mathematics

MDPI

Article

Exact Thresholds for Global Univalence, Convexity, and Positive Real Part of a Modified Second Einstein Function

Amr M. Y. Abdelaty *, Hesham M. A. Mohamed and Ibrahim S. Elshazly *

Department of Basic Sciences, Common First Year Deanship, King Saud University, Riyadh 12373, Saudi Arabia; hmmohamed@ksu.edu.sa

* Correspondence: ayoussef@ksu.edu.sa (A.M.Y.A.); iali2.c@ksu.edu.sa (I.S.E.)

Abstract

We consider the parameterized modified second Einstein mapping in the unit disk and ask where its main geometric properties actually change as the parameter varies. The answer gives three different sharp thresholds. One determines global univalence, a second one convexity, and the third positive real part. Each threshold is necessary and sufficient. After normalization, the function is of bounded turning on the whole univalence range and is therefore close-to-convex there. The endpoint behavior is also exact: the derivative and convexity criteria reach equality at the positive real boundary point, while the positive-real-part condition reaches equality at the negative real boundary point. Earlier work used decimal cutoffs and plots to describe the admissible range. We replace these decimal cutoffs by exact analytic thresholds, including necessity and boundary equalities; the numerical refinement is small, especially at the upper endpoint. For the proof, we combine Bernoulli expansions with explicit tail estimates and harmonic minimum arguments. Real-axis points give necessity, while monotonicity in the parameter and a Sturm calculation settle the convexity endpoint. In this way, the three transitions and their endpoint behavior are determined exactly for this family.

Keywords: second Einstein function; univalence; convexity; positive real part; close-to-convexity; exact threshold

MSC: 30C45; 30C55

1. Introduction

We start with one question: as the parameter varies, do univalence, convexity, and positive real part change at the same value? For the modified second Einstein family, there is no reason that they should. Earlier papers used a practical decimal interval in later class constructions, but that interval does not decide whether the endpoints are sharp or whether necessity holds. It also leaves the boundary cases open. Here we determine the three thresholds separately, compare their parameter ranges, and identify the equality at each critical endpoint.

Let

$$E_2(z) = \frac{z^2 e^z}{(e^z - 1)^2}, \qquad G_\kappa(z) = E_2(z) + \kappa z, \qquad \kappa > 0. \tag{1}$$

with the removable value at the origin understood. A 2022 *Symmetry* paper is the closest earlier point of reference for the second modified map [1]. It includes the fixed function $E_2(z) + \frac{1}{2}z$ as a companion object, while the paper itself develops the modified E_1 branch

Received: 3 October 2026
Revised: 15 October 2026
Accepted: 26 October 2026
Published: 5 November 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

and leaves the modified second branch open. The present family starts from that same second map and allows its linear coefficient to vary. Ibrahim S. Elshazly, a corresponding author of the present article, also co-authored this 2022 study and the 2024 study cited below. The present work determines exact mapping thresholds and boundary equalities, whereas those studies concern bi-univalent coefficient problems.

In 2022, El-Qadeem, Saleh, and Mamon used the parameterized modified second Einstein map in bi-univalent coefficient and Fekete–Szegő problems [2]. Their work adopts the interval $[0.28, 0.92]$ and states convexity and positive real part there, with representative plots used to indicate failure outside the two cutoffs. The same interval appears again in the 2024 *Symmetry* paper on λ-pseudo bi-starlike classes [3]. It is also retained in the 2026 downstream q-generalized distribution-series construction [4]. For the present paper, these are direct comparators because the numerical mapping range enters their setup.

The present theorem system resolves these questions exactly. Write

$$\kappa_{\text{local}} = \frac{e(3-e)}{(e-1)^3}, \qquad \kappa_{\text{conv}} = \frac{2e(2e-5)}{(e-1)^4}, \qquad \kappa_{\text{pos}} = \frac{e}{(e-1)^2}.$$

These thresholds are strictly ordered. Global univalence of G_κ holds exactly from κ_{local} onward. On the same range, the normalized companion is of bounded turning and is close-to-convex. The convexity range begins exactly at κ_{conv}, whereas positive real part holds exactly up to κ_{pos}. Hence convexity and positive real part hold simultaneously exactly between the latter two thresholds. The three endpoints are distinguished by different criterion-specific boundary equalities: derivative equality at the global-univalence threshold, equality in the logarithmic-derivative convexity criterion at the convexity threshold, and positive-real-part equality at the final threshold.

The derivative-to-univalence step is classical, not a new implication: positive real part of the derivative on the disk gives global univalence through the Noshiro–Warschawski criterion [5–7]. Likewise, bounded turning and the resulting close-to-convexity of the normalized companion use standard class definitions and implications [8,9], while the convexity criterion and the Carathéodory terminology belong to the classical and modern background [10,11]. What is determined here is the object-specific sharp parameter geometry and its necessity and endpoint-equality structure.

The proofs follow one common route, but the last threshold needs an additional step. Bernoulli expansions together with explicit tail estimates give the boundary inequalities, and harmonic minimum arguments carry them into the disk. Necessity is read from suitable points on the real axis. For convexity, we also use monotonicity in the parameter, which reduces the remaining check to the endpoint; the resulting polynomial positivity is then verified by an exact Sturm argument [12]. No numerical plot is needed for these proofs.

The next sections establish the three exact thresholds, separate the seven parameter regimes, classify the boundary equalities, and finally compare the exact range with the earlier decimal prescription.

2. Object, Notation, and Exact Thresholds

Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

The nearest nonremovable singularities of E_2 occur at $\pm 2\pi i$. Hence every function used below extends analytically to a neighborhood of the closed unit disk. Define

$$A(z) = E_2'(z), \qquad B(z) = E_2'(z) + zE_2''(z).$$

The Bernoulli expansion is

$$E_2(z) = 1 + \sum_{m\geq 1} (-1)^m b_m z^{2m}, \qquad b_m = \frac{(2m-1)|B_{2m}|}{(2m)!} > 0, \tag{2}$$

and therefore

$$A(z) = \sum_{m\geq 1} (-1)^m c_m z^{2m-1}, \qquad c_m = 2m b_m > 0, \tag{3}$$

$$B(z) = \sum_{m\geq 1} (-1)^m (2m) c_m z^{2m-1}. \tag{4}$$

Using $|B_{2m}| = 2(2m)!\zeta(2m)/(2\pi)^{2m}$, together with $\zeta(2m) < 2$ and $\pi > 3$, gives the uniform majorants

$$b_m < \frac{4(2m-1)}{36^m}, \qquad c_m < \frac{8m(2m-1)}{36^m}. \tag{5}$$

Because the nearest nonremovable singularities are at $\pm 2\pi i$, the series above and their differentiated forms converge absolutely and uniformly on the closed unit disk. Termwise differentiation and the boundary Fourier rearrangements used below are therefore justified. These estimates provide the exact tail bounds used below and in the Proof Supplement.

Definition 1. *Set*

$$\begin{aligned} \kappa_{\text{local}} &= -A(1) = \frac{e(3-e)}{(e-1)^3}, \\ \kappa_{\text{conv}} &= -B(1) = \frac{2e(2e-5)}{(e-1)^4}, \\ \kappa_{\text{pos}} &= E_2(1) = E_2(-1) = \frac{e}{(e-1)^2}. \end{aligned} \tag{6}$$

Lemma 1 (Exact threshold order)**.** *The thresholds defined in Equation* (6) *satisfy*

$$0 < \kappa_{\text{local}} < \kappa_{\text{conv}} < \kappa_{\text{pos}}.$$

Proof. Direct simplification gives

$$\kappa_{\text{conv}} - \kappa_{\text{local}} = \frac{e(e^2-7)}{(e-1)^4} > 0,$$

because $e > 2.7$, and

$$\kappa_{\text{pos}} - \kappa_{\text{conv}} = \frac{e(e^2-6e+11)}{(e-1)^4} = \frac{e((e-3)^2+2)}{(e-1)^4} > 0.$$

Positivity of κ_{local} follows from $e < 3$. For an explicit enclosure, put $L_{40} = \sum_{n=0}^{40} 1/n!$. The first omitted term is $1/41!$, and each later ratio is at most $1/42$, so

$$L_{40} < e < L_{40} + \frac{42}{41 \cdot 41!}.$$

https://doi.org/10.3390/math14214064

Substitution with outward rational bounds gives

$$\frac{1509475787}{10^{10}} < \kappa_{\text{local}} < \frac{1509475788}{10^{10}},$$
$$\frac{2722666621}{10^{10}} < \kappa_{\text{conv}} < \frac{2722666622}{10^{10}},$$
$$\frac{9206735942}{10^{10}} < \kappa_{\text{pos}} < \frac{9206735943}{10^{10}}.$$

The full fractions and interval formulas are given in Supplementary Section S3. □

For later reference define

$$L = \{\kappa > 0 : G'_\kappa \neq 0 \text{ on } \mathbb{D}\}, \qquad U = \{\kappa > 0 : G_\kappa \text{ is univalent on } \mathbb{D}\},$$

$$C = \{\kappa > 0 : G_\kappa \text{ is convex on } \mathbb{D}\},$$
$$J = \{\kappa > 0 : \operatorname{Re} G_\kappa(z) > 0 \text{ for every } z \in \mathbb{D}\}.$$

The symbols L and U distinguish local univalence (critical-point freedom) from global injectivity. Their equality below is a conclusion specific to this family, so both are retained. For $\kappa > 0$, introduce the normalized companion

$$F_\kappa(z) = \frac{G_\kappa(z) - 1}{\kappa}, \qquad F_\kappa(0) = 0, \qquad F'_\kappa(0) = 1. \tag{7}$$

We use the standard normalized bounded-turning class [9]

$$\mathcal{R} = \{f \text{ analytic on } \mathbb{D} : f(0) = 0,\ f'(0) = 1,\ \operatorname{Re} f'(z) > 0 \text{ on } \mathbb{D}\},$$

and the normalized Carathéodory class [11]

$$\mathcal{P} = \{p \text{ analytic on } \mathbb{D} : p(0) = 1,\ \operatorname{Re} p(z) > 0 \text{ on } \mathbb{D}\}.$$

3. Main Results and the Three-Threshold Phase Portrait

Theorem 1 (Exact global mapping-property theorem). *For the family G_κ in (1),*

$$U = L = [\kappa_{\text{local}}, \infty), \qquad C = [\kappa_{\text{conv}}, \infty), \qquad J = (0, \kappa_{\text{pos}}],$$

and hence

$$C \cap J = [\kappa_{\text{conv}}, \kappa_{\text{pos}}].$$

Moreover,

$$F_\kappa \in \mathcal{R} \iff F_\kappa \text{ is close-to-convex on } \mathbb{D} \iff \kappa \geq \kappa_{\text{local}},$$

and

$$G_\kappa \in \mathcal{P} \iff 0 < \kappa \leq \kappa_{\text{pos}}.$$

All displayed endpoints are exact.

Theorem 2 (Complete critical boundary-equality theorem). *At each critical parameter, the corresponding real-part criterion is strictly positive in $\mathbb{D}$ and vanishes at exactly one point of $\partial\mathbb{D}$, as specified below.*

(a) *At $\kappa = \kappa_{\text{local}}$, $\operatorname{Re} G'_\kappa(z) > 0$ for $z \in \mathbb{D}$, while the unique unit-circle derivative equality is*

$$G'_{\kappa_{\text{local}}}(1) = 0.$$

(b) *At* $\kappa = \kappa_{\mathrm{conv}}$,

$$\mathrm{Re}\left(1 + \frac{zG_\kappa''(z)}{G_\kappa'(z)}\right) > 0 \quad (z \in \mathbb{D}),$$

while the unique unit-circle equality of the convexity criterion occurs at $z = 1$.

(c) *At* $\kappa = \kappa_{\mathrm{pos}}$, $\mathrm{Re}\, G_\kappa(z) > 0$ *throughout* $\mathbb{D}$, *while the unique unit-circle equality of the positive-real-part criterion is*

$$\mathrm{Re}\, G_{\kappa_{\mathrm{pos}}}(-1) = 0.$$

We use the phrase "critical boundary equality" only as a convenient label for these three cases. They are equalities for different boundary quantities. At the corresponding critical parameter, no further nonreal equality occurs.

Corollary 1 (Exact phase portrait)**.** *The parameter axis decomposes into the seven regimes shown in Table 1.*

Table 1. Exact three-threshold phase portrait. The composite column $U/\mathcal{R}/\mathrm{CTC}$ records, in order, global univalence of G_κ, bounded turning of F_κ, and close-to-convexity of F_κ.

Parameter Regime	$U/\mathcal{R}$/CTC	Convex	$G_\kappa \in \mathcal{P}$	Equality/Failure Feature
$0 < \kappa < \kappa_{\mathrm{local}}$	no	no	yes	real critical point in $(0,1)$
$\kappa = \kappa_{\mathrm{local}}$	yes	no	yes	derivative equality at $z = 1$
$\kappa_{\mathrm{local}} < \kappa < \kappa_{\mathrm{conv}}$	yes	no	yes	no critical boundary equality
$\kappa = \kappa_{\mathrm{conv}}$	yes	yes	yes	convexity-criterion equality at $z = 1$
$\kappa_{\mathrm{conv}} < \kappa < \kappa_{\mathrm{pos}}$	yes	yes	yes	strict interior regime
$\kappa = \kappa_{\mathrm{pos}}$	yes	yes	yes	positive-real equality at $z = -1$
$\kappa > \kappa_{\mathrm{pos}}$	yes	yes	no	negative real part near $z = -1$

Sections 4–7 contain the proofs of Theorems 1 and 2. After the three exact parameter sets have been established, Corollary 1 is simply their ordered combination.

4. Exact Global-Univalence and Bounded-Turning Threshold

The first threshold controls more than critical-point freedom. The same sharp inequality gives $\mathrm{Re}\, G_\kappa'(z) > 0$, which yields the exact global-univalence threshold.

Proposition 1 (Exact positive-derivative-real-part threshold)**.** *For* $\kappa > 0$,

$$\mathrm{Re}\, G_\kappa'(z) > 0 \quad \textit{for every } z \in \mathbb{D} \iff \kappa \geq \kappa_{\mathrm{local}}.$$

Consequently,

$$L = [\kappa_{\mathrm{local}}, \infty).$$

At $\kappa = \kappa_{\mathrm{local}}$, *the sole unit-circle derivative equality is* $z = 1$. *If* $0 < \kappa < \kappa_{\mathrm{local}}$, *there is a real point* $x_\kappa \in (0,1)$ *with* $G_\kappa'(x_\kappa) = 0$.

Proof. For $z = e^{it}$, the expansion (3) gives

$$\mathrm{Re}\, A(e^{it}) - A(1) = \sum_{m \geq 1} (-1)^m c_m \big(\cos((2m-1)t) - 1\big). \tag{8}$$

The first term contributes $c_1(1 - \cos t)$ with $c_1 = 1/6$. Discarding favorable odd terms and using

$$1 - \cos((2m-1)t) \leq (2m-1)^2(1 - \cos t)$$

for the adverse even terms yields

$$\operatorname{Re} A(e^{it}) - A(1) \geq \left(\frac{1}{6} - S_{\text{even}}\right)(1 - \cos t),$$

where

$$S_{\text{even}} = \sum_{j \geq 1} c_{2j}(4j-1)^2.$$

The finite part and the analytic remainder satisfy

$$\sum_{j=1}^{6} c_{2j}(4j-1)^2 = \frac{3861696291570651165019}{25359685841593958400000},$$

$$\sum_{j \geq 7} c_{2j}(4j-1)^2 < \sum_{j \geq 7} \frac{16j(4j-1)^3}{1296^j} < \frac{36}{10^{17}}.$$

The tail uses (5); its rational evaluation follows by applying $q\,d/dq$ to the geometric series. Thus

$$S_{\text{even}} < \frac{152277}{10^6}, \qquad \frac{4}{25} - S_{\text{even}} > \frac{7723}{10^6}.$$

Supplementary Sections S2 and S4 give the full rational tail and subtraction. In particular,

$$S_{\text{even}} < \frac{4}{25}, \qquad \frac{1}{6} - S_{\text{even}} > \frac{1}{150}.$$

Hence

$$\operatorname{Re} A(e^{it}) - A(1) \geq \frac{1}{150}(1 - \cos t), \tag{9}$$

with strict inequality away from $t \equiv 0 \pmod{2\pi}$. By harmonic minimum,

$$\operatorname{Re} A(z) > A(1) = -\kappa_{\text{local}} \qquad (z \in \mathbb{D}).$$

Therefore for $\kappa \geq \kappa_{\text{local}}$,

$$\operatorname{Re} G'_\kappa(z) = \kappa + \operatorname{Re} A(z) > \kappa - \kappa_{\text{local}} \geq 0,$$

and the inequality is strict in $\mathbb{D}$; thus no critical point occurs there. At $\kappa = \kappa_{\text{local}}$, equality on the unit circle can occur only at $z = 1$, where $G'_\kappa(1) = \kappa + A(1) = 0$.

Conversely, if $0 < \kappa < \kappa_{\text{local}}$, then $G'_\kappa(0) = \kappa > 0$ while

$$G'_\kappa(1) = \kappa - \kappa_{\text{local}} < 0.$$

Continuity on the real interval gives some $x_\kappa \in (0,1)$ for which $G'_\kappa(x_\kappa) = 0$. Thus strict positive derivative real part fails below κ_{local}, proving the stated equivalence and the identity for L. □

Corollary 2 (Exact global univalence, bounded turning, and close-to-convexity)**.** *For $\kappa > 0$, the following are equivalent:*

$$\kappa \geq \kappa_{\text{local}}, \qquad \operatorname{Re} G'_\kappa(z) > 0 \ \ (z \in \mathbb{D}), \qquad G_\kappa \text{ is univalent on } \mathbb{D}.$$

Equivalently ,

$$U = L = [\kappa_{\text{local}}, \infty).$$

https://doi.org/10.3390/math14214064

For the normalized companion in (7),

$$F_\kappa \in \mathcal{R} \iff F_\kappa \text{ is close-to-convex on } \mathbb{D} \iff \kappa \geq \kappa_{\text{local}}.$$

Proof. The first implication is Proposition 1. Since $\mathbb{D}$ is convex, the Noshiro–Warschawski criterion yields global univalence from $\operatorname{Re} G'_\kappa > 0$ [5,6]. Conversely, for $0 < \kappa < \kappa_{\text{local}}$, Proposition 1 supplies an interior zero of G'_κ, which is impossible for a univalent analytic function.

From (7),

$$F'_\kappa(z) = \frac{G'_\kappa(z)}{\kappa},$$

so $F_\kappa \in \mathcal{R}$ exactly when $\kappa \geq \kappa_{\text{local}}$. The condition $\operatorname{Re} F'_\kappa > 0$ also implies close-to-convexity (with respect to the identity map) by the classical close-to-convex criterion [8]. Below κ_{local}, the interior critical point rules out close-to-convexity because every close-to-convex function is univalent. □

5. Exact Positive-Real-Part Threshold

Proposition 2 (Exact positive-real-part/Carathéodory set)**.** *The exact set is*

$$J = (0, \kappa_{\text{pos}}],$$

equivalently,

$$G_\kappa \in \mathcal{P} \iff 0 < \kappa \leq \kappa_{\text{pos}}.$$

At $\kappa = \kappa_{\text{pos}}$, the unique unit-circle equality of the real-part criterion is $z = -1$.

Proof. On $|z| = 1$, (2) gives

$$\operatorname{Re} E_2(e^{it}) - E_2(1) = \sum_{m \geq 1} (-1)^m b_m \big(\cos(2mt) - 1\big). \tag{10}$$

The first term is $b_1(1 - \cos 2t)$ with $b_1 = 1/12$. We use

$$1 - \cos(2mt) \leq m^2(1 - \cos 2t)$$

to bound the even indices $m \geq 2$, the only adverse terms. We retain the conservative sum

$$S_{\text{pos}} := \sum_{m \geq 2} m^2 b_m,$$

which also includes the favorable odd indices. Its finite part and remainder are

$$\sum_{m=2}^{12} m^2 b_m = \frac{42939076777171070142938291}{2352533523238532874240000},$$
$$\sum_{m \geq 13} m^2 b_m < \sum_{m \geq 13} \frac{4m^2(2m-1)}{36^m} < \frac{103}{10^{18}}.$$

Consequently $S_{\text{pos}} < 18253/10^6$ and $1/50 - S_{\text{pos}} > 1747/10^6$. The remainder follows from (5) and the geometric-series moment formula; full rational values are in Supplementary Sections S2 and S5.

$$S_{\text{pos}} < \frac{1}{50}, \qquad \frac{1}{12} - S_{\text{pos}} > \frac{19}{300}.$$

Consequently

$$\operatorname{Re} E_2(e^{it}) - E_2(1) \geq \frac{19}{300}(1 - \cos 2t), \tag{11}$$

strictly unless $t \equiv 0$ or $\pi \pmod{2\pi}$. Thus

$$\operatorname{Re} E_2(e^{it}) \geq E_2(1) = \kappa_{\text{pos}},$$

with equality only at $z = \pm 1$. For $0 < \kappa \leq \kappa_{\text{pos}}$,

$$\operatorname{Re} G_\kappa(e^{it}) \geq \kappa_{\text{pos}} + \kappa \cos t \geq \kappa_{\text{pos}} - \kappa \geq 0. \tag{12}$$

If $\kappa < \kappa_{\text{pos}}$, the boundary lower bound is strictly positive. At $\kappa = \kappa_{\text{pos}}$, equality in (12) requires $\cos t = -1$; combined with (11), this leaves only $z = -1$. Harmonic minimum then yields $\operatorname{Re} G_\kappa(z) > 0$ for every $z \in \mathbb{D}$. Strictness follows from the strong minimum principle and $\operatorname{Re} G_\kappa(0) = 1$; a boundary zero therefore does not exclude the endpoint $\kappa = \kappa_{\text{pos}}$.

For necessity, if $\kappa > \kappa_{\text{pos}}$, then on the negative real radius

$$G_\kappa(-r) \longrightarrow E_2(-1) - \kappa = \kappa_{\text{pos}} - \kappa < 0 \qquad (r \uparrow 1).$$

Hence $\operatorname{Re} G_\kappa$ is negative at an interior point for all sufficiently large $r < 1$. Therefore $J = (0, \kappa_{\text{pos}}]$. □

6. Exact Convexity Threshold

The convexity endpoint requires the most substantial exact argument. For $\kappa \geq \kappa_{\text{conv}} > \kappa_{\text{local}}$, Corollary 2 gives $\operatorname{Re} G'_\kappa(z) > 0$ on $\mathbb{D}$. Hence the denominator in the logarithmic-derivative criterion is nonzero throughout the candidate range. We may therefore apply the classical convexity criterion [10,11]:

$$G_\kappa \text{ convex on } \mathbb{D} \iff \operatorname{Re} Q_\kappa(z) > 0 \quad (z \in \mathbb{D}), \tag{13}$$

where

$$Q_\kappa(z) = 1 + \frac{zE_2''(z)}{\kappa + E_2'(z)} = \frac{\kappa + B(z)}{\kappa + A(z)}. \tag{14}$$

The boundary estimate from Proposition 1 also gives $\operatorname{Re}(\kappa + A(e^{it})) \geq \kappa - \kappa_{\text{local}} > 0$ for $\kappa \geq \kappa_{\text{conv}}$, so the denominator in the boundary form below is nonzero on the unit circle as well.

6.1. Boundary Reduction and Monotonicity in the Parameter

Put $z = e^{it}$. Then

$$\begin{aligned} \operatorname{Re} Q_\kappa(e^{it}) &= \frac{F_\kappa(t)}{|\kappa + A(e^{it})|^2}, \\ F_\kappa(t) &= \operatorname{Re}\Big[(\kappa + B(e^{it}))(\kappa + \overline{A(e^{it})})\Big] \\ &= \kappa^2 + \kappa U(t) + V(t), \end{aligned} \tag{15}$$

where, with A and B evaluated at e^{it},

$$U(t) = \operatorname{Re}\Big(A(e^{it}) + B(e^{it})\Big), \qquad V(t) = \operatorname{Re}\Big(B(e^{it})\overline{A(e^{it})}\Big).$$

The first two harmonics of U are

$$-\frac{1}{2}\cos t + \frac{1}{12}\cos 3t = \frac{x^3}{3} - \frac{3x}{4}, \qquad x = \cos t.$$

This cubic has minimum $-\sqrt{3}/4 > -7/16$, while, with $T_U = \sum_{m\geq 3}(2m+1)c_m$,

$$\sum_{m=3}^{12}(2m+1)c_m = \frac{4530810083056196491913}{6137043973665737932800000},$$
$$\sum_{m\geq 13}(2m+1)c_m < \sum_{m\geq 13}\frac{8m(2m-1)(2m+1)}{36^m} < \frac{427}{10^{18}}.$$

Hence $T_U < 7383/10^6$, leaving margin $1/100 - T_U > 2617/10^6$. The exact sums are in Supplementary Section S6.1. In particular,

$$\sum_{m\geq 3}(2m+1)c_m < \frac{1}{100}.$$

Thus

$$U(t) > -\frac{179}{400} > -\frac{1}{2}. \tag{16}$$

Since $\kappa_{\mathrm{conv}} > 27/100$,

$$\frac{\partial F_\kappa(t)}{\partial \kappa} = 2\kappa + U(t) > 2\frac{27}{100} - \frac{1}{2} = \frac{1}{25} > 0 \tag{17}$$

for every $\kappa \geq \kappa_{\mathrm{conv}}$. It therefore suffices to certify the endpoint $\kappa = \kappa_{\mathrm{conv}}$.

6.2. Exact Endpoint Verification

At $\kappa = \kappa_{\mathrm{conv}} = -B(1)$, one has $F_{\kappa_{\mathrm{conv}}}(0) = 0$. Define, for $r \geq 1$,

$$V_r = \sum_{n\geq 1}(4n+2r)c_n c_{n+r} > 0.$$

The nonconstant Fourier coefficients of $F_{\kappa_{\mathrm{conv}}}$ are

$$C_{2m-1} = \kappa_{\mathrm{conv}}(-1)^m(2m+1)c_m, \qquad C_{2r} = (-1)^r V_r.$$

With $x = \cos t$ and

$$R_j(x) = \frac{1-T_j(x)}{1-x} = \left(\frac{\sin(jt/2)}{\sin(t/2)}\right)^2 \geq 0, \qquad R_j(1) = j^2,$$

we obtain the factorization

$$F_{\kappa_{\mathrm{conv}}}(t) = (1-x)H(x), \qquad H(x) = -\sum_{j\geq 1} C_j R_j(x). \tag{18}$$

The first six harmonics are

$$H_6(x) = \frac{\kappa_{\mathrm{conv}}}{2}R_1 + V_1 R_2 - \frac{\kappa_{\mathrm{conv}}}{12}R_3 - V_2 R_4 + \frac{\kappa_{\mathrm{conv}}}{144}R_5 + V_3 R_6.$$

The bounds in Section 2 and Supplementary Section S3 yield the following exact rational enclosure

$$\kappa_* = \frac{13613333}{50000000}, \qquad \delta = \frac{1}{400000000}, \qquad |\kappa_{\mathrm{conv}} - \kappa_*| < \delta,$$

together with

$$V_1 > \frac{336653}{20000000} = 0.01683265, \quad V_2 < \frac{26641}{20000000} = 0.00133205,$$
$$V_3 > \frac{31}{400000} = 0.0000775.$$

Here the bounds can be reproduced from

$$M_{n,r} = \frac{64(4n+2r)n(n+r)(2n-1)(2n+2r-1)}{36^r 1296^n},$$

since $(4n+2r)c_n c_{n+r} < M_{n,r}$. Write $v_{r,N} = \sum_{n=1}^{N}(4n+2r)c_n c_{n+r}$. Then

r	N	$v_{r,N}$	$\sum_{n>N} M_{n,r}$ is less than
$ine1$	3	183247/10886400	$1433/10^{11}$
2	3	5582933/4191264000	$649/10^{12}$
3	2	59/760320	$1126/10^{11}$

For the lower bounds the finite parts suffice. Exact subtraction, using the full rational tails in Supplementary Section S6.2, gives

$$V_1 - \frac{336653}{20000000} > \frac{3}{10^9}, \quad \frac{26641}{20000000} - V_2 > \frac{8}{10^9},$$
$$V_3 - \frac{31}{400000} > \frac{9}{10^8}, \quad \delta - |\kappa_{\mathrm{conv}} - \kappa_*| > \frac{3}{10^{10}}.$$

Because $0 \le R_j(x) \le j^2$ for $-1 \le x \le 1$, the κ-enclosure propagates uniformly through the three κ-terms as

$$\left|(\kappa_{\mathrm{conv}} - \kappa_*)\left[\frac{1}{2}R_1 - \frac{1}{12}R_3 + \frac{1}{144}R_5\right]\right| \le \delta\left(\frac{1}{2}R_1 + \frac{1}{12}R_3 + \frac{1}{144}R_5\right) \le \delta\frac{205}{144}.$$

Substituting the three rational V_r bounds in their favorable directions and using this uniform error bound gives, for $-1 \le x \le 1$,

$$H_6(x) - \frac{27}{1000} > \kappa_*\left(\frac{1}{2}R_1 - \frac{1}{12}R_3 + \frac{1}{144}R_5\right) + \frac{336653}{20000000}R_2 - \frac{26641}{20000000}R_4 + \frac{31}{400000}R_6 - \delta\frac{205}{144} - \frac{27}{1000} = S(x),$$

where the final equality is the exact degree-five polynomial identity

$$S(x) = \frac{1}{57600000000}\left(142848000x^5 + 1885354624x^4 + 1057273984x^3 - 6348379168x^2 - 3715097248x + 7036155579\right). \tag{19}$$

Its Sturm sequence has signs

$$(+,+,-,-,+,-) \quad \text{at } x = -1,$$

$$(+,-,-,-,+,-) \quad \text{at } x = 1,$$

https://doi.org/10.3390/math14214064

so the variation count is three at both endpoints and S has no zero in $[-1,1]$. Moreover,

$$S(-1)=\frac{1696035433}{19200000000}>0,\qquad S(1)=\frac{19385257}{19200000000}>0.$$

In particular,

$$S(1)-\frac{1}{1000}=\frac{185257}{19200000000}>0.$$

Supplementary Section S6.4 displays all six primitive-integer Sturm polynomials, obtained using only positive normalizations. The same exact chain has variations 3 and 2 at $1011/1000$ and $1013/1000$, respectively, and variation 3 at 1. Thus the nearby exterior root lies in $(1011/1000, 1013/1000)$, at distance greater than $11/1000$ from $[-1,1]$. This diagnostic isolation is separate from the exact zero-root count on $[-1,1]$; no approximate root is used as proof.

Hence

$$H_6(x)>\frac{27}{1000}\qquad(-1\le x\le 1). \tag{20}$$

For the omitted harmonics, $R_j\le j^2$. The odd contribution $E_{\mathrm{odd}}=\sum_{m\ge4}|C_{2m-1}|(2m-1)^2$ is bounded by

$$E_{\mathrm{odd}}<\frac{273}{1000}\left(\frac{13628529362854364649833}{61370439736657379328000}+\frac{269}{10^{15}}\right)<\frac{606251}{10^8},$$

$$\sum_{m\ge13}\frac{8m(2m-1)^3(2m+1)}{36^m}<\frac{269}{10^{15}}.$$

The fraction in the first line is the exact sum of $(2m+1)c_m(2m-1)^2$ for $4\le m\le 12$. For the even contribution, split $E_{\mathrm{even}}=\sum_{r\ge4}4r^2V_r$ into a finite block and two strips. The coefficient majorant above gives

$$\sum_{r=4}^{11}\sum_{n=1}^{8}4r^2(4n+2r)c_nc_{n+r}<\frac{259031884}{10^{12}},$$

$$\sum_{r=4}^{11}\sum_{n\ge9}4r^2M_{n,r}<\frac{55}{10^{26}},$$

$$\sum_{r\ge12}\sum_{n\ge1}4r^2M_{n,r}<\frac{573}{10^{16}}.$$

Thus $E_{\mathrm{even}}<259032/10^9$, and

$$E_{\mathrm{odd}}+E_{\mathrm{even}}<\frac{6321542}{10^9}<\frac{1}{150},\qquad\frac{1}{150}-\frac{6321542}{10^9}>\frac{345}{10^6}.$$

The finite blocks and both geometric-moment evaluations are displayed exactly in Supplementary Section S6.5. Therefore,

$$\sum_{j\ge7}|C_j|j^2<\frac{1}{150}. \tag{21}$$

Combining (20) and (21),

$$H(x)>\frac{27}{1000}-\frac{1}{150}=\frac{61}{3000}>0. \tag{22}$$

Thus (18) gives $F_{\kappa_{\mathrm{conv}}}(t) > 0$ for $t \not\equiv 0 \pmod{2\pi}$, with the sole boundary zero at $t = 0$. Since the denominator in (15) is nonzero, $\operatorname{Re} Q_{\kappa_{\mathrm{conv}}}$ is positive on the unit circle except at $z = 1$, where $Q_{\kappa_{\mathrm{conv}}}(1) = 0$. Harmonic minimum yields

$$\operatorname{Re} Q_{\kappa_{\mathrm{conv}}}(z) > 0 \qquad (z \in \mathbb{D}). \tag{23}$$

By the strict monotonicity (17), the same conclusion holds for every $\kappa > \kappa_{\mathrm{conv}}$.

6.3. Necessity

If $0 < \kappa < \kappa_{\mathrm{local}}$, Proposition 1 supplies an interior critical point, which is incompatible with convexity. Now suppose $\kappa_{\mathrm{local}} \le \kappa < \kappa_{\mathrm{conv}}$. Along the real radius $r \uparrow 1$, Proposition 1 gives

$$\kappa + A(r) > 0,$$

whereas

$$\kappa + B(r) \longrightarrow \kappa + B(1) = \kappa - \kappa_{\mathrm{conv}} < 0.$$

Therefore for $r < 1$ sufficiently close to one,

$$Q_\kappa(r) = \frac{\kappa + B(r)}{\kappa + A(r)} < 0,$$

contradicting the convexity criterion (13). Hence

$$C = [\kappa_{\mathrm{conv}}, \infty).$$

Together with Proposition 2, this completes the proof of Theorem 1.

7. Critical Boundary-Equality Classification and Uniqueness

We now collect the strictness statements already obtained in the preceding sections. Here "equality" is criterion-specific: derivative equality, equality in the convexity logarithmic-derivative criterion, and positive-real-part equality refer to different boundary quantities.

At $\kappa = \kappa_{\mathrm{local}}$, inequality (9) is strict for every nonzero boundary angle, while $G'_\kappa(1) = 0$. Hence $z = 1$ is the unique derivative boundary equality and no nonreal derivative equality exists.

At $\kappa = \kappa_{\mathrm{conv}}$, Equations (18) and (22) give strict positivity for every boundary angle except $t = 0$. At $z = 1$, the numerator in (14) is

$$\kappa_{\mathrm{conv}} + B(1) = 0,$$

while the denominator is

$$\kappa_{\mathrm{conv}} + A(1) = \kappa_{\mathrm{conv}} - \kappa_{\mathrm{local}} > 0.$$

Thus $Q_{\kappa_{\mathrm{conv}}}(1) = 0$, and $z = 1$ is the unique boundary equality of the convexity criterion.

At $\kappa = \kappa_{\mathrm{pos}}$, inequality (11) is strict away from $z = \pm 1$; the additional factor in (12) forces equality only when $\cos t = -1$. Therefore $z = -1$ is the unique boundary equality of the positive-real-part criterion. These three strictness arguments prove Theorem 2 and exhaust the designated critical boundary-equality system.

8. Exact Thresholds and the Earlier Decimal Interval

The comparison includes our co-author Ibrahim S. Elshazly's earlier 2024 work [3]. The 2022 and 2024 same-family papers use the decimal interval $[0.28, 0.92]$ for the parameterized modified second Einstein map. They state convexity and positive real part on this interval and use representative plots to assert loss of convexity below 0.28 and failure of positive real part above 0.92 [2,3]. The same numerical premise remains in a 2026 downstream class construction [4]. The exact thresholds proved here show that these two decimal mapping cutoffs are not sharp: Indeed, the rational enclosures in Section 2 give the exact comparisons

$$\kappa_{\mathrm{conv}} < \frac{273}{1000} < \frac{7}{25} = 0.28, \qquad 0.92 = \frac{23}{25} < \frac{9206735942}{10^{10}} < \kappa_{\mathrm{pos}}.$$

$$[0.28, 0.92] \subsetneq [\kappa_{\mathrm{conv}}, \kappa_{\mathrm{pos}}].$$

The discrepancy occurs in two narrow, nonempty slabs:

$$\kappa \in [\kappa_{\mathrm{conv}}, 0.28)$$

where convexity already holds, and

$$\kappa \in (0.92, \kappa_{\mathrm{pos}}]$$

where positive real part still holds. Thus both properties hold simultaneously on the full exact interval $[\kappa_{\mathrm{conv}}, \kappa_{\mathrm{pos}}]$.

The widths of the two refinements satisfy

$$\frac{77333378}{10^{10}} < 0.28 - \kappa_{\mathrm{conv}} < \frac{77333379}{10^{10}}, \qquad \frac{6735942}{10^{10}} < \kappa_{\mathrm{pos}} - 0.92 < \frac{6735943}{10^{10}}.$$

The upper refinement is therefore less than 0.000674. The contribution is the exact analytic thresholds, necessity, and equality classification, rather than a large numerical change. The coefficient and Fekete–Szegő theorems in the cited papers retain their stated hypotheses; extending them would require checking their class definitions and proofs.

9. Limitations and Open Directions

The results are sharp for the family considered here, and we do not read them beyond that setting.

- No larger Einstein-type family or additional parameter set is treated here.
- Coefficient, Fekete–Szegő, Hankel-determinant, and higher-order extremal problems for bi-univalent classes remain outside the present problem.
- The Sturm step is used only for the endpoint problem required in this paper. A general result for broader special-function families would need a separate argument.

It would be useful to know how far the same proof route extends to other fixed kernels. That question is not answered by the calculations above, so we leave it open.

For a compact summary of the supplementary exact-arithmetic material and the proof-dependency structure, see Appendices A and B.

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/math14214064/s1: Exact-Arithmetic Proof Supplement: Modified Second Einstein Mapping: Thresholds, Tails, and Sturm Verification (Sections S1–S7).

Author Contributions: Conceptualization: A.M.Y.A., H.M.A.M. and I.S.E.; Methodology: A.M.Y.A. and I.S.E.; Formal analysis: H.M.A.M. and I.S.E.; Investigation: A.M.Y.A. and I.S.E.; Software: A.M.Y.A.; Validation: H.M.A.M. and I.S.E.; Data curation: A.M.Y.A.; Resources: H.M.A.M.;

Writing—original draft: H.M.A.M. and I.S.E.; Writing—review and editing: A.M.Y.A.; Supervision: I.S.E.; Project administration: A.M.Y.A.; Funding acquisition: H.M.A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Ongoing Research Funding Program, King Saud University, Riyadh, Saudi Arabia, under project number ORF-2026-2209.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors would like to extend their sincere appreciation to the Ongoing Research Funding Program (ORF-2026-2209), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A. Supplementary Exact-Arithmetic Proof

The main proofs now include the finite sums, analytic tail majorants, and resulting bounds; the Supplement retains the full rational evaluations. The Supplement records the threshold identities, a rational enclosure of e, the two tail estimates used for the first thresholds, the denominator check for convexity, the required V_r bounds, the Sturm polynomial and its root count, and the omitted-harmonic tail estimate. These are the finite calculations on which the proof depends. They are written out explicitly, so the mathematical argument can be checked from the manuscript and Supplement without external code or a machine-readable certificate. The detailed locations are Supplementary Sections S2 and S3 (moments and thresholds), S4 and S5 (the first two boundary estimates), and S6.1–S6.5 (monotonicity, coefficient bounds, polynomial, Sturm chain, and omitted harmonics). An optional `verify_exact.py` script is included with the Supplement to reproduce these rational calculations using SymPy version 1.14.0; the displayed proof does not depend on running it.

Appendix B. Proof Dependency Synopsis

The exact logical order is:

$$\text{Bernoulli expansion/tails} \Longrightarrow \text{derivative boundary inequality},$$

$$\operatorname{Re} G_\kappa' > 0 \iff \kappa \geq \kappa_{\text{local}} \Longrightarrow U = L = [\kappa_{\text{local}}, \infty),$$

with the same step giving exact bounded-turning and close-to-convex membership of the normalized companion F_κ;

$$\begin{aligned}\text{Bernoulli expansion/tails} &\Longrightarrow \text{positive-real boundary inequality}\\ &\Longrightarrow J = (0, \kappa_{\text{pos}}] \iff G_\kappa \in \mathcal{P},\end{aligned}$$

$$U = L \text{ and } \kappa_{\text{conv}} > \kappa_{\text{local}} \Longrightarrow Q_\kappa \text{ analytic for } \kappa \geq \kappa_{\text{conv}},$$

$$\begin{aligned}&\text{boundary reduction} + \text{monotonicity} + \text{Sturm/tail verification}\\ &\Longrightarrow C = [\kappa_{\text{conv}}, \infty).\end{aligned}$$

The strict forms of the three boundary inequalities then give the complete criterion-specific boundary-equality theorem.

References

1. El-Qadeem, A.H.; Mamon, M.A.; Elshazly, I.S. Application of Einstein Function on Bi-Univalent Functions Defined on the Unit Disc. *Symmetry* **2022**, *14*, 758. [CrossRef]
2. El-Qadeem, A.H.; Saleh, S.A.; Mamon, M.A. Bi-Univalent Function Classes Defined by Using a Second Einstein Function. *J. Funct. Spaces* **2022**, *2022*, 6933153. [CrossRef]
3. El-Qadeem, A.H.; Murugusundaramoorthy, G.; Halouani, B.; Elshazly, I.S.; Vijaya, K.; Mamon, M.A. On Lambda-Pseudo Bi-Starlike Functions Related to Second Einstein Function. *Symmetry* **2024**, *16*, 1429. [CrossRef]
4. Olatunji, S.O.; Oluwayemi, M.; Akgül, A.; Abdulwahab, R. Bi-univalency of q-generalized distribution series conjugated with second Einstein number. *J. Class. Anal.* **2026**, *28*, 121–136. [CrossRef]
5. Noshiro, K. On the Theory of Schlicht Functions. *J. Fac. Sci. Hokkaido Imp. Univ. Ser. I Math.* **1934**, *2*, 129–155. [CrossRef]
6. Warschawski, S.E. On the Higher Derivatives at the Boundary in Conformal Mapping. *Trans. Am. Math. Soc.* **1935**, *38*, 310–340. [CrossRef]
7. Goodman, A.W. A Note on the Noshiro–Warschawski Theorem. *J. D'Analyse Math.* **1972**, *25*, 401–408. [CrossRef]
8. Kaplan, W. Close-to-Convex Schlicht Functions. *Mich. Math. J.* **1952**, *1*, 169–185. [CrossRef]
9. MacGregor, T.H. Functions Whose Derivative Has a Positive Real Part. *Trans. Am. Math. Soc.* **1962**, *104*, 532–537. [CrossRef]
10. Alexander, J.W. Functions Which Map the Interior of the Unit Circle Upon Simple Regions. *Ann. Math.* **1915**, *17*, 12–22. [CrossRef]
11. Duren, P.L. *Univalent Functions*; *Grundlehren der mathematischen Wissenschaften*; Springer: New York, NY, USA, 1983; Volume 259.
12. Basu, S.; Pollack, R.; Roy, M.F. *Algorithms in Real Algebraic Geometry*, 2nd ed.; Algorithms and Computation in Mathematics; Springer: Berlin/Heidelberg, Germany, 2006; Volume 10. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.