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Frenet Apparatus of Curves in n-Dimensional Generalized Dual Space and Its Application to Rational Space Curves

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Abstract

This paper is devoted to describing and investigating the geometry of a generalized dual curve in the generalized dual space $\mathbb{D}_p^n$. Analogous to the classical theory of curves, we construct a dual frame for generalized dual curves, which we call the generalized dual Frenet apparatus. Moreover, using the framework developed in the paper, we propose an algorithm that symbolically computes the similarities between two rational algebraic curves in the generalized 3-dimensional dual space $\mathbf{D}_p^3$. The algorithm relies on the relation between the dual curvatures and torsions of the curves. We provide an implementation of the algorithm in the computer algebra system Maple, and perform extensive experimentation of the proposed method.

Keywords: generalized dual space; curve; similarity detection; rational space curve

MSC: 53A04; 53A17; 14H50; 14Q20

1. Introduction

The differential geometry of curves provides a fundamental tool for describing and comparing geometric shapes. In the three-dimensional Euclidean space, the local geometry of a regular curve is characterized by its Frenet frame, curvature, and torsion. More generally, a regular curve in the n-dimensional Euclidean space $\mathbb{E}^n$ admits a Frenet n-frame $\{V_1, V_2, \ldots, V_n\}$ and $n-1$ successive curvature functions $k_1, k_2, \ldots, k_{n-1}$. Given the arc-length parameter s, the corresponding Frenet equations are

$$\begin{aligned} V_1' &= k_1 V_2, \\ V_i' &= -k_{i-1} V_{i-1} + k_i V_{i+1}, \qquad 2 \le i \le n-1, \\ V_n' &= -k_{n-1} V_{n-1}. \end{aligned} \tag{1}$$

When $n = 3$, k_1 and k_2 reduce up to the adopted sign conventions, to the ordinary curvature and torsion [1].

Dual numbers extend the classical geometry of curves by incorporating infinitesimal and line-geometric information. The dual-number algebra is defined by

$$\mathbb{D} = \left\{ x + \varepsilon y : x, y \in \mathbb{R}, \quad \varepsilon^2 = 0 \right\}.$$

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In a 3-dimensional setting, the study mapping establishes a one-to-one correspondence between unit dual vectors and oriented lines in 3-dimensional Euclidean space. This relation has led to applications of dual-number methods in line geometry, ruled surfaces, spatial kinematics, mechanism theory, robotics, and rigid-body motion [2–4]. The geometry of curves in the classical dual space $\mathbb{D}^3$ has also been applied to rectifying curves, spherical curves, helices, and other special curves [5–8]. More recently, López [9] developed a systematic Frenet theory for dual curves, including dual curvature, dual torsion, and an existence theorem for curves with prescribed invariants. In the paper [9], López investigated regular curves in the 3-dimensional dual space $\mathbb{D}^3$ from a differential geometric perspective, with the primary objective of establishing the Frenet formulas that define curvature and torsion. As evidenced by the references cited in his work, several studies on this topic exist in the literature. Lopez noted a fundamental conceptual ambiguity in relation to the nature of the curve parameter. Specifically, this issue arises from whether the curves in $\mathbb{D}^3$ are considered functions of a real variable or a dual variable—two settings that correspond to intrinsically distinct mathematical objects. Consequently, Lopez demonstrated that the standard arc-length reparameterization commonly adopted in the literature cannot be directly applied to dual curves without mathematical inconsistencies, and subsequently introduced a rigorous definition of admissible curves necessary for the proper construction of the Frenet equations.

Otherwise, for a fixed $p \in \mathbb{R}$, Soylu, Çolakoğlu, and Özdemir [10] introduced the generalized dual-number algebra

$$\mathbb{D}_p = \left\{ x + y\varepsilon_p : x, y \in \mathbb{R}, \quad \varepsilon_p^2 = -p^2 + 2p\varepsilon_p \right\}$$

and constructed the generalized 3-dimensional dual Euclidean space $\mathbb{D}_p^3$. They established its basic algebraic and geometric properties and introduced the corresponding inner product, norm, unit vectors, generalized dual angles, and study mapping. Nevertheless, the explicit representation of geometric quantities in the generalized basis requires a careful treatment of the inner product, normalization, orthogonality, and invertibility conditions.

In this regard, one of the main objectives of this paper is to employ this geometric structure in the n-dimensional generalized dual space $\mathbb{D}_p^n$ and, following the framework introduced by López [9], to formulate the Frenet theory of generalized dual-parametric curves $\boldsymbol{\alpha} : I \longrightarrow \mathbb{D}_p^n$. Assuming that the curves are regular, we construct the generalized dual Frenet n-frame $\{W_1, W_2, \ldots, W_n\}$, define the $n-1$ generalized dual curvatures $K_1, K_2, \ldots, K_{n-1}$, and derive the corresponding higher-dimensional Frenet equations. The relationship between the ordinary and generalized dual Frenet apparatus is also investigated through the algebra isomorphism $\mathbb{D}_p \cong \mathbb{D}$. In the three-dimensional case, K_1 and K_2 correspond to generalized dual curvature and torsion, respectively. In this paper, these curvatures are denoted by $\kappa_{\boldsymbol{\alpha}}$ and $\tau_{\boldsymbol{\alpha}}$, respectively.

A further objective of the paper is to apply the above theory to the problem of detecting similarities between properly parametrized rational curves in $\mathbb{D}_p^3$. In the classical Euclidean setting, Alcázar, Hermoso, and Muntingh proposed that curvature and torsion can be used to detect symmetries and similarities of rational algebraic space curves [11,12]. Their method relies on the fact that the parameter correspondence induced by a similarity between two properly parametrized rational curves is a Möbius transformation

$$\varphi(t) = \frac{at + b}{ct + d}, \qquad ad - bc \neq 0.$$

Using this idea, one can reduce the problem to solving algebraic equations whose factors yield the desired Möbius reparameterizations. Taking inspiration from this idea, we address

the problem of detecting dual similarities between rational algebraic curves in $\mathbb{D}_p^3$ using the dual curvature and torsion.

The paper is organized as follows. Section 2 reviews generalized dual numbers and introduces the generalized dual space $\mathbb{D}_p^n$. Section 3 develops the generalized dual Frenet n-frame and derives the corresponding Frenet equations. Section 4 presents an algorithm for detecting generalized dual similarities between rational curves in $\mathbb{D}_p^3$ using the previous ideas. An illustrative example and concluding remarks complete the section.

2. Preliminaries

Let $\mathbb{R}$ denote the field of real numbers. For a fixed $p \in \mathbb{R}$, the set

$$\mathbb{D}_p = \left\{ x + y\varepsilon_p : x, y \in \mathbb{R}, \quad \varepsilon_p^2 = -p^2 + 2p\varepsilon_p \right\}$$

is called the set of generalized dual numbers. An element $z = x + y\varepsilon_p \in \mathbb{D}_p$ is referred to as a generalized dual number. Equipped with the usual addition and the multiplication induced by the relation

$$\varepsilon_p^2 = -p^2 + 2p\varepsilon_p,$$

the set $\mathbb{D}_p$ forms a commutative and associative real algebra with identity; see [10].

The three-dimensional generalized dual space $\mathbb{D}_p^3$, together with its associated algebraic and metric structures, was introduced in [10]. In particular, vector addition, scalar multiplication, the generalized dual scalar product, and the corresponding norm were established therein. In this paper, we extend these notions to the n-dimensional setting.

Let $\mathbb{E}^n$ denote the n-dimensional Euclidean space. We define the n-dimensional generalized dual space by

$$\mathbb{D}_p^n = \mathbb{E}^n + \varepsilon_p \mathbb{E}^n = \left\{ x + \varepsilon_p x^* : x, x^* \in \mathbb{R}^n \right\}.$$

Thus, every $z \in \mathbb{D}_p^n$ can be written as

$$z = x + \varepsilon_p x^*, \qquad x, x^* \in \mathbb{R}^n.$$

The space $\mathbb{D}_p^n$ may therefore be regarded as an extension of $\mathbb{E}^n$ over the generalized dual number algebra $\mathbb{D}_p$. Vector addition and scalar multiplication are naturally induced by the corresponding operations in $\mathbb{E}^n$ together with the defining relation for ε_p.

For

$$z = x + \varepsilon_p x^*, \qquad w = y + \varepsilon_p y^*$$

in $\mathbb{D}_p^n$, the generalized dual scalar product is defined by

$$\langle z, w \rangle_p = \langle x, y \rangle - p^2 \langle x^*, y^* \rangle + \varepsilon_p (\langle x, y^* \rangle + \langle x^*, y \rangle + 2p \langle x^*, y^* \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{E}^n$.

For convenience, we introduce the linear map

$$H_p : \mathbb{D}_p^n \longrightarrow \mathbb{E}^n, \qquad H_p(z) = x + px^*,$$

for $z = x + \varepsilon_p x^* \in \mathbb{D}_p^n$. In terms of H_p, the generalized dual scalar product can equivalently be expressed as

$$\langle z, w \rangle_p = \langle H_p(z), H_p(w) \rangle + (\varepsilon_p - p)\big(\langle H_p(z), y^* \rangle + \langle x^*, H_p(w) \rangle\big).$$

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For $z = x + \varepsilon_p x^* \in \mathbb{D}_p^n$ satisfying $H_p(z) \neq 0$, the generalized dual norm of z is defined by

$$\|z\|_p = \|H_p(z)\| + (\varepsilon_p - p)\frac{\langle H_p(z), x^* \rangle}{\|H_p(z)\|}, \tag{2}$$

where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{E}^n$. Accordingly, the normalized vector associated with z is given by

$$\frac{z}{\|z\|_p} = \frac{H_p(z)}{\|H_p(z)\|} + \frac{\varepsilon_p - p}{\|H_p(z)\|^3}\Big(\|H_p(z)\|^2 x^* - \langle H_p(z), x^* \rangle H_p(z)\Big). \tag{3}$$

In the three-dimensional case, we also consider the generalized dual vector product, which will be used later in the study of rational curves in $\mathbb{D}_p^3$. For

$$z = x + \varepsilon_p x^*, \qquad w = y + \varepsilon_p y^* \in \mathbb{D}_p^3,$$

the generalized dual vector product is defined by

$$z \times_p w = H_p(z) \times H_p(w) + (\varepsilon_p - p)\big(H_p(z) \times y^* + x^* \times H_p(w)\big),$$

where $\times$ denotes the standard vector product in $\mathbb{E}^3$.

The notions introduced above provide the algebraic and metric framework necessary to develop generalized dual Frenet apparatus in $\mathbb{D}_p^n$. The generalized dual vector product, which is specific to the three-dimensional setting, will be used only in the subsequent investigation of similarities between rational curves in $\mathbb{D}_p^3$. Further properties of the generalized dual scalar product, norm, and vector product can be found in [10].

3. The Generalized Dual Frenet Frame of Curves in $\mathbb{D}_p^n$

In [9], the author investigated the problem of arc-length parametrization of dual curves in the 3-dimensional dual space $\mathbb{D}^3$ and constructed their Frenet n-frame. In this paper, we investigate this problem in the n-dimensional generalized dual space $\mathbb{D}_p^n$. To this end, we adopt the definition of a dual curve given in [9].

3.1. Generalized Dual Parametric Curves

Let $\mathbb{E}^n$ denote the n-dimensional Euclidean space.

Definition 1. *Let $I = (a, b)$ be an open interval of $\mathbb{R}$. A C^∞-mapping $\boldsymbol{x} : I \to \mathbb{E}^n$ is called a parametric curve in $\mathbb{E}^n$.*

We write $\mathbf{x}(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t))$, where $\mathbf{x}_i : I \to \mathbb{E}^n$, $i = 1, 2, \ldots, n$ are differentiable real functions. Denote the first derivative of $\mathbf{x}(t)$ by $\mathbf{x}'(t)$.

Definition 2. *A parametric curve $\boldsymbol{x}(t)$ in $\mathbb{E}^n$ is called a regular parametric curve if $\boldsymbol{x}'(t) \neq 0$ for all $t \in I$.*

Definition 3. *Let $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ be two parametric curves in $\mathbb{E}^n$. A C^∞-mapping $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ is called a generalized dual parametric curve (GDP-curve, for short) in $\mathbb{D}_p^n$ if $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p \boldsymbol{y}(t) = (\boldsymbol{x}(t) + p\boldsymbol{y}(t)) + (\varepsilon_p - p)\boldsymbol{y}(t)$ for all $t \in I$.*

Definition 4. *A parametric curve $\boldsymbol{\alpha}(t)$ in $\mathbb{D}_p^n$ is called a regular generalized dual parametric curve (RGDP-curve, for short) if $\boldsymbol{\alpha}'(t) \neq 0$ for all $t \in I$.*

Definition 5. *Let $\boldsymbol{\alpha}(t)$ be a RGDP-curve in $\mathbb{D}_p^n$. Then, $\boldsymbol{\alpha}(t)$ will be called parametrized by arc-length if $\|\boldsymbol{\alpha}'(t)\|_p = 1$ for all $t \in I$.*

Recall the linear map $H_p : \mathbb{D}_p^n \to \mathbb{E}^n$ such that $H_p(z) = x + px^*$ for all $z = x + x^*\varepsilon_p \in \mathbb{D}_p^n$. Then, for the GDP-curve $\boldsymbol{\alpha}(t) = \mathbf{x}(t) + \varepsilon_p y(t)$, we have $H_p(\boldsymbol{\alpha}(t)) = \mathbf{x}(t) + py(t)$ for all $t \in I$. Clearly, $H_p(\boldsymbol{\alpha}(t))$ is a parametric curve in $\mathbb{E}^n$.

Proposition 1. *Let $\boldsymbol{\alpha}(t)$ be a parametric curve in $\mathbb{D}_p^n$ such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p y(t)$. If $H_p(\boldsymbol{\alpha}'(t)) \neq 0$ and $< H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t) >= 0$ for all $t \in I$, then there exists a reparametrization of $\boldsymbol{\alpha}(t)$ with arc-length.*

Proof. By Definition 5, if $\boldsymbol{\alpha}$ is parametrized with arc-length, then we have $\|\boldsymbol{\alpha}'\|_p = 1$. Hence, the equality implies $< H_p(\boldsymbol{\alpha}'(t)), H_p(\boldsymbol{\alpha}'(t)) >= 1$ and $< H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t) >= 0$ for all $t \in I$. That is, $H_p(\boldsymbol{\alpha})$ is a regular curve in $\mathbb{E}^n$ and the velocities of the real and generalized dual parts are orthogonal. Now, fix t_0 in the domain I of $\boldsymbol{\alpha}$ and consider the arc-length function $s = s(t) = \int_{t_0}^t \|\boldsymbol{\alpha}'(u)\|_p du = \int_{t_0}^t \|H_p(\boldsymbol{\alpha}'(u))\| du$. This the derivative $\frac{ds}{dt}$ of the function $s = s(t)$ is the speed function $v = v(t) = \|H_p(\boldsymbol{\alpha}'(t))\|$ of $\boldsymbol{\alpha}$. Since $H_p(\boldsymbol{\alpha})$ is regular, we have $\frac{ds}{dt} > 0$. By the fundamental theorem of calculus, the function s has an inverse function $t = t(s)$ such that $\frac{dt}{ds} > 0$. Now, let $\boldsymbol{\beta}$ be the reparametrization $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s))$ of $\boldsymbol{\alpha}$. We claim that $\boldsymbol{\beta}$ has unit speed. Thus, we have $\boldsymbol{\beta}'(s) = \frac{dt}{ds}(s)\boldsymbol{\alpha}'(t(s)) = \frac{dt}{ds}(s)(\boldsymbol{x}'(t(s)) + \varepsilon_p \boldsymbol{y}'(t(s)))$. Hence, the speed of $\boldsymbol{\beta}$ is $\|\boldsymbol{\beta}'(s)\|_p = \|H_p(\boldsymbol{\beta}'(s))\| = \frac{dt}{ds}(s)\|H_p(\boldsymbol{\alpha}'(t(s)))\| = \frac{dt}{ds}(s)\frac{ds}{dt}(t(s)) = 1$. □

Remark 1. *By Proposition 1, there exist generalized dual curves in $\mathbb{D}_p^n$ that do not admit arc-length parametrization.*

The following examples illustrate the statement in Remark 1.

Example 1. *For all $p \in \mathbb{R}$, we consider the curve $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p y(t) = (-pt, 0, 1) + \varepsilon_p(t, 0, 0) \in \mathbb{D}_p^3$ for all $t \in \mathbb{R}$. Then we have $H_p(\boldsymbol{\alpha}'(t)) = (0, 0, 0)$ and $< H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t) >= 0$ for all $t \in \mathbb{R}$. In this case, we obtain a contradiction with Proposition 1. That is, the curve $\boldsymbol{\alpha}(t)$ cannot be parametrized with arc-length.*

Example 2. *For $p \neq 1$, we consider the curve $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p \boldsymbol{y}(t) = ((1-p)t, 0, 0) + \varepsilon_p(t, t, 0) \in \mathbb{D}_p^3$ for all $t \in \mathbb{R}$. Then we have $H_p(\boldsymbol{\alpha}'(t)) = (1, 0, 0)$ and $< H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t) >\neq 0$ for all $t \in \mathbb{R}$. In this case, we obtain a contradiction with Proposition 1. That is, the curve $\boldsymbol{\alpha}(t)$ cannot be parametrized with arc-length.*

Example 3. *For $p \neq 0$, we consider the curve $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p y(t) = (1, -2pt, 0) + \varepsilon_p(t, t, 0) \in \mathbb{D}_p^3$ for all $t \in \mathbb{R}$. Then we have $H_p(\boldsymbol{\alpha}'(t)) \neq 0$ and $< H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t) >= 0$ for all $t \in \mathbb{R}$. By Proposition 1, the curve $\boldsymbol{\alpha}(t)$ can be reparametrized with arc-length. Let $\overline{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}(\phi(s))$ for all $s \in \mathbb{R}$, where $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(s) = \frac{s}{p\sqrt{2}} = t$ is a diffeomorphism. Then, we have $\overline{\boldsymbol{\alpha}}(s) = (1, -\sqrt{2}s, 0) + \varepsilon_p(\frac{s}{p\sqrt{2}}, \frac{s}{p\sqrt{2}}, 0)$.*

3.2. The Frenet Formulas

In order to define the Frenet frame of a regular parametric curve in $\mathbb{E}^n$, we require that the regular parametric curve to be parametrized by arc-length. We assume that $\boldsymbol{\alpha}(s)$ is parametrized with arc-length in $\mathbb{D}_p^n$ such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha})(s) + (\varepsilon_p - p)\boldsymbol{y}(s)$. By Proposition 1, the curve $H_p(\boldsymbol{\alpha}(s)) = \boldsymbol{x}(s) + p\boldsymbol{y}(s)$ is a regular parametric curve in $\mathbb{E}^n$. The Frenet equations for the curve $H_p(\boldsymbol{\alpha}(s))$ in $\mathbb{E}^n$ are given in Theorem 1 of [1]. In the case $n = 3$, let $\kappa_{H_p(\boldsymbol{\alpha}(s))}$ denote the curvature of $H_p(\boldsymbol{\alpha}(s))$. Then,

by Theorem 3.2 in [13], we assume that $\kappa_{H_p(\boldsymbol{\alpha}(s))} > 0$. Throughout this section, we assume that $H_p(\alpha'(s))$, $H_p(\alpha''(s))$, ..., $H_p(\alpha^{(n-1)}(s))$ are linearly independent in $\mathbb{E}^n$ for all $s \in I$. We thus obtain the following theorem which is a modified version of the result given in Theorem 1 of [1].

Theorem 1. *Let $H_p(\boldsymbol{\alpha}) : I \to \mathbb{E}^n$ be a C^∞-mapping with arc-length parametrization. We assume that at each point $H_p(\boldsymbol{\alpha}(s))$, the vectors $H_p(\boldsymbol{\alpha}'(s)), H_p(\boldsymbol{\alpha}''(s)), \ldots, H_p(\boldsymbol{\alpha}^{(n-1)}(s))$ are linearly independent. Then there exists an orthonormal set of linearly independent vectors, called the Frenet n-frame $V_1(s), V_2(s), \ldots, V_n(s)$ satisfying the relations:*

$$\begin{aligned} V_1'(s) &= k_1(s)V_2(s) \\ V_i'(s) &= -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s) \\ V_n'(s) &= -k_{n-1}(s)V_{n-1}(s) \end{aligned} \tag{4}$$

Here the Frenet n-frame can be given as follows:

$$\begin{aligned} V_1(s) &= H_p(\boldsymbol{\alpha}'(s)) \\ V_i(s) &= \frac{E_i(s)}{\|E_i(s)\|}, \end{aligned} \tag{5}$$

for $i = 2, 3, \ldots, n-1$. $V_n(s)$ is the unique orientation-preserving unit vector satisfying the equalities $< V_n(s), V_i(s) >= 0$ for all $i = 2, 3, \ldots, n-1$ where

$$E_i(s) = H_p(\boldsymbol{\alpha}^i(s)) - \sum_{j=1}^{i-1} < H_p(\boldsymbol{\alpha}^i(s)), V_j(s) > V_j(s) \tag{6}$$

for all $i = 1, 2, \ldots, n-1$

Remark 2. *The n-th Frenet vector $V_n(s) \in \mathbb{E}^n$ is not constructed from an n-th derivative of the curve, as $H_p(\alpha^{(n)}(s))$ may not be linearly independent of the preceding derivatives (see [1]). The vector $V_n(s)$ is the unique unit vector in $\mathbb{E}^n$ orthogonal to $V_1(s), \ldots, V_{n-1}(s)$ such that the basis $\{V_1(s), \ldots, V_{n-1}(s), V_n(s)\}$ satisfies the standard positive orientation $\det(V_1(s), V_2(s), \ldots, V_n(s)) = 1$ for all $s \in I$. Moreover, the Frenet frame and the algorithm for arc-length parametrizations are presented in [14].*

We call the above relations the Frenet formulas, and $k_i(s)$ denotes the i-th curvature of $H_p(\boldsymbol{\alpha}(s))$. In the special case $n = 3$, $k_2(s)$ is also known as the torsion of $H_p(\boldsymbol{\alpha}(s))$.

Now we consider the curve $\boldsymbol{\alpha}(s)$ in $\mathbb{D}_p^n$ parametrized with arc-length such that $\boldsymbol{\alpha}(s) = \mathbf{x}(s) + \varepsilon_p \mathbf{y}(s)$. We investigate the generalized dual Frenet n-frame and the generalized dual Frenet equations of $\boldsymbol{\alpha}(s)$. Assume further that, at each point $\boldsymbol{\alpha}(s)$ the vectors $\boldsymbol{\alpha}'(s), \boldsymbol{\alpha}''(s), \ldots, \boldsymbol{\alpha}^{(n-1)}(s)$ are linearly independent. First, we define the vector
$F_1 = W_1 = \boldsymbol{\alpha}'(s) = H_p(\boldsymbol{\alpha}'(s)) + (\varepsilon_p - p)\mathbf{y}'(s) = E_1 + (\varepsilon_p - p)E_1^* = V_1 + (\varepsilon_p - p)V_1^*$ such that $< W_1, W_1 >_p = 1$, where V_1 is the Frenet vector of the curve $H_p(\boldsymbol{\alpha}(s))$ in (5). Now we consider the vector
$F_2 = \boldsymbol{\alpha}''(s) - < \boldsymbol{\alpha}''(s), F_1 >_p F_1$ such that $< F_1, F_2 >_p = 0$. Using the equality, we have $F_2 = E_2 + (\varepsilon_p - p)E_2^*$, where

$$E_2 = H_p(\boldsymbol{\alpha}''(s)) - < H_p(\boldsymbol{\alpha}''(s)), V_1 > V_1$$

and

$$E_2^* = y''(s) - < H_p(\boldsymbol{\alpha}''(s), V_1 > V_1^* - \left(< H_p(\boldsymbol{\alpha}''(s), V_1^* > + < y''(s), V_1 >\right) V_1$$

By the equality (2), since E_2 has the form given in (6) such that $\|E_2\| \neq 0$, we obtain $\|F_2\|_p = \|E_2\| + (\varepsilon_p - p)\frac{<E_2, E_2^*>}{\|E_2\|} \neq 0$. Using the equality (3), we have $W_2 = \frac{F_2}{\|F_2\|_p} = \frac{E_2}{\|E_2\|} + \frac{(\varepsilon_p - p)}{\|E_2\|}\left(E_2^* - \frac{<E_2, E_2^*>}{\|E_2\|^2} E_2\right) = V_2 + (\varepsilon_p - p)V_2^*$ such that $< W_1, W_2 >_p = 0$ and $< W_2, W_2 >_p = 1$, where V_2 is the Frenet vector of the curve $H_p(\boldsymbol{\alpha}(s))$ in (5) and $V_2^* = \frac{E_2^*}{\|E_2\|} - \frac{<E_2, E_2^*> E_2}{\|E_2\|^3}$.

Here, by continuing the Gram–Schmidt orthogonalization procedure, the following generalized dual Frenet n-frame is obtained:

$$\begin{aligned} W_1(s) &= V_1(s) + (\varepsilon_p - p)V_1^*(s) \\ W_i(s) &= V_i(s) + (\varepsilon_p - p)V_i^*(s) \end{aligned} \tag{7}$$

for $i = 2, 3, \ldots, n-1$.

$W_n(s)$ is a unique vector satisfying the equalities $< W_n(s), W_i(s) >_p = 0$ for all $i = 1, 2, \ldots, n-1$ where $V_i(s)$ has the form given in (6) and

$$V_i^* = \frac{E_i^*}{\|E_i\|} - \frac{< E_i, E_i^* > E_i}{\|E_i\|^3}$$

for all $i = 2, \ldots, n-1$. Here $V_i(s)$ is the form (6) and

$$V_i^* = \frac{E_i^*}{\|E_i\|} - \frac{< E_i, E_i^* > E_i}{\|E_i\|^3} \tag{8}$$

and

$$E_i^*(s) = y^{(i)}(s) - \sum_{j=1}^{i-1}\left(< H_p(\boldsymbol{\alpha}^{(i)}(s), E_j > E_j^* + \left(< H_p(\boldsymbol{\alpha}^{(i)}(s), E_j^* > + < y^{(i)}(s), E_j >\right) E_j\right) \tag{9}$$

for all $i = 2, \ldots, n-1$. From (1), $W_n(s)$ is a unique vector satisfying the equalities $< W_n(s), W_i(s) >_p = 0$ for all $i = 1, 2, \ldots, n-1$. Moreover, we set $W_n(s) = V_n(s) + (\varepsilon_p - p)V_n^*(s)$, where $V_n(s)$ is the unique orientation-preserving unit vector in $\mathbb{E}^n$ defined in (1). To satisfy the dual orthonormality conditions $\langle W_n(s), W_i(s)\rangle_p = 0$ for all $i = 1, 2, \ldots, n-1$ and $\langle W_n(s), W_n(s)\rangle_p = 1$, the vector $V_n^*(s) \in \mathbb{E}^n$ is uniquely determined by the equations $\langle V_n^*(s), V_i(s)\rangle = -\langle V_n(s), V_i^*(s)\rangle$ and $\langle V_n^*(s), V_n(s)\rangle = 0$ for $i = 1, 2, \ldots, n-1$.

The following theorem gives the generalized dual Frenet equations of the curve $\boldsymbol{\alpha}(s)$ parametrized by arc length curve in $\mathbb{D}_p^n$.

Theorem 2. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be a C^∞-mapping with arc-length parametrization. We assume that at each point $\boldsymbol{\alpha}(s)$, the vectors $\boldsymbol{\alpha}'(s), \boldsymbol{\alpha}''(s), \ldots, \boldsymbol{\alpha}^{(n-1)}(s)$ are linearly independent. Then the generalized dual Frenet n-frame $W_1(s), W_2(s), \ldots, W_n(s)$ of the curve $\boldsymbol{\alpha}(s)$ in (7) satisfies following relations:*

$$\begin{aligned} W_1'(s) &= K_1(s)W_2(s) \\ W_i'(s) &= -K_{i-1}(s)W_{i-1}(s) + K_i(s)W_{i+1}(s) \\ W_n'(s) &= -K_{n-1}(s)W_{n-1}(s) \end{aligned} \tag{10}$$

for all $i = 2, 3, \ldots, n-1$ and where $K_j : I \to \mathbb{D}_p$ is a dual function for $j = 1, 2, \ldots, n-1$.

Proof. The construction of Frenet frame and the derivation of the curvatures in n-dimensional Euclidean space are given in [1] and [14]. Similar proofs can be given in this space. We consider the generalized dual Frenet frame $W_1(s), \dots W_n(s)$ of the regular parametric curve $\boldsymbol{\alpha}(s)$ in (7). Hence formulas for the derivatives W_i' can be obtained by first differentiating the orthonormality relations $< W_i(s), W_j(s) >_p = \delta_{ij}$, to get $< W_i'(s), W_j(s) >_p = - < W_i(s), W_j'(s) >_p$. From [15], let $W_i'(s) = \sum_{k=1}^{i+1} \lambda_{ik} W_k(s)$ and $W_j'(s) = \sum_{l=1}^{j+1} \lambda_{jl} W_l(s)$ for $\lambda_{ik}, \lambda_{jl} \in \mathbb{D}_p$. Using the equality $< W_i'(s), W_j(s) >_p = - < W_i(s), W_j'(s) >_p$, we have $\lambda_{ij} + \lambda_{ji} = 0$ for all i, j, $\lambda_{ij} = 0$ for $j < i-1$ and $\lambda_{ji} = 0$ for $j > i+1$, where $i, j = 1, 2, \dots, n-1$. Hence $W_i'(s) = \lambda_{ii-1} W_{i-1}(s) + \lambda_{ii+1} W_{i+1}(s) = -K_{i-1}(s) W_{i-1}(s) + K_i(s) W_{i+1}(s)$, where $K_i(s) = \lambda_{ii+1} = -\lambda_{i+1i}$ and for all $i = 1, 2, \dots, n-1$. Moreover, since $W_i^{(n)}(s)$ is a linear combination of $W_1(s), \dots W_n(s)$, we have $W_i^{(n)}(s) = -K_{n-1}(s) W_{n-1}(s)$. □

Definition 6. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be a C^∞-mapping with arc-length parametrization such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}(s)) + (\varepsilon_p - p)\boldsymbol{y}(s)$. Let $k_i(s)$ denote the i th curvatures of the curve $H_p(\boldsymbol{\alpha}(s))$. Then the functions $K_i : I \to \mathbb{D}_p$ such that $K_i(s) = k_i(s) + (\varepsilon_p - p)k_i^*$ are called i-th generalized dual curvatures of $\boldsymbol{\alpha}(s)$.*

Theorem 3. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be a C^∞-mapping with arc-length parametrization such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}(s)) + (\varepsilon_p - p)\boldsymbol{y}(s)$. Assume that $V_1(s), V_2(s), \dots, V_n(s)$ are the Frenet n-frame of the curve $H_p(\boldsymbol{\alpha}(s))$ and $W_1(s), W_2(s), \dots, W_n(s)$ are the generalized dual Frenet n-frame of the curve $\boldsymbol{\alpha}(s)$. Then, the generalized dual Frenet n-frame $W_i(s) = V_i(s) + (\varepsilon_p - p)V_i^*(s)$ and the generalized dual curvatures $K_i(s) = k_i(s) + (\varepsilon_p - p)k_i^*(s)$ are given by the following expressions, where $V_i'(s)$ is the form in (1) and*

$$\begin{aligned} (V_i^*)'(s) &= \left(-k_{i-1}(s)V_{i-1}^*(s) + k_i(s)V_{i+1}^*(s)\right) + \left(-k_{i-1}^*(s)V_{i-1}(s) + k_i^*(s)V_{i+1}(s)\right) \\ k_i(s) &= < V_i'(s), V_{i+1}(s) > \\ k_i^*(s) &= < V_i'(s), V_{i+1}^*(s) > + < (V_i^*)'(s), V_{i+1}(s) > \end{aligned} \tag{11}$$

for all $i = 1, 2, \dots, n-1$.

Proof. The proof follows from the equality (7), Theorem 1 and Theorem 2. To clarify the theorem, we verify the stated formulas explicitly for $n = 3$. In this case, we have $i = 1, 2$. In the case $i = 1$, using the equality (8), we obtain the equality $(V_1^*)'(s)$. From the equalities (10), we have $K_1 = < W_1', W_2 >_p = < V_1', V_2 > + (\varepsilon_p - p)(< V_1', V_2^* > + < (V_1^*)', V_2 >)$. Hence we obtain $k_1(s) = < V_1', V_2 >$ and $k_1^*(s) = < V_1', V_2^* > + < (V_1^*)', V_2 >$. We consider the case $i = 2$. Then, using the equalities (10), we have $K_2 = < W_2', W_3 >_p = < V_2', V_3 > + (\varepsilon_p - p)(< V_2', V_3^* > + < (V_2^*)', V_3 >)$. Hence we obtain $k_2(s) = < V_2', V_3 >$ and $k_2^*(s) = < V_2', V_3^* > + < (V_2^*)', V_3 >$. Similarly, using the equalities (7) and (10), we have $(V_2^*)'(s) = -k_1(s)V_1^*(s) + k_2(s)V_3^*(s)$. The remaining identities in theorem follow in same way. □

In the case $n = 3$, the following corollary follows from direct computations based on Theorems 2 and 3. In this case, we denote the generalized dual Frenet frame of $\boldsymbol{\alpha}$ by $\{T_{\boldsymbol{\alpha}}, N_{\boldsymbol{\alpha}}, B_{\boldsymbol{\alpha}}\}$ and $\kappa_{\boldsymbol{\alpha}}$ and $\tau_{\boldsymbol{\alpha}}$ denote its curvature and torsion, respectively. In addition, we denote the Frenet frame of $H_p(\boldsymbol{\alpha})$ by $\{T_{H_p(\boldsymbol{\alpha})}, N_{H_p(\boldsymbol{\alpha})}, B_{H_p(\boldsymbol{\alpha})}\}$ and $\kappa_{H_p(\boldsymbol{\alpha})}$ and $\tau_{H_p(\boldsymbol{\alpha})}$ denote the curvature and torsion of $H_p(\boldsymbol{\alpha})$, respectively.

Corollary 1. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^3$ be a C^∞-mapping with arc-length parametrization such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}) + (\varepsilon_p - p)\boldsymbol{y}(s)$. Then*

$$\begin{aligned} T_{\boldsymbol{\alpha}} =& T_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\boldsymbol{y}' \\ N_{\boldsymbol{\alpha}} =& N_{H_p(\boldsymbol{\alpha})} + \frac{(\varepsilon_p - p)}{\kappa_{H_p(\boldsymbol{\alpha})}}\left(\boldsymbol{y}'' - \kappa^*_{\boldsymbol{\alpha}} N_{H_p(\boldsymbol{\alpha})}\right) \\ B_{\boldsymbol{\alpha}} =& B_{H_p(\boldsymbol{\alpha})} + \frac{(\varepsilon_p - p)}{\kappa_{H_p(\boldsymbol{\alpha})}}\left(T_{\boldsymbol{\alpha}} \times y'' + \kappa_{H_p(\boldsymbol{\alpha})}\boldsymbol{y}' \times N_{H_p(\boldsymbol{\alpha})} - \kappa^*_{\boldsymbol{\alpha}} B_{H_p(\boldsymbol{\alpha})}\right)' \\ \kappa_{\boldsymbol{\alpha}} =& \kappa_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\kappa^*_{\boldsymbol{\alpha}} \\ \tau_{\boldsymbol{\alpha}} =& \tau_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\tau^*_{\boldsymbol{\alpha}} \end{aligned} \tag{12}$$

where $\kappa^*_{\boldsymbol{\alpha}} = \left\langle \boldsymbol{y}'', N_{H_p(\boldsymbol{\alpha})} \right\rangle$ *and* $\tau^*_{\boldsymbol{\alpha}} = \frac{\left\langle \boldsymbol{y}''' - \frac{\kappa'_{H_p(\boldsymbol{\alpha})}}{\kappa_{H_p(\boldsymbol{\alpha})}}\boldsymbol{y}'' + \kappa^2_{H_p(\boldsymbol{\alpha})}\boldsymbol{y}', B_{H_p(\boldsymbol{\alpha})} \right\rangle}{\kappa_{H_p(\boldsymbol{\alpha})}} - \tau_{H_p(\boldsymbol{\alpha})}\kappa^*_{\boldsymbol{\alpha}}$.

Proof. Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^3$ be a C^∞-mapping with arc-length parametrization such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}) + (\varepsilon_p - p)\boldsymbol{y}(s)$. In Proposition 2.8 of [9], we put $\alpha(s) := H_p(\boldsymbol{\alpha})(s)$ and $\beta(s) := \boldsymbol{y}(s)$. The result follows by the same argument as in the proof of Proposition 2.8 in [9]. □

3.3. Some Special RGDP-Curves

The following theorem about curves of constant curvature in $\mathbb{D}_p^2$ is a generalization of Theorem 3.1 in [9].

Theorem 4. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^2$ be a C^∞-mapping with arc-length parametrization such that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}(s)) + (\varepsilon_p - p)\boldsymbol{y}(s)$. If the curvature of the curve $\boldsymbol{\alpha}(s)$ is a constant, then*

$$H_p(\boldsymbol{\alpha}(s)) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right) \tag{13}$$

and

$$\boldsymbol{y}(s) = \left(-r(as+b)\sin\frac{s}{r} - ar^2\cos\frac{s}{r}, r(as+b)\cos\frac{s}{r} - ar^2\sin\frac{s}{r}\right) \tag{14}$$

for all $a, b, p \in \mathbb{R}$ and $r > 0$.

The following theorems generalize the results presented in Section 4, 5 and 6 of [9], and the following theorems were obtained:

Theorem 5. *Let $K : I \to \mathbb{D}_p$ be a generalized dual C^∞-function such that $K = k_1 + (\varepsilon_p - p)k_1^*$, where $k_1, k_1^* : I \to \mathbb{R}$ such that $k_1 > 0$. Then there exists a generalized dual curve $\boldsymbol{\alpha} : I \to \mathbb{D}_p^2$ whose curvature is K. In this case, for the curve $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}(s)) + (\varepsilon_p - p)\boldsymbol{y}(s)$, the components of the curve $\boldsymbol{\alpha}$ are given by the following expressions:*

$$H_p(\boldsymbol{\alpha}(s)) = \left(\int_{s_0}^{s} \cos\theta(u)du, \int_{s_0}^{s} \sin\theta(u)du\right) \tag{15}$$

and

$$\boldsymbol{y}(s) = -\int_{s_0}^{s}\left[\int_{s_0}^{u} k_1^*(w)dw\right](-\sin\theta(u), \cos\theta(u))du, \tag{16}$$

where the curvature of the curve $H_p(\boldsymbol{\alpha}(s)$ is $\theta'(s)$.

Theorem 6. *Let $K_1, K_2 : I \to \mathbb{D}_p$ be generalized dual C^∞-functions such that $K_i = k_i + (\varepsilon_p - p)k_i^*$ for $i = 1, 2$, where $k_i, k_i^* : I \to \mathbb{R}$ such that $k_1 > 0$. Then there exists a generalized dual curve*

$\boldsymbol{\alpha} : I \to \mathbb{D}_p^3$ whose curvature and torsion are K_1 and K_2 respectively. In this case, the dual parts of the curve $\boldsymbol{\alpha}(s) = \boldsymbol{x}(s) + \varepsilon_p \boldsymbol{y}(s) = H_p(\boldsymbol{\alpha}(s)) + (\varepsilon_p - p)\boldsymbol{y}(s)$ take the following forms:

$$\begin{aligned} k_1^*(s) &= H_p(\boldsymbol{\alpha}'(s)) - \boldsymbol{y}k_2(s) \\ k_2^*(s) &= \frac{k_1\boldsymbol{y}'' + k_1(k_2 H_p(\boldsymbol{\alpha}') + H_p(\boldsymbol{\alpha})k_2') - k_1'(k_2 H_p(\boldsymbol{\alpha}) + \boldsymbol{y}') + k_1^3\boldsymbol{y}}{k_1^2} \end{aligned} \tag{17}$$

In this case, the curvature and the torsion of the curve $H_p(\boldsymbol{\alpha})$ are $k_1(s)$ and $k_2(s)$, respectively. Moreover, $H_p(\boldsymbol{\alpha})$ is unique up to a rigid motion of $\mathbb{E}^3$.

3.4. Arbitrary-Speed RGDP-Curves

Since the Frenet formulas in this space are valid only for unit-speed *RGDP*-curves, we have thus far examined these formulas alongside some of their fundamental properties. Now, we aim to define curves of arbitrary speed and derive the corresponding Frenet formulas for the case $n = 3$.

In this section, the results obtained for unit-speed generalized dual parametric curves are extended to arbitrary-speed curves. In the case $n = 3$, this extension was described in [13] for regular curves in Euclidean space. By applying this method, the following results are obtained for generalized dual regular parametric curves.

Definition 7. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be an RGDP- curve such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p y(t) = H_p(\boldsymbol{\alpha}) + (\varepsilon_p - p)y(t)$. The curve $\boldsymbol{\alpha}(t)$ is called an arbitrary-speed RGDP-curve if $\|H_p(\boldsymbol{\alpha}')\| \neq 0$ and $< H_p(\boldsymbol{\alpha}'), \boldsymbol{y}'(t) >= 0$ for all $t \in I$.*

Let $v(t)$ be the speed function of the *RGDP*-curve $\boldsymbol{\alpha}(t)$. From Definition 7, we have $v(t) = \|\boldsymbol{\alpha}'(t)\|_p = \|H_p(\boldsymbol{\alpha}'(t))\| + (\varepsilon_p - p)\frac{\langle H_p(\boldsymbol{\alpha}'(t)), \boldsymbol{y}'(t)\rangle}{\|H_p(\boldsymbol{\alpha}'(t))\|} = \|H_p(\boldsymbol{\alpha}'(t))\| > 0$ and $\frac{ds}{dt} = v(t)$. Clearly, the speed function of $\boldsymbol{\alpha}$ is equal to the speed function of the Euclidean regular parametric curve $H_p(\boldsymbol{\alpha})$.

Theorem 7. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be an arbitrary-speed RGDP-curve such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p y(t) = H_p(\boldsymbol{\alpha}(t)) + (\varepsilon_p - p)y(t)$. Assume that $V_1(t), V_2(t), \ldots, V_n(t)$ are the Frenet n-frame of the curve $H_p(\boldsymbol{\alpha}(t))$ and $W_1(t), W_2(t), \ldots, W_n(t)$ are the generalized dual Frenet n-frame of the curve $\boldsymbol{\alpha}(t)$. Then, the generalized dual Frenet n-frame $W_i(t) = v(t)\big(V_i(t) + (\varepsilon_p - p)V_i^*(t)\big)$ and the generalized dual curvatures $K_i(t) = v(t)\big(k_i(t) + (\varepsilon_p - p)k_i^*(t)\big)$ are given by the following expressions, where $V_i'(t)$ is the form in (1) and*

$$\begin{aligned} (V_i^*)'(t) =& v(t)\big[\big(-k_{i-1}(t)V_{i-1}^*(t) + k_i(t)V_{i+1}^*(t)\big)\big] + \big(-k_{i-1}^*(t)V_{i-1}(t) + k_i^*(t)V_{i+1}(t)\big) \\ k_i(t) =& \frac{< V_i'(t), V_{i+1}(s) >}{v(t)} \\ k_i^*(t) =& \frac{< V_i'(t), V_{i+1}^*(t) > + < (V_i^*)'(t), V_{i+1}(t) >}{v(t)} \end{aligned}$$

for all $i = 1, 2, \ldots, n-1$.

Proof. Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be an arbitrary-speed *RGDP*-curve such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p \boldsymbol{y}(t) = H_p(\boldsymbol{\alpha}(t)) + (\varepsilon_p - p)\boldsymbol{y}(t)$. Then the speed of $\boldsymbol{\alpha}$ in $\mathbb{D}_p^n$ is equal to the speed of $H_p(\boldsymbol{\alpha})$ $v(t) = \|\boldsymbol{\alpha}'(t)\|_p = \|H_p(\boldsymbol{\alpha}'(t))\| > 0$ in $\mathbb{E}^n$. As in Section 2.4 of [13], we transfer to $\boldsymbol{\alpha}$ the generalized dual Frenet apparatus of a unit-speed reparametrization $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$. Clearly, if s is an arc-length function for $\boldsymbol{\alpha}$ as in Proposition 1, then $\boldsymbol{\alpha}(t) = \boldsymbol{\beta}(s(t))$ for all $t \in I$. In this case, let $W_i^\beta, K_i^\beta, i = 1, 2, \ldots, n$ be the generalized dual Frenet apparatus of $\boldsymbol{\beta}$. We define for $\boldsymbol{\alpha}$ the generalized dual Frenet apparatus $W_i = W_i^\beta(s), K_i = K_i^\beta(s), i = 1, 2, \ldots, n$.

By the chain rule, we have $W_1' = (W_1^\beta)'(s)\frac{ds}{dt} = v(t)(W_1^\beta)'(s)$. Hence by the generalized dual Frenet equations in (10), we have $(W_1^\beta)' = K_1^\beta W_2^\beta = K_1 W_2$. From these equalities, we obtain $W_1' = v(t)K_1 W_2$. Then, in the case $i = 1$, using this equality, Equation (7) and the proof of Theorem 3, we obtain the equalities $(V_1^*)'(t) = v(t)\big(k_1(t)V_2^*(t) + k_1^*(t)V_2(t)\big)$, $k_1(t) = \frac{<V_1'(t),V_2(s)>}{v(t)}$ and $k_1^*(t) = \frac{<V_1'(t),V_2^*(t)>+<(V_1^*)'(t),V_2(t)>}{v(t)}$ in the theorem. The remaining identities follow similarly. □

We consider an arbitrary-speed $RGDP$-curve $\boldsymbol{\alpha}(t)$ for the case $n = 3$. The following corollary can be obtained by computations from Theorem 7. In this case, we denote the generalized dual Frenet frame of an arbitrary-speed curve $\boldsymbol{\alpha}$ by $\{T_{\boldsymbol{\alpha}}, N_{\boldsymbol{\alpha}}, B_{\boldsymbol{\alpha}}\}$ and $\kappa_{\boldsymbol{\alpha}}$ and $\tau_{\boldsymbol{\alpha}}$ stand for its curvature and torsion, resp. In addition, we denote the Frenet frame of $H_p(\boldsymbol{\alpha})$ by $\{T_{H_p(\boldsymbol{\alpha})}, N_{H_p(\boldsymbol{\alpha})}, B_{H_p(\boldsymbol{\alpha})}\}$ and $\kappa_{H_p(\boldsymbol{\alpha})}$ and $\tau_{H_p(\boldsymbol{\alpha})}$ stand for $H_p(\boldsymbol{\alpha})$ curvature and torsion, resp.

Corollary 2. *Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be an arbitrary-speed RGDP-curve such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p \boldsymbol{y}(t) = H_p(\boldsymbol{\alpha}(t)) + (\varepsilon_p - p)\boldsymbol{y}(t)$ and $v = v(t) = \|H_p(\boldsymbol{\alpha}'(t))\|$. Then*

$$\begin{aligned}
T_{\boldsymbol{\alpha}} =& T_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\frac{\boldsymbol{y}'}{v}\\
N_{\boldsymbol{\alpha}} =& N_{H_p(\boldsymbol{\alpha})} + \frac{(\varepsilon_p - p)}{v^2\kappa_{H_p(\boldsymbol{\alpha})}}\left(\boldsymbol{y}'' - \frac{v'}{v}\boldsymbol{y}' - v^2\kappa^*_{H_p(\boldsymbol{\alpha})}N_{H_p(\boldsymbol{\alpha})}\right)\\
B_{\boldsymbol{\alpha}} =& B_{H_p(\boldsymbol{\alpha})} + \frac{(\varepsilon_p - p)}{v^2\kappa_{H_p(\boldsymbol{\alpha})}}\left[T_{\boldsymbol{\alpha}} \times (\boldsymbol{y}'' - \frac{v'}{v}\boldsymbol{y}') + v\kappa_{H_p(\boldsymbol{\alpha})}(\boldsymbol{y}' \times N_{H_p(\boldsymbol{\alpha})}) - v^2\kappa^*_{\boldsymbol{\alpha}}B_{H_p(\boldsymbol{\alpha}}\right]'\\
\kappa_{\boldsymbol{\alpha}} =& \kappa_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\kappa^*_{\boldsymbol{\alpha}}\\
\tau_{\boldsymbol{\alpha}} =& \tau_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\tau^*_{\boldsymbol{\alpha}}
\end{aligned}$$

where

$$\kappa^*_{\boldsymbol{\alpha}} = \frac{1}{v^2}\left\langle \boldsymbol{y}'' - \frac{v'}{v}\boldsymbol{y}', N_{H_p(\boldsymbol{\alpha})}\right\rangle$$

and

$$\tau^*_{\boldsymbol{\alpha}} = \frac{\left\langle \boldsymbol{y}''' - \left(\frac{\kappa'_{H_p(\boldsymbol{\alpha})}}{\kappa_{H_p(\boldsymbol{\alpha})}} + \frac{3v'}{v}\right)\boldsymbol{y}'' + \left(v^2\kappa^2_{H_p(\boldsymbol{\alpha})} - \frac{v''}{v} + 3\left(\frac{v'}{v}\right)^2 + \frac{v'\kappa'_{H_p(\boldsymbol{\alpha})}}{v\kappa_{H_p(\boldsymbol{\alpha})}}\right)\boldsymbol{y}', B_{H_p(\boldsymbol{\alpha})}\right\rangle}{v^3\kappa_{H_p(\boldsymbol{\alpha})}} - \frac{\tau_{H_p(\boldsymbol{\alpha})}\kappa^*_{\boldsymbol{\alpha}}}{\kappa_{H_p(\boldsymbol{\alpha})}}$$

Proof. Let $\boldsymbol{\alpha} : I \to \mathbb{D}_p^n$ be an arbitrary-speed $RGDP$-curve such that $\boldsymbol{\alpha}(t) = \boldsymbol{x}(t) + \varepsilon_p\boldsymbol{y}(t) = H_p(\boldsymbol{\alpha}(t)) + (\varepsilon_p - p)\boldsymbol{y}(t)$ and $v = v(t) = \|H_p(\boldsymbol{\alpha}'(t))\|$. Since $\boldsymbol{\alpha} = \boldsymbol{\beta}(s)$, where s is the arc-length function of $\boldsymbol{\alpha}$, we have $\boldsymbol{\alpha}' = \boldsymbol{\beta}'(s)\frac{ds}{dt} = vW_1^\beta = vT_{\boldsymbol{\alpha}}$. Hence we obtain $T_{\boldsymbol{\alpha}} = \frac{\boldsymbol{\alpha}'}{v} = \frac{H_p(\boldsymbol{\alpha}')}{v} + (\varepsilon_p - p)\frac{\boldsymbol{y}'}{v}$. Here, by putting $\frac{H_p(\boldsymbol{\alpha}')}{v} = T_{H_p(\boldsymbol{\alpha})}$, we have the equality $T_{\boldsymbol{\alpha}} = T_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)\frac{\boldsymbol{y}'}{v}$. Differentiating $T_{\boldsymbol{\alpha}}$, we obtain

$$T'_{\boldsymbol{\alpha}} = T'_{H_p(\boldsymbol{\alpha})} + (\varepsilon_p - p)(\frac{\boldsymbol{y}'}{v})'. \tag{18}$$

In addition, using the equality (10), we have $T'_{\boldsymbol{\alpha}} = v\kappa_{\boldsymbol{\alpha}}N_{\boldsymbol{\alpha}}$. Here, by putting $\kappa_{\boldsymbol{\alpha}} = \kappa + (\varepsilon_p - p)\kappa^*$ and $N_{\boldsymbol{\alpha}} = N + (\varepsilon_p - p)N^*$, we have

$$T'_{\boldsymbol{\alpha}} = v(\kappa + (\varepsilon_p - p)\kappa^*)(N + (\varepsilon_p - p)N^*) = v\kappa N + (\varepsilon_p - p)(v\kappa^* N + v\kappa N^*. \tag{19}$$

The equalities (18) and (19) imply $\kappa = \kappa_{H_p(\boldsymbol{\alpha})}$, $\kappa^* = \frac{1}{v^3} < \boldsymbol{y}''v - \boldsymbol{y}'v', N_{H_p(\boldsymbol{\alpha})} >$, $N = N_{H_p(\boldsymbol{\alpha})}$ and $N^* = \frac{1}{v\kappa_{H_p(\boldsymbol{\alpha})}}\left((\frac{\boldsymbol{y}'}{v})' - < (\frac{\boldsymbol{y}'}{v})', N_{H_p(\boldsymbol{\alpha})} > N_{H_p(\boldsymbol{\alpha})}\right)$. Other equalities in the theorem are derived from the proofs of Corollary 1 and Theorem 7 in the same way. □

Remark 3. *In Corollary 2, we find effectively computable expressions for the Frenet apparatus in* $\mathbb{D}_p^3$. *The curvature and torsion of an arbitrary-speed RGDP-curve* $\boldsymbol{\alpha}(t)$ *can also be expressed as follows* $\kappa_{\boldsymbol{\alpha}} = \frac{\|\boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}''\|_p}{\|\boldsymbol{\alpha}'\|_p}$ *and* $\tau_{\boldsymbol{\alpha}} = \frac{\langle \boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'', \boldsymbol{\alpha}''' \rangle_p}{\|\boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}''\|_p^2}$. *Indeed, these equalities show that for RGDP-curves,* $\|\boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}''\|_p \neq 0$ *is equivalent to the usual condition* $\kappa_{\boldsymbol{\alpha}} \neq 0$. *In addition,* $\|\boldsymbol{\alpha}'\|_p = \|H_p(\boldsymbol{\alpha}')\| > 0$.

4. Similarity Detection of Rational Algebraic Generalized Dual Space Curves

In this section, we propose an algorithm for determining whether two rational algebraic generalized dual curves are equivalent under a similarity transformation. The proposed method, inspired by Alcázar et al. [12], relies on the dual curvature and dual torsion obtained in the preceding section.

4.1. Similarities and Isometries of Rational Algebraic Generalized Dual Space Curves

Let $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{D}_p^3$ be two rational algebraic generalized dual curves parametrized by

$$\begin{aligned} \boldsymbol{\alpha}_j : \mathbb{R} &\longrightarrow \boldsymbol{C}_j \subset \mathbb{D}_p^3, \\ t &\longmapsto \boldsymbol{\alpha}_j(t) = H_p(\boldsymbol{\alpha}_j(t)) + (\varepsilon_p - p)\boldsymbol{y}_j(t), \qquad j \in \{1,2\}, \end{aligned} \tag{20}$$

where

$$H_p(\boldsymbol{\alpha}_j), \boldsymbol{y}_j : \mathbb{R} \longrightarrow \mathbb{R}^3$$

are rational algebraic parametrizations. We further assume that the following conditions hold:

(i) The parametrizations $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, respectively, are *proper*. In particular, the corresponding real parametrizations $H_p(\boldsymbol{\alpha}_j)$ and $\boldsymbol{y}_j$ are proper, i.e., birational onto their images. Thus, except possibly for finitely many points, each point of the corresponding curve is attained by a unique parameter value.
(ii) The parametrizations $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ have the same degree n.
(iii) Neither of the curves parametrized by $H_p(\boldsymbol{\alpha}_j)$, $j \in \{1,2\}$, is a line.
(iv) For each $j \in \{1,2\}$, the velocity vectors $H_p(\boldsymbol{\alpha}_j'(t))$ and $\boldsymbol{y}_j'(t)$ are orthogonal for every admissible value of t, that is,

$$\left\langle H_p(\boldsymbol{\alpha}_j'(t)), \boldsymbol{y}_j'(t) \right\rangle = 0.$$

(v) For each $j \in \{1,2\}$, the numerator and denominator of every component of both $H_p(\boldsymbol{\alpha}_j(t))$ and $\boldsymbol{y}_j(t)$ are relatively prime.

Definition 8. *A similarity transformation of the generalized dual space* $\mathbb{D}_p^3$ *is a mapping that preserves ratios of generalized dual distances. Equivalently, a mapping*

$$f : \mathbb{D}_p^3 \longrightarrow \mathbb{D}_p^3$$

is a similarity transformation if and only if it can be expressed as

$$f(\boldsymbol{\alpha}) = \lambda \boldsymbol{Q}\boldsymbol{\alpha} + \boldsymbol{b}, \qquad \lambda \in \mathbb{R}^+, \quad \boldsymbol{b} \in \mathbb{D}_p^3, \quad \boldsymbol{Q} \in \mathbb{D}_p^{3\times 3}, \quad \boldsymbol{Q}^T\boldsymbol{Q} = I_3. \tag{21}$$

where $\mathbb{D}_p^{3\times 3}$ *is the set of generalized dual matrices of size* 3×3.

Writing

$$\boldsymbol{Q} = \boldsymbol{Q}_1 + (\varepsilon_p - p)\boldsymbol{Q}_2,$$

where $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathbb{R}^{3\times 3}$ *and* $\mathbb{R}^{3\times 3}$ *is the set of real matrices of size* 3×3*, the orthogonality condition* $\boldsymbol{Q}^T\boldsymbol{Q} = I_3$ *yields*

$$\boldsymbol{Q}_1^T\boldsymbol{Q}_1 = \boldsymbol{Q}_1\boldsymbol{Q}_1^T = I_3, \tag{22}$$

together with

$$\boldsymbol{Q}_1^T\boldsymbol{Q}_2 + \boldsymbol{Q}_2^T\boldsymbol{Q}_1 = 0_3, \tag{23}$$

and

$$\boldsymbol{Q}_1\boldsymbol{Q}_2^T + \boldsymbol{Q}_2\boldsymbol{Q}_1^T = 0_3. \tag{24}$$

Consequently, $\boldsymbol{Q}_1$ *is an orthogonal matrix, while* $\boldsymbol{Q}_1^T\boldsymbol{Q}_2$ *and* $\boldsymbol{Q}_1\boldsymbol{Q}_2^T$ *are skew-symmetric matrices. The positive scalar* λ *is called the similarity ratio.*

A similarity f satisfies the following equations,

$$\| f(\boldsymbol{\alpha}_1) - f(\boldsymbol{\alpha}_2) \|_p = \lambda \| \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 \|_p, \qquad \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{D}_p^3, \tag{25}$$

If $\lambda = 1$ then f is a *dual isometry*, i.e., f preserves dual distances.

This section considers similar algebraic dual space curves (20), that is, one is the image of the other under a similarity. We say that $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are related by a similarity f when $f(\boldsymbol{C}_1) = \boldsymbol{C}_2$. Moreover, a rational dual curve $\boldsymbol{C} \subset \mathbb{D}_p^3$ has a symmetry/self-similarity if there exists a similarity f such that $f(\boldsymbol{C}) = \boldsymbol{C}$.

Our goal is to analyze whether the curves $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are similar. We recall the following result from [12], which is central to our algorithm. We first recall that a *Möbius transformation* is a rational function

$$\varphi(t) = \frac{at + b}{ct + d}, \qquad \Delta := ad - bc \neq 0. \tag{26}$$

It is well known that the birational functions on the line are the Möbius transformations [16].

Theorem 8. *[12] Let* $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{D}_p^3$ *be rational algebraic generalized dual space curves with proper parametrizations* $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 : \mathbb{R} \to \mathbb{D}_p^3$*. Then* $\boldsymbol{C}_1, \boldsymbol{C}_2$ *are similar if and only if there exists a similarity* f *and a Möbius transformation* φ *such that*

$$\boldsymbol{\alpha}_2(\varphi(t)) = f(\boldsymbol{\alpha}_1(t)). \tag{27}$$

Notice that the coefficients of φ, i.e., a, b, c, d in (26) are defined up to a common multiple.

For brevity, we write Equation (27) as $\lambda\boldsymbol{Q}\boldsymbol{\alpha}_1 + \boldsymbol{b} = \boldsymbol{\alpha}_2 \circ \varphi$.

Given that $\boldsymbol{Q} := \boldsymbol{Q}_1 + (\varepsilon_p - p)\boldsymbol{Q}_2$, $\boldsymbol{\alpha}_j(t) := H_p(\boldsymbol{\alpha}_j)(t) + (\varepsilon_p - p)\boldsymbol{y}_j(t), j \in \{1, 2\}$ and $\boldsymbol{b} := \boldsymbol{b}_1 + (\varepsilon_p - p)\boldsymbol{b}_2$, the following two equations are obtained from Equation (27);

$$\lambda\boldsymbol{Q}_1 H_p(\boldsymbol{\alpha}_1) + \boldsymbol{b}_1 = H_p(\boldsymbol{\alpha}_2) \circ \varphi, \tag{28}$$

$$\lambda(\boldsymbol{Q}_2 H_p(\boldsymbol{\alpha}_1) + \boldsymbol{Q}_1\boldsymbol{y}_1) + \boldsymbol{b}_2 = \boldsymbol{y}_2 \circ \varphi. \tag{29}$$

Lemma 1. *Consider proper parametrizations $\boldsymbol{\alpha}_j, j \in \{1,2\}$ as in (20), a similarity f as in (21), and a Möbius transformation φ, that satisfy (27). Then*

$$\lambda \parallel \boldsymbol{\alpha}_1^{'}(t) \parallel_p - \parallel (\boldsymbol{\alpha}_2 \circ \varphi)^{'}(t) \parallel_p = 0. \tag{30}$$

Proof. From the short form of Equation (27), $\lambda \boldsymbol{Q}\boldsymbol{\alpha}_1 + \boldsymbol{b} = \boldsymbol{\alpha}_2 \circ \varphi$. Differentiating and taking dual norms gives $\lambda \boldsymbol{Q} \parallel \boldsymbol{\alpha}_1^{'}(t) \parallel_p = \parallel (\boldsymbol{\alpha}_2 \circ \varphi)^{'}(t) \parallel_p$. The orthogonality of $\boldsymbol{Q}$ yields Equation (30). □

4.2. Differential Invariants

In the remainder of this section, we examine how the dual curvature $\kappa_{\boldsymbol{\alpha}}$ and the dual torsion $\tau_{\boldsymbol{\alpha}}$ of a parametrized dual curve $\boldsymbol{\alpha}$ transform under similarities and Möbius reparametrizations. These quantities are defined in Remark (3) by the following equations:

$$\kappa_{\boldsymbol{\alpha}} := \frac{\left\| \boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'' \right\|_p}{\left\| \boldsymbol{\alpha}' \right\|_p^3}, \tag{31}$$

and

$$\tau_{\boldsymbol{\alpha}} := \frac{\left\langle \boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'', \boldsymbol{\alpha}''' \right\rangle_p}{\left\| \boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'' \right\|_p^2}, \tag{32}$$

respectively. It is worth noting that, for a rational parametrization $\boldsymbol{\alpha}$, both $\kappa_{\boldsymbol{\alpha}}^2$ and $\tau_{\boldsymbol{\alpha}}$ are rational functions, whereas the dual curvature $\kappa_{\boldsymbol{\alpha}}$ itself is, in general, not rational. Notice that, since the parametrization $\boldsymbol{\alpha}(t)$ is regular, the denominator of $\kappa_{\boldsymbol{\alpha}}$ is nonzero for any parameter value, and similarly since we assume $H_p(\boldsymbol{\alpha}_j)$, $j \in \{1,2\}$, is a nondegenerate, i.e., is not a line, the denominator of $\tau_{\boldsymbol{\alpha}}$ is also nonzero for all admissible parameter values. Thus both the curvature and torsion are well-defined.

Lemma 2. *For a similarity $f(\boldsymbol{\alpha}) = \lambda \boldsymbol{Q}\boldsymbol{\alpha} + \boldsymbol{b}$ as in (21) and $\boldsymbol{\alpha}$ as in (20),*

$$\lambda \kappa_{f \circ \boldsymbol{\alpha}} = \kappa_{\boldsymbol{\alpha}}, \qquad \lambda \tau_{f \circ \boldsymbol{\alpha}} = det(\boldsymbol{Q}) \tau_{\boldsymbol{\alpha}}. \tag{33}$$

Proof. A straightforward calculation yields, for any invertible dual matrix $\boldsymbol{M} \in \mathbb{D}_p^{3\times 3}$ and dual vectors $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{D}_p^3$, the identity

$$\boldsymbol{M}_{\boldsymbol{\beta}} \times_p \boldsymbol{M}_{\boldsymbol{\gamma}} = det(\boldsymbol{M}) \boldsymbol{M}^{-1} (\boldsymbol{\beta} \times_p \boldsymbol{\gamma}).$$

Using $(f \circ \boldsymbol{\alpha})^{(n)} = \lambda \boldsymbol{Q} \boldsymbol{\alpha}^{(n)}$, for $n \in \{1,2,3\}$ and the fact that $\boldsymbol{Q}$ is orthogonal,

$$\lambda \kappa_{f \circ \boldsymbol{\alpha}} = \frac{\parallel \boldsymbol{Q}\boldsymbol{\alpha}^{'} \times_p \boldsymbol{Q}\boldsymbol{\alpha}^{''} \parallel_p}{\parallel \boldsymbol{Q}\boldsymbol{\alpha}^{'} \parallel_p^3} = \frac{\parallel det(\boldsymbol{Q}) \boldsymbol{Q}(\boldsymbol{\alpha}^{'} \times_p \boldsymbol{\alpha}^{''}) \parallel_p}{\parallel \boldsymbol{Q}\boldsymbol{\alpha}^{'} \parallel_p^3} = \frac{\parallel \boldsymbol{\alpha}^{'} \times_p \boldsymbol{\alpha}^{''} \parallel_p}{\parallel \boldsymbol{\alpha}^{'} \parallel_p^3} = \kappa_{\boldsymbol{\alpha}},$$

$$\tau_{f \circ \boldsymbol{\alpha}} := \frac{\langle \boldsymbol{Q}\boldsymbol{\alpha}^{'} \times_p \boldsymbol{Q}\boldsymbol{\alpha}^{''}, \boldsymbol{Q}\boldsymbol{\alpha}^{'''} \rangle_p}{\parallel \boldsymbol{Q}\boldsymbol{\alpha}^{'} \times_p \boldsymbol{Q}\boldsymbol{\alpha}^{''} \parallel_p^2} = \frac{\langle det(\boldsymbol{Q}) \boldsymbol{Q}(\boldsymbol{\alpha}^{'} \times_p \boldsymbol{\alpha}^{''}), \boldsymbol{Q}\boldsymbol{\alpha}^{'''} \rangle_p}{\parallel det(\boldsymbol{Q}) \boldsymbol{Q}\boldsymbol{\alpha}^{'} \times_p \boldsymbol{\alpha}^{''} \parallel_p^2} = det(\boldsymbol{Q}) \tau_{\boldsymbol{\alpha}}.$$

□

Next, we present a lemma that describes the behavior of the dual curvature and dual torsion under reparametrization, in particular, under a Möbius transformation.

Lemma 3. *Let $\boldsymbol{\alpha}$ be a generalized rational dual parametrization as in (21) and let $\phi \subset C^3(U)$, with $U \subset \mathbb{R}$ open. Then,*

$$\kappa_{\boldsymbol{\alpha}\circ\boldsymbol{\phi}} = \kappa_{\boldsymbol{\alpha}} \circ \phi, \qquad \tau_{\boldsymbol{\alpha}\circ\boldsymbol{\phi}} = \tau_{\boldsymbol{\alpha}} \circ \phi, \tag{34}$$

whenever both sides are defined.

Proof. Let $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \phi$, where $\phi'(t) \neq 0$. By the chain rule,

$$\boldsymbol{\beta}' = \phi'(\boldsymbol{\alpha}' \circ \phi), \qquad \boldsymbol{\beta}'' = \phi''(\boldsymbol{\alpha}' \circ \phi) + (\phi')^2(\boldsymbol{\alpha}'' \circ \phi).$$

Consequently,

$$\boldsymbol{\beta}' \times_p \boldsymbol{\beta}'' = (\phi')^3(\boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'') \circ \phi.$$

Substituting into the curvature formula gives $\kappa_{\boldsymbol{\beta}} = \kappa_{\boldsymbol{\alpha}} \circ \phi$. Moreover, the chain rule yields

$$\langle \boldsymbol{\beta}' \times_p \boldsymbol{\beta}'', \boldsymbol{\beta}''' \rangle_p = (\phi')^6 \Big(\langle \boldsymbol{\alpha}' \times_p \boldsymbol{\alpha}'', \boldsymbol{\alpha}''' \rangle_p \circ \phi \Big).$$

Since the denominator of the torsion formula is multiplied by the same factor $(\phi')^6$, we obtain $\tau_{\boldsymbol{\beta}} = \tau_{\boldsymbol{\alpha}} \circ \phi$. This completes the proof. □

The following lemma relates the dual curvatures and the dual torsions of similar curves.

Lemma 4. *Suppose the dual parametrizations $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ define dual curves with $f(\boldsymbol{C}_1) = \boldsymbol{C}_2$ under a similarity f with ratio λ. Then there exists a Möbius transformation φ such that*

$$\kappa_{\boldsymbol{\alpha}_2} \circ \varphi = \kappa_{\boldsymbol{\alpha}_2 \circ \varphi} = \kappa_{f \circ \boldsymbol{\alpha}_1} = \frac{1}{\lambda} \kappa_{\boldsymbol{\alpha}_1}, \tag{35}$$

$$\tau_{\boldsymbol{\alpha}_2} \circ \varphi = \tau_{\boldsymbol{\alpha}_2 \circ \varphi} = \tau_{f \circ \boldsymbol{\alpha}_1} = \frac{det(\boldsymbol{Q})}{\lambda} \tau_{\boldsymbol{\alpha}_1}. \tag{36}$$

Proof. By Theorem 8, there exists a Möbius transformation φ such that $f \circ \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 \circ \varphi$. The statement follows from Lemmas 2 and 3. □

4.3. Similarity Invariants

In this subsection, we consider two rational algebraic generalized dual curves $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ defined by the proper parametrizations $\boldsymbol{\alpha}_1 = H_p(\boldsymbol{\alpha}_1) + (\varepsilon_p - p)\boldsymbol{y}_1$ and $\boldsymbol{\alpha}_2 = H_p(\boldsymbol{\alpha}_2) + (\varepsilon_p - p)\boldsymbol{y}_2$, satisfying the assumptions in the Section 4.1. It follows from Theorem 8 and Lemma 4 that the dual curves $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are similar if and only if there exists a Möbius transformation φ satisfying

$$\kappa_{\boldsymbol{\alpha}_2} \circ \varphi = \frac{1}{\lambda} \kappa_{\boldsymbol{\alpha}_1}, \qquad \tau_{\boldsymbol{\alpha}_2} \circ \varphi = \frac{\det(\boldsymbol{Q})}{\lambda} \tau_{\boldsymbol{\alpha}_1}, \tag{37}$$

where $\boldsymbol{Q}$ is the dual orthogonal matrix corresponding to the similarity transformation and λ denotes its similarity ratio.

By Corollary (2), the dual curvature $\kappa_{\boldsymbol{\alpha}_j} = \kappa_{H_p(\boldsymbol{\alpha}_j)} + (\varepsilon_p - p)\kappa^*_{\boldsymbol{\alpha}_j}$ and the dual torsion $\tau_{\boldsymbol{\alpha}_j} = \tau_{H_p(\boldsymbol{\alpha}_j)} + (\varepsilon_p - p)\tau^*_{\boldsymbol{\alpha}_j}$, for $j \in \{1, 2\}$, satisfy the following relations:

$$\kappa_{H_p(\boldsymbol{\alpha}_2)} \circ \varphi = \frac{1}{\lambda} \kappa_{H_p(\boldsymbol{\alpha}_1)}, \tag{38}$$

$$\kappa^*_{\boldsymbol{\alpha}_2} \circ \varphi = \frac{1}{\lambda}\kappa^*_{\boldsymbol{\alpha}_1}, \tag{39}$$

$$\tau_{H_p(\boldsymbol{\alpha}_2)} \circ \varphi = \frac{det(\boldsymbol{Q})}{\lambda}\tau_{H_p(\boldsymbol{\alpha}_1)}, \tag{40}$$

$$\tau^*_{\boldsymbol{\alpha}_2} \circ \varphi = \frac{det(\boldsymbol{Q})}{\lambda}\tau^*_{\boldsymbol{\alpha}_1}. \tag{41}$$

Thus, $\dfrac{\kappa^*_{\boldsymbol{\alpha}_1}}{\kappa_{H_p(\boldsymbol{\alpha}_1)}} = \dfrac{\kappa^*_{\boldsymbol{\alpha}_2} \circ \varphi}{\kappa_{H_p(\boldsymbol{\alpha}_2)} \circ \varphi} = \dfrac{\kappa^*_{\boldsymbol{\alpha}_2}}{\kappa_{H_p(\boldsymbol{\alpha}_2)}} \circ \varphi$ and $\dfrac{\tau^*_{\boldsymbol{\alpha}_1}}{\tau_{H_p(\boldsymbol{\alpha}_1)}} = \dfrac{\tau^*_{\boldsymbol{\alpha}_2} \circ \varphi}{\tau_{H_p(\boldsymbol{\alpha}_2)} \circ \varphi} = \dfrac{\tau^*_{\boldsymbol{\alpha}_2}}{\tau_{H_p(\boldsymbol{\alpha}_2)}} \circ \varphi$.
Since we want to guarantee the rationality of $\kappa_{H_p(\boldsymbol{\alpha}_j)}$ and $\kappa^*_{\boldsymbol{\alpha}_1}$, we will use their squares.

Theorem 9. *Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two rational algebraic generalized dual curves properly parametrized by $\boldsymbol{\alpha} = H_p(\boldsymbol{\alpha}) + (\varepsilon_p - p)\boldsymbol{y}$ satisfying the hypotheses in Section 4.1. Let*

$$S_1(\boldsymbol{\alpha}) := \frac{\kappa^{*2}_{\boldsymbol{\alpha}}}{\kappa^2_{H_p(\boldsymbol{\alpha})}}, \qquad S_2(\boldsymbol{\alpha}) := \frac{\tau^*_{\boldsymbol{\alpha}}}{\tau_{H_p(\boldsymbol{\alpha})}}. \tag{42}$$

Then, each $S_j, j \in \{1,2\}$, commutes with Möbius transformations, i.e., $S_j(\boldsymbol{\alpha} \circ \varphi) = S_j(\boldsymbol{\alpha}) \circ \varphi, j \in \{1,2\}$.

Proof. By Lemma 3, the generalized dual curvature and torsion are invariant under reparametrizations:

$$\kappa_{\boldsymbol{\alpha}\circ\varphi} = \kappa_{\boldsymbol{\alpha}} \circ \varphi, \qquad \tau_{\boldsymbol{\alpha}\circ\varphi} = \tau_{\boldsymbol{\alpha}} \circ \varphi.$$

Comparing their components gives

$$\kappa_{H_p(\boldsymbol{\alpha}\circ\varphi)} = \kappa_{H_p(\boldsymbol{\alpha})} \circ \varphi, \qquad \kappa^*_{\boldsymbol{\alpha}\circ\varphi} = \kappa^*_{\boldsymbol{\alpha}} \circ \varphi,$$

and analogous identities for the torsion components. Therefore, substituting into Equation (42) yields

$$S_j(\boldsymbol{\alpha} \circ \varphi) = S_j(\boldsymbol{\alpha}) \circ \varphi, \qquad j = 1,2.$$

□

Let us write $\boldsymbol{u}_1 := H_p(\boldsymbol{\alpha}_1)(t)$ and $\boldsymbol{v}_1 := (H_p(\boldsymbol{\alpha}_2) \circ \varphi)(t)$, so that $\lambda\boldsymbol{Q}_1\boldsymbol{u}_1 + \boldsymbol{b}_1 = \boldsymbol{v}_1$ from Equation (28). Consequently, differentiating this equation up to order three with respect to t yields $\lambda\boldsymbol{Q}D(\boldsymbol{u}_1) = D(\boldsymbol{v}_1)$, where

$$D(\boldsymbol{u}_1) := \left[\boldsymbol{u}_1', \boldsymbol{u}_1'', \boldsymbol{u}_1'''\right], \qquad D(\boldsymbol{v}_1) := \left[\boldsymbol{v}_1', \boldsymbol{v}_1'', \boldsymbol{v}_1'''\right]. \tag{43}$$

That is, $D(\boldsymbol{u}_1)$ and $D(\boldsymbol{v}_1)$ are matrices whose columns consist of the first three derivatives of $\boldsymbol{u}_1, \boldsymbol{v}_1$ with respect to t. If $\boldsymbol{u}_1, \boldsymbol{v}_1$ are not contained in a hyperplane, $D(\boldsymbol{u}_1)$ and $D(\boldsymbol{v}_1)$ are invertible.

Finally, observe that once φ is determined, then, by Equation (37), we can determine the similarity ratio $\lambda = \dfrac{\kappa_{\boldsymbol{u}_1}}{\kappa_{\boldsymbol{v}_1}}$.

Remark 4. *Here, we assume that neither of the curves properly parametrized by $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ is contained in a hyperplane. This condition guarantees that the determinants of $D(\boldsymbol{u}_1)$ and $D(\boldsymbol{v}_1)$ do not vanish identically, that is, except for a finite number of parameter values $D(\boldsymbol{u}_1)$ and $D(\boldsymbol{v}_1)$ are invertible matrices.*

Lemma 5. *Suppose that neither of the curves properly parametrized by $\boldsymbol{u}_1$ $\boldsymbol{v}_1$ is contained in a hyperplane. Then $\boldsymbol{Q}_1 = \frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1}$ is a constant orthogonal matrix.*

Proof. Please see Appendix A. □

Therefore, we have two cases depending on the input curves:

i. If $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ do not lie in a hyperplane, then $\boldsymbol{Q}_1 = \frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1}$.

ii. If $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ lie in a hyperplane, then the orthogonal matrix $\boldsymbol{Q}_1$ of the real similarity between the parametrizations $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ can be obtained by solving the system of linear equations $\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{'} = \boldsymbol{v}_1^{'}$, $\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{''} = \boldsymbol{v}_1^{''}$, $\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{'''} = \boldsymbol{v}_1^{'''}$, $\boldsymbol{Q}_1^T \boldsymbol{Q}_1 = I_3$.

Moreover, $\boldsymbol{b}_1 = \boldsymbol{v}_1 - \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1$.

Similarly, let us write $\boldsymbol{u}_2 := \boldsymbol{y}_1(t)$ and $\boldsymbol{v}_2 := (\boldsymbol{y}_2 \circ \varphi)(t)$, so that by Equation (29), $\lambda(\boldsymbol{Q}_1 \boldsymbol{u}_2 + \boldsymbol{Q}_2 \boldsymbol{u}_1) + \boldsymbol{b}_2 = \boldsymbol{v}_2$. Differentiating this equation with respect to t yields

$$\lambda(\boldsymbol{Q}_1 \boldsymbol{u}_2^{'} + \boldsymbol{Q}_2 \boldsymbol{u}_1^{'}) = \boldsymbol{v}_2^{'}. \tag{44}$$

After computing $\boldsymbol{Q}_1$, to determine $\boldsymbol{Q}_2$, if any, we assume that Equations (23), (24) and (44) hold. First, let us find a unique parametric solution for the matrix $\boldsymbol{Q}_2$. Using Equation (23) or (24), one can see that $\boldsymbol{Q}_1^T \boldsymbol{Q}_2$ is a skew-symmetric matrix. Then for a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, using the skew-symmetric cross product representation, we can write $\boldsymbol{Q}_1^T \boldsymbol{Q}_2 = [\boldsymbol{\omega}]_\times$, where

$$[\boldsymbol{w}]_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Multiplying the latter on the left by $\boldsymbol{Q}_1$, we have $\boldsymbol{Q}_2 = \boldsymbol{Q}_1 [\boldsymbol{\omega}]_\times$. Substituting the last equation in Equation (44) yields

$$\lambda(\boldsymbol{Q}_1 \boldsymbol{u}_2^{'} + \boldsymbol{Q}_1 [\boldsymbol{\omega}]_\times \boldsymbol{u}_1^{'}) = \boldsymbol{v}_2^{'} \tag{45}$$

$$\lambda\left(\boldsymbol{Q}_1 \boldsymbol{u}_2^{'} + \boldsymbol{Q}_1\left(\boldsymbol{\omega} \times \boldsymbol{u}_1^{'}\right)\right) = \boldsymbol{v}_2^{'} \tag{46}$$

Multiplying the latter on the left by $\frac{1}{\lambda} \boldsymbol{Q}_1^T$, we get

$$\boldsymbol{u}_2^{'} + \boldsymbol{\omega} \times \boldsymbol{u}_1^{'} = \frac{1}{\lambda} \boldsymbol{Q}_1^T \boldsymbol{v}_2^{'} \tag{47}$$

$$\boldsymbol{\omega} \times \boldsymbol{u}_1^{'} = \frac{1}{\lambda} \boldsymbol{Q}_1^T \boldsymbol{v}_2^{'} - \boldsymbol{u}_2^{'} \tag{48}$$

Setting $\boldsymbol{\Theta} = \frac{1}{\lambda} \boldsymbol{Q}_1^T \boldsymbol{v}_2^{'} - \boldsymbol{u}_2^{'}$ yields $\boldsymbol{\omega} \times \boldsymbol{u}_1^{'} = \boldsymbol{\Theta}$. Differentiating $\boldsymbol{\omega} \times \boldsymbol{u}_1^{'} = \boldsymbol{\Theta}$ twice with respect to t, we have a system of equations

$$\begin{aligned} \boldsymbol{\omega} \times \boldsymbol{u}_1^{'} &= \boldsymbol{\Theta} \\ \boldsymbol{\omega} \times \boldsymbol{u}_1^{''} &= \boldsymbol{\Theta}' \\ \boldsymbol{\omega} \times \boldsymbol{u}_1^{'''} &= \boldsymbol{\Theta}'', \end{aligned}$$

Rewriting the above system using skew-symmetric cross product representation, we have

$$\begin{aligned}\left[\boldsymbol{u}_1'\right]_\times \boldsymbol{\omega} &= -\boldsymbol{\Theta}\\ \left[\boldsymbol{u}_1''\right]_\times \boldsymbol{\omega} &= -\boldsymbol{\Theta}'\\ \left[\boldsymbol{u}_1'''\right]_\times \boldsymbol{\omega} &= -\boldsymbol{\Theta}''.\end{aligned}$$

Setting

$$\mathcal{M} = \begin{bmatrix}\left[\boldsymbol{u}_1'\right]_\times\\ \left[\boldsymbol{u}_1''\right]_\times\\ \left[\boldsymbol{u}_1'''\right]_\times\end{bmatrix} \text{ and } \mathcal{B} = -\begin{bmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\Theta}'\\ \boldsymbol{\Theta}''\end{bmatrix}, \tag{49}$$

we have a matrix equation $\mathcal{M}\boldsymbol{\omega} = \mathcal{B}$. The coefficient matrix has full column rank since $\boldsymbol{u}_1$ is not contained in a hyperplane, that is, $\|\boldsymbol{u}_1' \, \boldsymbol{u}_1'' \, \boldsymbol{u}_1'''\|$ is not identically zero. Thus, $\mathcal{M}\boldsymbol{\omega} = \mathcal{B}$ possesses a unique $\boldsymbol{w}$ solution. This solution could be obtained using the pseudoinverse of $\mathcal{M}$ as

$$\boldsymbol{\omega} = \left(\mathcal{M}^T\mathcal{M}\right)^{-1}\mathcal{M}^T\mathcal{B}.$$

Ultimately, the matrix $\boldsymbol{Q}_2$ can be computed as $\boldsymbol{Q}_2 = \boldsymbol{Q}_1[w]_\times$.

We verify that the resulting matrix $\boldsymbol{Q} = \boldsymbol{Q}_1 + (\epsilon_p - p)\boldsymbol{Q}_2$ satisfies the orthogonality condition $\boldsymbol{Q}^T\boldsymbol{Q} = I_3$. Using $\boldsymbol{Q}_1^T\boldsymbol{Q}_1 = I_3$, $\boldsymbol{Q}_1^T\boldsymbol{Q}_2 + \boldsymbol{Q}_2^T\boldsymbol{Q}_1 = 0_3$ and $(\epsilon_p - p)^2 = 0$, we obtain

$$\begin{aligned}\boldsymbol{Q}^T\boldsymbol{Q} &= \left(\boldsymbol{Q}_1 + (\epsilon_p - p)\boldsymbol{Q}_2\right)^T\left(\boldsymbol{Q}_1 + (\epsilon_p - p)\boldsymbol{Q}_2\right)\\ &= \boldsymbol{Q}_1^T\boldsymbol{Q}_1 + (\epsilon_p - p)\left(\boldsymbol{Q}_1^T\boldsymbol{Q}_2 + \boldsymbol{Q}_2^T\boldsymbol{Q}_1\right) + (\epsilon_p - p)^2\boldsymbol{Q}_2^T\boldsymbol{Q}_2\\ &= I_3.\end{aligned}$$

Therefore, $\boldsymbol{Q}$ is a generalized dual orthogonal matrix.

Finally, the second component of the translation vector is given by

$$\boldsymbol{b}_2 = \boldsymbol{v}_2 - \lambda(\boldsymbol{Q}_1\boldsymbol{u}_2 + \boldsymbol{Q}_2\boldsymbol{u}_1). \tag{50}$$

Since Equation (44) holds identically, differentiating the above expression gives $\boldsymbol{b}_2' = \boldsymbol{0}$. Thus, $\boldsymbol{b}_2$ is a constant vector.

We therefore obtain the following result.

Theorem 10. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{D}_p^3$ be two rational algebraic dual curves properly parametrized by $\boldsymbol{\alpha}_1 = H_p(\boldsymbol{\alpha}_1)(t) + (\varepsilon_p - p)\boldsymbol{y}_1(t), \boldsymbol{\alpha}_2 = H_p(\boldsymbol{\alpha}_2)(t) + (\varepsilon_p - p)\boldsymbol{y}_2(t)$, respectively, satisfying the hypothesis in Section 4.1. If $\boldsymbol{C}_1, \boldsymbol{C}_2$ are related by a similarity f given in Equation (21), then there exists a Möbius transformation $s = \varphi(t) = \dfrac{at+b}{ct+d}$, with $ad - bc \neq 0$ satisfying,*

$$S_j(\boldsymbol{\alpha}_1)(t) = S_j(\boldsymbol{\alpha}_2)(s), \tag{51}$$

for $j \in \{1, 2\}$. The similarity ratio λ is determined by

$$\lambda = \frac{\kappa_{H_p(\boldsymbol{\alpha}_1)}(t)}{\kappa_{H_p(\boldsymbol{\alpha}_2)}(\varphi(t))}. \tag{52}$$

Proof. Since C_1 and C_2 are similar, Theorem 8 guarantees the existence of a similarity f and a Möbius transformation φ such that

$$\boldsymbol{\alpha}_2 \circ \varphi = f \circ \boldsymbol{\alpha}_1.$$

By Lemma 2, both components of the curvature are multiplied by $\frac{1}{\lambda}a$, while those of the torsion are multiplied by $\frac{det(\boldsymbol{Q})}{\lambda}$. Consequently,

$$S_j(f \circ \boldsymbol{\alpha}_1) = S_j(\boldsymbol{\alpha}_1), \qquad j = 1, 2.$$

Using Theorem 9, we obtain

$$\begin{aligned} S_j(\boldsymbol{\alpha}_2) \circ \varphi &= S_j(\boldsymbol{\alpha}_2 \circ \varphi) \\ &= S_j(f \circ \boldsymbol{\alpha}_1) \\ &= S_j(\boldsymbol{\alpha}_1), \end{aligned}$$

which proves Equation (51). Moreover, the transformation law for the real curvature gives

$$\lambda = \frac{\kappa_{H_p(\boldsymbol{\alpha}_1)}(t)}{\kappa_{H_p(\boldsymbol{\alpha}_2)}(\varphi(t))},$$

as stated in Equation (52). □

Furthermore, let

$$\boldsymbol{Q} = \boldsymbol{Q}_1 + (\varepsilon_p - p)\boldsymbol{Q}_2 \in \mathbb{D}_p^{3\times 3}.$$

If the curves are not contained in a hyperplane, then $\boldsymbol{Q}_1$ is given by

$$\boldsymbol{Q}_1 = \frac{1}{\lambda} D\big(H_p(\boldsymbol{\alpha}_2)(s)\big) D\big(H_p(\boldsymbol{\alpha}_1)(t)\big)^{-1}.$$

If, on the other hand, the curves lie in a hyperplane, then $\boldsymbol{Q}_1$ can be determined by solving the following system of equations:

$$\begin{cases} \lambda \boldsymbol{Q}_1 H_p(\boldsymbol{\alpha}_1')(t) = H_p(\boldsymbol{\alpha}_2')(s), \\ \lambda \boldsymbol{Q}_1 H_p(\boldsymbol{\alpha}_1'')(t) = H_p(\boldsymbol{\alpha}_2'')(s), \\ \lambda \boldsymbol{Q}_1 H_p(\boldsymbol{\alpha}_1''')(t) = H_p(\boldsymbol{\alpha}_2''')(s), \\ \boldsymbol{Q}_1^T \boldsymbol{Q}_1 = I_3, \end{cases}$$

where the matrix I_3 is the unit matrix of size 3×3. In either case, the first component of the translation vector

$$\boldsymbol{b} = \boldsymbol{b}_1 + (\varepsilon_p - p)\boldsymbol{b}_2 \in \mathbb{D}_p^3$$

is obtained as

$$\boldsymbol{b}_1 = H_p(\boldsymbol{\alpha}_2)(s) - \lambda \boldsymbol{Q}_1 H_p(\boldsymbol{\alpha}_1)(t).$$

Once $\boldsymbol{Q}_1$ has been determined, $\boldsymbol{Q}_2$ can be obtained by solving

$$\begin{cases} \lambda\big(\boldsymbol{Q}_1 \boldsymbol{y}_1'(t) + \boldsymbol{Q}_2 H_p(\boldsymbol{\alpha}_1')(t)\big) = \boldsymbol{y}_2'(s), \\ \boldsymbol{Q}_1^T \boldsymbol{Q}_2 + \boldsymbol{Q}_2^T \boldsymbol{Q}_1 = 0_3, \\ \boldsymbol{Q}_1 \boldsymbol{Q}_2^T + \boldsymbol{Q}_2 \boldsymbol{Q}_1^T = 0_3, \end{cases}$$

where the matrix 0_3 is the zero matrix of size 3×3. Finally, the second component of the translation vector is given by

$$\boldsymbol{b}_2 = \boldsymbol{y}_2(s) - \lambda\big(\boldsymbol{Q}_1\boldsymbol{y}_1(t) + \boldsymbol{Q}_2 H_p(\boldsymbol{\alpha}_1)(t)\big).$$

4.4. The Algorithm

In this section, we will see how to turn the previous results into an algorithm detecting similarities. We will assume that the similarity invariants $S_j, j \in \{1,2\}$ defined in Equation (42) are not constant.

First, by Theorem 10, we know that

$$S_j(\boldsymbol{\alpha}_1)(t) = S_j(\boldsymbol{\alpha}_2)(\varphi(t)), j \in \{1,2\},$$

where $\varphi(t) = \dfrac{at+b}{ct+d}$, $ad - bc \neq 0$. Let us denote $s = \varphi(t)$, and consider the expressions

$$S_j(\boldsymbol{\alpha}_1)(t) - S_j(\boldsymbol{\alpha}_2)(s) = 0, j \in \{1,2\}. \tag{53}$$

Clearing the denominators, one can see that the expressions in Equation (53) define two algebraic curves in the (t,s) plane. By Equation (53), all the (t,s) points satisfying $s = \varphi(t)$ belong to both curves. Thus, these two curves have infinitely many points in common, so by Bezout's Theorem they must share a common factor. This factor is precisely the result of clearing the denominators in $s - \varphi(t) = 0$. Below, we formulate this idea.

Let us write

$$S_j(\boldsymbol{\alpha}_1)(t) = \frac{U_j(t)}{V_j(t)}, \qquad S_j(\boldsymbol{\alpha}_2)(t) = \frac{\widehat{U}_j(s)}{\widehat{V}_j(s)}, \tag{54}$$

and let

$$E_j(t,s) := U_j(t)\widehat{V}_j(s) - V_j(t)\widehat{U}_j(s), \tag{55}$$

for $j \in \{1,2\}$. From the previous discussions, we know that the algebraic curves defined by $E_1(t,s)$ and $E_2(t,s)$ must share a common factor. Thus, let

$$G(t,s) := gcd(E_1(t,s), E_2(t,s)), \tag{56}$$

where gcd denotes the greatest common divisor. Finally, for an arbitrary Möbius transformation $\varphi(t) = \dfrac{at+b}{ct+d} = s$, $ad - bc \neq 0$, we call

$$F(t,s) = s(ct+d) - (at+b) \tag{57}$$

the associated Möbius factor of φ. Notice that the condition $ad - bc \neq 0$ guarantees that F is irreducible. Then we have the following result.

Theorem 11. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{D}_p^3$ be two rational algebraic dual curves properly parametrized by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ satisfying the hypotheses in Section (4.1), and let G be as in Equation (56). If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are similar, then there exists a Möbius factor of F such that F divides G.*

Proof. From Theorem (10), the zero set of F is included in the zero set of the expressions in Equation (51), and therefore in the zero sets of E_1, E_2. Thus, the zero set of F is included in the zero set of G, and therefore, by Study's Lemma (see Section 6.13 in [17]), F divides G. □

The above result allows us construct our computational method to determine the Möbius transformation φ by computing the polynomial $G(t,s)$ in Equation (56), factoring it, and looking for the Möbius factors. Once φ is found, the components λ, $\boldsymbol{Q}$ and $\boldsymbol{b}$ of the similarity transformation are determined in the manner described in the previous subsection.

This leads to the following Algorithm 1, `DualSim3D`, to check whether or not two given rational algebraic dual curves are similar. We emphasize again that we assume that the similarity invariants S_1, S_2 are not constant.

Algorithm 1 `DualSim3D`

Input: Two parametrizations $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, satisfying the hypotheses in Section 4.1, such that both the similarity invariants S_1, S_2 are not constant.
Output: Either the list of Möbius transformations and similarities or the warning: "The curves are not similar."

procedure `DualSim3D`($\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$)
  Compute the factors FS of the polynomial G that is defined at (56).
  Check FS to find the set MB of Möbius factors.
  Compute the set MBS of Möbius transformations corresponding to MB
  for $\varphi \in MBS$ **do**
    Compute the similarity ratio $\boldsymbol{\alpha}$, the dual orthogonal matrix $\boldsymbol{Q}$ and the translation vector $\boldsymbol{b}$ corresponding to the similarity transformation.
  end for
  if $MB = \emptyset$ **then return** "These curves are not similar."
  end if
end procedure

The algorithm `DualSim3D` has been implemented in Maple $^{TM}(2021)$ [18]. Technical details and the source code for the procedures are provided in the third author's personal website [19]. Below we provide a detailed example to illustrate the method.

Example 4. *Consider the following two rational parametrizations (see Figure 1)*

$$\boldsymbol{\alpha}_1(t) = \begin{bmatrix} \dfrac{-67t^{10} - 305t^8 + 530t^6 - 530t^4 + 305t^2 + 67}{15(t^2+1)^5} \\ \dfrac{150t^9 + 680t^7 - 467t^5 + 680t^3 + 150t}{15(t^2+1)^5} \\ \dfrac{-32t^5 + 32t^3}{(t^2+1)^4} \end{bmatrix}$$

$$+ (\varepsilon_2 - 2) \begin{bmatrix} \dfrac{26t^{10} + 190t^8 - 220t^6 + 220t^4 - 190t^2 - 26}{15(t^2+1)^5} \\ \dfrac{-60t^9 - 400t^7 + 88t^5 - 400t^3 - 60t}{15(t^2+1)^5} \\ \dfrac{-2t^7 + 14t^5 - 14t^3 + 2t}{(t^2+1)^4} \end{bmatrix},$$

$$\boldsymbol{\alpha}_2(t) = \begin{bmatrix} \dfrac{\begin{array}{c}693565t^{10}-436930t^9+6147080t^8+10887760t^7\\ +12730480t^6-30243360t^5+46997440t^4+14733440t^3\\ +69924080t^2+18453280t+33424768\end{array}}{90(5t^2+2t+10)^5} \\ -\dfrac{\begin{array}{c}14(13331t^{10}+19195t^9-86210t^8+92660t^7+286160t^6\\ -860160t^5-3232600t^4-1895360t^3-973760t^2\\ -57520t-882592)\end{array}}{45(5t^2+2t+10)^5} \\ \dfrac{\begin{array}{c}2125012t^{10}+3949115t^9+23018645t^8+29571520t^7\\ +97916560t^6+75121200t^5+199811920t^4+148117280t^3\\ +201379520t^2+55604560t+62637040\end{array}}{45(5t^2+2t+10)^5} \end{bmatrix}$$

$$+(\varepsilon_2-2)\begin{bmatrix} \dfrac{\begin{array}{c}-1521161t^{10}-3459340t^9-3669640t^8-26814860t^7\\ -66860780t^6-48336960t^5+142277800t^4+99962960t^3\\ +23275040t^2-39911360t+34983712\end{array}}{45(5t^2+2t+10)^5} \\ -\dfrac{\begin{array}{c}777065t^{10}+10485850t^9+18315820t^8+5418740t^7\\ +132849500t^6+315682080t^5+219243080t^4+257149360t^3\\ +81227200t^2+30495200t-23685472\end{array}}{90(5t^2+2t+10)^5} \\ \dfrac{\begin{array}{c}2(675100t^{10}+2113865t^9+6679415t^8+9076390t^7\\ +39066370t^6+46824120t^5+39881380t^4+40382360t^3\\ +41930120t^2+16009360t+5451232)\end{array}}{45(5t^2+2t+10)^5} \end{bmatrix},$$

which parametrize two generalized dual curves. Applying Algorithm `DualSim3D`*, we get* $G = \gcd(E_1, E_2)$ *as*

$$\begin{aligned} G(t,s) =& s^4t^4+5s^3t^4-\frac{17}{4}s^4t^2+\frac{13}{4}s^2t^4+5s^3t^2-\frac{15}{2}st^4+s^4-\frac{39}{2}s^2t^2 \\ &+\frac{9}{4}t^4+5s^3-25st^2+\frac{13}{4}s^2-\frac{41}{2}t^2-\frac{15}{2}s+\frac{9}{4}. \end{aligned}$$

Factoring G, we get the set MF of Möbius factors as follows:

$$MF = \{st-2s+3t+1, st-1/2s-1/2t-\frac{3}{2}, st+2s+3t-1, st+\frac{1}{2}s-\frac{1}{2}t+\frac{3}{2}\},$$

which correspond to the following four Möbius transformations

$$\varphi_1(t) = -\frac{3t-1}{t+2}, \qquad \varphi_2(t) = \frac{t-3}{1+2t}, \qquad \varphi_3(t) = -\frac{3t+1}{t-2}, \qquad \varphi_4(t) = \frac{t+3}{2t-1}.$$

Thus, we obtain $\lambda = \frac{1}{2}$ *and subsequently, four similarities* $f_i(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{Q}^i\boldsymbol{\alpha}_1 + \boldsymbol{b}^i$ *for* $i \in \{1, 2, \cdots, 4\}$*, where orthogonal matrices are*

$$Q^1 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} + (\varepsilon_2 - 2) \begin{bmatrix} -2 & -4 & -3 \\ -1 & 3 & -4 \\ 0 & 2 & -1 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} + (\varepsilon_2 - 2) \begin{bmatrix} 2 & -4 & 3 \\ 1 & 3 & 4 \\ 0 & 2 & -1 \end{bmatrix},$$

$$Q^3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} + (\varepsilon_2 - 2) \begin{bmatrix} -2 & 4 & 3 \\ -1 & -3 & 4 \\ 0 & -2 & 1 \end{bmatrix},$$

$$Q^4 = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} + (\varepsilon_2 - 2) \begin{bmatrix} 2 & 4 & -3 \\ 1 & -3 & -4 \\ 0 & -2 & -1 \end{bmatrix}.$$

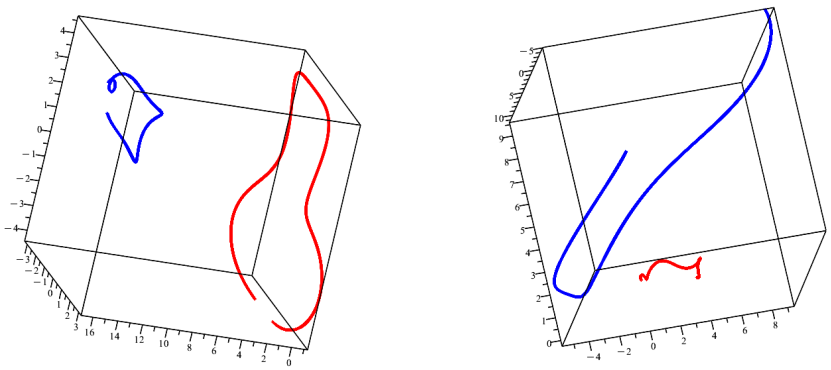
The translations are as follows:

$$\boldsymbol{b}^1 = \boldsymbol{b}^2 = \boldsymbol{b}^3 = \boldsymbol{b}^4 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + (\varepsilon_2 - 2) \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}.$$

Consider the parametrizations

$$\boldsymbol{\alpha}_j(t) = H_p(\boldsymbol{\alpha}_j(t)) + (\varepsilon_2 - 2)\boldsymbol{y}_j(t), \qquad j \in \{1, 2\},$$

given in Example 4.



(**a**) (**b**)

Figure 1. Plots of the curves in Example 4. (**a**) The curves parametrized by $H_p(\boldsymbol{\alpha}_1(t))$ (red) and its image under a similarity transformation, parametrized by $H_p(\boldsymbol{\alpha}_2(t))$ (red). (**b**) The curves parametrized by $\boldsymbol{y}_1(t)$ (red) and its image under a similarity transformation, parametrized by $\boldsymbol{y}_2(t)$ (red).

Example 5. *Consider the following two rational parametrizations (see Figure 2)*

$$\boldsymbol{\alpha}_1(t) = \begin{bmatrix} -\dfrac{8t^2+4t-15}{t+1} \\ \dfrac{-2t^4-8t^3-14t^2-12t-3}{(t+1)^2} \\ \dfrac{-4t^4-20t^3-36t^2-28t-7}{(t+1)^3} \end{bmatrix} + (\varepsilon_{-2}+2)\begin{bmatrix} \dfrac{-2t^2-t+4}{t+1} \\ \dfrac{1}{2}t^2+t+1 \\ t+2 \end{bmatrix},$$

$$\boldsymbol{\alpha}_2(t) = \begin{bmatrix} \dfrac{-29t^5-354t^4-1711t^3-4087t^2-4818t-2239}{2(t+2)^3(t+3)^2} \\ \dfrac{(4t^5+47t^4+214t^3+470t^2+492t+191)\sqrt{6}+4(t+2)^3(t+3)^2}{4(t+2)^3(t+3)^2} \\ \dfrac{(-20t^5-291t^4-1628t^3-4420t^2-5862t-3053)\sqrt{2}+28(t+2)^3(t+3)^2}{4(t+2)^3(t+3)^2} \end{bmatrix}$$
$$+ (\varepsilon_{-2}+2)\begin{bmatrix} \dfrac{((48t^5+704t^4+3963t^3+10815t^2+14409t+7537)\sqrt{3}-12t^5-141t^4-642t^3-1410t^2-1476t-573)\sqrt{2}}{4(t+2)^3(t+3)^2} + \dfrac{18(t+2)^2(t^3+\frac{25}{3}t^2+\frac{415}{18}t+\frac{190}{9})}{4(t+2)^3(t+3)^2} \\ \dfrac{(-2(t+2)^3(t^2+6t+\frac{17}{2})\sqrt{2}-100t^5-1224t^4-5932t^3-14204t^2-16776t-7804)\sqrt{3}}{8(t+2)^3(t+3)^2} - \dfrac{-98(t+2)^2(t^2+\frac{302}{49}t+9)(t+3)}{8(t+2)^3(t+3)^2} \\ \dfrac{(10t^5+144t^4+797t^3+2138t^2+2796t+1432)\sqrt{2}+(-258t^5-3154t^4-15258t^3-36450t^2-42924t-19892)\sqrt{3}-16(t+2)^3(t+3)^2}{8(t+2)^3(t+3)^2} \end{bmatrix},$$

which parametrize two generalized dual curves. Applying Algorithm `DualSim3D`*, we get* $G = \gcd(E_1, E_2)$ *as*

$$G(t,s) = s^2t^2 + 2s^2t + 6st^2 + 12st + 9t^2 + 2s + 18t + 51.$$

Factoring G, we get the set MF of Möbius factors as follows:

$$MF = \{st + 2s + 3t + 5, st + 3t + 1\},$$

which correspond to the following two Möbius transformations

$$\varphi_1(t) = -\frac{3t+5}{t+2}, \qquad \varphi_2(t) = -\frac{3t+1}{t}.$$

Thus, we get $\lambda = \frac{\sqrt{3}}{2}$ *and, afterwards, four similarities* $f(z) = \frac{\sqrt{3}}{2}\boldsymbol{Q}\boldsymbol{\alpha}_1 + \boldsymbol{b}$, *where orthogonal matrices are*

$$\boldsymbol{Q}^1 = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} + (\varepsilon_{-2}+2)\begin{bmatrix} \frac{5}{\sqrt{6}} & -\frac{3}{\sqrt{6}}+\sqrt{2} & -\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{6}} \\ -1-\frac{3}{2\sqrt{3}} & 1 & -1 \\ -\frac{5}{2} & 2 & -3 \end{bmatrix},$$

$$\boldsymbol{Q}^2 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} + (\varepsilon_{-2}+2)\begin{bmatrix} \frac{5}{\sqrt{6}} & -\frac{3}{\sqrt{6}}+\sqrt{2} & \frac{3}{\sqrt{2}}+\frac{3}{\sqrt{6}} \\ 1+\frac{3}{2\sqrt{3}} & 1 & 1 \\ \frac{5}{2} & 2 & 3 \end{bmatrix}.$$

The translations are as follows:

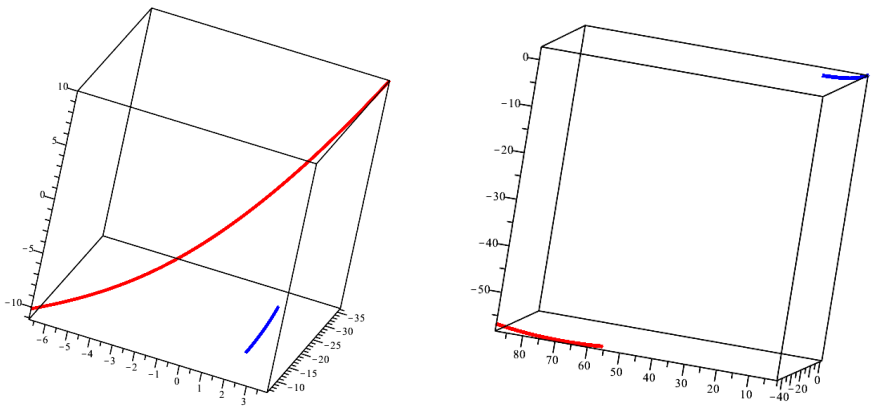
$$\boldsymbol{b}^1 = \begin{bmatrix} -16 \\ 2\sqrt{6}-1 \\ -10\sqrt{2}+3 \end{bmatrix} + (\varepsilon_{-2}+2)\begin{bmatrix} 5+(24\sqrt{3}-6)\sqrt{2} \\ -\frac{\sqrt{6}}{2}-16\sqrt{3}-19 \\ -42\sqrt{3}+\frac{5\sqrt{2}}{2}-2 \end{bmatrix},$$

$$\boldsymbol{b}^2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + (\varepsilon_{-2}+2)\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Consider the parametrizations

$$\boldsymbol{\alpha}_j(t) = H_p(\boldsymbol{\alpha}_j(t)) + (\varepsilon_{-2}+2)\boldsymbol{y}_j(t), \qquad j \in \{1,2\},$$

given in Example 5.



(**a**) (**b**)

Figure 2. Plots of the curves in Example 5. (**a**) The curves parametrized by $H_p(\boldsymbol{\alpha}_1(t))$ (red) and its image under a similarity transformation, parametrized by $H_p(\boldsymbol{\alpha}_2(t))$ (red). (**b**) The curve parametrized by $\boldsymbol{y}_1(t)$ (red) and its image under a similarity transformation, parametrized by $\boldsymbol{y}_2(t)$ (red).

Example 6. *Consider the following two rational parametrizations (see Figure 3)*

$$\boldsymbol{\alpha}_1(t) = \begin{bmatrix} \dfrac{-t^4-4t^2+1}{(t^2+1)^2} \\ \dfrac{-2t^3+2t}{(t^2+1)^2} \\ \dfrac{t^2(2t^6+6t^4-3t^3+6t^2+t+2)}{(t^2+1)^4} \end{bmatrix} + (\varepsilon_{(\frac{1}{2})} - \frac{1}{2}) \begin{bmatrix} -\dfrac{t^2}{t^2+1} \\ \dfrac{t}{t^2+1} \\ \dfrac{2t^2}{t^2+1} \end{bmatrix},$$

$$\boldsymbol{\alpha}_2(t) = \begin{bmatrix} \dfrac{-21167t^8-95496t^7-233748t^6-253584t^5-50706t^4}{14(5t^2+6t+9)^4} \\ \dfrac{351864t^3+556956t^2+489888t+164025}{14(5t^2+6t+9)^4} \\ \dfrac{-16609t^8-66252t^7-214992t^6-373356t^5-560682t^4}{14(5t^2+6t+9)^4} \\ \dfrac{-663876t^3-600696t^2-306180t-85293}{14(5t^2+6t+9)^4} \\ \dfrac{21297t^8+117216t^7+455436t^6+1035504t^5+1746198t^4}{14(5*t^2+6t+9)^4} \\ \dfrac{1912896t^3+1548396t^2+734832t+216513}{14(5*t^2+6t+9)^4} \end{bmatrix}$$
$$+ \left(\varepsilon_{\frac{1}{2}} - \frac{1}{2}\right) \begin{bmatrix} \dfrac{-70967t^8-179976t^7-68076t^6+768312t^5+2021598t^4}{7(5t^2+6t+9)^4} \\ \dfrac{1988712t^3+936036t^2-157464t-413343}{7(5t^2+6t+9)^4} \\ \dfrac{25549t^8+122487t^7+326871t^6+447579t^5+357453t^4}{7(5t^2+6t+9)^4} \\ \dfrac{-9963t^3-252963t^2-325863t-91854}{7(5t^2+6t+9)^4} \\ \dfrac{90897t^8+419886t^7+1099548t^6+1477926t^5+1108566t^4}{7(5t^2+6t+9)^4} \\ \dfrac{-190998t^3-1076004t^2-1246590t-413343}{7(5t^2+6t+9)^4} \end{bmatrix},$$

which parametrize two generalized dual curves. Applying Algorithm `DualSim3D`*, we get* $G = \gcd(E_1, E_2)$ *as*

$$G(t,s) = st + 2s + 3t.$$

which corresponds to the following Möbius transformation

$$\varphi_1(t) = -\frac{3t}{t+2}.$$

Thus, we get $\lambda = \dfrac{3}{2}$ *and, afterwards, four similarities* $fz) = \dfrac{3}{2}\boldsymbol{Q}\boldsymbol{\alpha}_1 + \boldsymbol{b}$*, where the orthogonal matrix is*

$$\boldsymbol{Q} = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \end{bmatrix} + \left(\varepsilon_{(\frac{1}{2})} - \frac{1}{2}\right) \begin{bmatrix} -\frac{11}{3} & -5 & -4 \\ -2 & \frac{2}{3} & 1 \\ -6 & 2 & 3 \end{bmatrix}.$$

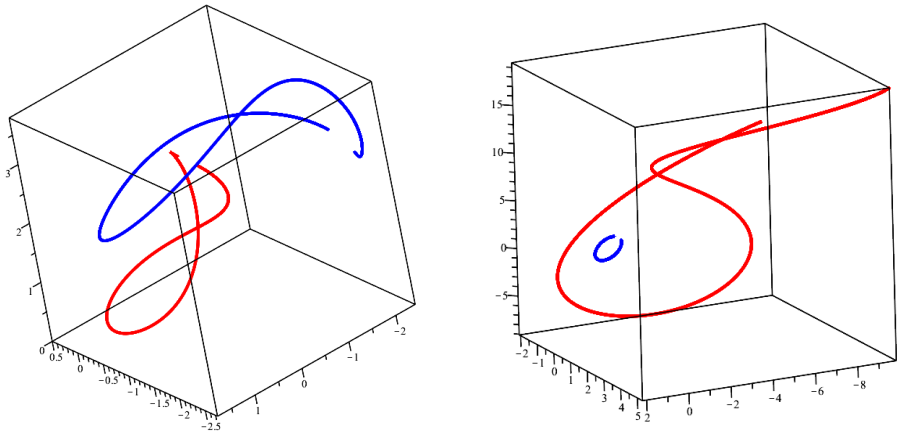
The translations are as follows:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \left(\varepsilon_{(\frac{1}{2})} - \frac{1}{2} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Consider the parametrizations

$$\boldsymbol{\alpha}_j(t) = H_p(\boldsymbol{\alpha}_j(t)) + (\varepsilon_{(\frac{1}{2})} - \frac{1}{2})\boldsymbol{y}_j(t), \qquad j \in \{1,2\},$$

given in Example 6.



(**a**) (**b**)

Figure 3. Plots of the curves in Example 6. (**a**) The curves parametrized by $H_p(\boldsymbol{\alpha}_1(t))$ (red) and its image under a similarity transformation, parametrized by $H_p(\boldsymbol{\alpha}_2(t))$ (red). (**b**) The curve parametrized by $\boldsymbol{y}_1(t)$ (red) and its image under a similarity transformation, parametrized by $\boldsymbol{y}_2(t)$ (red).

Remark 5. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{D}_p^3$ be two rational algebraic generalized dual curves with proper parametrizations*

$$\boldsymbol{\alpha}_j = H_p(\boldsymbol{\alpha}_j) + (\varepsilon_p - p)\boldsymbol{y}_j, \qquad j \in \{1,2\},$$

satisfying the hypotheses of Section 4.1. When the dual similarity invariants S_j, $j \in \{1,2\}$, defined in Equation (42), are constant, similarity detection proceeds by first determining the similarities between the real curves parametrized by $H_p(\boldsymbol{\alpha}_1)$ and $H_p(\boldsymbol{\alpha}_2)$.

More precisely, consider a generalized dual similarity with ratio λ, generalized dual orthogonal matrix

$$\boldsymbol{Q} = \boldsymbol{Q}_1 + (\varepsilon_p - p)\boldsymbol{Q}_2,$$

and translation vector

$$\boldsymbol{b} = \boldsymbol{b}_1 + (\varepsilon_p - p)\boldsymbol{b}_2.$$

Applying the algorithm presented in [12] to the real parametrizations $H_p(\boldsymbol{\alpha}_1), H_p(\boldsymbol{\alpha}_2)$ yields candidate values of λ, the real orthogonal matrix $\boldsymbol{Q}_1$, and the translation vector $\boldsymbol{b}_1$, satisfying Equation (28). For each candidate, the remaining components $\boldsymbol{Q}_2$ and $\boldsymbol{b}_2$ of the similarity transformation are then determined, when they exist, from Equation (29) using the procedure described in Section 4.3.

5. Conclusions and Future Work

In this paper, we developed a generalized dual Frenet apparatus for regular curves in the n-dimensional generalized dual space $\mathbb{D}_p^n$. We derived the corresponding Frenet equations and established the relation between the generalized dual Frenet apparatus and the classical Frenet apparatus. In the three-dimensional case, these results were then used to develop a symbolic method for detecting similarities between rational algebraic curves. The proposed algorithm, based on the generalized dual curvature and torsion, determines the possible Möbius transformations and the corresponding similarity transformations. Its implementation in Maple demonstrates the applicability of the proposed approach through explicit examples.

The proposed method has two main limitations. First, it does not apply when the curvatures are constant. Second, it requires the generalized dual curves to admit an arc-length parametrization satisfying the conditions stated in the paper. Consequently, the method cannot be applied to generalized dual curves that do not satisfy these conditions.

As future work, we plan to investigate similarity detection in cases where the similarity invariants are constant, as well as to extend the proposed approach to higher-dimensional generalized dual spaces and other equivalence problems for rational curves.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

In this appendix, we provide a proof of Lemma 5. Assume that $\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1 + \boldsymbol{b}_1 = \boldsymbol{v}_1$. Let us differentiate this equation with respect to t up to the fourth order. We have

$$\begin{aligned} \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1' &= \boldsymbol{v}_1' \\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1'' &= \boldsymbol{v}_1'' \\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1''' &= \boldsymbol{v}_1''' \\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(4)} &= \boldsymbol{v}_1^{(4)}. \end{aligned} \tag{A1}$$

We define $\Delta(\boldsymbol{u}_1) = det(D(\boldsymbol{u}_1)) = \left\| \boldsymbol{u}_1' \, \boldsymbol{u}_1'' \, \boldsymbol{u}_1''' \right\|$, and similarly $\Delta(\boldsymbol{v}_1) = \left\| \boldsymbol{v}_1' \, \boldsymbol{v}_1'' \, \boldsymbol{v}_1''' \right\|$. One can easily verify that $\Delta(\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1) = \lambda^3 det(\boldsymbol{Q}_1) \Delta(\boldsymbol{u}_1) = \pm \lambda^3 \Delta(\boldsymbol{u}_1)$. Also, using Equation (A1), $\Delta(\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1) = \Delta(\boldsymbol{v}_1)$. Setting

$$I_1(\boldsymbol{u}_1) = \frac{\left\| \boldsymbol{u}_1^{(4)} \, \boldsymbol{u}_1'' \, \boldsymbol{u}_1''' \right\|}{\Delta(\boldsymbol{u}_1)}, \; I_2(\boldsymbol{u}_1) = \frac{\left\| \boldsymbol{u}_1' \, \boldsymbol{u}_1^{(4)} \, \boldsymbol{u}_1''' \right\|}{\Delta(\boldsymbol{u}_1)}, \; I_3(\boldsymbol{u}_1) = \frac{\left\| \boldsymbol{u}_1' \, \boldsymbol{u}_1'' \, \boldsymbol{u}_1^{(4)} \right\|}{\Delta(\boldsymbol{u}_1)}, \tag{A2}$$

we obtain

$$
\begin{aligned}
I_1(\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1) &= \frac{\left\| \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(4)}\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1''\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1''' \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = \frac{\pm \lambda^3 \left\| \boldsymbol{u}_1^{(4)}\ \boldsymbol{u}_1''\ \boldsymbol{u}_1''' \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = I_1(\boldsymbol{u}_1) \\
I_2(\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1) &= \frac{\left\| \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1'\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(4)}\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1''' \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = \frac{\pm \lambda^3 \left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1^{(4)}\ \boldsymbol{u}_1''' \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = I_2(\boldsymbol{u}_1) \\
I_3(\lambda \boldsymbol{Q}_1 \boldsymbol{u}_1) &= \frac{\left\| \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1'\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1''\ \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(4)} \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = \frac{\pm \lambda^3 \left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1''\ \boldsymbol{u}_1^{(4)} \right\|}{\pm \lambda^3 \Delta(\boldsymbol{u}_1)} = I_3(\boldsymbol{u}_1).
\end{aligned} \tag{A3}
$$

Then, for $i \in \{1,2,3\}$ Equations (A1) and (A3) yield

$$
I_i(\boldsymbol{u}_1) = I_i(\boldsymbol{v}_1). \tag{A4}
$$

We now establish several results based on the preceding setup.

1. $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = \begin{bmatrix} 0 & 0 & I_1(\boldsymbol{u}_1) \\ 1 & 0 & I_2(\boldsymbol{u}_1) \\ 0 & 1 & I_3(\boldsymbol{u}_1) \end{bmatrix}$ and $D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)' = \begin{bmatrix} 0 & 0 & I_1(\boldsymbol{v}_1) \\ 1 & 0 & I_2(\boldsymbol{v}_1) \\ 0 & 1 & I_3(\boldsymbol{v}_1) \end{bmatrix}$.

Proof. We only prove $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = \begin{bmatrix} 0 & 0 & I_1(\boldsymbol{u}_1) \\ 1 & 0 & I_2(\boldsymbol{u}_1) \\ 0 & 1 & I_3(\boldsymbol{u}_1) \end{bmatrix}$, since the other identity follows similarly. Let $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = M$ for an unknown matrix M. We set $M = [\boldsymbol{m}_1\, \boldsymbol{m}_2\, \boldsymbol{m}_3]$, where $\boldsymbol{m}_i = (m_{i1}, m_{i2}, m_{i3})^T$ for $i \in \{1,2,3\}$. Since $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = M$ and $D(\boldsymbol{u}_1)' = \left[\boldsymbol{u}_1''\ \boldsymbol{u}_1'''\ \boldsymbol{u}_1^{(4)} \right]$, we have $D(\boldsymbol{u}_1) M = D(\boldsymbol{u}_1)'$, and hence $D(\boldsymbol{u}_1) M = \left[\boldsymbol{u}_1''\ \boldsymbol{u}_1'''\ \boldsymbol{u}_1^{(4)} \right]$. The latter yields three systems of equations $D(\boldsymbol{u}_1) \boldsymbol{m}_i = \boldsymbol{u}_1^{(i+1)}$ for $i \in 1,2,3$. Since $D(\boldsymbol{u}_1)$ is invertible, each system of equations has a unique solution $\boldsymbol{m}_i$. Solving each system using Cramer's rule, for $\boldsymbol{m}_1$, we have

$$
m_{11} = \frac{\left\| \boldsymbol{u}_1''\ \boldsymbol{u}_1''\ \boldsymbol{u}_1'''' \right\|}{\Delta(\boldsymbol{u}_1)} = 0, \quad m_{21} = \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1''\ \boldsymbol{u}_1'''' \right\|}{\Delta(\boldsymbol{u}_1)} = 1, \quad m_{31} = \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1''\ \boldsymbol{u}_1''' \right\|}{\Delta(\boldsymbol{u}_1)} = 0,
$$

for $\boldsymbol{m}_2$, we have

$$
m_{12} = \frac{\left\| \boldsymbol{u}_1'''\ \boldsymbol{u}_1''\ \boldsymbol{u}_1'''' \right\|}{\Delta(\boldsymbol{u}_1)} = 0, \quad m_{22} = \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1'''\ \boldsymbol{u}_1'''' \right\|}{\Delta(\boldsymbol{u}_1)} = 0, \quad m_{32} = \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1''\ \boldsymbol{u}_1'''' \right\|}{\Delta(\boldsymbol{u}_1)} = 1,
$$

and for $\boldsymbol{m}_3$, we have

$$
\begin{aligned}
m_{13} &= \frac{\left\| \boldsymbol{u}_1^{(4)}\ \boldsymbol{u}_1''\ \boldsymbol{u}_1''' \right\|}{\Delta(\boldsymbol{u}_1)} = I_1(\boldsymbol{u}_1), \quad m_{23} = \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1^{(4)}\ \boldsymbol{u}_1''' \right\|}{\Delta(\boldsymbol{u}_1)} = I_2(\boldsymbol{u}_1), \\
m_{33} &= \frac{\left\| \boldsymbol{u}_1'\ \boldsymbol{u}_1''\ \boldsymbol{u}_1^{(4)} \right\|}{\Delta(\boldsymbol{u}_1)} = I_3(\boldsymbol{u}_1).
\end{aligned}
$$

The above solutions yield

$$
M = \begin{bmatrix} 0 & 0 & I_1(\boldsymbol{u}_1) \\ 1 & 0 & I_2(\boldsymbol{u}_1) \\ 0 & 1 & I_3(\boldsymbol{u}_1) \end{bmatrix}. \tag{A5}
$$

□

2. $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)'$.

Proof. Using the previous result, and the fact that $I_i(\boldsymbol{u}_1) = I_i(\boldsymbol{v}_1)$, we have

$$\begin{bmatrix} 0 & 0 & I_1(\boldsymbol{u}_1) \\ 1 & 0 & I_2(\boldsymbol{u}_1) \\ 0 & 1 & I_3(\boldsymbol{u}_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_1(\boldsymbol{v}_1) \\ 1 & 0 & I_2(\boldsymbol{v}_1) \\ 0 & 1 & I_3(\boldsymbol{v}_1) \end{bmatrix}$$
$$D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)'$$

□

3. $\frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} = \boldsymbol{R}$, where $\boldsymbol{R}$ is a 3×3 constant matrix.

Proof. Let us differentiate $\frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} = \boldsymbol{Q}_1$ with respect to t. Then, it is obtained

$$\begin{aligned} \boldsymbol{R}' &= \left[\frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} \right]' \\ &= \frac{1}{\lambda} \left[D(\boldsymbol{v}_1)' D(\boldsymbol{u}_1)^{-1} - D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' D(\boldsymbol{u}_1)^{-1} \right] \\ &= \frac{1}{\lambda} \left[D(\boldsymbol{v}_1) D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)' D(\boldsymbol{u}_1)^{-1} - D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' D(\boldsymbol{u}_1)^{-1} \right] \\ &= \frac{1}{\lambda} \left[D(\boldsymbol{v}_1) \Big(D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)' - D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' \Big) D(\boldsymbol{u}_1)^{-1} \right]. \end{aligned}$$

Since $D(\boldsymbol{u}_1)^{-1} D(\boldsymbol{u}_1)' = D(\boldsymbol{v}_1)^{-1} D(\boldsymbol{v}_1)'$, we have $\boldsymbol{R}' = \boldsymbol{0}_3$. Therefore, $\boldsymbol{R}$ is a constant matrix. □

4. $\lambda^2 D(\boldsymbol{u}_1)^T D(\boldsymbol{u}_1) = D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1)$.

Proof. Let us compute $D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1)$. This matrix product is given by

$$D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1) = \begin{bmatrix} < \boldsymbol{v}_1', \boldsymbol{v}_1' > & < \boldsymbol{v}_1', \boldsymbol{v}_1'' > & < \boldsymbol{v}_1', \boldsymbol{v}_1''' > \\ < \boldsymbol{v}_1'', \boldsymbol{v}_1' > & < \boldsymbol{v}_1'', \boldsymbol{v}_1'' > & < \boldsymbol{v}_1'', \boldsymbol{v}_1''' > \\ < \boldsymbol{v}_1''', \boldsymbol{v}_1' > & < \boldsymbol{v}_1''', \boldsymbol{v}_1'' > & < \boldsymbol{v}_1''', \boldsymbol{v}_1''' > \end{bmatrix}. \tag{A6}$$

Using Equation (A1), we have $< \boldsymbol{v}_1^{(i)}, \boldsymbol{v}_1^{(j)} > = < \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(i)}, \lambda \boldsymbol{Q}_1 \boldsymbol{u}_1^{(j)} > = \lambda^2 < \boldsymbol{u}_1^{(i)}, \boldsymbol{u}_1^{(j)} >$ for $i, j \in \{1, 2, 3\}$. Using this in Equation (A6), we get

$$D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1) = \lambda^2 D(\boldsymbol{u}_1)^T D(\boldsymbol{u}_1).$$

□

5. $\frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} = \boldsymbol{R}$, where $\boldsymbol{R}$ is an orthogonal matrix.

Proof. Let us compute $\boldsymbol{R}\boldsymbol{R}^T$. Then, we have

$$\begin{aligned} \boldsymbol{R}^T \boldsymbol{R} &= \left[\frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} \right]^T \frac{1}{\lambda} D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} \\ &= \frac{1}{\lambda^2} \left[D(\boldsymbol{u}_1)^T \right]^{-1} D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1) D(\boldsymbol{u}_1)^{-1} \end{aligned}$$

Here, using $\lambda^2 D(\boldsymbol{u}_1)^T D(\boldsymbol{u}_1) = D(\boldsymbol{v}_1)^T D(\boldsymbol{v}_1)$, we get

$$\begin{aligned} \boldsymbol{R}^T \boldsymbol{R} &= \frac{1}{\lambda^2} \left[D(\boldsymbol{u}_1)^T \right]^{-1} \lambda^2 D(\boldsymbol{u}_1)^T D(\boldsymbol{u}_1) D(\boldsymbol{u}_1)^{-1} \\ &= \left[D(\boldsymbol{u}_1)^T \right]^{-1} D(\boldsymbol{u}_1)^T D(\boldsymbol{u}_1) D(\boldsymbol{u}_1)^{-1} = \boldsymbol{I}_3 \end{aligned}$$

□

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