mathbb{F}_5$, of the same characteristics, is of Type A.

(4) When R/P has characteristic 0, Type B holds exactly when -3 is a square in $\mathrm{Frac}(R/P)$*: there, t is a root of t^2+t+1 precisely when $(2t+1)^2=-3$, and $t=1$ is not a root because $3\neq 0$. Thus, a prime with residue domain $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{C}$ is of Type B, while one with residue domain $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ is of Type A.*

(5) Type B is a property of the fraction field $\mathrm{Frac}(R/P)$ *(equivalently, of the homogeneous equation in R/P of Definition 4); it asks for a root of t^2+t+1 in* $\mathrm{Frac}(R/P)$*, not in R/P itself. A root $\omega\neq 1$ lying in R/P makes P of Type B, and the converse holds whenever R/P is integrally closed (for instance a field, a unique factorization domain or a Dedekind domain), because ω is integral over R/P, being a root of the monic polynomial t^2+t+1. The converse fails in general: in $R=\mathbb{Z}[\sqrt{-3}]$, the prime $P=(0)$ is of Type B, since* $\mathrm{Frac}(R)=\mathbb{Q}(\sqrt{-3})$ *contains $\omega=(-1+\sqrt{-3})/2$ (take $\xi=-1+\sqrt{-3}$ and $\eta=2$ in Definition 4), whereas $\omega\notin\mathbb{Z}[\sqrt{-3}]$, so that t^2+t+1 has no root in $R/P=R$.*

For a radical ideal, cdf-primarity is decided by the types of its minimal primes and by which of them contain 3: a prime of Type B is tolerated only if every other minimal prime contains 3. This is the content of Theorem 1, which requires neither the finiteness of the set of minimal primes nor any comaximality. Its proof compares R/I with the residue domains of the primes involved, and we first record what that comparison yields.

Lemma 6. *Let $\mathcal{F}$ be a nonempty family of distinct prime ideals of R, and let $I = \bigcap_{P\in\mathcal{F}} P$. For $P \in \mathcal{F}$ put $D_P = R/P$, a domain, and for $x \in R$, write $x_P \in D_P$ for the image of x under the quotient map $R \to D_P$. Given $a, b \in R$, say that P satisfies Δ_P if $a_P = b_P$, and that it satisfies Q_P if $a_P^2 + a_P b_P + b_P^2 = 0$. Then, the following apply:*

(1) $a - b \in I$ if and only if every $P \in \mathcal{F}$ satisfies Δ_P, and $a^2 + ab + b^2 \in I$ if and only if every $P \in \mathcal{F}$ satisfies Q_P;

(2) $a^3 - b^3 \in I$ if and only if every $P \in \mathcal{F}$ satisfies Δ_P or Q_P.

Moreover, for every prime ideal P of every ring A, with $D = A/P$:

(3) P is of Type B if and only if there are $\xi, \eta \in D$ with $\xi \neq \eta$ and $\xi^2 + \xi\eta + \eta^2 = 0$;

(4) $3 \notin P$ if and only if there is a $\xi \in D$ with $3\xi^2 \neq 0$.

Informally, Δ_P records equality of a and b at P, and Q_P records vanishing of the quadratic factor $a^2 + ab + b^2$ there; the proof below simply unwinds Definitions 1 and 4 one prime at a time.

Proof. An element of R lies in I exactly when it lies in every $P \in \mathcal{F}$, that is, when its image in every D_P vanishes. Applied to $a - b$ and to $a^2 + ab + b^2$, this criterion gives (1). Applied to $a^3 - b^3$, it says that $a_P^3 = b_P^3$ for every P; since D_P is a domain and $a_P^3 - b_P^3 = (a_P - b_P)(a_P^2 + a_P b_P + b_P^2)$, this holds precisely when each P satisfies Δ_P or Q_P, which is (2).

Part (3) is the first formulation of Definition 4 with the requirement $\eta \neq 0$ dropped; the paragraph following that definition shows the requirement to be redundant.

For (4), $3 \notin P$ says exactly that $3 \neq 0$ in D. If it holds, take $\xi = 1$, so that $3\xi^2 = 3 \neq 0$; if it fails, $3 = 0$ in D and $3\xi^2 = 0$ for every ξ. □

We say that a nonempty set $\mathcal{F}$ of prime ideals of R satisfies the *$\mathcal{B}$-condition* if at most one member of $\mathcal{F}$ is of Type B and, when $P_0 \in \mathcal{F}$ is of Type B, every $P' \in \mathcal{F}$ with $P' \neq P_0$ contains 3. Since a prime containing 3 is of Type A (Remark 9(1)), this is the same as demanding that for every Type B member P of $\mathcal{F}$, every other member contains 3. Throughout, Type B refers to the fraction field of the residue domain, equivalently to the homogeneous equation of Definition 4 in R/P; this is weaker than asking for a root of $t^2 + t + 1$ in R/P itself, as Remark 9(5) explains. We can now prove the criterion announced above.

Theorem 1. *Let R be a ring, let I be a proper radical ideal of R, and let $\mathrm{Min}(I)$ be the set of minimal primes of I. The following are equivalent.*

(1) I is cdf-primary.

(2) For every $P \in \mathrm{Min}(I)$ of Type B, every $P' \in \mathrm{Min}(I)$ with $P' \neq P$ contains 3.

(3) The set $\mathcal{B} = \{P \in \mathrm{Min}(I) : P \text{ is of Type B}\}$ satisfies $|\mathcal{B}| \leq 1$, and if $\mathcal{B} = \{P_0\}$, then $3 \in P'$ for all $P' \in \mathrm{Min}(I) \setminus \{P_0\}$.

Thus, I is cdf-primary if and only if $\mathrm{Min}(I)$ satisfies the $\mathcal{B}$-condition. Moreover, if $I = \bigcap_{P\in\mathcal{F}} P$ for some nonempty set $\mathcal{F}$ of primes that satisfies the $\mathcal{B}$-condition, then I is cdf-primary; here, $\mathcal{F}$ need be neither finite nor consist of minimal primes, and its members need not be comaximal.

Proof. Since I is proper and radical, the cdf-primary condition on I reads: $a^3 - b^3 \in I \Rightarrow a - b \in I$ or $a^2 + ab + b^2 \in I$. Every prime containing I contains a minimal prime of I, so the radical ideal I, the intersection of the primes containing it, is the intersection of $\mathrm{Min}(I)$; and $\mathrm{Min}(I) \neq \emptyset$ because I is proper.

Sufficiency is explained next in the general form. Let $\mathcal{F}$ be a nonempty set of primes with $I = \bigcap_{P\in\mathcal{F}} P$ satisfying the $\mathcal{B}$-condition, and use the notation of Lemma 6. Let $a, b \in R$ with $a^3 - b^3 \in I$; by Lemma 6(2), every $P \in \mathcal{F}$ satisfies Δ_P or Q_P. Suppose toward a contradiction that neither $a - b \in I$ nor $a^2 + ab + b^2 \in I$. Then, by Lemma 6(1), some

$P \in \mathcal{F}$ fails Δ_P and some $P' \in \mathcal{F}$ fails $Q_{P'}$. Since every member satisfies Δ or Q, at P, it is Q_P that holds and at P' it is $\Delta_{P'}$; the latter gives $a_{P'} = b_{P'}$, so $a_{P'}^2 + a_{P'}b_{P'} + b_{P'}^2 = 3a_{P'}^2$, which is nonzero because $Q_{P'}$ fails. Necessarily $P' \neq P$, since by the same disjunction, no member can fail both. Now, $a_P \neq b_P$ and $a_P^2 + a_P b_P + b_P^2 = 0$, so P is of Type B by Lemma 6(3); and $3a_{P'}^2 \neq 0$ gives $3 \notin P'$ by Lemma 6(4). This violates the $\mathscr{B}$-condition for $\mathcal{F}$. Hence, $a - b \in I$ or $a^2 + ab + b^2 \in I$, and I is cdf-primary. Applied to $\mathcal{F} = \mathrm{Min}(I)$, this shows that the $\mathscr{B}$-condition for $\mathrm{Min}(I)$ implies (1).

(1) ⇒ (2): Suppose I is cdf-primary, and suppose toward a contradiction that (2) fails: some $P \in \mathrm{Min}(I)$ is of Type B while some $P' \in \mathrm{Min}(I)$, $P' \neq P$, has $3 \notin P'$. The ideals P and P' are distinct minimal primes of I and hence incomparable. Let $S = R \setminus (P \cup P')$; it is multiplicatively closed, since P and P' are prime, and $1 \in S$, $0 \notin S$. As $I \subseteq P$, we have $I \cap S = \emptyset$, so $J = S^{-1}I$ is cdf-primary in $T = S^{-1}R$ by Theorem 8 (whose proof is elementary and does not use the results of this section, so there is no circularity). Put $M = S^{-1}P$ and $N = S^{-1}P'$.

The primes of T containing J correspond to the primes $\mathfrak{p}$ of R with $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \cap S = \emptyset$, i.e., $\mathfrak{p} \subseteq P \cup P'$. By prime avoidance, $\mathfrak{p} \subseteq P$ or $\mathfrak{p} \subseteq P'$, and the minimality of P and P' over I then forces $\mathfrak{p} = P$ or $\mathfrak{p} = P'$. So, M and N are the only primes of T containing J; since P and P' are incomparable, neither contains the other, so both are maximal and $M + N = T$. Consequently, $\sqrt{J} = M \cap N$. Moreover, J is radical: if $(x/s)^n \in J$, then $tx^n \in I$ for some $t \in S$; hence, $(tx)^n \in I$, so $tx \in I$ and $x/s \in J$. Thus, $J = M \cap N$.

The residue ring T/M is the localization of R/P at the image of S, which does not contain 0; it is a domain lying between R/P and $\mathrm{Frac}(R/P)$ and has the same fraction field. So, M is of Type B, because P is. Also, the preimages of M and N in R are P and P', whence $3 \notin N$, as $3 \notin P'$. By Lemma 6(3),(4), applied to the primes M and N of T, choose $\xi \neq \eta$ in T/M with $\xi^2 + \xi\eta + \eta^2 = 0$ and $\zeta \in T/N$ with $3\zeta^2 \neq 0$. Since $M + N = T$ and $M \cap N = J$, the Chinese Remainder Theorem gives an isomorphism $T/J \cong T/M \times T/N$, and so elements $a, b \in T$ whose images are (ξ, ζ) and (η, ζ), respectively. Then, $a^3 - b^3$ maps to $\big((\xi - \eta)(\xi^2 + \xi\eta + \eta^2),\, 0\big) = (0, 0)$, so $a^3 - b^3 \in J$; $a - b$ maps to $(\xi - \eta, 0) \neq 0$, so $a - b \notin J$; and $a^2 + ab + b^2$ maps to $(0, 3\zeta^2) \neq 0$, so $a^2 + ab + b^2 \notin J = \sqrt{J}$. This contradicts the cdf-primarity of J. Hence, (2) holds.

(2) ⇒ (3): Suppose (2) holds. If $|\mathscr{B}| \geq 2$, say $P_1, P_2 \in \mathscr{B}$ with $P_1 \neq P_2$, then (2) applied to P_1 requires $3 \in P_2$; but a prime containing 3 is of Type A (Remark 9(1)), so $P_2 \notin \mathscr{B}$, which is a contradiction. So, $|\mathscr{B}| \leq 1$. If $\mathscr{B} = \{P_0\}$, condition (2) applied to P_0 is exactly the remaining clause of (3).

(3) ⇒ (1): Condition (3) says that $\mathrm{Min}(I)$ satisfies the $\mathscr{B}$-condition, and $I = \bigcap \mathrm{Min}(I)$, so I is cdf-primary by the sufficiency argument above. This closes the cycle (1) ⇒ (2) ⇒ (3) ⇒ (1). □

Remark 10. *When the minimal primes are finitely many and pairwise comaximal, as happens in a ZPI-ring, the localization in the proof of (1) ⇒ (2) is not needed: for $P + P' = R$, the Chinese Remainder Theorem yields $a, b \in R$ directly, and it does so over $\mathbb{Z}$ in the explicit form of Remark 23. More generally, if $I = P_1 \cap \ldots \cap P_r$ with $P_1, \ldots, P_r$ pairwise incomparable primes, then $\mathrm{Min}(I) = \{P_1, \ldots, P_r\}$: a prime containing $I \supseteq P_1 \ldots P_r$ contains some P_i, so every minimal prime of I is among the P_i, and each P_i is minimal because any prime $\mathfrak{p}$ with $I \subseteq \mathfrak{p} \subseteq P_i$ contains some P_k, forcing $P_k \subseteq P_i$, hence $k = i$ and $\mathfrak{p} = P_i$. Pairwise comaximal distinct primes are pairwise incomparable, so Theorem 1 applies to the intersection of finitely many pairwise comaximal primes, with $\{P_1, \ldots, P_r\}$ in place of $\mathrm{Min}(I)$. For infinite families, the intersection of the members need not have them as minimal primes, which is why the last sentence of Theorem 1 states sufficiency only.*

Depending on the size of $\mathscr{B}$ and on whether the remaining minimal primes contain 3, four mutually exclusive and exhaustive cases arise, each of which Theorem 1 decides.

Corollary 2. *Let I be a proper radical ideal of R and $\mathscr{B}$ the set of Type B primes in* $\mathrm{Min}(I)$*:*

(1) If every minimal prime of I is of Type A, then I is cdf-primary.
(2) If exactly one minimal prime P_0 of I is of Type B and every other minimal prime contains 3*, then I is cdf-primary.*
(3) If two or more minimal primes of I are of Type B, then I is not cdf-primary.
(4) If exactly one minimal prime P_0 of I is of Type B and some other minimal prime does not contain 3*, then I is not cdf-primary.*

Proof. Immediate from the equivalence (1) ⇔ (3) of Theorem 1: cases (1) and (2) are the two ways the $\mathscr{B}$-condition can hold, and cases (3) and (4) are the two ways it can fail. □

Remark 11. *In case (1) of Corollary 2, where every minimal prime of I is of Type A, the first alternative of Definition 1 always holds. The argument applies to any nonempty family $\mathscr{F}$ of Type A primes with $I = \bigcap_{P\in\mathscr{F}} P$, in the notation of Lemma 6; take $\mathscr{F} = \mathrm{Min}(I)$. Let $a, b \in R$ with $a^3 - b^3 \in I$. If Δ_P failed for some $P \in \mathscr{F}$, so that $a_P \neq b_P$, then part (2) of that lemma would force Q_P there, that is, $a_P^2 + a_P b_P + b_P^2 = 0$; part (3) would then show P to be of Type B, contrary to the assumption. So, Δ_P holds for every $P \in \mathscr{F}$, and $a - b \in I$ by part (1).*

Example 9. *In $\mathbb{Z}$, the prime (p), for a prime number p, is of Type B exactly when $p \equiv 1 \pmod 3$ by Remark 9(2), the residue field at (p) being the prime field $\mathbb{F}_p$; the minimal primes of a nonzero radical ideal of $\mathbb{Z}$ being the primes (p) with p dividing it, Corollary 2 recovers Examples 3, 4 and 5 at once. Six representative cases are collected in Table 2.*

By Remark 11, the first alternative always holds for $10\mathbb{Z}$, which is therefore even cdf-absorbing. In $42\mathbb{Z}$, the presence of (3) *does not rescue the ideal, since case (2) asks that every prime other than the Type B one contain* 3*; in $21\mathbb{Z}$, the only such prime is* (3) *itself, which is of Type A because it contains* 3*. Theorem 5 below turns these case-by-case verdicts into a complete criterion for squarefree n.*

Table 2. Six representative cases for cdf-primary ideals over the integers.

Ideal	Minimal Primes	Case of Corollary 2	Cdf-Primary
$10\mathbb{Z}$	(2), (5), both of Type A	(1)	yes
$14\mathbb{Z}$	(2), (7); (7) of Type B, $3 \notin (2)$	(4)	no
$35\mathbb{Z}$	(5), (7); (7) of Type B, $3 \notin (5)$	(4)	no
$42\mathbb{Z}$	(2), (3), (7); (7) of Type B, $3 \notin (2)$	(4)	no
$91\mathbb{Z}$	(7), (13), both of Type B	(3)	no
$21\mathbb{Z}$	(3), (7); (7) of Type B, $3 \in (3)$	(2)	yes

Theorem 1 also reaches non-radical ideals through their radicals: an ideal I and its radical $\sqrt{I}$ lie in the same primes and hence have the same minimal primes.

Corollary 3. *Let I be a proper ideal of a ring R. Then, $\sqrt{I}$ is cdf-primary if and only if the set* $\mathrm{Min}(I)$ *of minimal primes of I satisfies the $\mathscr{B}$-condition; when I is itself radical, this says that I is cdf-primary if and only if* $\mathrm{Min}(I)$ *satisfies the $\mathscr{B}$-condition.*

Proof. The ideal $\sqrt{I}$ is proper and radical, and $\mathrm{Min}(\sqrt{I}) = \mathrm{Min}(I)$, the primes containing I being those containing $\sqrt{I}$. Theorem 1 applied to $\sqrt{I}$ gives the first assertion; when I is radical, we have $I = \sqrt{I}$, which gives the second. □

A primary ideal is cdf-primary in any ring (Proposition 2), whereas Corollary 3 only gives information about its radical, which is prime. The hypothesis that I be primary cannot, however, be weakened to "$\sqrt{I}$ is prime": an embedded prime can destroy cdf-primarity, as the next example shows.

Example 10. *Let F be a field with $\mathrm{char}(F) \neq 3$. The ideal $I = (x^2, xy)$ in $R = F[x, y]$ has prime radical and is not primary, and it is not cdf-primary. Here, $I = (x) \cap (x, y)^2$, so $\sqrt{I} = (x)$ is prime, but I is not primary, since it has the embedded prime $(x, y) \supsetneq (x)$; indeed, $xy \in I$, $x \notin I$, yet $y \notin \sqrt{I} = (x)$, so the primary condition fails.*

Take $a = x + y$ and $b = y$. Then,

$$a^3 - b^3 = x^3 + 3x^2y + 3xy^2 = x^2(x + 3y) + 3y(xy) \in (x^2, xy) = I,$$

while $a - b = x \notin I$ (as I is generated in degree 2 and x has degree 1), and $a^2 + ab + b^2 = x^2 + 3xy + 3y^2$ takes the value $3y^2 \neq 0$ at $x = 0$, so $x \nmid a^2 + ab + b^2$, i.e., $a^2 + ab + b^2 \notin \sqrt{I} = (x)$. This last step is the only one that uses $\mathrm{char}(F) \neq 3$, and it must, because in characteristic 3, every proper ideal is cdf-primary (Proposition 3). Thus, (a, b) is a cdf-zero pair, and I is not cdf-primary despite having the single minimal prime $\sqrt{I} = (x)$.

Remark 12. *In the second assertion of Corollary 3, the restriction to radical I cannot be lifted: for non-radical I, the $\mathscr{B}$-condition need not suffice for I itself to be cdf-primary, even when $\mathscr{B} = \emptyset$. For example, $I = 45\mathbb{Z}$ has $\sqrt{I} = 15\mathbb{Z}$ with minimal primes $P_1 = (3)$ and $P_2 = (5)$. Since $3 \in P_1$ and $5 \equiv 2 \pmod 3$, both primes are of Type A (Remark 9(1) and (2)) and $\mathscr{B} = \emptyset$. Yet $45\mathbb{Z}$ is not cdf-primary: the pair $(a, b) = (16, 1)$ of Example 6, read in $\mathbb{Z}$, has $a^3 - b^3 = 4095 = 45 \cdot 91 \in 45\mathbb{Z}$, while $a - b = 15 \notin 45\mathbb{Z}$ and $a^2 + ab + b^2 = 273 \notin 15\mathbb{Z} = \sqrt{45\mathbb{Z}}$.*

Here, $45\mathbb{Z} = 9\mathbb{Z} \cap 5\mathbb{Z}$, and the two alternatives fail for different reasons. The first fails at the primary component $9\mathbb{Z}$: the two entries of the pair $(16, 1)$ are congruent modulo P_1, since $a - b = 15 \in P_1$, but $15 \notin 9\mathbb{Z}$. The second fails at P_2, where $a \equiv b$ as well, so that $a^2 + ab + b^2 \equiv 3a^2 \equiv 3 \not\equiv 0 \pmod 5$ because $3 \notin P_2$; hence, $a^2 + ab + b^2 \notin P_2 \supseteq \sqrt{45\mathbb{Z}}$.

Necessity is a different matter. By Theorem 2, the cdf-primarity of I implies that of $\sqrt{I}$ in every ring, so by Corollary 3, the $\mathscr{B}$-condition on $\mathrm{Min}(I)$ is necessary for I itself. It is not sufficient: $45\mathbb{Z}$ shows this with $\mathscr{B} = \emptyset$, and the ideal $63\mathbb{Z}$ with $\mathscr{B} \neq \emptyset$ (Remark 18, with Remark 24). For $\mathbb{Z}$, the non-radical case is nevertheless settled completely by Theorem 6.

Example 11. *Theorem 1 applies in rings whose minimal primes are not comaximal, and they are even infinite in number.*

(a) Let F be a field containing a primitive cube root of unity ω (for instance $F = \mathbb{C}$; then, $\mathrm{char}(F) \neq 3$), and let $I = (xy) = (x) \cap (y)$ in $R = F[x, y]$. Then, $\mathrm{Min}(I) = \{(x), (y)\}$, and these primes are not comaximal, since $(x) + (y) = (x, y) \neq R$. Their residue domains $F[y]$ and $F[x]$ have fraction fields containing ω, so both primes are of Type B and I is not cdf-primary by Corollary 2(3). An explicit cdf-zero pair is $a = x + y$, $b = x + \omega y$: one finds $a - b = (1 - \omega)y$, $a - \omega b = (1 - \omega)(x - \omega^2 y)$ and $a - \omega^2 b = (1 - \omega^2)x$, so that

$$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b) = 3(1 - \omega)\, xy\, (x - \omega^2 y) \in I,$$

using $(1-\omega)(1-\omega^2) = 3$, while $a - b \notin I$ and $a^2 + ab + b^2 = (a - \omega b)(a - \omega^2 b) = 3x(x - \omega^2 y) \notin I = \sqrt{I}$, because $x^2 \notin (xy)$ and $3 \neq 0$.

(b) Let $I = (xy)$ in $\mathbb{F}_2[x, y]$. Again, $\mathrm{Min}(I) = \{(x), (y)\}$ and these primes are not comaximal. Their residue domains $\mathbb{F}_2[y]$ and $\mathbb{F}_2[x]$ are of Type A: if $f/g \in \mathbb{F}_2(y)$, in lowest terms, satisfied $t^2 + t + 1 = 0$, then $f^2 + fg + g^2 = 0$ would give $g \mid f^2$ and $f \mid g^2$, so f and g would be constants in $\mathbb{F}_2$, and $t^2 + t + 1$ has no root in $\mathbb{F}_2$. So, I is cdf-primary by Corollary 2(1) and indeed cdf-absorbing by Remark 11.

(c) Let $R = \mathbb{Q}[x_1, x_2, \ldots]/(x_i x_j : i \neq j)$ with $I = (0)$. For each i, let $P_i = (x_j : j \neq i)$; the map $R \to \mathbb{Q}[x_i]$ sending $x_i \mapsto x_i$ and $x_j \mapsto 0$ for $j \neq i$ is well defined and surjective with kernel P_i, so P_i is prime with residue domain $\mathbb{Q}[x_i]$. Every element of R has the form $c + g_1 + g_2 + \ldots$ (finite sum) with $c \in \mathbb{Q}$ and $g_i \in x_i\mathbb{Q}[x_i]$, and it lies in every P_i only if $c + g_i = 0$ for all i, which forces it to be 0; so $(0) = \bigcap_i P_i$ is radical. A prime $\mathfrak{p}$ of R cannot omit two of the x_j, since $x_i x_j = 0$, so it contains P_i for some i; and $x_i \notin P_i$ shows that the P_i are pairwise incomparable. Hence, $\mathrm{Min}(0) = \{P_i : i \geq 1\}$ is infinite, and these primes are not pairwise comaximal, as they all lie in $(x_1, x_2, \ldots) \neq R$. The residue fields $\mathbb{Q}(x_i)$ contain no primitive cube root of unity by the argument of (b) together with the fact that $t^2 + t + 1$ has no root in $\mathbb{Q}$. So, every minimal prime is of Type A, and (0) is cdf-primary by Corollary 2(1); it is not primary, since $x_1 x_2 = 0$ while neither factor is nilpotent.

Theorem 2. *Let I be a proper ideal of an arbitrary ring R. If I is cdf-primary, then $\mathrm{Min}(I)$ satisfies the $\mathscr{B}$-condition. Consequently, $\sqrt{I}$ is cdf-primary and indeed cdf-absorbing.*

Proof. Once the $\mathscr{B}$-condition for $\mathrm{Min}(I)$ is known, Corollary 3 shows that $\sqrt{I}$ is cdf-primary, and Lemma 2 shows that, being radical, it is cdf-absorbing. So, suppose the $\mathscr{B}$-condition fails for $\mathrm{Min}(I)$: there are distinct $P, P' \in \mathrm{Min}(I)$ with P of Type B and $3 \notin P'$. We use the localization $T = S^{-1}R$ at $S = R \setminus (P \cup P')$ and its ideals $J = S^{-1}I$, $M = S^{-1}P$ and $N = S^{-1}P'$ from the proof of (1) $\Rightarrow$ (2) of Theorem 1. The following facts were established there without using that I is radical: J is a proper cdf-primary ideal of T by Theorem 8; M and N are the only primes of T containing J, and they are maximal and comaximal; T/M is a field with the same fraction field as R/P, so that M is of Type B, and $3 \notin N$. Only the final assertion $J = M \cap N$ of that proof used radicality, and it is not needed here.

Step 1: T/J is a product of two local rings with nil maximal ideals. Put $C = T/J$, and write $\bar{M}, \bar{N}$ for the images of M, N; these are the only primes of C, both maximal, so $\mathrm{Nil}(C) = \bar{M} \cap \bar{N}$. As $\bar{M} + \bar{N} = C$, write $1 = m + n$ with $m \in \bar{M}$ and $n \in \bar{N}$. Then, $mn \in \mathrm{Nil}(C)$, so $(mn)^k = 0$ for some $k \geq 1$. Expanding $(m+n)^{2k-1} = 1$ shows that $(m^k) + (n^k) = C$, say $um^k + vn^k = 1$. Put $e = um^k \in \bar{M}$; then, $1 - e = vn^k \in \bar{N}$, and $e(1-e) = uv(mn)^k = 0$, so e is an idempotent and $C \cong A \times B$ with $A = C/eC$ and $B = C/(1-e)C$. The primes of A correspond to the primes of C containing e, that is, to $\bar{M}$ alone (as $1 - e \in \bar{N}$ shows $e \notin \bar{N}$), and likewise B has the single prime $\bar{N}$. Thus, A and B are local rings whose maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ are nil with residue fields $A/\mathfrak{m}_A = T/M =: \kappa$ and $B/\mathfrak{m}_B = T/N$.

Step 2: lifting a root of $t^2 + t + 1$ to A. Let $g(t) = t^2 + t + 1$. Since M is of Type B, there is $\omega \in \kappa$ with $g(\omega) = 0$ and $\omega \neq 1$. Moreover, $3 \neq 0$ in κ, because $3 \notin P$ by Remark 9(1) and M contracts to P; hence, $g'(\omega) = 2\omega + 1 \neq 0$, since $(2\omega+1)^2 = 4g(\omega) - 3 = -3 \neq 0$. Choose $\Omega_0 \in A$ lifting ω and put $x = g(\Omega_0) \in \mathfrak{m}_A$; as $\mathfrak{m}_A$ is nil, $x^r = 0$ for some $r \geq 1$. Define $\Omega_{j+1} = \Omega_j - g(\Omega_j)/g'(\Omega_j)$. We claim that for every j, Ω_j is defined, $\Omega_j \equiv \Omega_0 \pmod{\mathfrak{m}_A}$ and $g(\Omega_j) \in x^{2^j}A$. For $j = 0$, this is clear. If it holds for j, then $g'(\Omega_j) \equiv 2\omega + 1 \not\equiv 0 \pmod{\mathfrak{m}_A}$, so $g'(\Omega_j)$ is a unit of the local ring A and Ω_{j+1} is defined; the correction $h = g(\Omega_j)/g'(\Omega_j)$ lies in $x^{2^j}A \subseteq \mathfrak{m}_A$, so $\Omega_{j+1} \equiv \Omega_j \pmod{\mathfrak{m}_A}$; and since g is monic quadratic,

$$g(\Omega_{j+1}) = g(\Omega_j) - g'(\Omega_j)h + h^2 = h^2 \in x^{2^{j+1}}A.$$

Choose j with $2^j \geq r$ and put $\Omega = \Omega_j$: then $g(\Omega) = 0$ and $\Omega \equiv \omega \not\equiv 1 \pmod{\mathfrak{m}_A}$, so $\Omega - 1$ is a unit of A.

Step 3: a cdf-zero pair. In $C \cong A \times B$, put $c = (\Omega, 1)$ and $d = (1, 1)$. Then, $c^3 - d^3 = \big((\Omega - 1)g(\Omega), 0\big) = 0$ and $c - d = (\Omega - 1, 0) \neq 0$. Also, $c^2 + cd + d^2 = (g(\Omega), 3) = (0, 3)$, and $3 \notin \mathfrak{m}_B$ because $3 \notin P'$ and N contracts to P', so 3 is a unit of B and no power of $(0, 3)$ vanishes; thus, $c^2 + cd + d^2 \notin \mathrm{Nil}(C)$. So, (c, d) is a cdf-zero pair of the zero ideal of $C = T/J$, and J is not cdf-primary by Lemma 5(2). This contradicts the cdf-primarity of J and proves the theorem. □

5. ZPI-Rings, Dedekind Domains and the Integers

We now make the results of the previous section explicit for ZPI-rings and for $\mathbb{Z}$. Since every proper ideal of a ZPI-ring is a finite product of primes, Theorem 1 becomes a statement about the prime factorization of I; over $\mathbb{Z}$, where the residue field at the maximal ideal (p) is $\mathbb{F}_p$, Remark 9(2) turns the Type A/Type B dichotomy into a congruence condition modulo 3.

Proposition 9. *Let R be a ZPI-ring and $I = P_1^{k_1} \dots P_r^{k_r}$ a proper ideal, the P_i being its distinct minimal primes. If I is cdf-primary, then $\sqrt{I}$ is cdf-primary.*

Proof. The radical of I is the intersection $\sqrt{I} = P_1 \cap \dots \cap P_r$ of its minimal primes, and in a ZPI-ring, these are pairwise comaximal (Section 1). Their powers are comaximal as well, since raising $P_i + P_j = R$ to the power $k_i + k_j$ gives $P_i^{k_i} + P_j^{k_j} = R$. Comaximal ideals meet in their product, so $I = P_1^{k_1} \dots P_r^{k_r} = \bigcap_i P_i^{k_i}$ and the Chinese Remainder Theorem gives $R/I \cong \prod_i R/P_i^{k_i}$. This splitting is used twice below: its surjectivity produces the pair (c, d), and the equality $I = \bigcap_i P_i^{k_i}$ recognizes when $c^3 - d^3$ lies in I.

Since $P_1, \dots, P_r$ are the minimal primes of $\sqrt{I}$, Theorem 1 applies to $\sqrt{I}$ and says that $\sqrt{I}$ is cdf-primary if and only if the $\mathscr{B}$-condition holds for $\{P_1, \dots, P_r\}$. We show that this condition holds whenever I is cdf-primary.

Suppose for contradiction that the $\mathscr{B}$-condition fails. In its equivalent form (2) of Theorem 1, this means that there exists a prime P_{j_0} of Type B and a prime P_ℓ (with $\ell \neq j_0$) satisfying $3 \notin P_\ell$. We construct a cdf-zero pair of I itself, contradicting I cdf-primary.

We first produce $u, v \in R$ with

$$u^2 + uv + v^2 \equiv 0 \pmod{P_{j_0}^{k_{j_0}}}, \qquad u - v \not\equiv 0 \pmod{P_{j_0}}. \tag{†}$$

Note first that $3 \notin P_{j_0}$, since a prime containing 3 is of Type A (Remark 9(1)). Two cases arise, according to whether P_{j_0} is maximal.

Case 1: P_{j_0} is maximal. Then, R/P_{j_0} is a field, so it is its own fraction field and a root $\omega \neq 1$ of t^2+t+1—a priori guaranteed only in $\mathrm{Frac}(R/P_{j_0})$—already lies in R/P_{j_0}. We lift ω to a root of $g(t) = t^2 + t + 1$ in $A = R/P_{j_0}^{k_{j_0}}$. Write $N = P_{j_0}/P_{j_0}^{k_{j_0}}$, so that $N^{k_{j_0}} = 0$ and $A/N \cong R/P_{j_0}$. The root ω is simple: the derivative of g at ω is $2\omega + 1$, and $(2\omega+1)^2 = 4(\omega^2 + \omega + 1) - 3 = -3$, which is a unit of the field R/P_{j_0} because $3 \notin P_{j_0}$, so $2\omega + 1$ is a unit there. We may therefore lift ω by Newton's method. (This is Hensel's lemma for the simple root ω of g in the local ring A with nilpotent maximal ideal: $g'(\omega) = 2\omega + 1$ is a unit in $A/N \cong R/P_{j_0}$ because $(2\omega+1)^2 = -3$ and $3 \notin P_{j_0}$; hence, $g'(\Omega_0)$ is a unit of A for every lift Ω_0 of ω.) Start from any $\Omega_0 \in A$ reducing to ω and set $\Omega_{m+1} = \Omega_m - g(\Omega_m)/g'(\Omega_m)$. Suppose $\Omega_m \equiv \omega \pmod{N}$. Then, $g'(\Omega_m) \equiv 2\omega + 1 \pmod{N}$, a unit modulo the nilpotent ideal N and hence a unit of A, so the step is defined; and $g(\Omega_m) \equiv g(\omega) = 0 \pmod{N}$, so $\Omega_{m+1} \equiv \Omega_m \equiv \omega \pmod{N}$ and the induction continues. Since g is quadratic, writing $h = g(\Omega_m)/g'(\Omega_m)$ gives the exact identity

$$g(\Omega_{m+1}) = g(\Omega_m - h) = g(\Omega_m) - g'(\Omega_m)h + h^2 = h^2.$$

As $g'(\Omega_m)$ is a unit of A, the element h is a multiple of $g(\Omega_m)$, so h^2 is a multiple of $g(\Omega_m)^2$. Hence, $g(\Omega_m) \in N^{2^m}$ implies $g(\Omega_{m+1}) \in N^{2^{m+1}}$; since $g(\Omega_0) \in N$, induction gives $g(\Omega_m) \in N^{2^m}$ for every m. This vanishes once $2^m \geq k_{j_0}$; taking such an m and writing $\Omega = \Omega_m$, we obtain $\Omega \in A$ with $\Omega^2 + \Omega + 1 = 0$ in A and $\Omega \equiv \omega \not\equiv 1 \pmod{N}$. The iteration stops after finitely many steps, namely as soon as $2^m \geq k_{j_0}$, because $N^{k_{j_0}} = 0$; and the lifted root is not congruent to 1, because $\Omega \equiv \omega \pmod{N}$ while $\omega \neq 1$ in the field R/P_{j_0}. Let $u \in R$ be any preimage of Ω and put $v = 1$. Then, $u^2 + uv + v^2 = u^2 + u + 1 \in P_{j_0}^{k_{j_0}}$, while $u - v \notin P_{j_0}$ because $\Omega \not\equiv 1 \pmod{N}$; so (†) holds.

Case 2: P_{j_0} is not maximal. By the structure theorem of [2] recalled in Section 1, write $R = R_1 \times \ldots \times R_s$ with each R_σ a Dedekind domain or a special principal ideal ring. Every prime of R has the form $Q \times \prod_{\sigma \neq \sigma_0} R_\sigma$ for some index σ_0 with Q a prime of R_{σ_0}. Such a prime is maximal unless $Q = (0)$ with R_{σ_0} a Dedekind domain that is not a field, since the nonzero primes of a Dedekind domain and the unique prime of a special principal ideal ring are maximal. Write P_{j_0} in this form; as it is not maximal, $Q = (0)$, so $P_{j_0} = (0) \times \prod_{\sigma \neq \sigma_0} R_\sigma$; products of ideals in a finite product of rings are formed componentwise, and $(0)^k = (0)$, so $P_{j_0}^k = P_{j_0}$ for every $k \geq 1$, and no lifting through a nilpotent ideal is required here. Definition 4 supplies $\bar{u}, \bar{v} \in R/P_{j_0}$ with $\bar{u}^2 + \bar{u}\bar{v} + \bar{v}^2 = 0$, $\bar{v} \neq 0$ and $\bar{u} \neq \bar{v}$; lifting them to $u, v \in R$ gives (†), since $P_{j_0}^{k_{j_0}} = P_{j_0}$ makes the first congruence a statement modulo P_{j_0}.

In either case, we now have $u, v \in R$ satisfying (†). Choose $c, d \in R$ such that

$$c \equiv u,\ d \equiv v \pmod{P_{j_0}^{k_{j_0}}}; \qquad c \equiv d \equiv 1 \pmod{P_\ell^{k_\ell}};$$

$$c \equiv d \equiv 0 \pmod{P_i^{k_i}} \text{ for all } i \neq j_0, \ell.$$

Each of c and d is prescribed by one residue modulo each of the ideals $P_1^{k_1}, \ldots, P_r^{k_r}$, so such a pair exists by the surjectivity of $R \to \prod_i R/P_i^{k_i}$ recorded at the start of the proof. We check that (c, d) is a cdf-zero pair of I.

$c^3 - d^3 \in I$. Indeed, for modulo $P_{j_0}^{k_{j_0}}$, we have $c^3 - d^3 \equiv (u - v)(u^2 + uv + v^2) \equiv 0$ by (†), and modulo $P_i^{k_i}$ for $i \neq j_0$, we have $c \equiv d$; so $c^3 - d^3$ lies in every $P_i^{k_i}$, hence in $I = \bigcap_i P_i^{k_i}$.

$c - d \notin I$. Indeed, modulo P_{j_0} we have $c - d \equiv u - v \not\equiv 0$ by (†), and $I \subseteq P_{j_0}$.

$c^2 + cd + d^2 \notin \sqrt{I}$. Indeed, modulo P_ℓ we have $c \equiv d \equiv 1$, so $c^2 + cd + d^2 \equiv 3$; since $3 \notin P_\ell$, this gives $c^2 + cd + d^2 \notin P_\ell$, and $\sqrt{I} \subseteq P_\ell$.

This contradicts the cdf-primarity of I. Hence, the $\mathscr{B}$-condition holds, and $\sqrt{I}$ is cdf-primary by Theorem 1. □

The converse fails: a cdf-primary radical does not force I itself to be cdf-primary, as Remark 24 shows.

Remark 13. *Theorem 2 shows that the ZPI hypothesis of Proposition 9 can be dropped altogether: the cdf-primarity of I passes to $\sqrt{I}$ in every ring, in particular whenever R/I is Artinian, or more generally a finite product of local rings with nil maximal ideals (for instance a semiperfect ring of dimension 0, such as the non-Artinian ring $F[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ over a field F). The proof of Proposition 9 rests on two mechanisms: at an idempotent prime no lifting is needed, and at a maximal prime, a simple root of $t^2 + t + 1$ lifts through a nilpotent ideal. In the proof of Theorem 2, the same lifting is carried out in the two local factors of a localization at two minimal primes, where nilpotency can be replaced by nilness, and this is what removes the global splitting of R/I. The proof above is kept because it exhibits an explicit cdf-zero pair of I in R itself.*

Remark 14. *In a ZPI-ring, the radical of a cdf-primary ideal I is in fact cdf-absorbing, since $\sqrt{I}$ is cdf-primary by Proposition 9 and radical, so that Lemma 2 applies. This is a cubic analogue of Theorem 1 [10], which states that an sdf-primary ideal has an sdf-absorbing radical. By Theorem 2, we obtain the conclusion in every ring: the radical of a cdf-primary ideal is cdf-absorbing, which is the full cubic analogue of Theorem 1 [10].*

We now describe the cdf-primary ideals of a ZPI-ring in which 3 is a unit; the description is complete.

Proposition 10. *Let R be a ZPI-ring with $3 \in U(R)$ and $I = P_1^{k_1} \ldots P_r^{k_r}$ a proper ideal, the P_i being its distinct minimal primes.*

(1) *If I is cdf-primary, then either $r = 1$, in which case $I = P_1^{k_1}$ is primary (case (a)), or $r \geq 2$ and the set $\mathscr{B} = \{i : P_i \text{ is of Type B}\}$ is empty, so that every P_i is of Type A (case (b)).*
(2) *The converse of (1) holds in part: if $\mathscr{B} = \emptyset$ and I is radical, then I is cdf-primary; and in case (a), the ideal I is primary and hence cdf-primary by Proposition 2.*
(3) *If $r \geq 2$, then I is cdf-primary if and only if I is radical and $\mathscr{B} = \emptyset$.*

In particular, every cdf-primary ideal in a ZPI-ring with $3 \in U(R)$ is either primary or radical with at least two minimal primes, which are all of Type A.

Proof. Since $3 \in U(R)$, no prime ideal of R contains 3; in particular, $3 \notin P_i$ for every i.

Part (1): If $r = 1$, then $I = P_1^{k_1}$ is a prime power, and such an ideal is primary in a ZPI-ring. Indeed, passing to the component in which P_1 lives, we find that $R/P_1^{k_1}$ is either a domain (when P_1 is the zero prime of a Dedekind component) or a local ring whose maximal ideal is nilpotent, so that every zero-divisor is nilpotent; in both cases, the zero ideal is primary. So, I is primary, which is case (a), whatever the type of P_1.

If $r \geq 2$, we show $\mathscr{B} = \emptyset$. Since I is cdf-primary, $\sqrt{I}$ is cdf-primary by Proposition 9. The radical $\sqrt{I} = P_1 \cap \ldots \cap P_r$ is the intersection of the minimal primes of I, and in a ZPI-ring, these are pairwise comaximal (Section 1), so Theorem 1 applies and gives the $\mathscr{B}$-condition for $\{P_1, \ldots, P_r\}$ in its equivalent form (2): for every Type B prime P_{j_0}, every other P_i $(i \neq j_0)$ must satisfy $3 \in P_i$. But $3 \notin P_i$ for every i, as recorded at the outset; since $r \geq 2$, some P_i with $i \neq j_0$ always exists for any candidate P_{j_0}, and it cannot satisfy $3 \in P_i$. So, no P_{j_0} can be of Type B, i.e., $\mathscr{B} = \emptyset$. Case (b) holds.

Part (2): If $\mathscr{B} = \emptyset$ and I is radical, then $I = \sqrt{I} = P_1 \cap \ldots \cap P_r$ is the intersection of its minimal primes, and the P_i are distinct and pairwise comaximal (Section 1), so Theorem 1 gives I cdf-primary directly. The assertion about case (a) is the first argument of Part (1): for $r = 1$, the ideal $I = P_1^{k_1}$ is primary; hence, it is cdf-primary by Proposition 2.

Part (3): Let $r \geq 2$. If I is radical and $\mathscr{B} = \emptyset$, then I is cdf-primary by Part (2). Conversely, let I be cdf-primary; then, $\mathscr{B} = \emptyset$ by Part (1), and it remains to show that I is radical. Suppose not. Then, $\mathrm{Nil}(R/I) = \sqrt{I}/I \neq 0$. The ideals $P_i^{k_i}$ are pairwise comaximal, so $R/I \cong \prod_i R/P_i^{k_i}$ by the Chinese Remainder Theorem, and a nonzero nilpotent element of the product has a nonzero nilpotent component in some factor $R/P_i^{k_i}$. Taking a suitable power of that component, we obtain an index i and an element $x \in R$ with $x \notin P_i^{k_i}$ and $x^2 \in P_i^{k_i}$ (let $m \geq 2$ be least with the m-th power of the component zero, and lift its $(m-1)$-st power). Since $r \geq 2$, there is an index $j \neq i$. By the Chinese Remainder Theorem, choose $a, b \in R$ with $a \equiv x$ and $b \equiv 0 \pmod{P_i^{k_i}}$, and $a \equiv b \equiv 1 \pmod{P_l^{k_l}}$ for all $l \neq i$. Then, $a^3 - b^3 \in I$, because $x^3 = x \cdot x^2 \in P_i^{k_i}$ and $1 - 1 = 0$ in the other components; $a - b \equiv x \not\equiv 0 \pmod{P_i^{k_i}}$, so $a - b \notin I$; and $a^2 + ab + b^2 \equiv 3 \pmod{P_j^{k_j}}$, so $a^2 + ab + b^2 \in \sqrt{I} \subseteq P_j$ would force $3 \in P_j$, which is contrary to $3 \in U(R)$. Thus, (a, b) is a cdf-zero pair of I, which is a contradiction. Hence, I is radical. □

Remark 15. *For sdf-primary ideals, $2 \in U(R)$ forces primarity (Proposition 2 [10]). The cubic analogue fails, and Proposition 10 shows exactly how. In a ZPI-ring, a proper ideal is primary precisely when $r = 1$: one direction is the argument for case (a) in the proof above, which uses neither $3 \in U(R)$ nor cdf-primarity, and the other holds because a primary ideal has prime radical [1] and hence a single minimal prime. Cases (a) and (b) are therefore, respectively, the primary and the non-primary cdf-primary ideals. Case (b) does occur. In $R = \mathbb{Z}[1/3]$, where $3 \in U(R)$, the ideal $I = 10\mathbb{Z}[1/3]$ has minimal primes $2\mathbb{Z}[1/3]$ and $5\mathbb{Z}[1/3]$, which are both of Type A since $2 \equiv 5 \equiv 2 \pmod 3$; it is radical with $\mathscr{B} = \emptyset$, so Theorem 1 makes it cdf-primary, while $r = 2$ keeps it from being primary.*

For a ZPI-ring R with $3 \in U(R)$, case (b) is vacuous exactly when no two distinct comaximal primes of R are both of Type A: the minimal primes of a proper ideal are pairwise comaximal, and

conversely two distinct comaximal primes P and Q are the minimal primes of the proper radical ideal $PQ = P \cap Q$, which Proposition 10(2) makes cdf-primary when both are of Type A.

The radical hypothesis in Proposition 10(2) cannot be dropped (this is Proposition 10(3)). In $R = \mathbb{Z}[1/3]$, the ideal $I = 20R$ has $r = 2$ and $\mathscr{B} = \emptyset$, yet $(a, b) = (12, 2)$ is a cdf-zero pair of I. Indeed, $a^3 - b^3 = 1720 \in I$, while $a - b = 10 \notin 20R$ and $a^2 + ab + b^2 = 172 \notin 10R = \sqrt{I}$ (compare Example 13).

Three configurations of 3 are worth treating individually—$3 = 0$ in R, $3 \in U(R)$, and (3) maximal—although, as Remark 19 notes below, they do not exhaust the possibilities. The first is settled by Proposition 3, since then every proper ideal is cdf-primary. In the second, Proposition 10 describes the cdf-primary ideals completely. In the third, (3) is itself of Type A (Remark 9(1)), and its maximality forces every prime containing 3 to equal (3); this turns the companion clause of Theorem 1(3) into a bound on r and on the shape of I. This gives the following.

Theorem 3. *Let R be a ZPI-ring in which (3) is a maximal ideal, and let $I = P_1^{k_1} \dots P_r^{k_r}$ be a proper ideal, the P_i being its distinct minimal primes.*

(i) If I is cdf-primary, then $\mathscr{B} = \{i : P_i \text{ is of Type B}\}$ satisfies $|\mathscr{B}| \leq 1$, and if $\mathscr{B} = \{j_0\}$, then $3 \in P_i$, hence $P_i = (3)$ by the maximality of (3) for all $i \neq j_0$. In that case, $r \leq 2$, and $I = P_{j_0}^{k_{j_0}}$ or $I = P_{j_0}^{k_{j_0}}(3)^{k_{i_0}}$, where $i_0 \neq j_0$ is the remaining index and $k_{i_0} \geq 1$.

(ii) If I is radical, the $\mathscr{B}$-condition is sufficient as well: I is cdf-primary if and only if the $\mathscr{B}$-condition of Theorem 1(3) holds for $\{P_1, \dots, P_r\}$.

Proof. In a ZPI-ring, the minimal primes of a proper ideal are pairwise comaximal (Section 1), so the minimal primes $P_1, \dots, P_r$ of $I = P_1^{k_1} \dots P_r^{k_r}$ are comaximal. Since $\sqrt{I}$ is the intersection of the minimal primes of I, we have $\sqrt{I} = P_1 \cap \dots \cap P_r$. By Theorem 1, $\sqrt{I}$ is cdf-primary if and only if the $\mathscr{B}$-condition holds.

Part (i): By Proposition 9, $\sqrt{I}$ is cdf-primary; by the equivalence recorded above, the $\mathscr{B}$-condition for $\{P_1, \dots, P_r\}$ therefore holds.

For the remaining assertions of (i): if $\mathscr{B} = \{j_0\}$, the $\mathscr{B}$-condition gives $3 \in P_i$ for every $i \neq j_0$, and (3) being maximal, this forces $P_i = (3)$. As the P_i values are distinct, at most one index differs from j_0, so $r \leq 2$: if $r = 1$, then $I = P_{j_0}^{k_{j_0}}$, and if $r = 2$, say with remaining index i_0, then $I = P_{j_0}^{k_{j_0}}(3)^{k_{i_0}}$ with $k_{i_0} \geq 1$; here, $P_{j_0}^{k_{j_0}}$ and $(3)^{k_{i_0}}$ are comaximal, being powers of the comaximal minimal primes P_{j_0} and (3) of I (Section 1), so their product equals their intersection.

Part (ii): If I is radical, then $I = \sqrt{I} = P_1 \cap \dots \cap P_r$, and Theorem 1 applies to I directly, its minimal primes being comaximal; so I is cdf-primary if and only if the $\mathscr{B}$-condition holds. □

Remark 16. *Part (ii) of Theorem 3 is restricted to radical ideals for a reason: for a non-radical ideal, the $\mathscr{B}$-condition may hold without implying cdf-primarity. The $\mathscr{B}$-condition sees only $\sqrt{I}$, that is, divisibility to the first power, whereas I prescribes the exponents k_i. Consider $I = 63\mathbb{Z} = 3^2 \cdot 7\mathbb{Z}$, whose minimal primes (3) and (7) satisfy the $\mathscr{B}$-condition, and write $d = a - b$ and $s = a^2 + ab + b^2$, so that $a^3 - b^3 = ds$. At the prime 3, the two factors can share the exponent. Indeed, $s = d^2 + 3ab$, so $3 \mid d$ implies $3 \mid s$ (Corollary 1(b)), and if moreover $3 \nmid ab$, then s is divisible by 3 but not by 9 (Remark 2). Hence, $9 \mid ds$ can hold with $v_3(d) = v_3(s) = 1$, that is, with neither $9 \mid d$ nor $9 \mid s$. At a prime $p \neq 3$ with $p \nmid b$, no such sharing is possible, since $p \mid d$ gives $s \equiv 3b^2 \not\equiv 0 \pmod{p}$; so 7 divides at most one of d and s. Now, the first alternative for I requires $9 \mid d$ and $7 \mid d$, and the second requires $3 \mid s$ and $7 \mid s$. For $(a, b) = (22, 1)$, we have $d = 21 = 3 \cdot 7$ and $s = 507 = 3 \cdot 13^2$,*

so $ds = 63 \cdot 169 \in 63\mathbb{Z}$, while $9 \nmid d$ and $7 \nmid s$: the factor 9 is split between d and s, the factor 7 lies entirely in d, and neither alternative holds. For a squarefree ideal, this cannot happen, because each prime is needed only to the first power and $3 \mid d$ if and only if $3 \mid s$. This is why the $\mathscr{B}$-condition suffices for radical ideals and why the exponent of 3 in a cdf-primary ideal $n\mathbb{Z}$ that is not primary is at most 1 (Theorem 6). The same example appears in Remark 18, and Remark 24 describes the obstruction at a general prime.

Remark 17. *The bound $r \leq 2$ in Theorem 3(i) rests on both features of that case. It needs $\mathscr{B} \neq \emptyset$: when $\mathscr{B} = \emptyset$, there is no bound on r at all. In $\mathbb{Z}$, for instance, $p_1 \dots p_t\mathbb{Z}$ is cdf-primary for any distinct primes $p_1, \dots, p_t$ with $p_i \not\equiv 1 \pmod 3$, by Theorem 5(i), so r may be taken arbitrarily large; $30\mathbb{Z} = 2 \cdot 3 \cdot 5\mathbb{Z}$, whose minimal primes include (3), already has $r = 3$. The bound also needs the maximality of (3): without it, several primes may contain 3 (Remark 19), and the companion clause then bounds r only by 1 plus the number of primes containing 3 (Proposition 11) — which gives no bound at all when infinitely many primes contain 3. In the ZPI-ring $R = \mathbb{Z} \times \mathbb{F}_3[x]$, the element $3 = (3, 0)$ lies in every prime $\mathbb{Z} \times (q)$, q irreducible; for distinct monic irreducibles $q_1, \dots, q_t$, the radical ideal $I = 7\mathbb{Z} \times (q_1 \dots q_t)$ has the $t + 1$ minimal primes $7\mathbb{Z} \times \mathbb{F}_3[x]$, of Type B, and $\mathbb{Z} \times (q_j)$, which contain 3; so I is cdf-primary by Theorem 1 with $r = t + 1$ arbitrary.*

Remark 18. *For non-radical I, the $\mathscr{B}$-condition remains necessary, by Theorem 3(i), but it is no longer sufficient. In $\mathbb{Z}$, the ideal $63\mathbb{Z} = 3^2 \cdot 7\mathbb{Z}$ has minimal primes (7), of Type B, and (3), which contains 3 and is therefore of Type A (Remark 9(1)), so the condition holds; yet $(22, 1)$ is a cdf-zero pair, since $22^3 - 1 = 10647 = 63 \cdot 169$ while $22 - 1 = 21 \notin 63\mathbb{Z}$ and $22^2 + 22 + 1 = 507 \notin 21\mathbb{Z} = \sqrt{63\mathbb{Z}}$. Remark 24 explains where the obstruction comes from.*

Remark 19. *The three configurations listed before Theorem 3 do not exhaust all ZPI-rings: the ideal (3) may be a nonzero proper ideal that is not maximal. For example, let $R = \mathbb{Z}[\sqrt{-5}]$, which is a Dedekind domain and hence a ZPI-ring. Here, $t^2 + 5 \equiv (t - 1)(t + 1) \pmod 3$, so Dedekind's factorization criterion, applicable because $\mathbb{Z}[\sqrt{-5}]$ is the full ring of integers of $\mathbb{Q}(\sqrt{-5})$, shows that 3 splits as $(3) = QQ'$ with $Q = (3, 1 + \sqrt{-5})$ and $Q' = (3, 1 - \sqrt{-5})$ distinct. None of the three configurations applies to such R: $\operatorname{char}(R) = 0$, 3 is not a unit, and (3) is not maximal.*

Proposition 11. *Let R be a ZPI-ring and I a proper ideal with distinct minimal primes $P_1, \dots, P_r$, and let $V(3)$ denote the set of primes of R that contain 3.*

(1) If I is radical, then I is cdf-primary if and only if the $\mathscr{B}$-condition holds for $\{P_1, \dots, P_r\}$: at most, one P_i is of Type B, and if P_{j_0} is, then $P_i \in V(3)$ for all $i \neq j_0$.

(2) If I is cdf-primary, the same $\mathscr{B}$-condition holds. In particular, if some P_{j_0} is of Type B, then all other P_i lie in $V(3)$, and so $r \leq 1 + |V(3)|$ whenever $V(3)$ is finite.

Proof. In a ZPI-ring, the minimal primes of I are pairwise comaximal, and $\sqrt{I} = P_1 \cap \dots \cap P_r$. Part (1) is Theorem 1 applied to $I = \sqrt{I}$. For (2), $\sqrt{I}$ is cdf-primary by Proposition 9, with the same minimal primes, so (1) applied to $\sqrt{I}$ gives the $\mathscr{B}$-condition; the bound holds because the P_i with $i \neq j_0$ are distinct elements of $V(3)$. □

Every prime containing 3 is of Type A (Remark 9(1)), so for a ZPI-ring R such as $\mathbb{Z}[\sqrt{-5}]$ in the previous remark, the primes Q and Q' above 3 play the protective role that (3) plays in Theorem 3, and the bound of Proposition 11(2) is attained there: for a prime P of $\mathbb{Z}[\sqrt{-5}]$ above 7, which is of Type B, the radical ideal $PQQ' = P \cap Q \cap Q'$ satisfies the $\mathscr{B}$-condition and has $r = 3 = 1 + |V(3)|$. For non-radical I, the $\mathscr{B}$-condition is only necessary, as in Remark 18. We do not attempt here a version of Theorem 3 covering an arbitrary factorization of (3).

The next result is a partial cubic analogue of the characterization of Dedekind domains by sdf-primary ideals in Theorem 4 [10]. It follows directly from Theorem 3(ii); we state it for products of distinct maximal ideals so that it mirrors the quadratic result. We prove it for a Dedekind domain in which 3 is a prime element; the converse fails (Remark 22).

Theorem 4. *Let R be a Dedekind domain in which 3 is a prime element, and let $I = P_1 \dots P_r$ be a proper ideal of R that is a product of finitely many distinct maximal ideals. Then, I is cdf-primary if and only if either no P_i is of Type B, or exactly one is and every other P_i, if there is one, equals (3).*

Proof. Since I is proper, we have $r \geq 1$, the empty product being R. The P_i, being distinct maximal ideals, are pairwise comaximal, so $I = P_1 \dots P_r = P_1 \cap \dots \cap P_r$, and I is radical, being an intersection of primes. By Remark 10, the P_i are precisely the distinct minimal primes of I, and the factorization $I = P_1 \dots P_r$, all of whose exponents are 1, is of the form required by Theorem 3. A Dedekind domain is a ZPI-ring, and since 3 is a prime element, the ideal (3) is a nonzero prime, hence maximal, the nonzero primes of a Dedekind domain being maximal; so R satisfies the hypothesis of Theorem 3. As I is radical, part (ii) of that theorem applies to I: it is cdf-primary if and only if the set $\mathscr{B} = \{i : P_i \text{ is of Type B}\}$ satisfies $|\mathscr{B}| \leq 1$ and, when $\mathscr{B} = \{j_0\}$, $3 \in P_i$ for every $i \neq j_0$.

Suppose I is cdf-primary. Then, the $\mathscr{B}$-condition holds, so $|\mathscr{B}| \leq 1$ and only two cases arise. If $\mathscr{B} = \emptyset$, no P_i is of Type B. If $\mathscr{B} = \{j_0\}$, then $3 \in P_i$ for every $i \neq j_0$, so $(3) \subseteq P_i$ and, (3) being maximal and P_i proper, $P_i = (3)$; the P_i being distinct, exactly one of them is then of Type B and every other equals (3). In either case, $P_1, \dots, P_r$ satisfy the condition of the theorem.

Conversely, suppose they satisfy it. If no P_i is of Type B, then $\mathscr{B} = \emptyset$. If exactly one is, say P_{j_0}, and every other equals (3), then $\mathscr{B} = \{j_0\}$, again because the P_i values are distinct, and $3 \in P_i$ for every $i \neq j_0$. In either case, the $\mathscr{B}$-condition holds, so I is cdf-primary. □

Remark 20. *In Theorem 4, the hypothesis on 3 cannot be dropped. In the Dedekind domain $R = \mathbb{Z}[\sqrt{-5}]$, the ideal (3) factors as QQ' with $Q = (3, 1+\sqrt{-5})$ and is therefore not prime (Remark 19); here, $R/Q \cong \mathbb{F}_3$. Since $-5 \equiv 3^2 \pmod 7$, the prime 7 splits as well; let P be a prime above it, so that $R/P \cong \mathbb{F}_7$; then, P is of Type B, since $7 \equiv 1 \pmod 3$ (Remark 9(2)). Now, P and Q are distinct maximal ideals, hence comaximal, and $R/PQ \cong \mathbb{F}_7 \times \mathbb{F}_3 \cong \mathbb{Z}_{21}$, so PQ is cdf-primary (Example 5 with Lemma 5(2)). Yet the prime remaining beside P is Q, which contains 3 without equaling (3); so the equivalence of Theorem 4 fails in a Dedekind domain in which 3 is not a prime element.*

Remark 21. *In Theorem 4, Type B cannot be replaced by the condition $\operatorname{char}(R/P_i) \equiv 1 \pmod 3$; the two agree whenever R/P_i is the prime field $\mathbb{F}_p$ (Remark 9(2)), but it may differ otherwise. In $R = \mathbb{Z}[i]$, where 3 is a prime element, the prime $P = (11)$ has $\operatorname{char}(R/P) = 11 \equiv 2 \pmod 3$, yet P is of Type B, since $R/P \cong \mathbb{F}_{121}$ and $3 \mid 121 - 1$ (Remark 9(3)). The maximal ideals (11) and $(1+i)$ are comaximal, so $I = (11)(1+i) = (11) \cap (1+i)$; since (11) is of Type B while $3 \notin (1+i)$, Corollary 2(4) shows that I is not cdf-primary, even though no residue characteristic is congruent to 1 modulo 3.*

Remark 22. *Say that a ring satisfies the criterion of Theorem 4 if the equivalence stated there holds in it for every proper ideal that is a product of finitely many distinct maximal ideals; by Remark 10, those ideals are then the minimal primes of the product, so the condition depends on the ideal alone. Theorem 4 says that a Dedekind domain in which 3 is a prime element satisfies the criterion. The converse of that statement is false, the reason being that the criterion constrains only ideals that are products of distinct maximal ideals—all of them radical—and a ring may in addition have very few of these.*

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Theorem 1 already decides cdf-primarity for every product of distinct maximal ideals in every ring. When (3) *is maximal, that theorem yields exactly the criterion; so the criterion can fail only when* (3) *is not maximal. This is what happens in Remark 20, where* (3) *is not even prime. It happens again in* $\mathbb{Z}[x]$*, where* 3 *is prime but* $(3) \subsetneq (3,x)$*. The ideal* $(7,x)(3,x) = (21,x)$ *has quotient* $\mathbb{Z}_{21}$ *and is cdf-primary for the same reason as in that remark, while* $(7,x)$ *is of Type B and the remaining prime* $(3,x)$ *is not* (3)*. The criterion is in any case a weak requirement. The next two examples show that it can hold for rings far from Dedekind and that it can do so in two opposite ways.*

Take first $R = \mathbb{Z}_{(3)}[[x]]$*, which is a Noetherian local domain that is not a field. Here,* 3 *is a prime element, since* $R/(3) \cong \mathbb{F}_3[[x]]$ *is a domain, and* $(3,x)$ *is the unique maximal ideal, so the only product of distinct maximal ideals is* $(3,x)$ *itself and the criterion holds with* $r = 1$*. Yet* $\dim R = 2$*, so* R *is not Dedekind.*

The ring $R = \mathbb{Z}[2i] \cong \mathbb{Z}[t]/(t^2+4)$ *misses being Dedekind for the opposite reason. It is a Noetherian domain of dimension* 1*, being a quotient of* $\mathbb{Z}[t]$ *that is finite over* $\mathbb{Z}$*, and it has infinitely many maximal ideals. Moreover,* $R/3R \cong \mathbb{F}_9$ *is a field, so* 3 *is a prime element and* (3) *is maximal, whence the criterion holds automatically. Yet* i *is integral over* R *without lying in* R*, so* R *is not integrally closed and hence not Dedekind.*

Neither $\dim R = 1$ *nor an infinite supply of maximal ideals will therefore rescue the converse. A characterization along these lines must say something about the non-radical ideals* $P_1^{k_1} \dots P_r^{k_r}$ *as well, and finding the right condition is the open question.*

For squarefree n, the ideal $n\mathbb{Z}$ is radical with pairwise comaximal minimal primes, so Theorem 1 applies directly and the $\mathscr{B}$-condition becomes an explicit condition on the prime factorization of n. This settles the squarefree case; Theorem 6 below extends the classification to every n, and its proof rests on the squarefree case treated here.

Theorem 5. *Let* $n \geq 2$ *be a squarefree integer. Then,* $n\mathbb{Z}$ *is cdf-primary if and only if one of the following holds:*

(i) *No prime factor of* n *is congruent to* 1 *modulo* 3*; or*
(ii) $n = q$ *or* $n = 3q$*, where* $q \equiv 1 \pmod 3$ *is prime.*

The two cases are mutually exclusive, since a prime factor $q \equiv 1 \pmod 3$ *rules out (i).*

Proof. Since $n \geq 2$, write $n = p_1 \dots p_r$ with $r \geq 1$ and the p_i distinct primes. Then, $n\mathbb{Z} = p_1\mathbb{Z} \cap \dots \cap p_r\mathbb{Z}$ is a radical ideal whose minimal primes $p_i\mathbb{Z}$ are pairwise comaximal, so Theorem 1 applies: $n\mathbb{Z}$ is cdf-primary if and only if the $\mathscr{B}$-condition holds for $\{p_1\mathbb{Z}, \dots, p_r\mathbb{Z}\}$.

Each residue domain $\mathbb{Z}/p_i\mathbb{Z} = \mathbb{F}_{p_i}$ is a prime field, so $p_i\mathbb{Z}$ is of Type B exactly when $p_i \equiv 1 \pmod 3$ (Remark 9(2)), and it is $3 \in p_i\mathbb{Z}$ exactly when $p_i = 3$. The $\mathscr{B}$-condition therefore reads: at most, one prime factor of n is congruent to 1 modulo 3, and if exactly one is, say $p_{j_0} = q$, then every other prime factor equals 3.

If $\mathscr{B} = \emptyset$, this says that no prime factor is congruent to 1 modulo 3, which is (i). If $\mathscr{B} = \{j_0\}$, then $q \equiv 1 \pmod 3$ and every other prime factor is 3; the p_i being distinct, there is at most one such, so $n = q$ or $n = 3q$, which is (ii). Conversely, if (i) holds, then $\mathscr{B} = \emptyset$. If (ii) holds, then $q \equiv 1 \pmod 3$ puts its index in $\mathscr{B}$, while $3 \not\equiv 1 \pmod 3$ keeps the only other possible factor out of it, so $\mathscr{B} = \{j_0\}$ with $p_{j_0} = q$; and $3 \in 3\mathbb{Z}$ supplies the companion clause. In either case, the $\mathscr{B}$-condition holds. □

The proof of Theorem 1 constructs a cdf-zero pair whenever the $\mathscr{B}$-condition fails (directly by the Chinese Remainder Theorem when the two relevant minimal primes are comaximal, Remark 10); over $\mathbb{Z}$, it can be made explicit in the following normalized form with $b = 1$, which produces the witnesses used elsewhere in this paper.

Remark 23. *Suppose $n \geq 2$ is squarefree and satisfies neither (i) nor (ii). The failure of (i) gives a prime $q \mid n$ with $q \equiv 1 \pmod 3$; the failure of (ii) then means $n/q \notin \{1, 3\}$. Since n is squarefree, so is n/q; were 3 its only prime factor, we would have $n/q \in \{1, 3\}$, so n has a prime divisor $p \neq 3$ dividing n/q, and $p \neq q$ since $q^2 \nmid n$. Put $b = 1$ and let $\omega \in \{2, \ldots, q-1\}$ represent a primitive cube root of unity modulo q, which exists because $3 \mid q - 1$ and $\mathbb{F}_q^\times$ is cyclic (see the proof of Lemma 1(c)). By the Chinese Remainder Theorem, choose $a \in \mathbb{Z}$ with*

$$a \equiv \omega \pmod q, \qquad a \equiv 1 \pmod{p'} \text{ for every prime } p' \mid n \text{ with } p' \neq q.$$

Then, $q \mid a^2 + ab + b^2$, since $\omega^2 + \omega + 1 \equiv 0 \pmod q$, while $q \nmid a - b$ because $a - b \equiv \omega - 1 \not\equiv 0 \pmod q$. At p, we have $a \equiv b \pmod p$, so $a^2 + ab + b^2 \equiv 3 \pmod p$, which is nonzero because $p \neq 3$; and at every remaining prime $p' \mid n$, we have $a \equiv b$, hence $p' \mid a^3 - b^3$. Every prime factor of n therefore divides $a^3 - b^3$, and n is squarefree, so $n \mid a^3 - b^3$. On the other hand, $q \nmid a - b$ gives $n \nmid a - b$, and $p \nmid a^2 + ab + b^2$ gives $n \nmid a^2 + ab + b^2$. Since $\sqrt{n\mathbb{Z}} = n\mathbb{Z}$, the pair (a, b) is a cdf-zero pair of $n\mathbb{Z}$.

For $n = 2 \cdot 7$, this is the pair $(9, 1)$ of Example 4; for $n = 5 \cdot 7$, this is the pair $(16, 1)$ of Example 6, read in $\mathbb{Z}$ as in Remark 8; and for $n = 2 \cdot 5 \cdot 7$, the pair is $(51, 1)$.

The criterion is easy to apply; the smallest instances of its two cases—the second split according as $n = q$ or $n = 3q$—and of failure, are the following.

Example 12. *Among the ideals $n\mathbb{Z}$ with $n \geq 2$ squarefree:*

Cdf-primary, case (i) (all prime factors $\not\equiv 1 \pmod 3$): $n = 2, 3, 5, 6, 10, 11, 15, 17, 22, 23, 29, 30, 33, 34, \ldots$

Cdf-primary, case (ii) with $n = q$ ($q \equiv 1 \pmod 3$ prime): $n = 7, 13, 19, 31, 37, 43, \ldots$

Cdf-primary, case (ii) with $n = 3q$: $n = 21, 39, 57, 93, 111, 129, \ldots$

Not cdf-primary (a prime $\equiv 1 \pmod 3$ alongside another prime $\neq 3$): $n = 14, 26, 35, 38, 42, 62, 65, 70, 74, 77, 78, 86, \ldots$

In each case, the verdict is read off from the prime factorization of n by Theorem 5. Note that cdf-primarity is not preserved on passing from n to a multiple: $7\mathbb{Z}$ and $21\mathbb{Z}$ are cdf-primary while $14\mathbb{Z}$ and $42\mathbb{Z}$ are not, the prime 7 being protected by 3 but not by 2.

Remark 24. *The converse of Proposition 9 fails: $\sqrt{I}$ cdf-primary does not imply that I is cdf-primary—only that I is cdf-semiprimary (Lemma 3). The witness used earlier in the paper is $I = 63\mathbb{Z} = 3^2 \cdot 7\mathbb{Z}$ (Remark 18), whose radical $21\mathbb{Z}$ is cdf-primary by Theorem 5(ii). The smallest witness is $18\mathbb{Z} = 2 \cdot 3^2\mathbb{Z}$. Its radical $6\mathbb{Z}$ is cdf-primary by Theorem 5(i), while $18\mathbb{Z}$ is not cdf-primary by Theorem 6, and Case 2 of that theorem's proof supplies the witness $(a, b) = (7, 1)$. No smaller one exists. A witness must be non-squarefree, since otherwise $n\mathbb{Z} = \sqrt{n\mathbb{Z}}$; and the non-squarefree integers below 18, namely $4, 8, 9, 12$ and 16, are all cdf-primary. Remark 7 shows, by means of the pair $(22, 1)$ that $63\mathbb{Z}$ is not even weakly cdf-primary and hence not cdf-primary.*

The obstruction is one of exact divisibility, and it has nothing to do with the type of 7. By Corollary 1(b) and Remark 2, divisibility by 3 links $a - b$ and $a^2 + ab + b^2$ only at the first power, and here $3 \mid a - b$ while $3^2 \nmid a - b$, which is one power short of what $63\mathbb{Z}$ requires. The second alternative fails for a reason that operates at every prime $p \neq 3$ with $p \nmid b$: from $p \mid a - b$, one obtains $a^2 + ab + b^2 \equiv 3b^2 \not\equiv 0 \pmod p$. So, for $(a, b) = (22, 1)$, $7 \mid a - b$ forces $7 \nmid a^2 + ab + b^2$. The analogous pair $(16, 1)$ defeats $45\mathbb{Z} = 3^2 \cdot 5\mathbb{Z}$ for exactly the same reason (Remark 12), and there every prime is of Type A.

Before extending the criterion to all n, we record how the cubic condition compares with the quadratic one.

Remark 25. *The cubic and the quadratic conditions are incomparable. By Example 5 [10], for $n \geq 2$, the ideal $n\mathbb{Z}$ is sdf-primary in $\mathbb{Z}$ precisely when $n = p^k$ or $n = 2q^m$ for an odd prime q. Thus, $14\mathbb{Z}$ is sdf-primary but not cdf-primary (Example 4), while $21\mathbb{Z}$ is cdf-primary (Example 5) but not sdf-primary: for $a = 8$ and $b = 1$, we have $a^2 - b^2 = 63 \in 21\mathbb{Z}$, yet neither $a - b = 7$ nor $a + b = 9$ lies in $21\mathbb{Z} = \sqrt{21\mathbb{Z}}$. The discrepancy comes from the splitting of $t^2 + t + 1$ modulo a prime $q \equiv 1 \pmod 3$ together with the protective role of 3: by Theorem 1, a prime of Type B tolerates other primes only if they contain 3, whereas in the quadratic theory, that role falls to 2.*

We can now drop the squarefree hypothesis. The criterion below contains Theorem 5: for squarefree n, its form (iii) is condition (i) there, while forms (i) and (ii) restrict to $n = p$ and $n = 3p$ with $p \neq 3$ prime, which is condition (ii) there when $p \equiv 1 \pmod 3$ and condition (i) there otherwise.

Theorem 6. *Let $n \geq 2$ be an integer. Then, $n\mathbb{Z}$ is cdf-primary if and only if one of the following holds:*

(i) $n = p^k$ for a prime p and some $k \geq 1$;
(ii) $n = 3p^k$ for a prime $p \neq 3$ and some $k \geq 1$;
(iii) n is squarefree and no prime factor of n is congruent to 1 modulo 3.

The three forms overlap: $n = 3p$ with $p \neq 3$ a prime and $p \not\equiv 1 \pmod 3$ satisfies both (ii) and (iii), and a prime $n \not\equiv 1 \pmod 3$ satisfies both (i) and (iii).

Proof. Throughout, a and b denote arbitrary integers, and we write $d = a - b$ and $s = a^2 + ab + b^2$, so that $a^3 - b^3 = ds$; the ideal $n\mathbb{Z}$ is proper because $n \geq 2$. Recall that $\sqrt{n\mathbb{Z}} = \mathrm{rad}(n)\mathbb{Z}$, where $\mathrm{rad}(n)$ denotes the product of the distinct prime factors of n. We write v_p for the p-adic valuation on $\mathbb{Z}$.

Suppose first that n has one of the three forms. In case (i), the ideal $p^k\mathbb{Z}$ is primary, hence cdf-primary by Proposition 2, and in case (iii), the ideal $n\mathbb{Z}$ is cdf-primary by Theorem 5, whose condition (i) is exactly (iii). For case (ii), let $n = 3p^k$ with $p \neq 3$ and suppose $n \mid ds$. Then, $3 \mid ds$, so $3 \mid d$ or $3 \mid s$; these two are equivalent by Lemma 1(b), so both hold. If $p^k \mid d$, then $n = 3p^k$ divides d, because $3 \mid d$ and $\gcd(3, p^k) = 1$; the first alternative holds. Otherwise, $v_p(d) < k$, so $p^k \mid ds$ forces $p \mid s$, and together with $3 \mid s$, again by $\gcd(3, p) = 1$, this gives $\mathrm{rad}(n) = 3p \mid s$, the radical being $3p$ because $p \neq 3$; the second alternative holds.

Suppose now that n has none of the three forms. Since (i) fails, n has at least two distinct prime factors. We show that $n\mathbb{Z}$ fails to be cdf-primary in each of three cases.

Case 1: some prime $q \mid n$ with $q \equiv 1 \pmod 3$ coexists with a prime $p \mid n$, $p \notin \{3, q\}$. Suppose $n\mathbb{Z}$ were cdf-primary. Now, $\mathbb{Z}$ is a principal ideal domain, hence a ZPI-ring, and (3) is maximal because 3 is prime; writing $n = p_1^{k_1} \dots p_r^{k_r}$, the ideal $n\mathbb{Z}$ is proper and the (p_i) are its distinct minimal primes. So, Theorem 3(i) would give the $\mathscr{B}$-condition for those primes. But (q) is of Type B by Remark 9(2) while $3 \notin (p)$, so the condition fails.

Case 2: $9 \mid n$. The construction rests on Remark 2: divisibility by 3 links $a - b$ and $a^2 + ab + b^2$ only at the first power, so the exponent of 3 in n can be met by the two factors together without being met by either alone (compare Remark 16); the specializations are listed in Remark 26. Put $e = v_3(n) \geq 2$ and $m = n/3^e$; as n is not a power of 3, we have $m > 1$. Take

$$b = 1, \qquad a = 1 + 3^{e-1}m, \qquad \text{so that } d = 3^{e-1}m,\ s = d^2 + 3d + 3.$$

Since $e \geq 2$, we have $3 \mid d$, so 9 divides both d^2 and $3d$, whence $v_3(s) = 1$; equivalently, $s = 3\,(d^2/3 + d + 1)$ with $d^2/3$ and d both divisible by 3, so the bracket is $\equiv 1 \pmod 3$. Let ℓ be any prime dividing m; then, $\ell \neq 3$, by the choice $e = v_3(n)$, and $\ell \mid d$, so $s \equiv 3 \not\equiv 0 \pmod \ell$. Consequently, $v_3(ds) = (e - 1) + 1 = e = v_3(n)$, and $v_\ell(ds) = v_\ell(d) = v_\ell(n)$ for every such ℓ; as the primes dividing n are 3 and so are those dividing m, this gives $n \mid ds$.

On the other hand, $v_3(d) = e - 1 < e$ gives $n \nmid d$, while each $\ell \mid m$ divides $\mathrm{rad}(n)$ but not s, so $\mathrm{rad}(n) \nmid s$. Thus, (a, b) is a cdf-zero pair.

Case 3: $9 \nmid n$ *and Case 1 does not apply.* We claim that some prime $p \neq 3$ satisfies $v_p(n) = k \geq 2$ and that n has a further prime factor $\ell \notin \{p, 3\}$. Indeed, if some $q \mid n$ had $q \equiv 1 \pmod 3$, then every other prime factor of n would be 3, since Case 1 fails. As n is not a prime power, this gives $n = 3^\alpha q^\beta$ with $\alpha \geq 1$, and $9 \nmid n$ forces $\alpha = 1$, so $n = 3q^\beta$, which is form (ii), contrary to assumption. Hence, no prime factor of n is $\equiv 1 \pmod 3$, and since (iii) fails, n is not squarefree: some prime p has $v_p(n) = k \geq 2$, and $p \neq 3$ because $9 \nmid n$. If every prime factor of n other than p were 3, then, since $9 \nmid n$ bounds the exponent of 3 by 1, we would have $n = p^k$ or $n = 3p^k$, which are both excluded; so, ℓ exists.

Now, set $\tau = \lceil k/3 \rceil$ (the least integer with $3\tau \geq k$; Remark 26 lists the specializations) and $m = n/p^k$, and take

$$b = p^\tau, \qquad a = p^\tau(1 + m), \qquad \text{so that } d = p^\tau m,\ s = p^{2\tau}(m^2 + 3m + 3).$$

Here, $\tau < k$, since $\lceil k/3 \rceil < k$ precisely when $k \geq 2$. As $p \nmid m$, we obtain $v_p(d) = \tau$, so $v_p(ds) \geq \tau + 2\tau = 3\tau \geq k$; and every prime $\ell' \mid m$ satisfies $v_{\ell'}(ds) \geq v_{\ell'}(d) = v_{\ell'}(m) = v_{\ell'}(n)$, since $\ell' \neq p$. Hence, $n \mid ds$. Also, $v_p(d) = \tau < k$ gives $n \nmid d$. Finally, $\ell \mid m$, so $m^2 + 3m + 3 \equiv 3 \pmod \ell$, and $\ell \notin \{3, p\}$ gives $\ell \nmid s$; as $\ell \mid \mathrm{rad}(n)$, we obtain $\mathrm{rad}(n) \nmid s$. Thus, (a, b) is a cdf-zero pair.

The three cases are exhaustive: Case 1 covers every n having a prime factor $q \equiv 1 \pmod 3$ alongside a prime factor outside $\{3, q\}$, and if Case 1 does not apply, then $9 \mid n$ falls under Case 2 and $9 \nmid n$ under Case 3. Case 2 may also apply when Case 1 does; its construction does not presuppose that Case 1 fails. □

Remark 26. *The two constructions in the proof specialize to several of the witnesses used earlier in the paper. For* $n = 63 = 3^2 \cdot 7$, *Case 2 gives* $e = 2$ *and* $m = 7$, *hence* $(a, b) = (22, 1)$; *for* $n = 45 = 3^2 \cdot 5$, *it gives* $(16, 1)$, *and for* $n = 18 = 2 \cdot 3^2$, *it gives* $(7, 1)$. *For* $n = 20 = 2^2 \cdot 5$, *Case 3 gives* $p = 2$, $k = 2$, $\tau = 1$ *and* $m = 5$, *hence* $(a, b) = (12, 2)$.

Remark 27. *Two consequences are worth recording. First, form (ii) cannot be relaxed to* $3^\alpha p^k$ *with* $p \neq 3$ *and* $\alpha \geq 2$, *since by Case 2 of the proof, no such ideal is cdf-primary. Over* $\mathbb{Z}$, *this sharpens Theorem 3(i) when* $\mathscr{B} \neq \emptyset$, *where the exponent of* (3) *was only required to be at least* 1. *Second, a cdf-primary ideal of* $\mathbb{Z}$ *that is not primary is either of the form* $3p^k\mathbb{Z}$ *or squarefree with every prime factor* $\not\equiv 1 \pmod 3$.

Example 13. *The ideal* $20\mathbb{Z}$ *shows Case 3 of the proof at work. Here,* $20 = 2^2 \cdot 5$ *with* $2 \equiv 5 \equiv 2 \pmod 3$, *so both minimal primes are of Type A and the* $\mathscr{B}$*-condition holds; what fails is exact divisibility—not the type of a prime. Take* $a = 12$ *and* $b = 2$, *the pair of Example 7 with the roles interchanged, which is read in* $\mathbb{Z}$. *Then,* $a^3 - b^3 = 1720 \in 20\mathbb{Z}$, *while* $a - b = 10 \notin 20\mathbb{Z}$ *and* $a^2 + ab + b^2 = 172 \notin 10\mathbb{Z} = \sqrt{20\mathbb{Z}}$. *Here,* $a - b = 10$ *is divisible by* 2 *but not by* 4, *so the first alternative fails by one power of* 2, *and* $5 \nmid a^2 + ab + b^2$, *so the second fails, too. Thus,* $(12, 2)$ *is a cdf-zero pair and* $20\mathbb{Z}$ *is not cdf-primary, as Theorem 6 predicts.*

Remark 28. *Theorem 6 extends to every principal ideal domain* R *in which* 3 *is a prime element, for instance* $R = \mathbb{Z}[i]$. *For a nonzero non-unit* $n \in R$, *the ideal* nR *is cdf-primary if and only if, up to a unit factor, one of the following holds: (i)* $n = \pi^k$ *for a prime element* π *and some* $k \geq 1$; *(ii)* $n = 3\pi^k$ *for a prime element* π *not associated to* 3 *and some* $k \geq 1$; *(iii)* n *is a product of pairwise non-associate prime elements, none of which generates a prime ideal of Type B. Indeed,* R *is a unique factorization domain, so the* π*-adic valuations* v_π *replace the p-adic ones;* 3 *being a prime element,* $3 \mid a - b$ *and* $3 \mid a^2 + ab + b^2$ *are equivalent for* $a, b \in R$, *by the identity* $a^2 + ab + b^2 = (a - b)^2 + 3ab$

and the primality of 3*; and in the proof of Theorem 6, the condition* $p \equiv 1 \pmod 3$ *is replaced throughout by "*(p) *is of Type B" (over* $\mathbb{Z}$ *the two agree by Remark 9(2)). Case 1 uses Theorem 3(i), whose hypothesis holds since* (3) *is maximal in the one-dimensional domain* R*; Cases 2 and 3 are unchanged, with* $3^{e-1}m$ *and* $\pi^{\tau}m$ *in place of the displayed integers. For example, in* $\mathbb{Z}[i]$*, the ideal* $3(1+i)^5\mathbb{Z}[i]$ *is cdf-primary, though neither primary nor squarefree, while* $9(1+i)\mathbb{Z}[i]$ *is not; and* $(11)(1+i)$ *is not cdf-primary, because* (11) *is of Type B (Remark 21).*

6. Behavior Under Ring Constructions

We begin with the two directions in which cdf-primarity travels along a ring homomorphism. All ring homomorphisms are assumed to preserve the identity.

Proposition 12. *Let* $f : R \to R'$ *be a ring homomorphism.*

(1) If I' *is cdf-primary in* R'*, then* $f^{-1}(I')$ *is cdf-primary in* R*.*
(2) If f *is surjective and* I *is a cdf-primary ideal of* R *with* $\ker(f) \subseteq I$*, then* $f(I)$ *is cdf-primary in* R'*.*

Proof. (1) The ideal $f^{-1}(I')$ is proper, since $f(1) = 1 \notin I'$. Suppose $a^3 - b^3 \in f^{-1}(I')$. Then, $f(a)^3 - f(b)^3 = f(a^3 - b^3) \in I'$, so either $f(a-b) \in I'$ or $f(a^2+ab+b^2) \in \sqrt{I'}$. The first case gives $a - b \in f^{-1}(I')$. For the second case, we use that $f^{-1}(\sqrt{I'}) \subseteq \sqrt{f^{-1}(I')}$ always: if $f(z) \in \sqrt{I'}$, then $f(z)^n = f(z^n) \in I'$ for some n, so $z^n \in f^{-1}(I')$, i.e., $z \in \sqrt{f^{-1}(I')}$. Applying this to $z = a^2 + ab + b^2$ gives $a^2 + ab + b^2 \in \sqrt{f^{-1}(I')}$.

(2) First, $f(I)$ is proper: if $1 = f(z)$ with $z \in I$, then $1 - z \in \ker f \subseteq I$, forcing $1 \in I$. Now, let $a, b \in R'$ with $a^3 - b^3 \in f(I)$. Write $a = f(z)$ and $b = f(w)$ with $z, w \in R$ (using surjectivity). Then, $f(z^3 - w^3) = a^3 - b^3 \in f(I)$, so $z^3 - w^3 \in I + \ker f \subseteq I$ (since $\ker f \subseteq I$). By the cdf-primarity of I, either $z - w \in I$ or $z^2 + zw + w^2 \in \sqrt{I}$, giving $a - b = f(z-w) \in f(I)$ or $a^2 + ab + b^2 = f(z^2 + zw + w^2) \in f(\sqrt{I}) = \sqrt{f(I)}$. (*The equality* $f(\sqrt{I}) = \sqrt{f(I)}$ holds, since f is surjective with $\ker f \subseteq I$: the inclusion $f(\sqrt{I}) \subseteq \sqrt{f(I)}$ is immediate, and the reverse follows because if $f(z)^n \in f(I)$, then $z^n \in I + \ker f = I$.) □

The transfer results below involve two constructions whose notation we fix first. Recall that $R(+)M$ denotes the *idealization* $R \times M$ of an R-module M with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$, and $I(+)M$ its ideal $I \times M$ [13]. The *amalgamated algebra* of [14], attached to a ring homomorphism $f : R \to R'$ and an ideal J of R', is $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$, with $I^f_J = \{(r, f(r) + j) : r \in I, j \in J\}$.

Most of the standard constructions leave the quotient ring unchanged, and for those, cdf-primarity transfers for a single reason.

Theorem 7. *Let* I *be a proper ideal of* R*. Each of the following ideals is cdf-primary if and only if* I *is cdf-primary in* R*.*

(1) I/K *in* R/K *for any ideal* $K \subseteq I$*.*
(2) (I, x) *in the polynomial ring* $R[x]$ *for an indeterminate* x*.*
(3) $I(+)M$ *in the idealization* $R(+)M$ *for any* R*-module* M*.*
(4) I^f_J *in the amalgamated algebra* $R \bowtie^f J$ *for any ring homomorphism* $f : R \to R'$ *and any ideal* J *of* R'*.*
(5) $I \times T$ *in* $R \times T$*, for any ring* T*.*

Proof. In parts (2)–(5), the displayed ideal is, as the list below shows, the kernel of a surjection onto $R/I \neq 0$ and is therefore proper; in part (1), I/K is proper in R/K because I is proper in R. In particular, $I(+)M$ is proper in $R(+)M$ if and only if I is proper in R: the

unit $(1,0)$ lies in $I(+)M$ exactly when $1 \in I$. By Lemma 5(2), a proper ideal is cdf-primary precisely when the zero ideal of the quotient it determines is cdf-primary; so the property depends only on the isomorphism class of that quotient. In each part, the quotient by the displayed ideal is isomorphic to R/I:

(1) $(R/K)/(I/K) \cong R/I$ by the third isomorphism theorem.
(2) Evaluation at 0 gives a surjection $R[x] \to R/I, g \mapsto g(0)+I$, whose kernel is exactly (I,x).
(3) The first projection gives a surjective ring homomorphism $R(+)M \to R/I, (r,m) \mapsto r+I$ with kernel $I(+)M$.
(4) The first projection gives a surjection $R \bowtie^f J \to R/I, (r, f(r)+j) \mapsto r+I$ with kernel I_J^f.
(5) The first projection gives a surjection $R \times T \to R/I$ with kernel $I \times T$.

The conclusion then follows in each part. Part (1) contains Proposition 12(2): in the notation of that proposition, the isomorphism $R/\ker f \cong R'$ carries $I/\ker f$ to $f(I)$. □

Part (5), applied to the second factor, shows likewise that $R \times L$ is cdf-primary in $R \times T$ if and only if L is cdf-primary in T, for an ideal L of T. Taking f to be the identity in part (4) gives the *amalgamated duplication* $R \bowtie J = \{(r, r+j) : r \in R, j \in J\}$: the ideal $I \bowtie J = \{(r, r+j) : r \in I, j \in J\}$ is cdf-primary in $R \bowtie J$ if and only if I is cdf-primary in R.

Localization and products $I_1 \times I_2$ with both factors proper can change the quotient and so require separate arguments.

Theorem 8. *If I is cdf-primary in R and S is multiplicatively closed with $I \cap S = \emptyset$, then $S^{-1}I$ is cdf-primary in $S^{-1}R$.*

Proof. Since $I \cap S = \emptyset$, the ideal $S^{-1}I$ is proper in $S^{-1}R$: were $\frac{1}{1} = \frac{c}{s}$ with $c \in I$ and $s \in S$, some $t \in S$ would give $ts = tc \in I \cap S$. Let $\frac{a}{s}, \frac{b}{s'} \in S^{-1}R$ with $\left(\frac{a}{s}\right)^3 - \left(\frac{b}{s'}\right)^3 \in S^{-1}I$. Then, $\frac{s'^3a^3 - s^3b^3}{s^3s'^3} \in S^{-1}I$, so there exists $w \in S$ with $w(s'^3a^3 - s^3b^3) \in I$. Set $a' = s'a$ and $b' = sb$ in R. Then, $w(a'^3 - b'^3) \in I$, and since I is an ideal, $w^2 \cdot w(a'^3 - b'^3) = w^3(a'^3 - b'^3) \in I$. Since $w^3(a'^3 - b'^3) = (wa')^3 - (wb')^3$, the cdf-primarity of I yields one of the following two cases.

In the first case, $wa' - wb' = w(s'a - sb) \in I$. Then, $\frac{a}{s} - \frac{b}{s'} = \frac{s'a - sb}{ss'} = \frac{w(s'a - sb)}{wss'} \in S^{-1}I$.

In the second case, write $q' = a'^2 + a'b' + b'^2$. Then, $(wa')^2 + (wa')(wb') + (wb')^2 = w^2q' \in \sqrt{I}$, so there exists $n \geq 1$ with $(w^2q')^n = w^{2n}q'^n \in I$. Since $w^{2n}(ss')^{2n} \in S$, we obtain

$$\left(\frac{q'}{s^2s'^2}\right)^n = \frac{q'^n}{(ss')^{2n}} = \frac{w^{2n}q'^n}{w^{2n}(ss')^{2n}} \in S^{-1}I,$$

giving $\frac{q'}{s^2s'^2} \in \sqrt{S^{-1}I}$. Since

$$\left(\frac{a}{s}\right)^2 + \frac{a}{s} \cdot \frac{b}{s'} + \left(\frac{b}{s'}\right)^2 = \frac{a'^2 + a'b' + b'^2}{s^2s'^2} = \frac{q'}{s^2s'^2},$$

the second alternative holds for $S^{-1}I$. □

For an ideal A of R, write $Z_A(R) = \{r \in R : rz \in A \text{ for some } z \notin A\}$ for the set of A-zero-divisors of R. The converse of Theorem 8 holds under the regularity condition $S \cap Z_I(R) = \emptyset$, which ensures that the localization map reflects membership in I and $\sqrt{I}$. For a proper ideal, one has $I \subseteq Z_I(R)$, so the condition implies $I \cap S = \emptyset$; and for $I = R$, neither I nor $S^{-1}I = S^{-1}R$ is proper, so neither is cdf-primary. The two results therefore give an equivalence.

Proposition 13. *Let I be an ideal of R and $S \subseteq R$ multiplicatively closed. If $S \cap Z_I(R) = \emptyset$ and $S^{-1}I$ is cdf-primary, then I is cdf-primary.*

Proof. We first record that $S \cap Z_{\sqrt{I}}(R) = \emptyset$ as well. Suppose $v \in S$ and $vz \in \sqrt{I}$, so that $v^m z^m = (vz)^m \in I$ for some m. Since $v \notin Z_I(R)$, an induction on m shows that $v^m y \in I$ implies $y \in I$ for every $y \in R$; applied to $y = z^m$, this yields $z^m \in I$ and hence $z \in \sqrt{I}$.

Next, I is proper: if $1 \in I$, then $S^{-1}I = S^{-1}R$, which is not proper, contradicting the cdf-primarity of $S^{-1}I$. Now, let $a^3 - b^3 \in I$, so that $\left(\frac{a}{1}\right)^3 - \left(\frac{b}{1}\right)^3 \in S^{-1}I$. By cdf-primarity of $S^{-1}I$, one of the following two alternatives holds.

In the first case, $\frac{a-b}{1} \in S^{-1}I$, that is, $u(a-b) \in I$ for some $u \in S$. Since $u \notin Z_I(R)$, the implication $uz \in I \Rightarrow z \in I$ holds for every $z \in R$; applied to $z = a - b$, it gives $a - b \in I$.

In the second case, $\frac{a^2+ab+b^2}{1} \in \sqrt{S^{-1}I}$, that is, $\left(\frac{a^2+ab+b^2}{1}\right)^n \in S^{-1}I$ for some n, so there exists $v \in S$ with $v(a^2+ab+b^2)^n \in I \subseteq \sqrt{I}$. Since $v \notin Z_{\sqrt{I}}(R)$ by the observation recorded at the outset, the implication $vz \in \sqrt{I} \Rightarrow z \in \sqrt{I}$ holds for every $z \in R$; it gives $(a^2+ab+b^2)^n \in \sqrt{I}$, hence $a^2 + ab + b^2 \in \sqrt{I}$. □

Remark 29. *As noted before Proposition 13, if $S \cap Z_I(R) = \emptyset$, then I is cdf-primary if and only if $S^{-1}I$ is cdf-primary in $S^{-1}R$. The regularity condition cannot be dropped from Proposition 13. Take $R = \mathbb{Z}$, $I = 45\mathbb{Z}$ and $S = \mathbb{Z} \setminus 3\mathbb{Z}$. Then, $S^{-1}I = 9\mathbb{Z}_{(3)}$, because 5 is a unit of $\mathbb{Z}_{(3)}$, and this ideal is primary in the local ring $\mathbb{Z}_{(3)}$, hence cdf-primary. But $45\mathbb{Z} = 3^2 \cdot 5\mathbb{Z}$ is not cdf-primary (Theorem 6, Case 2; the pair $(16, 1)$ of Remark 26). Here, $5 \in S \cap Z_I(\mathbb{Z})$, since $5 \cdot 9 = 45 \in I$ while $9 \notin I$.*

The cdf-primarity of a product ideal passes to the factors; the converse needs more, and part (2) below assumes that I_1 is radical and contains 3.

Theorem 9. *Let I_i be proper ideals of R_i $(i = 1, 2)$ and $I = I_1 \times I_2 \subseteq R_1 \times R_2$.*

(1) If I is cdf-primary, then so are I_1 and I_2.
(2) If I_1 is radical with $3 \in I_1$ and I_2 is cdf-primary, then I is cdf-primary.

Proof. Throughout, we use that $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$.

(1) We show I_1 is cdf-primary; the argument for I_2 is symmetric. Let $a_1^3 - b_1^3 \in I_1$. Then, $(a_1, 0)^3 - (b_1, 0)^3 = (a_1^3 - b_1^3, 0) \in I$, so by the cdf-primarity of I either $(a_1 - b_1, 0) \in I$, giving $a_1 - b_1 \in I_1$, or $(a_1^2 + a_1 b_1 + b_1^2, 0) \in \sqrt{I}$, giving $a_1^2 + a_1 b_1 + b_1^2 \in \sqrt{I_1}$.

(2) Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ satisfy $a^3 - b^3 \in I$, so that $a_1^3 - b_1^3 \in I_1$ and $a_2^3 - b_2^3 \in I_2$. Since $3 \in I_1$, expanding the cube gives $(a_1 - b_1)^3 = a_1^3 - 3a_1^2 b_1 + 3a_1 b_1^2 - b_1^3 \equiv a_1^3 - b_1^3 \pmod{I_1}$, so $(a_1 - b_1)^3 \in I_1$ and therefore $a_1 - b_1 \in \sqrt{I_1} = I_1$. Now, use that I_2 is cdf-primary: if $a_2 - b_2 \in I_2$, then $(a_1 - b_1, a_2 - b_2) \in I$. If instead $a_2^2 + a_2 b_2 + b_2^2 \in \sqrt{I_2}$, then Lemma 4, applied to I_1 with $3 \in \sqrt{I_1}$ and $a_1 - b_1 \in \sqrt{I_1}$, gives $a_1^2 + a_1 b_1 + b_1^2 \in \sqrt{I_1}$; so the second alternative holds for I. □

Theorem 9 is not sharp: the converse of part (1) fails—both factors of $4\mathbb{Z} \times 2\mathbb{Z}$ are cdf-primary, $4\mathbb{Z}$ being primary and $2\mathbb{Z}$ prime, while the product is not (Example 2)—and the hypotheses of part (2) are far from necessary. The radicality of I_1 cannot be dropped, however. In $\mathbb{F}_3[x] \times \mathbb{Z}$, the ideal $(x^2) \times 2\mathbb{Z}$ has $3 = 0 \in (x^2)$ and a cdf-primary second factor, yet $a = (x, 1)$, $b = (0, 1)$ give $a^3 - b^3 = (x^3, 0) \in I$, $a - b = (x, 0) \notin I$, and $a^2 + ab + b^2 = (x^2, 3) \notin \sqrt{I} = (x) \times 2\mathbb{Z}$. Part (2) is the reduced characteristic-3 case of Proposition 14, as explained after the proof of that proposition. The exact criterion, which subsumes both parts, is the following.

Proposition 14. *Let I_i be proper ideals of R_i $(i = 1, 2)$ and write $\bar{R}_i = R_i/I_i$. Then, $I_1 \times I_2$ is cdf-primary in $R_1 \times R_2$ if and only if I_1 and I_2 are both cdf-primary and, for each of the two orderings $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$, cubing is injective on $\bar{R}_i$ or $3 \in \sqrt{I_j}$.*

Proof. By Lemma 5(2), the cdf-primarity of $I_1 \times I_2$ is the cdf-primarity of the zero ideal of $\bar{R}_1 \times \bar{R}_2$; cubes, the form $\zeta^2 + \zeta\eta + \eta^2$ and the nilradical are all computed componentwise, and $\mathrm{Nil}(\bar{R}_j) = \sqrt{I_j}/I_j$. A failure—that is, a cdf-zero pair of that zero ideal in the sense of Definition 3—is thus a pair of elements (ξ_1, ξ_2), (η_1, η_2) with $\xi_k^3 = \eta_k^3$ for $k = 1, 2$, such that $\xi_i \neq \eta_i$ for some index i and $\xi_j^2 + \xi_j\eta_j + \eta_j^2 \notin \mathrm{Nil}(\bar{R}_j)$ for some index j.

If $i = j$ can be taken, the two components at that index form a failure for I_i alone, again by Lemma 5(2); conversely, a failure for I_i extends to one for the product by putting $\xi_k = \eta_k = 0$ in the other component. So, $I_1 \times I_2$ cdf-primary forces I_1 and I_2 cdf-primary, and we may assume both from now on, in which case every failure has $i \neq j$.

Fix such an ordering (i, j). The demand at i is that $\bar{R}_i$ carry $\xi_i \neq \eta_i$ with $\xi_i^3 = \eta_i^3$, that is, that cubing not be injective on $\bar{R}_i$. At j, the second alternative fails, by the choice of j; since I_j is cdf-primary, the first alternative must hold there, so $\xi_j = \eta_j$ and $\xi_j^2 + \xi_j\eta_j + \eta_j^2 = 3\xi_j^2$. This element lies outside $\mathrm{Nil}(\bar{R}_j)$ for some choice of ξ_j precisely when $3 \notin \mathrm{Nil}(\bar{R}_j)$: for one direction, take $\xi_j = 1$; for the other, note that $3 \in \mathrm{Nil}(\bar{R}_j)$ forces $3\xi_j^2 \in \mathrm{Nil}(\bar{R}_j)$ for every ξ_j. Since $\mathrm{Nil}(\bar{R}_j) = \sqrt{I_j}/I_j$, this says $3 \notin \sqrt{I_j}$. The two demands sit in different components and may therefore be met at once; explicitly, a pair exhibiting the non-injectivity of cubing on $\bar{R}_i$, combined with $\xi_j = \eta_j = 1$ at j, is a failure. So, a failure at the ordering (i, j) exists exactly when cubing fails to be injective on $\bar{R}_i$ and $3 \notin \sqrt{I_j}$, which is the negation of the stated condition at that ordering. □

Theorem 9(2) is the case where $\bar{R}_1$ is reduced by characteristic 3. Every requirement of Proposition 14 is then met. The ideal I_2 is cdf-primary by hypothesis. So is I_1, because $\bar{R}_1$ has characteristic 3, so that Proposition 3 applies to its zero ideal and Lemma 5(2) transports the conclusion back to I_1. Finally, cubing on $\bar{R}_1$ is the Frobenius endomorphism, and it is injective because $\bar{R}_1$ is reduced: $\xi^3 = \eta^3$ gives $(\xi - \eta)^3 = 0$ and hence $\xi = \eta$. The ordering $(1, 2)$ is therefore covered by the first clause of the criterion, and the ordering $(2, 1)$ by the second, since $3 \in \sqrt{I_1}$. The criterion can be met in other ways: $2\mathbb{Z} \times 5\mathbb{Z}$ is cdf-primary in $\mathbb{Z} \times \mathbb{Z}$ although neither factor contains 3, because cubing is injective on $\mathbb{F}_2$ and on $\mathbb{F}_5$.

Proposition 15. *Let $s \geq 2$, let I_k be a proper ideal of R_k for $k = 1, \ldots, s$, and write $\bar{R}_k = R_k/I_k$. Then, $I_1 \times \ldots \times I_s$ is cdf-primary in $R_1 \times \ldots \times R_s$ if and only if every I_k is cdf-primary and, for every ordered pair (i, j) with $i \neq j$, cubing is injective on $\bar{R}_i$ or $3 \in \sqrt{I_j}$.*

Proof. By Lemma 5(2), we pass to the zero ideal of $\bar{R}_1 \times \ldots \times \bar{R}_s$, where everything is computed componentwise and $\mathrm{Nil}(\bar{R}_j) = \sqrt{I_j}/I_j$. Write $q_k = \xi_k^2 + \xi_k\eta_k + \eta_k^2$. A cdf-zero pair is then a pair of tuples (ξ_k), (η_k) with $\xi_k^3 = \eta_k^3$ for all k such that $\xi_i \neq \eta_i$ for some i and $q_j \notin \mathrm{Nil}(\bar{R}_j)$ for some j.

Suppose the stated condition holds and a cdf-zero pair exists with indices i and j as above. If $i = j$, the components at i form a cdf-zero pair of the zero ideal of $\bar{R}_i$, contradicting the cdf-primarity of I_i. If $i \neq j$, then cubing is not injective on $\bar{R}_i$ because $\xi_i^3 = \eta_i^3$ with $\xi_i \neq \eta_i$; so, the condition forces $3 \in \mathrm{Nil}(\bar{R}_j)$. The zero ideal of $\bar{R}_j$ is cdf-primary, and $\xi_j^3 = \eta_j^3$, so either $\xi_j = \eta_j$ or $q_j \in \mathrm{Nil}(\bar{R}_j)$. The second case is excluded by the choice of j. In the first, $q_j = 3\xi_j^2 \in \mathrm{Nil}(\bar{R}_j)$ since 3 is nilpotent, which is again a contradiction. So, the product is cdf-primary.

Conversely, suppose the product is cdf-primary. Each I_k is cdf-primary, because a cdf-zero pair of $\bar{R}_k$ extends to one of the product by putting $\xi_l = \eta_l = 0$ for $l \neq k$. Suppose that for some $i \neq j$, cubing is not injective on $\bar{R}_i$ and $3 \notin \sqrt{I_j}$. Choose $\xi_i \neq \eta_i$ in $\bar{R}_i$ with $\xi_i^3 = \eta_i^3$, put $\xi_j = \eta_j = 1$, and take all other coordinates equal to 0. Then, the cubes agree componentwise, $\xi_i \neq \eta_i$, and $q_j = 3 \notin \mathrm{Nil}(\bar{R}_j)$ because $3 \notin \sqrt{I_j}$. This is a cdf-zero pair, which is a contradiction. Hence, the stated condition holds. □

For $s = 2$, this is Proposition 14. Over $\mathbb{Z}$, it can be made explicit. Cubing is injective on $\mathbb{Z}/n\mathbb{Z}$ exactly when n is squarefree and has no prime factor congruent to 1 modulo 3: a prime $p \equiv 1 \pmod 3$ gives a nontrivial cube root of unity modulo p, and a prime power p^k with $k \geq 2$ gives the nonzero element $p^{\lceil k/3 \rceil}$ whose cube is 0 modulo p^k, while for squarefree n with all primes $\not\equiv 1 \pmod 3$, the cubing map is injective modulo each prime p, being the Frobenius map when $p = 3$ and a bijection of $\mathbb{F}_p^\times$ when $p \equiv 2 \pmod 3$, since then $\gcd(3, p-1) = 1$. By the Chinese Remainder Theorem, it suffices to look at the prime-power factors of n, and $3 \in \sqrt{n\mathbb{Z}}$ exactly when n is a power of 3. Together with Theorem 6, this decides every ideal $n_1\mathbb{Z} \times \ldots \times n_s\mathbb{Z}$ of $\mathbb{Z}^s$ with all $n_k \geq 2$. For instance, $7\mathbb{Z} \times 3\mathbb{Z}$ is cdf-primary, since $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}_{21}$, whereas $7\mathbb{Z} \times 2\mathbb{Z}$ is not, since $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_{14}$ (Examples 4 and 5). The three-factor ideal $2\mathbb{Z} \times 5\mathbb{Z} \times 11\mathbb{Z}$ is cdf-primary, cubing being injective on $\mathbb{F}_2$, $\mathbb{F}_5$, $\mathbb{F}_{11}$.

7. Open Questions

We record the remaining open questions.

(1) Does cdf-primarity imply the dual condition $a^3 - b^3 \in I \Rightarrow a - b \in \sqrt{I}$ or $a^2 + ab + b^2 \in I$ of Section 2 in an arbitrary ring? It does over $\mathbb{Z}$ and for primary ideals, and the dual condition does not imply cdf-primarity $(45\mathbb{Z})$; the general implication is open.

(2) What has to be added to the criterion of Theorem 4—some demand on the non-radical ideals, presumably—to characterize Dedekind domains this way? As stated, the criterion has no converse: it is automatic once (3) is maximal, and Remark 22 exhibits non-Dedekind rings satisfying it.

(3) By Theorem 2 and Corollary 3, the cdf-primarity of any proper ideal I forces the $\mathscr{B}$-condition on $\mathrm{Min}(I)$, but for non-radical ideals, this is not sufficient (Remark 18). A complete description of the cdf-primary ideals is available when R is a ZPI-ring with $3 \in U(R)$ (Proposition 10), when R is a principal ideal domain in which 3 is a prime element (Theorem 6 and Remark 28), and for finite products of such rings when the cubing condition of Proposition 15 can be evaluated, for instance in $\mathbb{Z}^s$. It is missing for a general ZPI-ring. Two cases are typical: ZPI-rings in which (3) factors into several maximal ideals, such as $\mathbb{Z}[\sqrt{-5}]$, where Proposition 11 settles only the radical case and bounds r by $1 + |V(3)|$; and Dedekind domains in which 3 is a prime element but which are not principal ideal domains, where Theorem 4 treats only products of distinct maximal ideals.

(4) Is there a useful S-version of the cdf-primary condition, relative to a multiplicatively closed set, and a counterpart for submodules, in the spirit of [15]?

8. Conclusions

We introduced cdf-primary ideals: whenever $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ lies in I, either the factor $a - b$ lies in I or the factor $a^2 + ab + b^2$ lies in $\sqrt{I}$. The condition is implied by both the primary and the cdf-absorbing conditions and implies the cdf-semiprimary one (Section 2), and it is incomparable with the quadratic sdf-primary condition of [10]: $14\mathbb{Z}$ is sdf-primary but not cdf-primary, while $21\mathbb{Z}$ is cdf-primary but not sdf-primary.

The central tool is the reduction to the zero ideal of R/I and, for radical ideals, the type of the minimal primes. A prime is of Type B when the fraction field of its residue domain contains a primitive cube root of unity, and a radical ideal in any ring is cdf-primary exactly when its minimal primes contain at most one Type B prime and, if one occurs, all the others contain 3 (Theorem 1). The prime 3 thus plays a protective role with no counterpart in the quadratic theory. By Theorem 2, the radical of any cdf-primary ideal is cdf-primary, indeed cdf-absorbing, so the $\mathscr{B}$-condition on the minimal primes is necessary for every

cdf-primary ideal. It is not sufficient for non-radical ideals, because the exponents impose divisibility conditions that the factors $a - b$ and $a^2 + ab + b^2$ can meet jointly without either meeting them alone; this is the obstruction of exact divisibility (Remarks 16 and 24).

With these results, the cdf-primary ideals are described completely in the following settings: ZPI-rings in which 3 is a unit (Proposition 10); the integers, and more generally principal ideal domains in which 3 is a prime element (Theorem 6 and Remark 28); and finite products, where the question reduces to the factors (Propositions 14 and 15). Cdf-primarity transfers through quotients, the ideal (I, x) of $R[x]$, idealization and amalgamation, and passes to localizations (Section 6). What remains is the non-radical case for ZPI-rings in which (3) factors, the converse direction of the characterization of Dedekind domains, and the questions listed in Section 7.

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