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Symplectic Hull Control via Nonstandard Two-Coordinate Extensions and Trace-Hermitian ACD Codes

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Abstract

Given a linear code C over $\mathbb{F}_q$ with generator matrix $\mathbf{G} = [\mathbf{X} \mid \mathbf{Z}]$, we study the nonstandard two-coordinate symplectic extension $\overline{\mathbf{G}}(\mathbf{u}, \mathbf{v}) = [\mathbf{X} \mid \mathbf{Xu} \mid \mathbf{Z} \mid \mathbf{Zv}]$ and its effects on the symplectic hull and minimum symplectic distance. The extension transforms the symplectic Gram matrix according to $\overline{\mathbf{M}} = \mathbf{M} + \mathbf{pf}^{\top} - \mathbf{fp}^{\top}$, where $\mathbf{p} = \mathbf{Xu}$ and $\mathbf{f} = \mathbf{Zv}$, and hence induces an alternating perturbation of rank at most two. We characterize all possible hull transitions, showing that the symplectic hull dimension can change only by -2, 0, or $+2$, and derive necessary and sufficient conditions for the corresponding cases. These results yield explicit criteria for symplectic LCD preservation, one-step hull reduction, a mother-code criterion for complete iterated hull reduction, and minimum-distance improvement. We further obtain exact counting formulas for admissible extension pairs, including inclusion–exclusion formulas under simultaneous structural and metric constraints, together with an automorphism-orbit reduction for eliminating equivalent extensions. In the binary case, the standard symplectic–quaternary correspondence transfers the framework to one-symbol extensions of quaternary trace-Hermitian additive complementary dual codes and yields a conditional comparison criterion with the Hermitian LCD subclass. Exhaustive computations over $\mathbb{F}_2$ and $\mathbb{F}_3$ provide finite-instance validation of the predicted hull transitions, distance behavior, and exact counts.

Keywords: Symplectic LCD codes; symplectic hulls; two-coordinate extensions; minimum symplectic distance; trace-Hermitian ACD codes; exact enumeration

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1. Introduction

The construction and systematic modification of codes with desirable algebraic and metric properties are central topics in coding theory. Beyond designing new code families, an important problem is how to transform an existing code in a controlled manner while preserving or improving its essential parameters. This question becomes particularly relevant for codes equipped with additional algebraic structures, such as symplectic codes, which play a fundamental role in quantum error-correcting code theory through the stabilizer formalism and related constructions [1]. In these settings, algebraic invariants such as duality relations and hull dimensions directly determine important structural properties of the resulting codes. Therefore, understanding how a structured modification affects these invariants not only is of theoretical interest in algebraic coding theory but also provides useful insights for the design and analysis of quantum coding structures.

In particular, one may ask whether a code can be extended in a controlled manner such that its duality properties, minimum distance, and other invariants can be precisely characterized. This perspective motivates the study of algebraically structured extensions and the development of general criteria describing the resulting changes in code parameters.

Linear complementary dual (LCD) codes, introduced by Massey [2], have attracted significant attention due to their applications in coding theory, cryptographic implementations, and the study of linear code structures [3,4]. An LCD code is characterized by the property that its intersection with its dual code is trivial, and many construction methods have been developed in different algebraic settings [5]. However, most existing works focus on constructing new LCD codes or improving their parameters, while the controlled modification of a given code and the exact evolution of its duality structure under such modifications remain less explored. In particular, it is natural to ask how a structured extension affects the hull dimension, when LCD properties can be preserved or created, and how many extension choices can achieve a desired property.

The fundamental concepts of linear coding theory, including generator matrices, dual codes, and minimum-distance parameters, are systematically developed in the classical textbooks by MacWilliams and Sloane [6] and Huffman and Pless [7].

Symplectic codes provide an important framework for studying such questions. They arise from non-Euclidean bilinear structures and are closely related to quantum error-correcting codes through the connection between symplectic spaces over $\mathbb{F}_2$ and additive codes over $\mathbb{F}_4$ [8,9]. Moreover, symplectic hulls play an essential role in understanding duality properties and several quantum coding constructions, including entanglement-assisted quantum codes [10]. Therefore, developing operations whose effects on the symplectic hull can be described exactly is of both theoretical and practical interest.

Let C be a $[2n,k]_q$ linear code with generator matrix $\mathbf{G} = [\mathbf{X} \mid \mathbf{Z}]$. The symplectic Gram matrix

$$\mathbf{M} = \mathbf{G}\mathbf{J}_{2n}\mathbf{G}^\top = \mathbf{X}\mathbf{Z}^\top - \mathbf{Z}\mathbf{X}^\top$$

determines the symplectic hull of C. In particular, C is symplectic LCD if and only if $\mathbf{M}$ is nonsingular.

Motivated by this problem, we introduce a nonstandard two-coordinate symplectic extension determined by the original generator matrix. For column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$, we define

$$\overline{\mathbf{G}}(\mathbf{u}, \mathbf{v}) = [\mathbf{X} \mid \mathbf{X}\mathbf{u} \mid \mathbf{Z} \mid \mathbf{Z}\mathbf{v}].$$

Unlike ordinary extensions, the appended coordinates $\mathbf{X}\mathbf{u}$ and $\mathbf{Z}\mathbf{v}$ are generated from the original code structure. Therefore, the extension is intrinsically coupled with the mother code. The corresponding symplectic Gram matrix satisfies

$$\overline{\mathbf{M}}(\mathbf{u}, \mathbf{v}) = \mathbf{M} + (\mathbf{X}\mathbf{u})(\mathbf{Z}\mathbf{v})^\top - (\mathbf{Z}\mathbf{v})(\mathbf{X}\mathbf{u})^\top,$$

which shows that the proposed extension changes the original symplectic Gram matrix by an alternating perturbation of rank at most two. This special structure allows for the resulting changes in hull dimension, duality properties, distance parameters, and extension counts to be analyzed exactly.

The decisive constraint is that the appended columns lie in $\mathrm{Im}(\mathbf{X})$ and $\mathrm{Im}(\mathbf{Z})$. Consequently, the possible rank changes of an alternating matrix do not by themselves determine which hull transitions the mother code can realize. The generalized Hermitian extension of Li et al. [1] maps $[m,k]_{q^2}$ codes to $[m+1,k+1]_{q^2}$ codes and characterizes Hermitian hulls and dual distances. By contrast, the present operation preserves dimension k while adding one symplectic coordinate pair.

The main objective of this paper is to establish a systematic theory for this nonstandard extension. More specifically, we investigate three fundamental questions: (i) how the

symplectic hull dimension changes under the extension; (ii) under what algebraic conditions LCD properties and minimum distance can be preserved or improved; and (iii) how many extension pairs satisfy a prescribed structural or metric requirement. Addressing these questions leads to a unified framework combining hull analysis, distance characterization, and exact enumeration of successful extensions.

The connection between binary symplectic codes and quaternary additive codes provides another interpretation of the proposed framework. The standard map from $\mathbb{F}_2^{2n}$ to $\mathbb{F}_4^n$ preserves symplectic weight and transforms binary symplectic orthogonality into trace-Hermitian orthogonality [8]. Hence, binary symplectic LCD codes correspond naturally to quaternary trace-Hermitian additive complementary dual codes. Trace-Hermitian and related additive complementary dual codes have been studied from structural, constructional, and computational perspectives [11–15].

For binary mother codes, the two-coordinate operation becomes a one-symbol extension of a quaternary additive code under the established symplectic–trace-Hermitian correspondence. This fixed-mother-code construction differs from the Plotkin-sum approach of Zhu et al. [16], which combines two constituent codes of length m to construct symplectic self-orthogonal and LCD codes of length $2m$ and dimension $k_1 + k_2$. Here the appended quaternary symbol is determined by the two generator blocks of a single mother code. The binary criteria therefore yield explicit conditions for trace-Hermitian additive complementary duality and Hamming-distance changes while preserving code cardinality.

Alternating-matrix rank bounds restrict possible hull changes and the matrix determinant lemma tests LCD preservation, but the realizability of a hull-reducing extension is governed by the hull-projected image spaces $\mathrm{Im}(\mathbf{KX})$ and $\mathrm{Im}(\mathbf{KZ})$. For a mother code with an initial hull of dimension $2r$, a complete reduction in r steps exists if and only if $\mathrm{rank}(\mathbf{K}_0\mathbf{X}_0) \geq r$ and $\mathrm{rank}(\mathbf{K}_0\mathbf{Z}_0) \geq r$, where the rows of $\mathbf{K}_0$ form a basis of the initial coefficient hull. Intersecting the attainable parameter sets with the complements of the minimum-weight forbidden subspaces yields exact counts for extensions that simultaneously reduce the hull and increase the minimum distance.

The main contributions of this paper are summarized as follows.

(1) We introduce a nonstandard two-coordinate symplectic extension and establish a complete description of its influence on symplectic hulls. In particular, we derive exact conditions determining all possible hull variations resulting from an alternating perturbation of rank at most two in the symplectic Gram matrix.
(2) We develop a unified theory describing the structural and metric evolution of the extended codes. This theory includes conditions for preserving and achieving LCD properties, a necessary and sufficient test for complete iterative hull reduction from the mother code, minimum-distance evolution, and forbidden-pair characterizations for simultaneous improvement of duality properties and minimum distance.
(3) We establish exact enumeration and symmetry-reduction methods for extension parameters and translate the obtained results to quaternary trace-Hermitian additive complementary dual codes through the binary symplectic correspondence. The theoretical results are further supported by exhaustive computational validation.

2. Preliminaries and Two-Coordinate Extensions

Let q be a prime power. For $n \geq 1$, set

$$\mathbf{J}_{2n} = \begin{pmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{pmatrix}.$$

The symplectic inner product on $\mathbb{F}_q^{2n}$ is $\langle \mathbf{x}, \mathbf{y} \rangle_s = \mathbf{x}\mathbf{J}_{2n}\mathbf{y}^\top$. When q is even, the entries 1 and -1 coincide, and the form is still alternating.

Let C be a $[2n,k]_q$ linear code with full-rank generator matrix $\mathbf{G} = [\mathbf{X} \mid \mathbf{Z}] \in \mathbb{F}_q^{k\times 2n}$, where $\mathbf{X}, \mathbf{Z} \in \mathbb{F}_q^{k\times n}$. Its symplectic dual and symplectic hull are

$$C^{\perp_s} = \{\mathbf{x} \in \mathbb{F}_q^{2n} : \langle \mathbf{x}, \mathbf{c}\rangle_s = 0 \text{ for all } \mathbf{c} \in C\},$$

$$\mathrm{Hull}_s(C) = C \cap C^{\perp_s}.$$

The symplectic Gram matrix of C is $\mathbf{M} = \mathbf{G}\mathbf{J}_{2n}\mathbf{G}^\top = \mathbf{X}\mathbf{Z}^\top - \mathbf{Z}\mathbf{X}^\top$. It is alternating. Since $\mathbf{G}$ has full row rank, the coefficient space $\mathcal{H}(C) = \{\mathbf{a} \in \mathbb{F}_q^k : \mathbf{a}\mathbf{M} = \mathbf{0}\}$ is isomorphic to $\mathrm{Hull}_s(C)$ by $\mathbf{a} \mapsto \mathbf{a}\mathbf{G}$. Hence $h_s(C) := \dim \mathrm{Hull}_s(C) = k - \mathrm{rank}(\mathbf{M})$. The code C is symplectic LCD if and only if $h_s(C) = 0$, equivalently $\det(\mathbf{M}) \neq 0$. In particular, a symplectic LCD code has even dimension.

For $\mathbf{x} = (x_1, \ldots, x_n \mid x_{n+1}, \ldots, x_{2n}) \in \mathbb{F}_q^{2n}$, define $\mathrm{wt}_s(\mathbf{x}) = \#\{i : (x_i, x_{n+i}) \neq (0,0)\}$. The symplectic minimum distance is $d_s(C) = \min\{\mathrm{wt}_s(\mathbf{c}) : \mathbf{0} \neq \mathbf{c} \in C\}$.

Definition 1 (Two-coordinate extension)**.** *Let $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$ be column vectors. The two-coordinate extension of C determined by $(\mathbf{u}, \mathbf{v})$ is the code $\overline{C}(\mathbf{u}, \mathbf{v})$ generated by $\overline{\mathbf{G}}(\mathbf{u}, \mathbf{v}) = [\mathbf{X} \mid \mathbf{X}\mathbf{u} \mid \mathbf{Z} \mid \mathbf{Z}\mathbf{v}] \in \mathbb{F}_q^{k\times(2n+2)}$.*

The generalized Hermitian extension in [1] and the symplectic Plotkin-sum construction in [16] have generator matrices of the respective forms

$$\mathbf{G}_{\mathrm{H}}^{\mathrm{gen}} = \begin{pmatrix} \mathbf{G}_{\mathrm{H}} & 0 \\ \mathbf{w} & a \end{pmatrix}, \qquad \mathbf{G}_{\mathrm{P}} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{G}_1 \\ 0 & \mathbf{G}_2 \end{pmatrix}.$$

where $\mathbf{G}_{\mathrm{H}}$ generates a Hermitian mother code over $\mathbb{F}_{q^2}$, and $\mathbf{G}_1, \mathbf{G}_2$ generate the two Plotkin constituents. The first operation increases the row dimension by one, while the second combines two codes and doubles the constituent length. The proposed extension instead retains the dimension and adds a symplectic coordinate pair computed from the two blocks of one fixed mother code.

The column-space restriction makes each new coordinate a linear functional of the corresponding original block. A new first coordinate defined by an appended column $\mathbf{p} \in \mathbb{F}_q^k$ depends only on the original first block precisely when $\ker(\mathbf{X}^\top) \subseteq \ker(\mathbf{p}^\top)$, equivalently $\mathbf{p} \in \mathrm{Im}(\mathbf{X})$. The analogous condition for the second coordinate is $\mathbf{f} \in \mathrm{Im}(\mathbf{Z})$. Thus, $\mathbf{a}\mathbf{X} = \mathbf{0}$ forces $\mathbf{a}\mathbf{p} = 0$, and $\mathbf{a}\mathbf{Z} = \mathbf{0}$ forces $\mathbf{a}\mathbf{f} = 0$. In the parameter representation, the forbidden conditions consequently take the separate forms $\mathbf{a}\mathbf{X}\mathbf{u} = 0$ and $\mathbf{a}\mathbf{Z}\mathbf{v} = 0$, leading to the product subspaces used in Theorem 7.

Write $r_X = \mathrm{rank}(\mathbf{X})$ and $r_Z = \mathrm{rank}(\mathbf{Z})$. The restricted family contains exactly $q^{r_X+r_Z}$ distinct appended-column pairs, compared with q^{2k} unrestricted pairs. Each restricted pair has exactly $q^{2n-r_X-r_Z}$ parameter preimages, allowing for searches to use one representative per appended-column pair while retaining this multiplicity in parameter counts. The candidate set is strictly smaller if $r_X < k$ or $r_Z < k$, and the two families coincide if $r_X = r_Z = k$. Allowing for each appended column to independently mix both original blocks recovers every column in $\mathbb{F}_q^k$ because $\mathrm{Im}(\mathbf{X}) + \mathrm{Im}(\mathbf{Z}) = \mathbb{F}_q^k$. Because a coefficient-hull basis matrix has full row rank, arbitrary appended columns can realize any two prescribed hull projections and hence always permit hull reduction when $h_s(C) \geq 2$. Under the blockwise restriction, the existence of such a pair is instead governed by the image-space conditions of Theorem 3. The restriction thus provides block-compatible extensions with explicit mother-code-dependent feasibility tests and a potentially smaller search domain.

Lemma 1 (Gram matrix)**.** *The symplectic Gram matrix of $\overline{C}(\mathbf{u}, \mathbf{v})$ is*

$$\overline{\mathbf{M}}(\mathbf{u}, \mathbf{v}) = \mathbf{M} + \mathbf{p}\mathbf{f}^\top - \mathbf{f}\mathbf{p}^\top, \qquad \mathbf{p} = \mathbf{X}\mathbf{u}, \quad \mathbf{f} = \mathbf{Z}\mathbf{v}.$$

Moreover, $\mathrm{rank}(\mathbf{p}\mathbf{f}^\top - \mathbf{f}\mathbf{p}^\top) \leq 2$.

Proof. By the standard symplectic coordinate order,

$$\mathbf{J}_{2n+2} = \begin{pmatrix} 0 & \mathbf{I}_{n+1} \\ -\mathbf{I}_{n+1} & 0 \end{pmatrix},$$

and hence

$$\begin{aligned} &\overline{\mathbf{G}}(\mathbf{u},\mathbf{v})\mathbf{J}_{2n+2}(\overline{\mathbf{G}}(\mathbf{u},\mathbf{v}))^{\top} \\ &= [\mathbf{X} \mid \mathbf{Xu}][\mathbf{Z} \mid \mathbf{Zv}]^{\top} - [\mathbf{Z} \mid \mathbf{Zv}][\mathbf{X} \mid \mathbf{Xu}]^{\top} \\ &= \mathbf{XZ}^{\top} - \mathbf{ZX}^{\top} + (\mathbf{Xu})(\mathbf{Zv})^{\top} - (\mathbf{Zv})(\mathbf{Xu})^{\top} \\ &= \mathbf{M} + \mathbf{pf}^{\top} - \mathbf{fp}^{\top}. \end{aligned}$$

The rank assertion follows from $\operatorname{rank}(\mathbf{pf}^{\top} - \mathbf{fp}^{\top}) \le \operatorname{rank}[\mathbf{p}\ \mathbf{f}] \le 2$. □

3. Hull Variation and LCD Constructions

The extension affects the hull only through the Gram perturbation in Lemma 1. Thus, this section is independent of minimum distance. The outcome is a structural classification of how the LCD property can be preserved or obtained.

3.1. The Three Hull Variations

Theorem 1 (Hull variation)**.** *Let $h_s(C) = \ell$. Then, $h_s(\overline{C}(\mathbf{u},\mathbf{v})) \in \{\ell - 2, \ell, \ell + 2\}$.*

Proof. By Lemma 1, $\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) - \mathbf{M} = \mathbf{pf}^{\top} - \mathbf{fp}^{\top}$, where $\mathbf{p} = \mathbf{Xu}$ and $\mathbf{f} = \mathbf{Zv}$. Hence $|\operatorname{rank}(\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})) - \operatorname{rank}(\mathbf{M})| \le 2$. Both $\mathbf{M}$ and $\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})$ are alternating. Hence their ranks are even, and

$$\operatorname{rank}(\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})) - \operatorname{rank}(\mathbf{M}) \in \{-2, 0, 2\}.$$

The result follows from $h_s(C) = k - \operatorname{rank}(\mathbf{M})$. □

Remark 1. *If $h_s(C) = 0$, then $h_s(\overline{C}(\mathbf{u},\mathbf{v})) \in \{0, 2\}$. If $h_s(C) = 1$, then $\overline{C}(\mathbf{u},\mathbf{v})$ cannot be symplectic LCD, since the dimension is odd. Thus, an odd-dimensional hull cannot be reduced to zero by extensions which preserve k.*

Let $h = h_s(C)$, and let $\mathbf{K} \in \mathbb{F}_q^{h\times k}$ have rows forming a basis of the coefficient hull

$$\mathcal{H}(C) = \{\mathbf{a} \in \mathbb{F}_q^k : \mathbf{aM} = \mathbf{0}\}.$$

If $h < k$, choose a matrix $\mathbf{P} \in \mathbb{F}_q^{(k-h)\times k}$ whose rows span a complement of $\mathcal{H}(C)$ in $\mathbb{F}_q^k$. Then,

$$\mathbf{S} = \begin{pmatrix} \mathbf{P} \\ \mathbf{K} \end{pmatrix}$$

is nonsingular, and

$$\mathbf{N} = \mathbf{PMP}^{\top}$$

is nonsingular. Indeed, $\mathcal{H}(C)$ is the radical of the alternating bilinear form represented by $\mathbf{M}$, and the restriction of this form to any vector-space complement of its radical is nondegenerate. If $h = k$, then $\mathcal{H}(C) = \mathbb{F}_q^k$, so $\mathbf{K}$ is nonsingular and $\mathbf{M} = 0$.

Lemma 2 (Exact hull cases)**.** *Let*

$$\mathbf{p} = \mathbf{Xu}, \qquad \mathbf{f} = \mathbf{Zv},$$

and put

$$s = \operatorname{rank}[\mathbf{Kp}\ \mathbf{Kf}].$$

If $h < k$ and $s = 0$, define

$$\lambda = (\mathbf{Pp})^{\top}\mathbf{N}^{-1}(\mathbf{Pf}).$$

Then, the following assertions hold.

(1) *A decrease by two occurs precisely when*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h - 2 \iff s = 2.$$

(2) *The hull dimension is preserved precisely when*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h$$

if and only if one of the following mutually exclusive conditions holds:

$$s = 1, \qquad s = 0 \text{ and } h = k, \qquad s = 0,\ h < k, \text{ and } \lambda \neq 1.$$

(3) *An increase by two occurs precisely when*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h + 2 \iff h < k, \quad s = 0, \quad \lambda = 1.$$

Proof. First, suppose that $h = k$. Then, $\mathbf{K}$ is nonsingular and $\mathbf{M} = 0$. Consequently,

$$s = \operatorname{rank}[\mathbf{Kp}\ \mathbf{Kf}] = \operatorname{rank}[\mathbf{p}\ \mathbf{f}].$$

If $s \leq 1$, then $\mathbf{p}$ and $\mathbf{f}$ are linearly dependent, and hence

$$\mathbf{pf}^{\top} - \mathbf{fp}^{\top} = 0.$$

Therefore,

$$\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = \mathbf{M} = 0,$$

so that

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = k = h.$$

If $s = 2$, then $\mathbf{p}$ and $\mathbf{f}$ are linearly independent. Thus, the alternating matrix

$$\mathbf{pf}^{\top} - \mathbf{fp}^{\top}$$

has rank 2. Since $\mathbf{M} = 0$,

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = 2,$$

and consequently

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = k - 2 = h - 2.$$

This proves all assertions when $h = k$.

Assume henceforth that $h < k$, and put

$$r = k - h.$$

Since the rows of $\mathbf{K}$ belong to the radical of the form represented by $\mathbf{M}$, we have

$$\mathbf{SMS}^{\top} = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{N} \in \mathbb{F}_q^{r\times r}$ is nonsingular and alternating.

Write

$$\mathbf{Sp} = \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \qquad \mathbf{Sf} = \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix},$$

where

$$\mathbf{a} = \mathbf{P}\mathbf{p}, \qquad \mathbf{b} = \mathbf{P}\mathbf{f}, \qquad \mathbf{c} = \mathbf{K}\mathbf{p}, \qquad \mathbf{d} = \mathbf{K}\mathbf{f}.$$

Then,

$$s = \operatorname{rank}[\mathbf{c}\ \mathbf{d}],$$

and

$$\mathbf{S}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})\mathbf{S}^{\top} = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}\begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix}^{\top} - \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}^{\top}.$$

Suppose first that $s = 2$. Then, $\mathbf{c}$ and $\mathbf{d}$ are linearly independent. Hence there exists $\mathbf{R} \in \mathbb{F}_q^{r\times h}$ satisfying

$$\mathbf{R}\mathbf{c} = -\mathbf{a}, \qquad \mathbf{R}\mathbf{d} = -\mathbf{b}.$$

Let

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}_r & \mathbf{R} \\ 0 & \mathbf{I}_h \end{pmatrix}.$$

The matrix $\mathbf{T}$ is nonsingular and

$$\mathbf{T}\begin{pmatrix} \mathbf{N} & 0 \\ 0 & 0 \end{pmatrix}\mathbf{T}^{\top} = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover,

$$\mathbf{T}\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{d} \end{pmatrix}.$$

Therefore, $\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})$ is congruent to

$$\begin{pmatrix} \mathbf{N} & 0 \\ 0 & \mathbf{c}\mathbf{d}^{\top} - \mathbf{d}\mathbf{c}^{\top} \end{pmatrix}.$$

Since $\mathbf{c}$ and $\mathbf{d}$ are linearly independent,

$$\operatorname{rank}\left(\mathbf{c}\mathbf{d}^{\top} - \mathbf{d}\mathbf{c}^{\top}\right) = 2.$$

Hence

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = r + 2 = \operatorname{rank}(\mathbf{M}) + 2,$$

and therefore

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h - 2.$$

Next, suppose that $s = 1$. First, assume $\mathbf{c} \neq 0$. Since $\operatorname{rank}[\mathbf{c}\ \mathbf{d}] = 1$, there exists $t \in \mathbb{F}_q$ such that

$$\mathbf{d} = t\mathbf{c}.$$

Choose $\mathbf{R} \in \mathbb{F}_q^{r\times h}$ such that

$$\mathbf{R}\mathbf{c} = -\mathbf{a},$$

and define

$$\mathbf{e} = \mathbf{b} - t\mathbf{a}.$$

Using the same matrix

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}_r & \mathbf{R} \\ 0 & \mathbf{I}_h \end{pmatrix},$$

we obtain

$$\mathbf{T}\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}, \qquad \mathbf{T}\begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ t\mathbf{c} \end{pmatrix}.$$

Thus, $\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})$ is congruent to

$$\begin{pmatrix} \mathbf{N} & -\mathbf{e}\mathbf{c}^{\top} \\ \mathbf{c}\mathbf{e}^{\top} & 0 \end{pmatrix}.$$

We note explicitly that $\mathbf{N}^{-1}$ is alternating. Indeed, for every column vector $\mathbf{x}$, putting $\mathbf{y} = \mathbf{N}^{-1}\mathbf{x}$ gives

$$\mathbf{x}^{\top}\mathbf{N}^{-1}\mathbf{x} = (\mathbf{N}\mathbf{y})^{\top}\mathbf{y} = \mathbf{y}^{\top}\mathbf{N}^{\top}\mathbf{y} = -\mathbf{y}^{\top}\mathbf{N}\mathbf{y} = 0.$$

This argument is valid in every characteristic, including characteristic 2.

Since $\mathbf{N}$ is nonsingular, the Schur complement of $\mathbf{N}$ is

$$\mathbf{c}\mathbf{e}^{\top}\mathbf{N}^{-1}\mathbf{e}\mathbf{c}^{\top} = \left(\mathbf{e}^{\top}\mathbf{N}^{-1}\mathbf{e}\right)\mathbf{c}\mathbf{c}^{\top} = 0.$$

Hence

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = \operatorname{rank}(\mathbf{N}) = r = \operatorname{rank}(\mathbf{M}),$$

and therefore

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h.$$

If $s = 1$, $\mathbf{c} = 0$, and $\mathbf{d} \neq 0$, the same argument with the roles of $\mathbf{p}$ and $\mathbf{f}$ interchanged gives the identical conclusion.

It remains to consider $s = 0$. Then,

$$\mathbf{c} = \mathbf{d} = 0,$$

and hence

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = \operatorname{rank}\left(\mathbf{N} + \mathbf{a}\mathbf{b}^{\top} - \mathbf{b}\mathbf{a}^{\top}\right).$$

Put

$$\mathbf{U} = [\mathbf{a}\ \mathbf{b}]$$

and

$$\mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then,

$$\mathbf{a}\mathbf{b}^{\top} - \mathbf{b}\mathbf{a}^{\top} = \mathbf{U}\mathbf{J}_2\mathbf{U}^{\top}.$$

By the matrix determinant lemma,

$$\begin{aligned} &\det\left(\mathbf{N} + \mathbf{a}\mathbf{b}^{\top} - \mathbf{b}\mathbf{a}^{\top}\right) \\ &\quad = \det(\mathbf{N})\det\left(\mathbf{I}_2 + \mathbf{J}_2\mathbf{U}^{\top}\mathbf{N}^{-1}\mathbf{U}\right). \end{aligned}$$

Since $\mathbf{N}^{-1}$ is alternating and

$$\lambda = \mathbf{a}^{\top}\mathbf{N}^{-1}\mathbf{b},$$

we have

$$\mathbf{U}^{\top}\mathbf{N}^{-1}\mathbf{U} = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}.$$

Consequently,

$$\mathbf{I}_2 + \mathbf{J}_2\mathbf{U}^{\top}\mathbf{N}^{-1}\mathbf{U} = (1-\lambda)\mathbf{I}_2,$$

and therefore

$$\det\left(\mathbf{N} + \mathbf{a}\mathbf{b}^{\top} - \mathbf{b}\mathbf{a}^{\top}\right) = \det(\mathbf{N})(1-\lambda)^2.$$

If $\lambda \neq 1$, the matrix is nonsingular, so

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = r = \operatorname{rank}(\mathbf{M}),$$

and hence

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h.$$

Finally, suppose that $\lambda = 1$. Then,

$$\mathbf{N} + \mathbf{a}\mathbf{b}^\top - \mathbf{b}\mathbf{a}^\top$$

is singular and alternating. Since the perturbation has rank at most two, the matrix rank can decrease by at most two,

$$\operatorname{rank}\left(\mathbf{N} + \mathbf{a}\mathbf{b}^\top - \mathbf{b}\mathbf{a}^\top\right) \geq r - 2.$$

On the other hand, the rank of an alternating matrix is even. Since $\mathbf{N}$ is a nonsingular alternating $r \times r$ matrix, r is even; singularity therefore implies

$$\operatorname{rank}\left(\mathbf{N} + \mathbf{a}\mathbf{b}^\top - \mathbf{b}\mathbf{a}^\top\right) \leq r - 2.$$

Hence its rank is exactly $r - 2$. It follows that

$$\operatorname{rank}\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = \operatorname{rank}(\mathbf{M}) - 2,$$

and therefore

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h + 2.$$

The three cases $s = 2$, $s = 1$, and $s = 0$ exhaust all possibilities, and the result follows. □

Lemma 2 shows that a Gram perturbation of rank at most two decreases the hull precisely when its appended columns remain independent after projection onto the coefficient hull. When the projected rank is zero and $h < k$, the scalar equation $\lambda = 1$ instead characterizes an increase in the hull by two.

Corollary 1 (Hull-neutral dependent extensions)**.** *If* $\operatorname{rank}[\mathbf{X}\mathbf{u}\ \mathbf{Z}\mathbf{v}] \leq 1$, *then*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h_s(C).$$

In particular, if C *is symplectic LCD, then* $\overline{C}(\mathbf{u},\mathbf{v})$ *is also symplectic LCD.*

Proof. Put $\mathbf{p} = \mathbf{X}\mathbf{u}$ and $\mathbf{f} = \mathbf{Z}\mathbf{v}$. If $\operatorname{rank}[\mathbf{p}\ \mathbf{f}] \leq 1$, then $\mathbf{p}\mathbf{f}^\top - \mathbf{f}\mathbf{p}^\top = 0$. Hence $\overline{\mathbf{M}}(\mathbf{u},\mathbf{v}) = \mathbf{M}$, and the two assertions follow. □

3.2. LCD-Preserving Extensions

Theorem 2 (LCD-preserving extension)**.** *Assume that* C *is symplectic LCD. Then,* $\overline{C}(\mathbf{u},\mathbf{v})$ *is symplectic LCD if and only if* $(\mathbf{X}\mathbf{u})^\top\mathbf{M}^{-1}(\mathbf{Z}\mathbf{v}) \neq 1$.

Proof. Put $\mathbf{p} = \mathbf{X}\mathbf{u}$ and $\mathbf{f} = \mathbf{Z}\mathbf{v}$. By the matrix determinant lemma,

$$\det(\overline{\mathbf{M}}(\mathbf{u},\mathbf{v})) = \det(\mathbf{M})\det\left(\mathbf{I}_2 + \mathbf{J}_2[\mathbf{p}\ \mathbf{f}]^\top\mathbf{M}^{-1}[\mathbf{p}\ \mathbf{f}]\right).$$

Let $a = \mathbf{p}^\top\mathbf{M}^{-1}\mathbf{f}$. Since $\mathbf{M}^{-1}$ is alternating,

$$[\mathbf{p}\ \mathbf{f}]^\top\mathbf{M}^{-1}[\mathbf{p}\ \mathbf{f}] = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.$$

Thus, $\mathbf{I}_2 + \mathbf{J}_2[\mathbf{p}\ \mathbf{f}]^\top\mathbf{M}^{-1}[\mathbf{p}\ \mathbf{f}] = (1-a)\mathbf{I}_2$. Since $\det(\mathbf{M}) \neq 0$, the result follows. □

Corollary 2 (Repeated LCD-preserving extensions)**.** *Let $C_0 = C$ be a symplectic LCD code of length $2n$, with generator matrix*

$$\mathbf{G}_0 = [\mathbf{X}_0 \mid \mathbf{Z}_0]$$

and symplectic Gram matrix $\mathbf{M}_0 = \mathbf{M}$. For $1 \le i \le s$, let

$$\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^{n+i-1},$$

let C_i be obtained from C_{i-1} by the corresponding two-coordinate extension, and write

$$\mathbf{G}_i = [\mathbf{X}_i \mid \mathbf{Z}_i]$$

and $\mathbf{M}_i$ for the generator matrix and symplectic Gram matrix of C_i.

For each i for which $\mathbf{M}_{i-1}$ is nonsingular, define

$$\lambda_i = (\mathbf{X}_{i-1}\mathbf{u}_i)^\top \mathbf{M}_{i-1}^{-1} (\mathbf{Z}_{i-1}\mathbf{v}_i).$$

Then,

$$C_i \text{ is symplectic LCD for every } 1 \le i \le s$$

if and only if λ_i is defined and

$$\lambda_i \neq 1 \qquad \text{for every } 1 \le i \le s.$$

In particular, whenever these equivalent conditions hold,

$$\prod_{i=1}^{s} (1 - \lambda_i) \neq 0.$$

Proof. Since C_0 is symplectic LCD, $\mathbf{M}_0$ is nonsingular, so λ_1 is defined. By Theorem 2, C_1 is symplectic LCD if and only if $\lambda_1 \neq 1$.

Proceed inductively. If C_{i-1} is symplectic LCD, then $\mathbf{M}_{i-1}$ is nonsingular, so λ_i is defined, and Theorem 2 gives

$$C_i \text{ is symplectic LCD} \iff \lambda_i \neq 1.$$

Thus, all the codes $C_1, \ldots, C_s$ are symplectic LCD if and only if every λ_i is defined and different from 1. The final product statement is immediate. □

3.3. Reduction from a Nonzero Hull to an LCD Code

Theorem 3 (One-step hull reduction)**.** *Let $h = h_s(C) \ge 2$, and let $\mathbf{K}$ have rows forming a basis of the coefficient hull $\mathcal{H}(C)$. Then,*

$$h_s(\overline{C}(\mathbf{u}, \mathbf{v})) = h - 2$$

if and only if

$$\operatorname{rank}[\mathbf{KXu}\ \mathbf{KZv}] = 2.$$

In particular, if $h = 2$ and

$$\operatorname{rank}[\mathbf{KXu}\ \mathbf{KZv}] = 2,$$

then $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD.

Moreover, put

$$\mathcal{U} = \operatorname{Im}(\mathbf{KX}), \qquad \mathcal{V} = \operatorname{Im}(\mathbf{KZ}),$$

and

$$\alpha = \dim \mathcal{U}, \qquad \beta = \dim \mathcal{V}, \qquad \gamma = \dim(\mathcal{U} \cap \mathcal{V}).$$

A hull-reducing pair $(\mathbf{u}, \mathbf{v})$ *exists if and only if there are*

$$\mathbf{p} \in \mathcal{U}, \qquad \mathbf{f} \in \mathcal{V}$$

which are linearly independent. Equivalently,

$$N_q(\alpha, \beta, \gamma) > 0,$$

or, equivalently,

$$\alpha \geq 1, \qquad \beta \geq 1, \qquad \alpha + \beta - \gamma \geq 2.$$

Proof. The first assertion is the $s = 2$ case of Lemma 2. Since

$$\{\mathbf{KXu} : \mathbf{u} \in \mathbb{F}_q^n\} = \mathrm{Im}(\mathbf{KX}) = \mathcal{U}$$

and

$$\{\mathbf{KZv} : \mathbf{v} \in \mathbb{F}_q^n\} = \mathrm{Im}(\mathbf{KZ}) = \mathcal{V},$$

a hull-reducing pair exists precisely when one can choose $\mathbf{p} \in \mathcal{U}$ and $\mathbf{f} \in \mathcal{V}$ which are linearly independent.

By Proposition 1, the number of such ordered pairs is

$$N_q(\alpha, \beta, \gamma) = q^{\alpha+\beta} - q^{\alpha} - q^{\beta} - (q-1)q^{\gamma} + q.$$

This number is positive exactly when both $\mathcal{U}$ and $\mathcal{V}$ are nonzero and

$$\dim(\mathcal{U} + \mathcal{V}) = \alpha + \beta - \gamma \geq 2.$$

The final assertion follows. □

Theorem 3 converts the pairwise hull test into an attainable-extension criterion for a prescribed mother code. Whenever the image-space conditions hold, select linearly independent vectors $\mathbf{s} \in \mathcal{U}$ and $\mathbf{t} \in \mathcal{V}$ and solve the two linear systems

$$\mathbf{KXu} = \mathbf{s}, \qquad \mathbf{KZv} = \mathbf{t}.$$

The resulting extension lowers the hull dimension by exactly two, while failure of the image-space conditions rules out every admissible pair.

Theorem 4 (Iterated reduction to LCD)**.** *Let* $h_s(C) = 2r$*. For a sequence* $C = C_0, C_1, \ldots, C_r$ *of two-coordinate extensions, write* $\mathbf{G}_i = [\mathbf{X}_i \mid \mathbf{Z}_i]$ *and let* $\mathbf{K}_i$ *be a matrix whose rows form a basis of* $\mathcal{H}(C_i)$*. Then,* $h_s(C_i) = h_s(C_{i-1}) - 2$ *for* $1 \leq i \leq r$*, and hence* $h_s(C_r) = 0$*, if and only if*

$$\mathrm{rank}[\mathbf{K}_{i-1}\mathbf{X}_{i-1}\mathbf{u}_i \; \mathbf{K}_{i-1}\mathbf{Z}_{i-1}\mathbf{v}_i] = 2, \qquad 1 \leq i \leq r.$$

More precisely, the existence of a complete reduction sequence can be decided from the initial mother code. Put

$$\alpha_0 = \mathrm{rank}(\mathbf{K}_0\mathbf{X}_0), \qquad \beta_0 = \mathrm{rank}(\mathbf{K}_0\mathbf{Z}_0).$$

There exists a sequence of exactly r successive two-coordinate extensions with a hull decrease of two at every step if and only if

$$\boxed{\alpha_0 \geq r \quad \textit{and} \quad \beta_0 \geq r.}$$

When these inequalities hold, all extension columns may be chosen in advance from the column spaces of the initial generator matrix. For $r = 0$*, the assertion refers to the empty sequence.*

Proof. Apply Theorem 3 at each step.

We prove the additional criterion based solely on the mother code. The case $r = 0$ is immediate, so assume $r \geq 1$. Let $\mathbf{M}_0$ be the initial symplectic Gram matrix. Since the initial generator matrix has full row rank and the rows of $\mathbf{K}_0$ form a basis of the coefficient hull,

$$\mathrm{Im}(\mathbf{K}_0\mathbf{X}_0) + \mathrm{Im}(\mathbf{K}_0\mathbf{Z}_0) = \mathbb{F}_q^{2r}.$$

For necessity, suppose that a complete reduction sequence exists. Write

$$\mathbf{p}_i = \mathbf{X}_{i-1}\mathbf{u}_i, \qquad \mathbf{f}_i = \mathbf{Z}_{i-1}\mathbf{v}_i, \qquad \mathbf{W} = [\mathbf{p}_1 \ \cdots \ \mathbf{p}_r \, \mathbf{f}_1 \ \cdots \ \mathbf{f}_r].$$

Every appended first-block column belongs to $\mathrm{Im}(\mathbf{X}_0)$, and every appended second-block column belongs to $\mathrm{Im}(\mathbf{Z}_0)$, by induction on the extension steps. The final Gram matrix satisfies

$$\mathbf{M}_r = \mathbf{M}_0 + \mathbf{W}\mathbf{J}_{2r}\mathbf{W}^\top, \qquad \mathbf{J}_{2r} = \begin{pmatrix} 0 & \mathbf{I}_r \\ -\mathbf{I}_r & 0 \end{pmatrix}.$$

If a coefficient row vector $\mathbf{a} \in \mathcal{H}(C_0)$ satisfies $\mathbf{a}\mathbf{W} = 0$, then $\mathbf{a}\mathbf{M}_r = 0$. As the final code is LCD, $\mathbf{M}_r$ is nonsingular, so $\mathbf{a} = 0$. Therefore, the linear map from the initial coefficient hull to $\mathbb{F}_q^{2r}$ given by $\mathbf{a} \mapsto \mathbf{a}\mathbf{W}$ is injective and hence bijective. Thus,

$$\mathrm{rank}(\mathbf{K}_0\mathbf{W}) = 2r.$$

Its first and last r columns belong, respectively, to $\mathrm{Im}(\mathbf{K}_0\mathbf{X}_0)$ and $\mathrm{Im}(\mathbf{K}_0\mathbf{Z}_0)$. Consequently, $\alpha_0 \geq r$ and $\beta_0 \geq r$.

Conversely, assume both inequalities. Set

$$\mathcal{U}_0 = \mathrm{Im}(\mathbf{K}_0\mathbf{X}_0), \qquad \mathcal{V}_0 = \mathrm{Im}(\mathbf{K}_0\mathbf{Z}_0), \qquad \mathcal{I} = \mathcal{U}_0 \cap \mathcal{V}_0.$$

Choose complements $\mathcal{U}'$ and $\mathcal{V}'$ such that

$$\mathcal{U}_0 = \mathcal{U}' \oplus \mathcal{I}, \qquad \mathcal{V}_0 = \mathcal{V}' \oplus \mathcal{I}.$$

Put $a = \dim \mathcal{U}'$, $b = \dim \mathcal{V}'$ and $c = \dim \mathcal{I}$. Since $\mathcal{U}_0 + \mathcal{V}_0 = \mathbb{F}_q^{2r}$, we have

$$a + b + c = 2r, \qquad 0 \leq a, b \leq r, \qquad c = (r - a) + (r - b).$$

Choose bases of $\mathcal{U}'$, $\mathcal{V}'$ and $\mathcal{I}$. Assign $r - a$ of the basis vectors of $\mathcal{I}$ to the $\mathcal{U}_0$-side and the remaining $r - b$ to the $\mathcal{V}_0$-side. This produces vectors

$$\mathbf{s}_1, \ldots, \mathbf{s}_r \in \mathcal{U}_0, \qquad \mathbf{t}_1, \ldots, \mathbf{t}_r \in \mathcal{V}_0$$

whose union is a basis of $\mathbb{F}_q^{2r}$. Solve the finite-field linear systems

$$\mathbf{K}_0\mathbf{X}_0\mathbf{u}_i^{(0)} = \mathbf{s}_i, \qquad \mathbf{K}_0\mathbf{Z}_0\mathbf{v}_i^{(0)} = \mathbf{t}_i, \qquad 1 \leq i \leq r.$$

At stage i, append $i - 1$ zeros to each of $\mathbf{u}_i^{(0)}$ and $\mathbf{v}_i^{(0)}$ to obtain the admissible parameters $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^{n+i-1}$. Then, $\mathbf{p}_i = \mathbf{X}_0\mathbf{u}_i^{(0)}$ and $\mathbf{f}_i = \mathbf{Z}_0\mathbf{v}_i^{(0)}$.

For each $1 \leq j \leq r$, set

$$\mathbf{W}_j = [\mathbf{p}_1 \ \cdots \ \mathbf{p}_j \, \mathbf{f}_1 \ \cdots \ \mathbf{f}_j], \qquad \mathbf{B}_j = \mathbf{K}_0\mathbf{W}_j.$$

The matrix $\mathbf{B}_j$ has full column rank $2j$. Choose a change of coefficient basis

$$\mathbf{S} = \begin{pmatrix} \mathbf{P} \\ \mathbf{K}_0 \end{pmatrix}$$

such that

$$\mathbf{S}\mathbf{M}_0\mathbf{S}^\top = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{S}\mathbf{W}_j = \begin{pmatrix} \mathbf{A}_j \\ \mathbf{B}_j \end{pmatrix},$$

where $\mathbf{N}$ is nonsingular of order $k - 2r$. Since $\mathbf{B}_j$ has full column rank, there is a matrix $\mathbf{R}_j$ with $\mathbf{R}_j\mathbf{B}_j = -\mathbf{A}_j$. The invertible shear

$$\mathbf{T}_j = \begin{pmatrix} \mathbf{I}_{k-2r} & \mathbf{R}_j \\ 0 & \mathbf{I}_{2r} \end{pmatrix}$$

preserves the displayed block form of the initial Gram matrix and sends $\mathbf{S}\mathbf{W}_j$ to $(0, \mathbf{B}_j)^\top$. Thus,

$$\mathbf{T}_j\mathbf{S}\mathbf{M}_j\mathbf{S}^\top\mathbf{T}_j^\top = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & \mathbf{B}_j\mathbf{J}_{2j}\mathbf{B}_j^\top \end{pmatrix}.$$

The lower-right block has rank $2j$ because $\mathbf{B}_j$ has a left inverse and $\mathbf{J}_{2j}$ is nonsingular. Therefore,

$$\operatorname{rank}(\mathbf{M}_j) = \operatorname{rank}(\mathbf{M}_0) + 2j, \qquad h_s(C_j) = 2(r - j).$$

When $k = 2r$, the nonsingular block is empty and the same rank identity follows directly. Hence the constructed sequence reduces the hull by two at every step and terminates at an LCD code. □

The initial rank inequalities are not automatic. For example, over any finite field, take $k = n = 4$ and

$$\mathbf{X}_0 = [\mathbf{e}_1\ \mathbf{e}_2\ 0\ 0], \qquad \mathbf{Z}_0 = [\mathbf{e}_2\ 0\ \mathbf{e}_3\ \mathbf{e}_4],$$

where the $\mathbf{e}_i$ are the standard basis vectors of $\mathbb{F}_q^4$. The generator matrix has full row rank, but $h_s(C_0) = 2$ and $\operatorname{rank}(\mathbf{K}_0\mathbf{X}_0) = 0$. Thus, no one-step hull reduction is possible. Conversely, when the initial rank inequalities hold, the proof constructs a complete sequence by jointly selecting all extension pairs; a locally admissible choice alone need not ensure that the remaining steps can be completed. The deterministic construction is summarized in Algorithm 1.

Algorithm 1. Deterministic iterative LCD construction

The input is a full-row-rank generator matrix $\mathbf{G}_0 = [\mathbf{X}_0 \mid \mathbf{Z}_0]$ over a finite field. The procedure returns an LCD generator matrix together with all extension parameter pairs, or a rank certificate excluding complete reduction.

(1) Compute the initial Gram matrix and a basis matrix of its left kernel

$$\mathbf{M}_0 = \mathbf{X}_0\mathbf{Z}_0^\top - \mathbf{Z}_0\mathbf{X}_0^\top, \qquad \mathbf{K}_0\mathbf{M}_0 = 0, \qquad h = \operatorname{rank}(\mathbf{K}_0).$$

If $h = 0$, return the input code. If h is odd, terminate because an alternating Gram matrix of the unchanged dimension cannot become nonsingular.

(2) Set $r = h/2$ and form the projected matrices

$$\mathbf{A} = \mathbf{K}_0\mathbf{X}_0, \qquad \mathbf{B} = \mathbf{K}_0\mathbf{Z}_0.$$

Compute their ranks $\alpha_0 = \operatorname{rank}(\mathbf{A})$ and $\beta_0 = \operatorname{rank}(\mathbf{B})$. If $\alpha_0 < r$ or $\beta_0 < r$, terminate with these ranks as a certificate that no complete reduction sequence exists.

Algorithm 1. *Cont.*

(3) Compute column-space bases for $\mathcal{U}_0 = \mathrm{Im}(\mathbf{A})$ and $\mathcal{V}_0 = \mathrm{Im}(\mathbf{B})$, as well as a basis of $\mathcal{I} = \mathcal{U}_0 \cap \mathcal{V}_0$. Choose complements $\mathcal{U}'$ and $\mathcal{V}'$ as in the proof of Theorem 4. Write $a = \dim \mathcal{U}'$ and $b = \dim \mathcal{V}'$. Combine a basis of $\mathcal{U}'$ with $r - a$ basis vectors of $\mathcal{I}$. Combine a basis of $\mathcal{V}'$ with the remaining $r - b$ intersection basis vectors. This yields

$$\mathbf{s}_1, \ldots, \mathbf{s}_r \in \mathcal{U}_0, \qquad \mathbf{t}_1, \ldots, \mathbf{t}_r \in \mathcal{V}_0$$

whose combined $2r$ vectors form a basis of $\mathbb{F}_q^{2r}$.

(4) Apply Gaussian elimination to obtain one solution of each pair of systems

$$\mathbf{A}\mathbf{u}_i^{(0)} = \mathbf{s}_i, \qquad \mathbf{B}\mathbf{v}_i^{(0)} = \mathbf{t}_i, \qquad 1 \le i \le r.$$

The elimination of each coefficient matrix can be reused for all right-hand sides.

(5) For $i = 1, \ldots, r$, extend the selected parameter vectors by $i - 1$ zero entries and construct the next code

$$\mathbf{u}_i = \begin{pmatrix} \mathbf{u}_i^{(0)} \\ \mathbf{0}_{i-1} \end{pmatrix}, \qquad \mathbf{v}_i = \begin{pmatrix} \mathbf{v}_i^{(0)} \\ \mathbf{0}_{i-1} \end{pmatrix},$$

$$\mathbf{G}_i = [\mathbf{X}_{i-1} \mid \mathbf{X}_{i-1}\mathbf{u}_i \mid \mathbf{Z}_{i-1} \mid \mathbf{Z}_{i-1}\mathbf{v}_i].$$

Return $\mathbf{G}_r$ and the parameter pairs after exactly r iterations.

The combined basis chosen in Step 3 makes every prefix of the appended projected columns linearly independent. The proof of Theorem 4 therefore gives

$$h_s(C_i) = 2(r - i), \qquad 0 \le i \le r.$$

Consequently, the procedure cannot become blocked after passing the initial rank test. An independently chosen intermediate code with positive even hull dimension h_i may instead fail the one-step image-space test in Theorem 3, or fail the full remaining reduction test when either projected image has rank below $h_i/2$. The algorithm controls the hull only and does not impose a minimum-distance increase.

With ordinary finite-field Gaussian elimination and matrix multiplication, forming the initial Gram matrix and coefficient hull costs $O(k^2n + k^3)$ field operations. Computing the projected images, their intersection and the preimages of all chosen targets costs at most $O(hkn + h^2n + h^3)$. The successive generator and Gram matrix updates cost $O(rk(n + r) + rk^2)$. As $h \le k \le 2n$ and $r = h/2$, the total construction cost is $O(k^2n + k^3)$ field operations with $O(kn + k^2)$ working storage when intermediate matrices are not retained. These bounds exclude optional independent verification of every intermediate rank and the separate problem of minimum-distance computation. Direct enumeration would inspect as many as $q^{2(n+i-1)}$ parameter pairs at iteration i, while the deterministic construction uses only linear systems and does not enumerate the q^k codewords.

4. Distance Evolution and LCD Conditions

The hull results depend on the alternating perturbation of $\mathbf{M}$. Distance depends instead on whether the appended coordinate pair is zero on a coefficient vector. The following results keep these two mechanisms separate and then combine them when LCD distance improvement is required.

For $r \ge d_s(C)$, define, with respect to the fixed generator matrix $\mathbf{G}$,

$$\mathcal{A}_r(C) = \left\{ \mathbf{a} \in \mathbb{F}_q^k \setminus \{\mathbf{0}\} : \mathrm{wt}_s(\mathbf{a}\mathbf{G}) = r \right\}.$$

Since multiplication by a nonzero scalar does not change symplectic weight, $\mathcal{A}_r(C)$ is invariant under the action of $\mathbb{F}_q^*$ by scalar multiplication. Define an equivalence relation on $\mathcal{A}_r(C)$ by

$$\mathbf{a} \sim \mathbf{b} \quad \Longleftrightarrow \quad \mathbf{b} = \lambda \mathbf{a} \quad \text{for some } \lambda \in \mathbb{F}_q^*.$$

Let $\mathbb{P}\mathcal{A}_r(C)$ denote any set containing exactly one representative from each equivalence class. Equivalently, it contains one nonzero representative from each one-dimensional subspace whose nonzero vectors belong to $\mathcal{A}_r(C)$.

For $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$, define the left kernel

$$\mathcal{K}_{\mathbf{u},\mathbf{v}} = \left\{ \mathbf{a} \in \mathbb{F}_q^k : \mathbf{aXu} = 0,\ \mathbf{aZv} = 0 \right\}.$$

Also define the subcode

$$C^0_{\mathbf{u},\mathbf{v}} = \{ \mathbf{aG} : \mathbf{a} \in \mathcal{K}_{\mathbf{u},\mathbf{v}} \} \subseteq C.$$

For a symplectic code $\mathcal{E}$, write its symplectic weight enumerator as

$$W_{\mathcal{E}}(Y) = \sum_{\mathbf{c} \in \mathcal{E}} Y^{\mathrm{wt}_s(\mathbf{c})} = \sum_i A_i(\mathcal{E}) Y^i.$$

Theorem 5 (Weight distribution of the extension)**.** *For every* $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$,

$$W_{\overline{C}(\mathbf{u},\mathbf{v})}(Y) = Y W_C(Y) + (1 - Y) W_{C^0_{\mathbf{u},\mathbf{v}}}(Y).$$

Equivalently, if

$$W_C(Y) = \sum_{i=0}^{n} A_i Y^i, \qquad W_{C^0_{\mathbf{u},\mathbf{v}}}(Y) = \sum_{i=0}^{n} B_i Y^i,$$

then

$$A_j(\overline{C}(\mathbf{u},\mathbf{v})) = A_{j-1} + B_j - B_{j-1}, \qquad 0 \le j \le n+1,$$

where $A_{-1} = B_{-1} = A_{n+1} = B_{n+1} = 0$.

Proof. For $\mathbf{a} \in \mathcal{K}_{\mathbf{u},\mathbf{v}}$, the appended coordinate pair contributes zero to the symplectic weight. For $\mathbf{a} \notin \mathcal{K}_{\mathbf{u},\mathbf{v}}$, it contributes one. Since $\mathbf{G}$ has full row rank,

$$\begin{aligned} W_{\overline{C}(\mathbf{u},\mathbf{v})}(Y) &= \sum_{\mathbf{a} \in \mathcal{K}_{\mathbf{u},\mathbf{v}}} Y^{\mathrm{wt}_s(\mathbf{aG})} + \sum_{\mathbf{a} \notin \mathcal{K}_{\mathbf{u},\mathbf{v}}} Y^{\mathrm{wt}_s(\mathbf{aG})+1} \\ &= W_{C^0_{\mathbf{u},\mathbf{v}}}(Y) + Y\Big(W_C(Y) - W_{C^0_{\mathbf{u},\mathbf{v}}}(Y) \Big). \end{aligned}$$

Comparing coefficients gives the second assertion. □

Theorem 6 (Distance of a two-coordinate extension)**.** *Let* $d = d_s(C)$. *Then,*

$$d_s(\overline{C}(\mathbf{u},\mathbf{v})) = \min_{\mathbf{a} \neq \mathbf{0}} \{ \mathrm{wt}_s(\mathbf{aG}) + \epsilon_{\mathbf{u},\mathbf{v}}(\mathbf{a}) \},$$

where

$$\epsilon_{\mathbf{u},\mathbf{v}}(\mathbf{a}) = \begin{cases} 0, & \mathbf{a} \in \mathcal{K}_{\mathbf{u},\mathbf{v}}, \\ 1, & \mathbf{a} \notin \mathcal{K}_{\mathbf{u},\mathbf{v}}. \end{cases}$$

Consequently, $d_s(\overline{C}(\mathbf{u},\mathbf{v})) \in \{d, d+1\}$.

Proof. Every nonzero codeword of $\overline{C}(\mathbf{u},\mathbf{v})$ is

$$\mathbf{a}\overline{\mathbf{G}}(\mathbf{u},\mathbf{v}) = (\mathbf{aX}, \mathbf{aXu} \mid \mathbf{aZ}, \mathbf{aZv}), \qquad \mathbf{a} \neq \mathbf{0}.$$

The appended pair contributes zero to the symplectic weight if and only if $\mathbf{aXu} = \mathbf{aZv} = 0$. This gives the formula and the two possible values. □

Corollary 3 (Distance preservation and increase)**.** *Let $d = d_s(C)$. Then,*

(1) *The distance is preserved precisely when $d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d \iff \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} \neq \emptyset$.*
(2) *The distance is increased by one precisely when $d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d+1 \iff \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} = \emptyset$.*

Proof. This follows from Theorem 6 by restricting to the coefficient vectors of minimum-weight codewords of C. □

Theorem 7 (Forbidden-pair characterization of distance improvement)**.** *Let $d = d_s(C)$. For $\mathbf{a} \in \mathcal{A}_d(C)$, define*

$$\mathcal{B}_{\mathbf{a}} = \{(\mathbf{u},\mathbf{v}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n : \mathbf{aXu} = 0,\ \mathbf{aZv} = 0\}.$$

Then,

$$d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d+1 \iff (\mathbf{u},\mathbf{v}) \notin \bigcup_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} \mathcal{B}_{\mathbf{a}}.$$

Moreover, put

$$\rho_{\mathbf{X}}(\mathbf{a}) = \operatorname{rank}(\mathbf{aX}), \qquad \rho_{\mathbf{Z}}(\mathbf{a}) = \operatorname{rank}(\mathbf{aZ}).$$

Since $\mathbf{aX}$ and $\mathbf{aZ}$ are row vectors, both $\rho_{\mathbf{X}}(\mathbf{a})$ and $\rho_{\mathbf{Z}}(\mathbf{a})$ belong to $\{0,1\}$. Each rank equals zero exactly when the corresponding row vector vanishes. If

$$\sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{2n-\rho_{\mathbf{X}}(\mathbf{a})-\rho_{\mathbf{Z}}(\mathbf{a})} < q^{2n},$$

then there exists a pair $(\mathbf{u},\mathbf{v})$ such that $d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d+1$.

Proof. By Corollary 3, the distance is increased by one if and only if no coefficient vector $\mathbf{a} \in \mathcal{A}_d(C)$ satisfies $\mathbf{aXu} = \mathbf{aZv} = 0$. Since $\mathcal{B}_{\lambda\mathbf{a}} = \mathcal{B}_{\mathbf{a}}$ for all $\lambda \in \mathbb{F}_q^*$, it is enough to take the union over $\mathbb{P}\mathcal{A}_d(C)$.

For fixed $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$, the equations $\mathbf{aXu} = 0$ and $\mathbf{aZv} = 0$ are independent in the two variables $\mathbf{u}$ and $\mathbf{v}$. Hence

$$|\mathcal{B}_{\mathbf{a}}| = q^{n-\rho_{\mathbf{X}}(\mathbf{a})} q^{n-\rho_{\mathbf{Z}}(\mathbf{a})} = q^{2n-\rho_{\mathbf{X}}(\mathbf{a})-\rho_{\mathbf{Z}}(\mathbf{a})}.$$

The stated inequality implies that the union of the forbidden sets is a proper subset of $\mathbb{F}_q^n \times \mathbb{F}_q^n$. □

Corollary 4 (One-sided parameter improving LCD extensions)**.** *Assume that C is symplectic LCD and let $d = d_s(C)$.*

(1) *If there exists $\mathbf{u} \in \mathbb{F}_q^n$ such that*

$$\mathbf{aXu} \neq 0, \qquad \text{for all } \mathbf{a} \in \mathcal{A}_d(C),$$

then $\overline{C}(\mathbf{u},\mathbf{0})$ is symplectic LCD and

$$d_s(\overline{C}(\mathbf{u},\mathbf{0})) = d+1.$$

(2) *If there exists $\mathbf{v} \in \mathbb{F}_q^n$ such that*

$$\mathbf{aZv} \neq 0, \qquad \text{for all } \mathbf{a} \in \mathcal{A}_d(C),$$

then $\overline{C}(\mathbf{0},\mathbf{v})$ is symplectic LCD and

$$d_s(\overline{C}(\mathbf{0},\mathbf{v})) = d+1.$$

Thus, an initial $[2n, k, d]_q$ *symplectic LCD code gives a* $[2n+2, k, d+1]_q$ *symplectic LCD code whenever either nonvanishing condition holds.*

Proof. For $(\mathbf{u}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{v})$, the two appended columns are linearly dependent, so Corollary 1 gives preservation of the LCD property. The stated nonvanishing condition is precisely the condition $\mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{0}} = \emptyset$ or $\mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{0},\mathbf{v}} = \emptyset$. The result follows from Corollary 3. □

Corollary 5 (One-sided counting criterion)**.** *Assume that* C *is symplectic LCD and let* $d = d_s(C)$.

(1) *The number of vectors* $\mathbf{u} \in \mathbb{F}_q^n$ *such that* $\overline{C}(\mathbf{u}, \mathbf{0})$ *is symplectic LCD and has distance* $d+1$ *is at least*

$$q^n - \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{n-\rho_{\mathbf{X}}(\mathbf{a})}.$$

In particular, such a vector exists if this lower bound is positive.

(2) *The number of vectors* $\mathbf{v} \in \mathbb{F}_q^n$ *such that* $\overline{C}(\mathbf{0}, \mathbf{v})$ *is symplectic LCD and has distance* $d+1$ *is at least*

$$q^n - \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{n-\rho_{\mathbf{Z}}(\mathbf{a})}.$$

In particular, such a vector exists if this lower bound is positive.

Proof. For $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$, put

$$\mathcal{H}_{\mathbf{a}}^{\mathbf{X}} = \{\mathbf{u} \in \mathbb{F}_q^n : \mathbf{a}\mathbf{X}\mathbf{u} = 0\}.$$

Then,

$$|\mathcal{H}_{\mathbf{a}}^{\mathbf{X}}| = q^{n-\rho_{\mathbf{X}}(\mathbf{a})}.$$

Hence, by the union bound,

$$\left| \bigcup_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} \mathcal{H}_{\mathbf{a}}^{\mathbf{X}} \right| \leq \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{n-\rho_{\mathbf{X}}(\mathbf{a})}.$$

Therefore, the number of vectors $\mathbf{u}$ satisfying

$$\mathbf{a}\mathbf{X}\mathbf{u} \neq 0 \qquad \text{for every } \mathbf{a} \in \mathcal{A}_d(C)$$

is at least

$$q^n - \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{n-\rho_{\mathbf{X}}(\mathbf{a})}.$$

Every such vector gives, by Corollary 4, a symplectic LCD extension of distance $d+1$. In particular, such a vector exists whenever the displayed lower bound is positive.

The proof of the second assertion is identical with $\mathbf{Z}$ and $\mathbf{v}$ in place of $\mathbf{X}$ and $\mathbf{u}$. □

Corollary 6 (Repeated distance change)**.** *Let* C_0 *be a symplectic code of length* $2n$*. For* $1 \leq i \leq s$*, let*

$$\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^{n+i-1},$$

and let C_i *be obtained from* C_{i-1} *by the corresponding two-coordinate extension. Write*

$$\mathbf{G}_i = [\mathbf{X}_i \mid \mathbf{Z}_i], \qquad d_i = d_s(C_i).$$

For $1 \leq i \leq s$*, put*

$$\mathcal{K}_i = \left\{\mathbf{a} \in \mathbb{F}_q^k : \mathbf{a}\mathbf{X}_{i-1}\mathbf{u}_i = 0,\ \mathbf{a}\mathbf{Z}_{i-1}\mathbf{v}_i = 0\right\}.$$

Then,

$$d_i \in \{d_{i-1}, d_{i-1}+1\},$$

and

$$d_i = d_{i-1}+1 \iff \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset.$$

Consequently,

$$d_s(C_s) = d_s(C_0) + \#\{i : \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset\}.$$

Proof. Apply Corollary 3 successively to C_{i-1} and its two-coordinate extension C_i for $i = 1, \ldots, s$. □

4.1. LCD Extensions with Controlled Distance

Theorem 8 (LCD-preserving extensions with distance control)**.** *Let C be symplectic LCD with $d = d_s(C)$. Then, the following assertions hold.*

(1) *The extension $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD with distance d if and only if*

$$(\mathbf{Xu})^\top \mathbf{M}^{-1}(\mathbf{Zv}) \neq 1, \qquad \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} \neq \emptyset.$$

(2) *The extension $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD with distance $d+1$ if and only if*

$$(\mathbf{Xu})^\top \mathbf{M}^{-1}(\mathbf{Zv}) \neq 1, \qquad \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} = \emptyset.$$

Proof. Combine Theorem 2 with Corollary 3. □

For integers m, r, t with $0 \le t \le \min\{m, r\}$, define

$$\Lambda_q(m, r, t) = \begin{cases} 0, & t = 0, \\ (q^m - q^{m-t})q^{r-1}, & t > 0. \end{cases}$$

Lemma 3 (Bilinear equation count)**.** *Let $\mathbf{A} \in \mathbb{F}_q^{m\times r}$ have rank t. Then,*

$$\left|\{(\mathbf{x}, \mathbf{y}) \in \mathbb{F}_q^m \times \mathbb{F}_q^r : \mathbf{x}^\top \mathbf{A}\mathbf{y} = 1\}\right| = \Lambda_q(m, r, t).$$

Proof. If $t = 0$, then $\mathbf{x}^\top \mathbf{A}\mathbf{y} = 0$ for all $(\mathbf{x}, \mathbf{y})$, so there is no solution. Suppose $t > 0$. The map $\mathbf{x} \mapsto \mathbf{x}^\top \mathbf{A}$ has kernel of dimension $m - t$. Thus, $q^m - q^{m-t}$ choices of $\mathbf{x}$ give a nonzero row vector $\mathbf{x}^\top \mathbf{A}$. For each such $\mathbf{x}$, the equation $(\mathbf{x}^\top \mathbf{A})\mathbf{y} = 1$ is a nonzero affine linear equation in $\mathbf{y}$ and has q^{r-1} solutions. This gives the formula. □

Corollary 7 (Counting LCD distance-increasing extensions)**.** *Let C be symplectic LCD with $d = d_s(C)$, and put $\mathbf{T} = \mathbf{X}^\top \mathbf{M}^{-1}\mathbf{Z}$ and $\tau = \operatorname{rank}(\mathbf{T})$. For $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$, put $m_\mathbf{a} = n - \rho_\mathbf{X}(\mathbf{a})$ and $r_\mathbf{a} = n - \rho_\mathbf{Z}(\mathbf{a})$. Let $\mathbf{E}_\mathbf{a}$ and $\mathbf{F}_\mathbf{a}$ be matrices whose columns form bases of $\ker(\mathbf{aX})$ and $\ker(\mathbf{aZ})$, respectively, and put*

$$\tau_\mathbf{a} = \operatorname{rank}(\mathbf{E}_\mathbf{a}^\top \mathbf{T}\mathbf{F}_\mathbf{a}).$$

The number of pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ such that $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD and $d_s(\overline{C}(\mathbf{u}, \mathbf{v})) = d + 1$ is at least

$$q^{2n} - \Lambda_q(n, n, \tau) - \sum_{\mathbf{a}\in\mathbb{P}\mathcal{A}_d(C)} \left(q^{m_\mathbf{a}+r_\mathbf{a}} - \Lambda_q(m_\mathbf{a}, r_\mathbf{a}, \tau_\mathbf{a})\right).$$

Consequently, if

$$\Lambda_q(n, n, \tau) + \sum_{\mathbf{a}\in\mathbb{P}\mathcal{A}_d(C)} \left(q^{m_\mathbf{a}+r_\mathbf{a}} - \Lambda_q(m_\mathbf{a}, r_\mathbf{a}, \tau_\mathbf{a})\right) < q^{2n},$$

then there exists $(\mathbf{u}, \mathbf{v}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ such that $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD and

$$d_s(\overline{C}(\mathbf{u}, \mathbf{v})) = d + 1.$$

Proof. The pairs which do not preserve the LCD property are precisely $\mathcal{L} = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u}^\top \mathbf{T}\mathbf{v} = 1\}$. By Lemma 3, $|\mathcal{L}| = \Lambda_q(n, n, \tau)$.

For fixed $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$, the forbidden set $\mathcal{B}_\mathbf{a}$ is $\ker(\mathbf{aX}) \times \ker(\mathbf{aZ})$ and has size $q^{m_\mathbf{a}+r_\mathbf{a}}$. Its intersection with $\mathcal{L}$ is the solution set of the restricted bilinear equation

$$\mathbf{u}^\top \mathbf{T}\mathbf{v} = 1, \qquad \mathbf{u} \in \ker(\mathbf{aX}), \quad \mathbf{v} \in \ker(\mathbf{aZ}).$$

The rank of this restricted bilinear form is $\tau_\mathbf{a}$, so $|\mathcal{B}_\mathbf{a} \cap \mathcal{L}| = \Lambda_q(m_\mathbf{a}, r_\mathbf{a}, \tau_\mathbf{a})$. Therefore,

$$\left| \mathcal{L} \cup \bigcup_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} \mathcal{B}_\mathbf{a} \right| \le |\mathcal{L}| + \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} |\mathcal{B}_\mathbf{a} \setminus \mathcal{L}|.$$

Hence the number of pairs outside this union is at least the displayed lower bound. Theorem 8 gives the assertion. □

Remark 2. *The use of $\mathbb{P}\mathcal{A}_d(C)$ removes the repeated forbidden sets coming from nonzero scalar multiples of the same minimum-weight coefficient vector. The term $\Lambda_q(m_\mathbf{a}, r_\mathbf{a}, \tau_\mathbf{a})$ further removes the part of $\mathcal{B}_\mathbf{a}$ which already fails the LCD-preserving condition. Hence Corollary 7 is sharper than the direct union bound using all vectors in $\mathcal{A}_d(C)$.*

Corollary 8 (Iterated LCD-preserving construction with prescribed distance gain)**.** *Let C_0 be a symplectic LCD code of length $2n$, and let t be an integer with $0 \le t \le s$. For $1 \le i \le s$, let*

$$\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^{n+i-1},$$

and let C_i be obtained from C_{i-1} by the corresponding two-coordinate extension. For $0 \le i \le s$, write

$$\mathbf{G}_i = [\mathbf{X}_i \mid \mathbf{Z}_i], \qquad d_i = d_s(C_i),$$

and let $\mathbf{M}_i$ denote the symplectic Gram matrix of C_i.

For $1 \le i \le s$, define

$$\mathcal{K}_i = \left\{ \mathbf{a} \in \mathbb{F}_q^k : \mathbf{a}\mathbf{X}_{i-1}\mathbf{u}_i = 0,\ \mathbf{a}\mathbf{Z}_{i-1}\mathbf{v}_i = 0 \right\}.$$

Whenever $\mathbf{M}_{i-1}$ is nonsingular, define

$$\lambda_i = (\mathbf{X}_{i-1}\mathbf{u}_i)^\top \mathbf{M}_{i-1}^{-1} (\mathbf{Z}_{i-1}\mathbf{v}_i).$$

Then,

$$C_i \text{ is symplectic LCD for every } 1 \le i \le s$$

and

$$d_s(C_s) = d_s(C_0) + t$$

if and only if λ_i is defined and

$$\lambda_i \ne 1, \qquad 1 \le i \le s,$$

and

$$t = \#\left\{ i : \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset \right\}.$$

In particular,

$$d_s(C_0) \le d_s(C_s) \le d_s(C_0) + s.$$

Proof. By Corollary 2, all the codes $C_1, \ldots, C_s$ are symplectic LCD if and only if each λ_i is defined and satisfies $\lambda_i \ne 1$.

Independently, Corollary 6 gives

$$d_s(C_s) - d_s(C_0) = \#\left\{ i : \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset \right\}.$$

Combining these two conditions proves the equivalence. Since each of the s extensions increases the minimum distance by either zero or one, the final bounds follow. □

Theorem 9 (Hull-reducing extensions with distance control)**.** *Assume* $h_s(C) \geq 2$ *and*

$$\operatorname{rank}[\mathbf{KXu}\ \mathbf{KZv}] = 2.$$

Then, $h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h_s(C) - 2$*. If* $d = d_s(C)$*, then*

(1) *The distance is preserved precisely when* $d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d \iff \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} \neq \emptyset$.
(2) *The distance is increased by one precisely when* $d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d+1 \iff \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} = \emptyset$.

Proof. Use Theorem 3 and Corollary 3. □

Proposition 1 (Number of hull-reducing pairs)**.** *Assume* $h = h_s(C) \geq 2$*. Let*

$$\mathcal{U} = \operatorname{Im}(\mathbf{KX}), \qquad \mathcal{V} = \operatorname{Im}(\mathbf{KZ}),$$

and put $\alpha = \dim \mathcal{U}$, $\beta = \dim \mathcal{V}$*, and* $\gamma = \dim(\mathcal{U} \cap \mathcal{V})$*. For* $0 \leq \gamma \leq \min\{\alpha, \beta\}$*, set*

$$N_q(\alpha,\beta,\gamma) = q^{\alpha+\beta} - q^{\alpha} - q^{\beta} - (q-1)q^{\gamma} + q.$$

Then, the number of pairs $(\mathbf{u},\mathbf{v}) \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ *satisfying*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h - 2$$

is

$$q^{2n-\alpha-\beta} N_q(\alpha,\beta,\gamma).$$

In particular, hull-reducing pairs exist if and only if $N_q(\alpha,\beta,\gamma) > 0$*, equivalently* $\mathcal{U}$ *and* $\mathcal{V}$ *contain two linearly independent vectors.*

Proof. By Theorem 3, a pair is hull-reducing if and only if **KXu** and **KZv** are linearly independent. Each element of $\mathcal{U}$ has $q^{n-\alpha}$ preimages under **KX**, and each element of $\mathcal{V}$ has $q^{n-\beta}$ preimages under **KZ**. It remains to count linearly independent ordered pairs $(\mathbf{p},\mathbf{f}) \in \mathcal{U} \times \mathcal{V}$. The number of dependent pairs is

$$q^{\beta} + (q^{\gamma} - 1)q + (q^{\alpha} - q^{\gamma}) = q^{\alpha} + q^{\beta} + (q-1)q^{\gamma} - q.$$

Subtracting this from $q^{\alpha+\beta}$ gives $N_q(\alpha,\beta,\gamma)$. □

Theorem 10 (Hull reduction with distance increase)**.** *Assume* $h = h_s(C) \geq 2$ *and let* $d = d_s(C)$*. With* $\mathcal{U}$, $\mathcal{V}$, α, β, *and* γ *as in Proposition 1, define for each* $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$

$$\mathcal{U}_{\mathbf{a}} = (\mathbf{KX})(\ker(\mathbf{aX})), \qquad \mathcal{V}_{\mathbf{a}} = (\mathbf{KZ})(\ker(\mathbf{aZ})).$$

Put $\alpha_{\mathbf{a}} = \dim \mathcal{U}_{\mathbf{a}}$, $\beta_{\mathbf{a}} = \dim \mathcal{V}_{\mathbf{a}}$*, and* $\gamma_{\mathbf{a}} = \dim(\mathcal{U}_{\mathbf{a}} \cap \mathcal{V}_{\mathbf{a}})$*. Set*

$$\begin{aligned}\Delta_{\mathrm{red},d+1} = {} & q^{2n-\alpha-\beta} N_q(\alpha,\beta,\gamma) \\ & - \sum_{\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)} q^{2n-\rho_{\mathbf{X}}(\mathbf{a})-\rho_{\mathbf{Z}}(\mathbf{a})-\alpha_{\mathbf{a}}-\beta_{\mathbf{a}}} \\ & \times N_q(\alpha_{\mathbf{a}},\beta_{\mathbf{a}},\gamma_{\mathbf{a}})\end{aligned}$$

Then, the number of pairs $(\mathbf{u},\mathbf{v})$ *satisfying*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h - 2, \qquad d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d + 1$$

is at least $\Delta_{\mathrm{red},d+1}$*. Consequently, if* $\Delta_{\mathrm{red},d+1} > 0$*, then there exists* $(\mathbf{u},\mathbf{v})$ *such that*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h - 2, \qquad d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d + 1.$$

In particular, if $h = 2$*, then* $\overline{C}(\mathbf{u},\mathbf{v})$ *is symplectic LCD with distance* $d+1$.

Proof. Let $\mathcal{R}$ be the set of all hull-reducing pairs. Proposition 1 gives $|\mathcal{R}| = q^{2n-\alpha-\beta}N_q(\alpha,\beta,\gamma)$. For fixed $\mathbf{a} \in \mathbb{P}\mathcal{A}_d(C)$, the set $\mathcal{R} \cap \mathcal{B}_{\mathbf{a}}$ is counted by the same argument after restricting the first domain to $\ker(\mathbf{aX})$ and the second domain to $\ker(\mathbf{aZ})$. These domains have dimensions $n - \rho_{\mathbf{X}}(\mathbf{a})$ and $n - \rho_{\mathbf{Z}}(\mathbf{a})$, respectively. Hence

$$|\mathcal{R} \cap \mathcal{B}_{\mathbf{a}}| = q^{2n-\rho_{\mathbf{X}}(\mathbf{a})-\rho_{\mathbf{Z}}(\mathbf{a})-\alpha_{\mathbf{a}}-\beta_{\mathbf{a}}}N_q(\alpha_{\mathbf{a}},\beta_{\mathbf{a}},\gamma_{\mathbf{a}}).$$

Therefore,

$$\left|\mathcal{R} \setminus \bigcup_{\mathbf{a}\in\mathbb{P}\mathcal{A}_d(C)} \mathcal{B}_{\mathbf{a}}\right| \geq \Delta_{\mathrm{red},d+1}.$$

If $\Delta_{\mathrm{red},d+1} > 0$, there is a hull-reducing pair outside the union. The hull statement follows from Theorem 3, and the distance statement follows from Theorem 7. □

Corollary 9 (Construction from an even-dimensional hull)**.** *Let C_0 be a symplectic code of length $2n$ satisfying*

$$h_s(C_0) = 2r,$$

and let t be an integer with $0 \leq t \leq r$. For $1 \leq i \leq r$, let

$$\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}_q^{n+i-1},$$

and let C_i be obtained from C_{i-1} by the corresponding two-coordinate extension.

For $0 \leq i \leq r$, write

$$\mathbf{G}_i = [\mathbf{X}_i \mid \mathbf{Z}_i], \qquad d_i = d_s(C_i).$$

For $0 \leq i \leq r-1$, let $\mathbf{K}_i$ be a matrix whose rows form a basis of the coefficient hull $\mathcal{H}(C_i)$. For $1 \leq i \leq r$, define

$$\mathcal{K}_i = \left\{\mathbf{a} \in \mathbb{F}_q^k : \mathbf{aX}_{i-1}\mathbf{u}_i = 0,\ \mathbf{aZ}_{i-1}\mathbf{v}_i = 0\right\}.$$

Then,

$$h_s(C_r) = 0$$

and

$$d_s(C_r) = d_s(C_0) + t$$

if and only if

$$\operatorname{rank}[\mathbf{K}_{i-1}\mathbf{X}_{i-1}\mathbf{u}_i\ \mathbf{K}_{i-1}\mathbf{Z}_{i-1}\mathbf{v}_i] = 2, \qquad 1 \leq i \leq r,$$

and

$$t = \#\left\{i : \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset\right\}.$$

In particular,

$$d_s(C_0) \leq d_s(C_r) \leq d_s(C_0) + r.$$

Without a prescribed distance gain, the existence of a complete hull-reducing sequence is equivalent, by Theorem 4, to

$$\operatorname{rank}(\mathbf{K}_0\mathbf{X}_0) \geq r, \qquad \operatorname{rank}(\mathbf{K}_0\mathbf{Z}_0) \geq r.$$

These initial rank inequalities alone do not imply a particular distance gain.

Proof. Suppose first that

$$\operatorname{rank}[\mathbf{K}_{i-1}\mathbf{X}_{i-1}\mathbf{u}_i\ \mathbf{K}_{i-1}\mathbf{Z}_{i-1}\mathbf{v}_i] = 2$$

for every $1 \leq i \leq r$. By Theorem 3, each extension decreases the hull dimension by two. Therefore,

$$h_s(C_i) = 2(r-i), \qquad 0 \leq i \leq r,$$

and in particular $h_s(C_r) = 0$.

Conversely, suppose that $h_s(C_r) = 0$. A two-coordinate extension can decrease the hull dimension by at most two. Starting from $h_s(C_0) = 2r$, reaching zero after exactly r extensions therefore forces every extension to decrease the hull dimension by exactly two. Hence, for every $1 \le i \le r$, one has

$$h_s(C_{i-1}) \ge 2$$

and Theorem 3 yields

$$\operatorname{rank}[\mathbf{K}_{i-1}\mathbf{X}_{i-1}\mathbf{u}_i \; \mathbf{K}_{i-1}\mathbf{Z}_{i-1}\mathbf{v}_i] = 2.$$

Finally, Corollary 6 gives

$$d_s(C_r) - d_s(C_0) = \#\{i : \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset\},$$

which proves the distance assertion and the stated bounds. □

Corollary 10 (Iterated hull reduction with full distance gain)**.** *Let C_0 be a $[2n, k, d]_q$ code with $h_s(C_0) = 2r$. If the lower bound $\Delta_{\mathrm{red},d+1}$ in Theorem 10 is positive at each step of a recursive hull-reducing construction, then there exists a sequence of r two-coordinate extensions whose final code C_r is symplectic LCD and has parameters*

$$[2(n + r), k, d + r]_q$$

with respect to the symplectic minimum distance. The initial rank inequalities in Theorem 4 are necessary for the complete hull reduction, but the stepwise positivity hypothesis is an additional requirement for obtaining the full distance gain.

Proof. At each step, Theorem 10 gives an extension decreasing the hull dimension by two and increasing the distance by one. After r steps, the hull dimension is zero and the distance is $d + r$. □

4.2. Exact Enumeration and Orbit Reduction

The preceding lower bounds are useful for existence. For deterministic parameter construction, one can refine them to exact formulas by inclusion-exclusion over the projective minimum-weight coefficient set. The resulting formulas count admissible parameters without generating them. Constructing an extension still requires a successful pair, and the search may be shortened by eliminating duplicate appended columns and exploiting automorphism orbits.

For $\mathcal{S} \subseteq \mathbb{P}\mathcal{A}_d(C)$, define

$$\mathcal{X}_{\mathcal{S}} = \{\mathbf{u} \in \mathbb{F}_q^n : \mathbf{aXu} = 0 \text{ for all } \mathbf{a} \in \mathcal{S}\},$$

$$\mathcal{Z}_{\mathcal{S}} = \{\mathbf{v} \in \mathbb{F}_q^n : \mathbf{aZv} = 0 \text{ for all } \mathbf{a} \in \mathcal{S}\}.$$

Put $x_{\mathcal{S}} = \dim \mathcal{X}_{\mathcal{S}}$ and $z_{\mathcal{S}} = \dim \mathcal{Z}_{\mathcal{S}}$. For $\mathcal{S} = \emptyset$, one has $\mathcal{X}_\emptyset = \mathcal{Z}_\emptyset = \mathbb{F}_q^n$.

Theorem 11 (Exact inclusion-exclusion counts)**.** *Let $d = d_s(C)$. The following assertions hold.*

(1) *The number of vectors $\mathbf{u} \in \mathbb{F}_q^n$ for which $\overline{C}(\mathbf{u}, \mathbf{0})$ has distance $d + 1$ is*

$$N_{\mathbf{X},d+1} = \sum_{\mathcal{S} \subseteq \mathbb{P}\mathcal{A}_d(C)} (-1)^{|\mathcal{S}|} q^{x_{\mathcal{S}}}.$$

The corresponding number for $\overline{C}(\mathbf{0}, \mathbf{v})$ is

$$N_{\mathbf{Z},d+1} = \sum_{\mathcal{S} \subseteq \mathbb{P}\mathcal{A}_d(C)} (-1)^{|\mathcal{S}|} q^{z_{\mathcal{S}}}.$$

(2) *The number of pairs* $(\mathbf{u},\mathbf{v})$ *for which* $\overline{C}(\mathbf{u},\mathbf{v})$ *has distance* $d+1$ *is*

$$N_{d+1} = \sum_{\delta \subseteq \mathbb{P}A_d(C)} (-1)^{|\delta|} q^{x_\delta + z_\delta}.$$

(3) *Assume further that* C *is symplectic LCD, and put* $\mathbf{T} = \mathbf{X}^\top \mathbf{M}^{-1} \mathbf{Z}$. *Let* $\mathbf{E}_\delta$ *and* $\mathbf{F}_\delta$ *be matrices whose columns form bases of* $\mathcal{X}_\delta$ *and* $\mathcal{Z}_\delta$, *respectively, and put*

$$t_\delta = \operatorname{rank}(\mathbf{E}_\delta^\top \mathbf{T} \mathbf{F}_\delta).$$

Then, the number of pairs $(\mathbf{u},\mathbf{v})$ *for which* $\overline{C}(\mathbf{u},\mathbf{v})$ *is symplectic LCD and has distance* $d+1$ *is*

$$N_{\mathrm{LCD},d+1} = \sum_{\delta \subseteq \mathbb{P}A_d(C)} (-1)^{|\delta|} \left(q^{x_\delta + z_\delta} - \Lambda_q(x_\delta, z_\delta, t_\delta) \right).$$

Proof. For a fixed $\mathbf{a} \in \mathbb{P}A_d(C)$, the corresponding forbidden conditions are $\mathbf{aXu} = 0$ and $\mathbf{aZv} = 0$. Thus, the intersection of the forbidden conditions indexed by δ is $\mathcal{X}_\delta$ in the one-sided $\mathbf{X}$ case, $\mathcal{Z}_\delta$ in the one-sided $\mathbf{Z}$ case, and $\mathcal{X}_\delta \times \mathcal{Z}_\delta$ in the two-sided case. Inclusion-exclusion gives the first two assertions.

Assume now that C is symplectic LCD. Inside $\mathcal{X}_\delta \times \mathcal{Z}_\delta$, the pairs which fail the LCD condition are the solutions of $\mathbf{u}^\top \mathbf{T} \mathbf{v} = 1$. By Lemma 3, their number is $\Lambda_q(x_\delta, z_\delta, t_\delta)$. Applying inclusion-exclusion to the admissible set outside the LCD failing pairs gives the stated formula. □

Theorem 12 (Exact count for hull reduction and distance increase)**.** *Assume* $h = h_s(C) \geq 2$ *and let* $d = d_s(C)$. *For* $\delta \subseteq \mathbb{P}A_d(C)$, *put*

$$\mathcal{U}_\delta = (\mathbf{KX})(\mathcal{X}_\delta), \qquad \mathcal{V}_\delta = (\mathbf{KZ})(\mathcal{Z}_\delta),$$

and let

$$\alpha_\delta = \dim \mathcal{U}_\delta, \quad \beta_\delta = \dim \mathcal{V}_\delta, \quad \gamma_\delta = \dim(\mathcal{U}_\delta \cap \mathcal{V}_\delta).$$

Then, the number of pairs $(\mathbf{u},\mathbf{v})$ *satisfying*

$$h_s(\overline{C}(\mathbf{u},\mathbf{v})) = h-2, \qquad d_s(\overline{C}(\mathbf{u},\mathbf{v})) = d+1$$

is

$$N_{\mathrm{red},d+1} = \sum_{\delta \subseteq \mathbb{P}A_d(C)} (-1)^{|\delta|} q^{x_\delta + z_\delta - \alpha_\delta - \beta_\delta} N_q(\alpha_\delta, \beta_\delta, \gamma_\delta).$$

Consequently, if $h = 2$ *and* $N_{\mathrm{red},d+1} > 0$, *then there exists a symplectic LCD* $[2n+2, k, d+1]_q$ *code.*

Proof. For fixed δ, the pair $(\mathbf{u},\mathbf{v})$ lies in all forbidden sets indexed by δ if and only if $(\mathbf{u},\mathbf{v}) \in \mathcal{X}_\delta \times \mathcal{Z}_\delta$. On this product space, the pair is hull-reducing precisely when $\mathbf{KXu}$ and $\mathbf{KZv}$ are linearly independent. Proposition 1, applied to the restricted domains $\mathcal{X}_\delta$ and $\mathcal{Z}_\delta$, gives

$$q^{x_\delta + z_\delta - \alpha_\delta - \beta_\delta} N_q(\alpha_\delta, \beta_\delta, \gamma_\delta)$$

such pairs. Inclusion-exclusion over $\mathbb{P}A_d(C)$ gives the formula. The final assertion follows from Theorem 3. □

Corollary 11 (Deterministic parameter improvement)**.** *Let* C *be a* $[2n,k,d]_q$ *code.*

(1) *If* C *is symplectic LCD and at least one of*

$$N_{\mathbf{X},d+1} > 0, \qquad N_{\mathbf{Z},d+1} > 0, \qquad N_{\mathrm{LCD},d+1} > 0$$

holds, then there exists a symplectic LCD $[2n+2, k, d+1]_q$ *code.*

(2) *If $h_s(C) = 2$ and $N_{\mathrm{red},d+1} > 0$, then there exists a symplectic LCD $[2n+2, k, d+1]_q$ code.*
(3) *If $h_s(C) = 2r$ and the corresponding value of $N_{\mathrm{red},d+1}$ is positive at each of r successive hull-reducing steps, then there exists a symplectic LCD $[2(n+r), k, d+r]_q$ code.*

In assertion 3, the positive counts must be verified along one consistently selected sequence of extensions. The mother-code rank criterion in Theorem 4 guarantees the existence of a hull-reducing sequence, but not that the distance-improving counts remain positive.

Proof. The first assertion follows from Theorem 11 and Corollary 4. The second assertion follows from Theorem 12. Repeating the second assertion gives the third assertion. □

Corollary 11 gives a deterministic first stage for parameter search. A positive exact count proves existence before any random search is used. If the relevant automorphism group is available, Theorem 13 allows for the search to test orbit representatives rather than every parameter pair. If the exact count is zero, the prescribed one-step improvement cannot be obtained from the selected mother code and the selected extension model.

Remark 3 (Use in constructive searches)**.** *For a fixed mother code, the preceding results give a finite verification procedure based only on ranks, kernels, and minimum-weight coefficient vectors. A constructive search can be organized as follows.*

(1) *Compute $\mathbf{M}$, the coefficient hull matrix $\mathbf{K}$, the minimum-weight coefficient set $\mathcal{A}_d(C)$, and the projective set $\mathbb{P}\mathcal{A}_d(C)$.*
(2) *Use Corollary 11, or the lower bounds in Corollaries 5, 7 and Theorem 10, to decide whether the required extension is certified to exist.*
(3) *If enumeration is required, restrict the search to the admissible pair sets and then to orbit representatives under $\Gamma_s(C)$ whenever this group is available.*
(4) *For every output code, verify the final parameter claim by the Gram rank condition and the kernel condition on $\mathcal{A}_d(C)$, independently of the search method used to find the pair.*

Thus, random or local search is used only to find candidates, while the correctness of a new parameter is certified by the deterministic criteria in this paper.

Corollary 12 (Bonferroni deterministic tests)**.** *Let I_i denote the sum of the i-fold intersection terms in any of the exact formulas in Theorems 11 and 12, with I_0 equal to the ambient term. For every integer $e \geq 0$ with $2e+1 \leq |\mathbb{P}\mathcal{A}_d(C)|$, the partial sum*

$$\sum_{i=0}^{2e+1} (-1)^i I_i$$

is a lower bound for the corresponding number of admissible extension pairs. In particular, if this lower bound is positive, then the corresponding extension exists.

Proof. This is the Bonferroni inequality applied to the union of the forbidden sets. Odd truncations give upper bounds on the forbidden union and therefore lower bounds on its complement. □

We next record an orbit reduction principle. Let $\mathrm{Mon}_n(q)$ denote the group of $n \times n$ monomial matrices over $\mathbb{F}_q$. Define

$$\Gamma_s(C) = \left\{ \mathbf{P} \in \mathrm{Mon}_n(q) : \exists\, \mathbf{A} \in GL_k(q) \text{ such that } \mathbf{AX} = \mathbf{XP},\ \mathbf{AZ} = \mathbf{Z}(\mathbf{P}^{-1})^{\top} \right\}.$$

The set $\Gamma_s(C)$ is a group. Indeed, the identity matrix belongs to $\Gamma_s(C)$. If $\mathbf{P}_1, \mathbf{P}_2 \in \Gamma_s(C)$ correspond to $\mathbf{A}_1, \mathbf{A}_2 \in GL_k(q)$, respectively, then

$$(\mathbf{A}_1\mathbf{A}_2)\mathbf{X} = \mathbf{X}\mathbf{P}_1\mathbf{P}_2$$

and

$$(\mathbf{A}_1\mathbf{A}_2)\mathbf{Z} = \mathbf{Z}(\mathbf{P}_1^{-1})^\top(\mathbf{P}_2^{-1})^\top = \mathbf{Z}((\mathbf{P}_1\mathbf{P}_2)^{-1})^\top.$$

Hence $\mathbf{P}_1\mathbf{P}_2 \in \Gamma_s(C)$. Moreover, if $\mathbf{P}$ corresponds to $\mathbf{A}$, then $\mathbf{P}^{-1}$ corresponds to $\mathbf{A}^{-1}$.

The group $\Gamma_s(C)$ acts on $\mathbb{F}_q^n \times \mathbb{F}_q^n$ by

$$\mathbf{P} \cdot (\mathbf{u}, \mathbf{v}) = \big(\mathbf{P}\mathbf{u}, (\mathbf{P}^{-1})^\top\mathbf{v}\big).$$

Let Ω_{d+1} denote the set of all pairs producing an extension of distance $d+1$, let $\Omega_{\mathrm{LCD},d+1}$ denote the set of all pairs producing a symplectic LCD extension of distance $d+1$, and let $\Omega_{\mathrm{red},d+1}$ denote the set of all pairs producing hull reduction by two and distance $d+1$.

Theorem 13 (Automorphism orbit reduction)**.** *Let*

$$\Omega \in \{\Omega_{d+1}, \Omega_{\mathrm{LCD},d+1}, \Omega_{\mathrm{red},d+1}\}.$$

Then, Ω is invariant under $\Gamma_s(C)$, and extensions arising from pairs in the same $\Gamma_s(C)$-orbit are symplectically and symplectic-weight equivalent. Consequently, one representative from each orbit is sufficient for construction.

The number of such orbits is

$$\frac{1}{|\Gamma_s(C)|}\sum_{\mathbf{P}\in\Gamma_s(C)} |\operatorname{Fix}_\Omega(\mathbf{P})|.$$

For $\Omega = \Omega_{d+1}$,

$$|\operatorname{Fix}_{\Omega_{d+1}}(\mathbf{P})| = \sum_{\delta\subseteq\mathbb{P}A_d(C)} (-1)^{|\delta|} q^{x_{\delta,\mathbf{P}}+z_{\delta,\mathbf{P}}},$$

where

$$x_{\delta,\mathbf{P}} = \dim(\mathcal{X}_\delta \cap \ker(\mathbf{P} - \mathbf{I}_n))$$

and

$$z_{\delta,\mathbf{P}} = \dim\big(\mathcal{Z}_\delta \cap \ker((\mathbf{P}^{-1})^\top - \mathbf{I}_n)\big).$$

The corresponding fixed point counts for $\Omega_{\mathrm{LCD},d+1}$ and $\Omega_{\mathrm{red},d+1}$ are obtained by imposing, inside the same fixed subspaces, respectively the LCD condition of Theorem 2 and the hull reduction condition of Theorem 3.

Proof. Let $\mathbf{P} \in \Gamma_s(C)$, and choose $\mathbf{A} \in GL_k(q)$ satisfying

$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{P}, \qquad \mathbf{A}\mathbf{Z} = \mathbf{Z}(\mathbf{P}^{-1})^\top.$$

Applying the row operation $\mathbf{A}$ to $\overline{\mathbf{G}}(\mathbf{u}, \mathbf{v})$ gives

$$[\mathbf{X}\mathbf{P} \mid \mathbf{X}\mathbf{P}\mathbf{u} \mid \mathbf{Z}(\mathbf{P}^{-1})^\top \mid \mathbf{Z}(\mathbf{P}^{-1})^\top\mathbf{v}].$$

Put

$$\mathbf{Q} = \begin{pmatrix} \mathbf{P}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(q).$$

On $\mathbb{F}_q^{2(n+1)}$, consider the split transformation

$$(\mathbf{x} \mid \mathbf{z}) \longmapsto \big(\mathbf{x}\mathbf{Q} \mid \mathbf{z}(\mathbf{Q}^{-1})^\top\big).$$

Its matrix is

$$\begin{pmatrix} \mathbf{Q} & 0 \\ 0 & (\mathbf{Q}^{-1})^\top \end{pmatrix},$$

and a direct calculation gives

$$\begin{pmatrix} \mathbf{Q} & 0 \\ 0 & (\mathbf{Q}^{-1})^{\top} \end{pmatrix} \mathbf{J}_{2(n+1)} \begin{pmatrix} \mathbf{Q}^{\top} & 0 \\ 0 & \mathbf{Q}^{-1} \end{pmatrix} = \mathbf{J}_{2(n+1)}.$$

Thus, the transformation preserves the symplectic inner product.

It also preserves symplectic weight. Indeed, $\mathbf{Q}$ is monomial. If

$$\mathbf{Q} = \mathbf{D}\mathbf{\Pi},$$

where $\mathbf{D}$ is nonsingular diagonal and $\mathbf{\Pi}$ is a permutation matrix, then the two halves of a vector are transformed by the same coordinate permutation, with reciprocal nonzero scalings on corresponding entries. Consequently, a coordinate pair is zero before the transformation if and only if the corresponding coordinate pair is zero afterwards. Hence symplectic weight is preserved.

Applying this transformation to the preceding generator matrix gives

$$[\mathbf{X} \mid \mathbf{XPu} \mid \mathbf{Z} \mid \mathbf{Z}(\mathbf{P}^{-1})^{\top}\mathbf{v}] = \overline{\mathbf{G}}\big(\mathbf{Pu}, (\mathbf{P}^{-1})^{\top}\mathbf{v}\big).$$

Therefore, the extensions determined by $(\mathbf{u}, \mathbf{v})$ and

$$\mathbf{P} \cdot (\mathbf{u}, \mathbf{v}) = \big(\mathbf{Pu}, (\mathbf{P}^{-1})^{\top}\mathbf{v}\big)$$

are equivalent under a symplectic-weight isometry.

It follows that minimum symplectic distance, hull dimension, and the symplectic LCD property are constant on $\Gamma_s(C)$-orbits. Hence each of Ω_{d+1}, $\Omega_{\mathrm{LCD},d+1}$, and $\Omega_{\mathrm{red},d+1}$ is invariant under the group action. Burnside's lemma therefore gives

$$|\Omega/\Gamma_s(C)| = \frac{1}{|\Gamma_s(C)|} \sum_{\mathbf{P} \in \Gamma_s(C)} |\operatorname{Fix}_{\Omega}(\mathbf{P})|.$$

It remains to verify the displayed fixed point count for Ω_{d+1}. A pair $(\mathbf{u}, \mathbf{v})$ is fixed by $\mathbf{P}$ precisely when

$$\mathbf{u} \in \ker(\mathbf{P} - \mathbf{I}_n), \qquad \mathbf{v} \in \ker((\mathbf{P}^{-1})^{\top} - \mathbf{I}_n).$$

For $\mathcal{S} \subseteq \mathbb{P}\mathcal{A}_d(C)$, the fixed pairs lying in every forbidden set indexed by $\mathcal{S}$ form

$$(\mathcal{X}_{\mathcal{S}} \cap \ker(\mathbf{P} - \mathbf{I}_n)) \times \big(\mathcal{Z}_{\mathcal{S}} \cap \ker((\mathbf{P}^{-1})^{\top} - \mathbf{I}_n)\big).$$

This set has cardinality

$$q^{x_{\mathcal{S},\mathbf{P}} + z_{\mathcal{S},\mathbf{P}}}.$$

Inclusion–exclusion over $\mathbb{P}\mathcal{A}_d(C)$ therefore yields

$$|\operatorname{Fix}_{\Omega_{d+1}}(\mathbf{P})| = \sum_{\mathcal{S} \subseteq \mathbb{P}\mathcal{A}_d(C)} (-1)^{|\mathcal{S}|} q^{x_{\mathcal{S},\mathbf{P}} + z_{\mathcal{S},\mathbf{P}}}.$$

The same argument, with the LCD or hull-reduction condition imposed inside these fixed subspaces, gives the remaining fixed point counts. □

The three counting and search methods have different computational costs. Write $r_X = \operatorname{rank}(\mathbf{X})$, $r_Z = \operatorname{rank}(\mathbf{Z})$, and $t = |\mathbb{P}\mathcal{A}_d(C)|$. The total number of parameter pairs is $H = q^{2n}$, whereas the number of distinct appended-column pairs is $E = q^{r_X + r_Z}$. Each appended-column pair has exactly $q^{2n - r_X - r_Z}$ parameter preimages, and all hull and distance tests in this work depend only on the appended columns. Hence a scan of the E effective pairs suffices for exact parameter counting after multiplication by this common fiber size.

Suppose the Gram data and projective minimum-weight coefficient vectors have been precomputed. The rank, LCD, and minimum-weight kernel tests then cost at most $C =$

$O(k^2 + tk)$ field operations per effective pair. A direct computation of the minimum-weight coefficient set requires at most $O(q^k kn)$ field operations, which is a common preliminary cost for all distance-sensitive methods. Put $S = O(kn + k^2 + tk)$ for shared storage, and let $D = O((n + k + t)^3)$ and $W = O((n + k + t)^2)$ bound the time and auxiliary space of one inclusion–exclusion intersection computed by classical finite-field elimination.

The automorphism relations in Theorem 13 induce the following action on effective appended-column pairs.

$$(\mathbf{p}, \mathbf{f}) \mapsto (\mathbf{Ap}, \mathbf{Af}).$$

Write $g = |\Gamma_s(C)|$ and let B be the number of orbits on all E effective pairs. The following estimates use a stored visited set and a known group, with classical matrix arithmetic. The resulting complexity bounds are collected in Table 1.

Table 1. Time and working memory bounds for extension counting and search.

Method	Time	Working Memory
Exhaustive effective-pair scan	$O(EC)$	$O(S)$
Inclusion–exclusion count	$O(2^t D)$	$O(S + W)$
Orbit-representative search	$O(gEk^2 + BC)$	$O(S + E + g(n + k^2))$

The inclusion–exclusion bound evaluates the formulas of Theorems 11 and 12 by processing subsets consecutively without storing all intersection terms. For orbit counting alone, the Burnside fixed-space formulas in Theorem 13 have the upper bound $O(g2^t D)$ when evaluated separately for every group element. This computes an orbit count but does not construct orbit representatives. Generating the automorphism group is excluded from these bounds. An exhaustive group search may inspect up to $(q-1)^n n!$ monomial matrices.

Inclusion–exclusion is attractive for exact counting when $2^t D < EC$ and the minimum-weight set is available. Its exponential dependence on t can dominate when many minimum-weight projective classes occur. Orbit reduction can save expensive candidate tests when $B \ll E$ and the cost of generating orbits is smaller than the cost of the tests avoided. The visited set may then require memory proportional to E. For finding only one admissible extension, direct search can stop at its first success, whereas a positive exact count or an orbit count does not by itself supply a successful pair.

5. Binary Symplectic LCD Codes and Quaternary Additive Codes

We now specialize to $q = 2$. The trace-Hermitian description is standard for additive quaternary quantum codes [8,9].

5.1. Binary Symplectic Codes

Over $\mathbb{F}_2$,

$$(\mathbf{Xu})(\mathbf{Zv})^\top - (\mathbf{Zv})(\mathbf{Xu})^\top = (\mathbf{Xu})(\mathbf{Zv})^\top + (\mathbf{Zv})(\mathbf{Xu})^\top.$$

If C is binary symplectic LCD, Theorem 2 becomes $(\mathbf{Xu})^\top \mathbf{M}^{-1} (\mathbf{Zv}) = 0$.

Corollary 13 (Binary LCD-preserving extension)**.** *Let C be a binary symplectic LCD code with $d = d_s(C)$. Then, the following assertions hold.*

(1) *The extension $\overline{C}(\mathbf{u}, \mathbf{v})$ is symplectic LCD with distance d if and only if*

$$(\mathbf{Xu})^\top \mathbf{M}^{-1} (\mathbf{Zv}) = 0, \qquad \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} \neq \emptyset.$$

(2) *The extension* $\overline{C}(\mathbf{u}, \mathbf{v})$ *is symplectic LCD with distance* $d + 1$ *if and only if*

$$(\mathbf{Xu})^{\top}\mathbf{M}^{-1}(\mathbf{Zv}) = 0, \qquad \mathcal{A}_d(C) \cap \mathcal{K}_{\mathbf{u},\mathbf{v}} = \emptyset.$$

Proof. Over $\mathbb{F}_2$, the condition in Theorem 2 is equivalent to $(\mathbf{Xu})^{\top}\mathbf{M}^{-1}(\mathbf{Zv}) = 0$. Combining this with Corollary 3 gives the two assertions. □

5.2. Equivalence with Quaternary Trace-Hermitian ACD Codes

Let

$$\mathbb{F}_4 = \mathbb{F}_2(\omega), \qquad \omega^2 + \omega + 1 = 0,$$

and define the $\mathbb{F}_2$-linear map

$$\phi : \mathbb{F}_2^{2n} \longrightarrow \mathbb{F}_4^n, \qquad \phi(\mathbf{x} \mid \mathbf{z}) = \mathbf{x} + \omega\mathbf{z}.$$

Since $\{1, \omega\}$ is an $\mathbb{F}_2$-basis of $\mathbb{F}_4$, the map ϕ is an $\mathbb{F}_2$-linear bijection.

For an additive code

$$\mathcal{D} \leq_{\mathbb{F}_2} \mathbb{F}_4^n,$$

define the trace-Hermitian inner product by

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathrm{trH}} = \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}\left(\sum_{i=1}^{n} x_i \overline{y_i} \right), \qquad \overline{\alpha} = \alpha^2.$$

Its trace-Hermitian dual is

$$\mathcal{D}^{\perp_{\mathrm{trH}}} = \{\mathbf{y} \in \mathbb{F}_4^n : \langle \mathbf{x}, \mathbf{y} \rangle_{\mathrm{trH}} = 0 \text{ for every } \mathbf{x} \in \mathcal{D}\}.$$

Throughout this subsection, an additive code $\mathcal{D}$ is called *trace-Hermitian additive complementary dual*, abbreviated trace-Hermitian ACD, if

$$\mathcal{D} \cap \mathcal{D}^{\perp_{\mathrm{trH}}} = \{\mathbf{0}\}.$$

For

$$\mathbf{x}, \mathbf{z}, \mathbf{y}, \mathbf{w} \in \mathbb{F}_2^n,$$

we have

$$\overline{\mathbf{y} + \omega\mathbf{w}} = \mathbf{y} + \omega^2\mathbf{w}.$$

Using

$$\mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(1) = 0, \qquad \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(\omega) = \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(\omega^2) = 1,$$

one obtains

$$\begin{aligned} \langle \phi(\mathbf{x} \mid \mathbf{z}), \phi(\mathbf{y} \mid \mathbf{w}) \rangle_{\mathrm{trH}} &= \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}\left(\sum_{i=1}^{n} (x_i + \omega z_i)(y_i + \omega^2 w_i) \right) \\ &= \mathbf{x}\mathbf{w}^{\top} + \mathbf{z}\mathbf{y}^{\top} \\ &= \langle (\mathbf{x} \mid \mathbf{z}), (\mathbf{y} \mid \mathbf{w}) \rangle_s. \end{aligned}$$

Moreover,

$$\mathrm{wt}_H(\phi(\mathbf{x} \mid \mathbf{z})) = \mathrm{wt}_s(\mathbf{x} \mid \mathbf{z}),$$

because the i-th quaternary coordinate $x_i + \omega z_i$ is zero if and only if $(x_i, z_i) = (0, 0)$.

Theorem 14 (Equivalence with quaternary trace-Hermitian ACD codes)**.** *Let*

$$C \subseteq \mathbb{F}_2^{2n}$$

be a binary symplectic code, and put

$$\mathcal{D} = \phi(C) \subseteq \mathbb{F}_4^n.$$

Then,

$$|\mathcal{D}| = 2^{\dim_{\mathbb{F}_2} C}, \qquad d_H(\mathcal{D}) = d_s(C),$$

and

$$\phi(C^{\perp_s}) = \mathcal{D}^{\perp_{\mathrm{trH}}}.$$

Consequently,

$$\phi(\mathrm{Hull}_s(C)) = \mathcal{D} \cap \mathcal{D}^{\perp_{\mathrm{trH}}},$$

and hence

$$h_s(C) = \dim_{\mathbb{F}_2}\big(\mathcal{D} \cap \mathcal{D}^{\perp_{\mathrm{trH}}}\big).$$

In particular,

$$C \text{ is symplectic LCD} \iff \mathcal{D} \text{ is trace-Hermitian ACD}.$$

Conversely, every $\mathbb{F}_2$-additive code $\mathcal{D} \subseteq \mathbb{F}_4^n$ is the image under ϕ of a unique binary linear code $C \subseteq \mathbb{F}_2^{2n}$. Thus, ϕ gives a bijective correspondence between binary symplectic LCD codes and quaternary trace-Hermitian ACD codes.

Proof. The map ϕ is an $\mathbb{F}_2$-linear bijection and preserves weight, so

$$|\mathcal{D}| = |C| = 2^{\dim_{\mathbb{F}_2} C}$$

and

$$d_H(\mathcal{D}) = d_s(C).$$

The identity

$$\big\langle \phi(\mathbf{x} \mid \mathbf{z}), \phi(\mathbf{y} \mid \mathbf{w}) \big\rangle_{\mathrm{trH}} = \langle (\mathbf{x} \mid \mathbf{z}), (\mathbf{y} \mid \mathbf{w}) \rangle_s$$

shows that, for every $\mathbf{c} \in \mathbb{F}_2^{2n}$,

$$\mathbf{c} \in C^{\perp_s} \iff \phi(\mathbf{c}) \in \mathcal{D}^{\perp_{\mathrm{trH}}}.$$

Hence

$$\phi(C^{\perp_s}) = \mathcal{D}^{\perp_{\mathrm{trH}}}.$$

Since ϕ is bijective,

$$\phi(C \cap C^{\perp_s}) = \phi(C) \cap \phi(C^{\perp_s}) = \mathcal{D} \cap \mathcal{D}^{\perp_{\mathrm{trH}}}.$$

The hull-dimension and complementary-duality statements follow. The converse follows from the bijectivity of $\phi : \mathbb{F}_2^{2n} \to \mathbb{F}_4^n$. □

Under ϕ, the binary two-coordinate extension determined by $(\mathbf{u}, \mathbf{v})$ corresponds to appending one quaternary coordinate. Indeed, for a coefficient vector $\mathbf{a} \in \mathbb{F}_2^k$, the appended quaternary symbol is

$$\mathbf{aXu} + \omega\, \mathbf{aZv}.$$

Corollary 14 (One-symbol extension of quaternary trace-Hermitian ACD codes)**.** *Let*

$$\mathcal{D} = \phi(C)$$

be a quaternary trace-Hermitian ACD code corresponding to a binary symplectic LCD code C with symplectic Gram matrix $\mathbf{M}$. The one-symbol additive extension determined by $(\mathbf{u}, \mathbf{v})$ is trace-Hermitian ACD if and only if

$$(\mathbf{Xu})^{\top} \mathbf{M}^{-1} (\mathbf{Zv}) = 0.$$

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Denote this extension by $\overline{\mathcal{D}}(\mathbf{u}, \mathbf{v})$*, and let*

$$d = d_H(\mathcal{D}).$$

Then,

(1) *The condition*

$$d_H(\overline{\mathcal{D}}(\mathbf{u}, \mathbf{v})) = d$$

holds if and only if there exists $\mathbf{a} \in \mathcal{A}_d(C)$ *such that*

$$\mathbf{aXu} + \omega\,\mathbf{aZv} = 0.$$

(2) *The condition*

$$d_H(\overline{\mathcal{D}}(\mathbf{u}, \mathbf{v})) = d + 1$$

holds if and only if

$$\mathbf{aXu} + \omega\,\mathbf{aZv} \neq 0$$

for every $\mathbf{a} \in \mathcal{A}_d(C)$.

Proof. By Theorem 14, the one-symbol quaternary extension is trace-Hermitian ACD precisely when the corresponding binary symplectic extension is LCD. Since the field is binary, Theorem 2 gives

$$(\mathbf{Xu})^{\top}\mathbf{M}^{-1}(\mathbf{Zv}) = 0.$$

Furthermore, because $\{1, \omega\}$ is an $\mathbb{F}_2$-basis of $\mathbb{F}_4$,

$$\mathbf{aXu} + \omega\,\mathbf{aZv} = 0$$

if and only if

$$\mathbf{aXu} = 0 \qquad \text{and} \qquad \mathbf{aZv} = 0.$$

The two distance assertions therefore follow from Corollary 3. □

Corollary 15 (One-symbol trace-Hermitian ACD parameter improvement)**.** *Let*

$$\mathcal{D} = \phi(C)$$

be a quaternary trace-Hermitian ACD code corresponding to a binary symplectic LCD code C*, and let*

$$d = d_H(\mathcal{D}).$$

If there exists $\mathbf{u} \in \mathbb{F}_2^n$ *such that*

$$\mathbf{aXu} \neq 0 \qquad \text{for every } \mathbf{a} \in \mathcal{A}_d(C),$$

then

$$\overline{\mathcal{D}}(\mathbf{u}, \mathbf{0})$$

is a trace-Hermitian ACD code of length $n + 1$*, cardinality* $|\mathcal{D}|$*, and Hamming distance* $d + 1$.

The same conclusion holds for $\overline{\mathcal{D}}(\mathbf{0}, \mathbf{v})$ *if there exists* $\mathbf{v} \in \mathbb{F}_2^n$ *satisfying*

$$\mathbf{aZv} \neq 0 \qquad \text{for every } \mathbf{a} \in \mathcal{A}_d(C).$$

Proof. This is the image under ϕ of Corollary 4. □

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Corollary 16 (Weight distribution of quaternary trace-Hermitian ACD extensions)**.** *Let*

$$\mathcal{D} = \phi(C), \qquad \mathcal{D}^0_{\mathbf{u},\mathbf{v}} = \phi(C^0_{\mathbf{u},\mathbf{v}}).$$

Then, the Hamming weight enumerator of the corresponding one-symbol additive extension satisfies

$$W_{\overline{\mathcal{D}}(\mathbf{u},\mathbf{v})}(Y) = YW_{\mathcal{D}}(Y) + (1-Y)W_{\mathcal{D}^0_{\mathbf{u},\mathbf{v}}}(Y).$$

In particular, the Hamming weight distribution of $\overline{\mathcal{D}}(\mathbf{u},\mathbf{v})$ *is determined by those of* $\mathcal{D}$ *and* $\mathcal{D}^0_{\mathbf{u},\mathbf{v}}$.

Proof. This follows from Theorem 5 and the weight-preserving property of ϕ. □

5.3. Comparison with Quaternary Hermitian LCD Codes

A quaternary Hermitian LCD code is an $\mathbb{F}_4$-linear code $\mathcal{D} \leq \mathbb{F}_4^n$ satisfying $\mathcal{D} \cap \mathcal{D}^{\perp_H} = \{\mathbf{0}\}$, where $\perp_H$ is defined by the Hermitian inner product $\sum_i x_i\overline{y_i}$ [17,18]. A quaternary additive complementary dual code is only required to be $\mathbb{F}_2$-linear and is defined using trace-Hermitian orthogonality [11–13].

If $\mathcal{D}$ is $\mathbb{F}_4$-linear, then $\mathcal{D}^{\perp_H} = \mathcal{D}^{\perp_{\mathrm{trH}}}$. Indeed, trace-Hermitian orthogonality to both $\mathbf{y}$ and $\omega\mathbf{y}$ determines Hermitian orthogonality to $\mathbf{y}$. Hence every $\mathbb{F}_4$-linear Hermitian LCD code is a quaternary additive complementary dual code after forgetting $\mathbb{F}_4$-linearity. The converse need not hold.

Consequently, for fixed length n and cardinality 4^r,

$$d_{\mathrm{ACD}}(n, 2^{2r}) \geq d_{\text{H-LCD}}(n, r),$$

where the left side is the best minimum distance among quaternary additive complementary dual codes of size 2^{2r} and the right side is the best minimum distance among quaternary Hermitian LCD $[n, r]$ codes.

Proposition 2 (Criterion for improving on Hermitian LCD codes)**.** *Let* $C \subseteq \mathbb{F}_2^{2n}$ *be a binary symplectic LCD code with* $\dim_{\mathbb{F}_2} C = 2r$*, and let* $\mathcal{D} = \phi(C) \subseteq \mathbb{F}_4^n$*. If*

$$d_s(C) > d_{\text{H-LCD}}(n, r),$$

then $\mathcal{D}$ *is a quaternary additive complementary dual code of length* n*, size* 4^r*, and minimum distance* $d_s(C)$ *which outperforms every quaternary Hermitian LCD* $[n, r]$ *code. If* $\mathcal{D}$ *is not closed under multiplication by* ω*, then this improvement is attained outside the* $\mathbb{F}_4$*-linear Hermitian LCD class. This is a conditional distance comparison; it does not establish the existence of a code produced by the present construction that satisfies the strict inequality.*

Proof. Theorem 14 gives that $\mathcal{D}$ is additive complementary dual, $|\mathcal{D}| = 2^{2r} = 4^r$, and $d_H(\mathcal{D}) = d_s(C)$. The strict inequality with $d_{\text{H-LCD}}(n, r)$ gives the claimed improvement. If $\omega\mathcal{D} \neq \mathcal{D}$, then $\mathcal{D}$ is not $\mathbb{F}_4$-linear and hence is not a Hermitian LCD linear code. □

Proposition 2 is a benchmark test, not an existence theorem: the proposed extension does not by itself imply a strict distance advantage over the Hermitian LCD class. In particular, failure of $\mathbb{F}_4$-linearity only proves that the constructed code lies outside the Hermitian LCD linear subclass. The explicit length-four comparison is established in Section 6.1.

Remark 4 (Use of numerical tables)**.** *Known ACD parameters from the literature should not be counted as new examples of the present construction. In particular, several low-dimensional ACD parameters in [19] are no longer strict improvements when compared with the updated Hermitian LCD table in [20]. When the current Hermitian LCD entry is an interval*

$$d_{\text{H-LCD}}(n, r) \in [L, U],$$

a computed ACD code with distance $d > L$ improves the best presently displayed Hermitian LCD lower bound, while a strict separation from all Hermitian LCD codes requires $d > U$. Thus, every numerical claim in this paper is certified against the upper endpoint when the table entry is not exact.

The theoretical criteria above also provide a deterministic verification procedure for constructed extensions. For a computed binary symplectic generator matrix $\mathbf{G} = [\mathbf{X} \mid \mathbf{Z}]$ of dimension $2r$, the additive complementary dual property is certified by

$$\operatorname{rank}(\mathbf{G}\mathbf{J}_{2n}\mathbf{G}^{\top}) = 2r,$$

while the minimum distance is certified by exhaustive enumeration of the nonzero coefficient vectors or by an independent distance computation. If comparison with Hermitian LCD codes is intended, the resulting distance must additionally be compared with a rigorously established value or bound for $d_{\text{H-LCD}}(n, r)$.

For an extension produced from a non-LCD mother code, write $d_i = d_s(C_i)$ and

$$\mathcal{K}_i = \{\mathbf{a} \in \mathbb{F}_2^k : \mathbf{a}\mathbf{X}_{i-1}\mathbf{u}_i = \mathbf{a}\mathbf{Z}_{i-1}\mathbf{v}_i = 0\}.$$

The corresponding extension is certified to reduce the hull by two and increase the distance by one whenever

$$\operatorname{rank}[\mathbf{K}_{i-1}\mathbf{X}_{i-1}\mathbf{u}_i \; \mathbf{K}_{i-1}\mathbf{Z}_{i-1}\mathbf{v}_i] = 2, \qquad \mathcal{A}_{d_{i-1}}(C_{i-1}) \cap \mathcal{K}_i = \emptyset.$$

These rank and kernel conditions are independent of the search heuristic and can be checked by exhaustive enumeration of the nonzero coefficient vectors. They therefore provide an independent verification layer for any code obtained by the proposed construction.

6. Computational Validation

Computational validation uses exact finite-field arithmetic to test the predictions for structured two-coordinate extensions. For each prescribed mother code, we enumerate all ordered extension parameter pairs and compute the symplectic Gram rank, minimum distance, and weight distribution of the resulting codes. The computed hull and distance frequencies are then compared with the rank and kernel criteria, the weight-enumerator identities, and the exact inclusion–exclusion counts.

The small binary and ternary cases are complemented by explicit families with increasing and mixed hull dimensions, for which every intermediate extension is checked. This evaluates the complete iterative reduction mechanism and records the associated changes in minimum distance. We also compare the final code parameters with published symplectic and Hermitian LCD constructions under matched code length, cardinality, and minimum distance, and examine computational costs as the code dimension and hull dimension grow. Algorithm 2 gives the exhaustive enumeration procedure used in these calculations.

Algorithm 2. Exhaustive enumeration of two-coordinate extensions

The input is a full-row-rank generator matrix $\mathbf{G} = [\mathbf{X} \mid \mathbf{Z}] \in \mathbb{F}_q^{k\times 2n}$ over a finite field. The output records the number of parameter pairs for each resulting hull dimension and minimum symplectic distance.

(1) Compute the initial Gram matrix and precompute the symplectic weight of every nonzero coefficient vector

$$\mathbf{M} = \mathbf{X}\mathbf{Z}^\top - \mathbf{Z}\mathbf{X}^\top, \qquad w_{\mathbf{a}} = \mathrm{wt}_s(\mathbf{a}\mathbf{G}), \quad \mathbf{0} \neq \mathbf{a} \in \mathbb{F}_q^k.$$

Set all output counters to zero and put $d = \min_{\mathbf{a}\neq\mathbf{0}} w_{\mathbf{a}}$.

(2) For every ordered pair of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_q^n$, compute

$$\mathbf{p} = \mathbf{X}\mathbf{u}, \qquad \mathbf{f} = \mathbf{Z}\mathbf{v}, \qquad \mathbf{M}' = \mathbf{M} + \mathbf{p}\mathbf{f}^\top - \mathbf{f}\mathbf{p}^\top.$$

(3) Determine the new hull dimension and minimum distance by exact finite-field arithmetic

$$h' = k - \mathrm{rank}(\mathbf{M}'),$$
$$d' = \min_{\mathbf{0}\neq\mathbf{a}\in\mathbb{F}_q^k} \left(w_{\mathbf{a}} + \mathbf{1}_{\{(\mathbf{ap},\mathbf{af})\neq(0,0)\}} \right).$$

Increase the counter indexed by (h', d') by one. When a weight enumerator is required, count the same updated weights for all coefficient vectors.

(4) After all pairs have been processed, return the counter table and any requested weight enumerators. Check that the sum of the counters is q^{2n}.

The scan retains kernel multiplicities and precomputes the mother-codeword weights once. Classical Gaussian elimination gives the following field-operation and storage bounds

$$T_{\mathrm{pre}} = O(q^k kn),$$
$$T_{\mathrm{scan}} = O\left(q^{2n}(kn + k^3 + q^k k)\right),$$
$$M_{\mathrm{work}} = O(q^k + kn + k^2).$$

The streamed counter table tests finite instances exactly and does not replace the polynomial-time hull-only construction in Algorithm 1.

The first experiment validates the hull-reduction criterion and its exact counting formula. The second experiment validates the LCD-preserving condition, the distance criterion, the weight-enumerator formula, and the exact inclusion-exclusion counts. The third experiment verifies that the hull-reduction and distance-increase mechanism is not restricted to the binary field.

6.1. Binary Non-LCD Mother Code: Simultaneous Hull Reduction and Distance Increase

Consider the binary symplectic code $C_1 \subseteq \mathbb{F}_2^6$ generated by

$$\mathbf{G}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = [\mathbf{X}_1 \mid \mathbf{Z}_1],$$

where

$$\mathbf{X}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{Z}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Its symplectic Gram matrix is

$$\mathbf{M}_1 = \mathbf{X}_1\mathbf{Z}_1^\top - \mathbf{Z}_1\mathbf{X}_1^\top = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$h_s(C_1) = 2.$$

Direct enumeration of the three nonzero coefficient vectors gives

$$d_s(C_1) = 2, \qquad W_{C_1}(Y) = 1 + 3Y^2.$$

For this example, the coefficient hull matrix can be taken as

$$\mathbf{K} = \mathbf{I}_2.$$

Therefore,

$$\mathcal{U} = \mathrm{Im}(\mathbf{K}\mathbf{X}_1) = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

and

$$\mathcal{V} = \mathrm{Im}(\mathbf{K}\mathbf{Z}_1) = \mathbb{F}_2^2.$$

Thus,

$$\alpha = 1, \qquad \beta = 2, \qquad \gamma = 1.$$

The exact hull-reducing count in Proposition 1 is

$$N_2(1, 2, 1) = 2^3 - 2^1 - 2^2 - (2 - 1)2^1 + 2 = 2.$$

Since each element of $\mathcal{U}$ has $2^{3-1} = 4$ preimages under $\mathbf{K}\mathbf{X}_1$ and each element of $\mathcal{V}$ has $2^{3-2} = 2$ preimages under $\mathbf{K}\mathbf{Z}_1$, the predicted number of hull-reducing extension pairs is

$$2^{6-1-2}N_2(1, 2, 1) = 16.$$

We now choose

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then,

$$\mathbf{p} = \mathbf{X}_1\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{f} = \mathbf{Z}_1\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that

$$\mathrm{rank}[\mathbf{p}\ \mathbf{f}] = 2.$$

The extended generator matrix is

$$\overline{\mathbf{G}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Its Gram matrix is

$$\overline{\mathbf{M}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which has rank two over $\mathbb{F}_2$. Hence

$$h_s(\overline{C}_1) = 0.$$

A direct enumeration of the three nonzero codewords gives

$$d_s(\overline{C}_1) = 3, \qquad W_{\overline{C}_1}(Y) = 1 + 3Y^3.$$

Thus, the construction realizes

$$\boxed{[6,2,2]_2 \longrightarrow [8,2,3]_2}$$

with the final code symplectic LCD.

A fixed-mother-code comparison isolates the effect of using two independently chosen appended columns. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard basis of $\mathbb{F}_2^3$. Zero padding, one-sided parity-coordinate extension, and repetition of one original symplectic coordinate pair have the respective generator matrices

$$\begin{aligned}
\mathbf{G}_{\text{pad}} &= [\mathbf{X}_1 \mid \mathbf{0} \mid \mathbf{Z}_1 \mid \mathbf{0}], \\
\mathbf{G}_{\text{X}} &= [\mathbf{X}_1 \mid \mathbf{X}_1\mathbf{e}_3 \mid \mathbf{Z}_1 \mid \mathbf{0}], \\
\mathbf{G}_{\text{Z}} &= [\mathbf{X}_1 \mid \mathbf{0} \mid \mathbf{Z}_1 \mid \mathbf{Z}_1\mathbf{e}_2], \\
\mathbf{G}_{\text{rep},j} &= [\mathbf{X}_1 \mid \mathbf{X}_1\mathbf{e}_j \mid \mathbf{Z}_1 \mid \mathbf{Z}_1\mathbf{e}_j], \qquad 1 \le j \le 3.
\end{aligned}$$

In the first three cases, at least one appended column is zero. For every repeated pair, the same property follows from $\mathbf{X}_1\mathbf{e}_j = \mathbf{0}$ or $\mathbf{Z}_1\mathbf{e}_j = \mathbf{0}$. Hence the Gram perturbation vanishes and all these extensions retain hull dimension 2. Since all three nonzero words of C_1 have symplectic weight 2, a single appended linear functional has a nonzero coefficient vector in its kernel, leaving the minimum distance equal to 2.

By contrast, the two appended columns $\mathbf{X}_1\mathbf{e}_3$ and $\mathbf{Z}_1\mathbf{e}_2$ are independent, so the appended coordinate pair is nonzero on every nonzero codeword. The same mother code therefore yields

$$\begin{aligned}
(h_s, d_s)_{\text{pad}} = (h_s, d_s)_{\text{X}} = (h_s, d_s)_{\text{Z}} = (h_s, d_s)_{\text{rep},j} = (2,2), \\
(h_s, d_s)_{\text{two-block}} = (0,3).
\end{aligned}$$

The unrestricted two-column family also contains this successful pair. It has 16 distinct appended-column pairs, whereas the blockwise restricted family has 8, of which exactly 2 produce both a zero hull and distance 3.

The complete enumeration of all

$$2^3 \cdot 2^3 = 64$$

extension pairs provides a stronger validation. The results are shown in Table 2.

Table 2. Exhaustive enumeration for the binary non-LCD mother code C_1.

Rank$[\mathbf{p}\ \mathbf{f}]$	Number of Pairs	$h_s(\overline{C}_1)$	$d_s(\overline{C}_1)$
0	8	2	2
1	40	2	2
2	16	0	3
Total	64	–	–

The sixteen pairs with rank two are exactly the pairs predicted by Proposition 1. Moreover, every one of them simultaneously increases the minimum distance from two to three. This example therefore validates both the rank-two hull-reduction criterion and the compatibility of the hull and distance conditions.

The same mother code also gives an explicit orbit reduction under Theorem 13. Let

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Direct multiplication gives $\mathbf{A}\mathbf{X}_1 = \mathbf{X}_1\mathbf{P}$ and $\mathbf{A}\mathbf{Z}_1 = \mathbf{Z}_1\mathbf{P}$. Since the nonzero columns of $\mathbf{X}_1$ occur only in its third position, any permutation in $\Gamma_s(C_1)$ fixes this position. These are the only two possibilities over the binary field, giving

$$\Gamma_s(C_1) = \{\mathbf{I}_3, \mathbf{P}\}.$$

Writing $\mathbf{u} = (u_1, u_2, u_3)^\top$ and $\mathbf{v} = (v_1, v_2, v_3)^\top$, the joint hull-reduction and distance-increase condition is

$$u_3 = 1, \qquad v_1 + v_2 = 1.$$

There are 16 successful parameter pairs, none fixed by $\mathbf{P}$. Indeed, a fixed pair has $v_1 = v_2$, which contradicts the displayed condition. Burnside's formula therefore gives

$$|\Omega_{\mathrm{red},3}/\Gamma_s(C_1)| = \frac{16 + 0}{2} = 8.$$

On the full parameter space, $\mathbf{P}$ fixes 16 of the 64 pairs. Burnside's formula gives

$$|(\mathbb{F}_2^3 \times \mathbb{F}_2^3)/\Gamma_s(C_1)| = \frac{64 + 16}{2} = 40.$$

Orbit representatives reduce the number of candidate tests from 64 to 40, or from 16 to 8 within the successful parameter set.

Parameter orbits need not represent distinct codes. For this mother code, the 64 parameters produce only 8 distinct appended-column pairs, including 2 successful column pairs. The induced automorphism action identifies these two successful column pairs, leaving one effective-column orbit. The parameter-orbit count therefore measures a guaranteed elimination of equivalent parameter choices rather than the number of inequivalent extended codes.

The binary-to-quaternary correspondence also gives a nontrivial ACD example. Under

$$\phi(\mathbf{x} \mid \mathbf{z}) = \mathbf{x} + \omega\mathbf{z},$$

the two generator rows of $\overline{\mathbf{G}}_1$ correspond to

$$(\omega, \omega, 0, \omega), \qquad (0, \omega, 1, \omega^2).$$

Hence, the resulting additive code is

$$\mathcal{D}_1 = \mathrm{span}_{\mathbb{F}_2}\{(\omega, \omega, 0, \omega), (0, \omega, 1, \omega^2)\} \subseteq \mathbb{F}_4^4.$$

It has cardinality 4 and Hamming weight enumerator

$$W_{\mathcal{D}_1}(Y) = 1 + 3Y^3.$$

Since the binary Gram matrix $\overline{\mathbf{M}}_1$ is nonsingular, Theorem 14 shows that $\mathcal{D}_1$ is additive complementary dual. Moreover,

$$\omega(\omega, \omega, 0, \omega) = (\omega^2, \omega^2, 0, \omega^2) \notin \mathcal{D}_1,$$

so $\mathcal{D}_1$ is not $\mathbb{F}_4$-linear. This confirms that the construction naturally produces additive complementary dual codes that need not belong to the $\mathbb{F}_4$-linear Hermitian LCD class.

This difference in linearity does not imply a strict distance advantage. Consider the parameter-matched Hermitian LCD code

$$\mathcal{D}_{\mathrm{H}} = \mathrm{span}_{\mathbb{F}_4}\{(1,1,1,0)\} \subseteq \mathbb{F}_4^4.$$

Its generator has Hermitian self-inner product $1+1+1=1$ in $\mathbb{F}_4$, so this one-dimensional code is Hermitian LCD. Its three nonzero words all have weight 3. Moreover, for every vector over $\mathbb{F}_4$, the Hermitian self-inner product equals its Hamming weight reduced modulo 2. Hence, a one-dimensional Hermitian LCD code of length four must have odd minimum distance, which is at most three. We obtain the exact benchmark

$$d_{\text{H-LCD}}(4,1) = 3 = d_H(\mathcal{D}_1).$$

Thus, the constructed additive code and the Hermitian LCD code have the same length, cardinality, and minimum distance. The example establishes a different algebraic structure, not a strict improvement in distance.

In fact, distance four is impossible even among all trace-Hermitian ACD codes of length four and cardinality four. Suppose that $\mathcal{E} = \{\mathbf{0}, \mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b}\}$ were such an ACD code with minimum distance four. For each coordinate i, all of a_i, b_i, a_i+b_i would be nonzero, so $a_i \neq b_i$ and $a_i\overline{b_i} \in \{\omega, \omega^2\}$. Since both of these elements have absolute trace one, $\mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(a_i\overline{b_i}) = 1$ at every coordinate. Therefore,

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathrm{trH}} = \sum_{i=1}^{4} \mathrm{Tr}_{\mathbb{F}_4/\mathbb{F}_2}(a_i\overline{b_i}) = 0.$$

The trace-Hermitian form is alternating, so its restriction to the two-dimensional binary space $\mathcal{E}$ would be identically zero, contradicting the ACD property. Since $\mathcal{D}_1$ attains distance three, this proves the stronger equality

$$d_{\mathrm{ACD}}(4,4) = d_{\text{H-LCD}}(4,1) = 3.$$

Larger initial hulls can be tested using disjoint copies of the same binary mother code. For a positive integer r, let

$$\mathbf{X}_0^{(r)} = \mathrm{diag}(\underbrace{\mathbf{X}_1, \dots, \mathbf{X}_1}_{r}), \qquad \mathbf{Z}_0^{(r)} = \mathrm{diag}(\underbrace{\mathbf{Z}_1, \dots, \mathbf{Z}_1}_{r}),$$
$$\mathbf{G}_0^{(r)} = [\mathbf{X}_0^{(r)} \mid \mathbf{Z}_0^{(r)}].$$

The initial Gram matrix vanishes and the projected column-space ranks are $\mathrm{rank}(\mathbf{X}_0^{(r)}) = r$ and $\mathrm{rank}(\mathbf{Z}_0^{(r)}) = 2r$. Thus, Theorem 4 guarantees a full reduction from hull dimension $2r$. At step i, use the parameters

$$\mathbf{u}_i = \begin{pmatrix} \mathbf{e}_{3i} \\ \mathbf{0}_{i-1} \end{pmatrix}, \qquad \mathbf{v}_i = \begin{pmatrix} \mathbf{e}_{3i-1} \\ \mathbf{0}_{i-1} \end{pmatrix}, \qquad 1 \le i \le r,$$

where the standard basis vectors have length $3r$. Each pair modifies only one constituent block and turns its zero Gram block into the nonsingular matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Consequently, after j steps, the exact hull dimension and weight enumerator are

$$h_s(C_j^{(r)}) = 2(r-j), \qquad W_{C_j^{(r)}}(Y) = (1+3Y^3)^j(1+3Y^2)^{r-j}.$$

The explicit factorization gives minimum distance two until the last step and distance three at completion. The resulting parameter change is

$$[6r, 2r, 2]_2^{\mathrm{s}} \longrightarrow [8r, 2r, 3]_2^{\mathrm{s}}.$$

Independent enumeration over $\mathbb{F}_2$ of every nonzero codeword at every step, together with Gram-rank calculations, gives the following results. The exact validation results are presented in Table 3.

Table 3. Exact multi-step hull and distance verification for larger binary mother codes.

r	h_0	Steps	Initial $[2n, k, d]$	Final $[2n, k, d]$	h_{final}	$\lvert\mathcal{C}\rvert$
2	4	2	[12, 4, 2]	[16, 4, 3]	0	16
3	6	3	[18, 6, 2]	[24, 6, 3]	0	64
4	8	4	[24, 8, 2]	[32, 8, 3]	0	256
6	12	6	[36, 12, 2]	[48, 12, 3]	0	4096
8	16	8	[48, 16, 2]	[64, 16, 3]	0	65,536

A mixed-hull family is obtained by adjoining a nondegenerate binary constituent to the preceding construction. Define

$$\mathbf{X}_{\mathrm{L}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z}_{\mathrm{L}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The associated two-dimensional code has hull dimension zero and weight enumerator $1 + 3Y^3$. For $r \geq 2$, let

$$\mathbf{X}_{\mathrm{mix}}^{(r)} = \mathrm{diag}(\mathbf{X}_0^{(r)}, \mathbf{X}_{\mathrm{L}}),$$
$$\mathbf{Z}_{\mathrm{mix}}^{(r)} = \mathrm{diag}(\mathbf{Z}_0^{(r)}, \mathbf{Z}_{\mathrm{L}}).$$

This mother code has $k = 2r + 2$ and $n = 3r + 3$. A basis of its coefficient hull consists of the first $2r$ standard coordinate vectors, giving

$$h_s(C_{\mathrm{mix},0}^{(r)}) = 2r < k,$$
$$\mathrm{rank}(\mathbf{K}_0 \mathbf{X}_{\mathrm{mix}}^{(r)}) = r,$$
$$\mathrm{rank}(\mathbf{K}_0 \mathbf{Z}_{\mathrm{mix}}^{(r)}) = 2r.$$

Applying the same successive extension pairs to the first r constituents leaves the nondegenerate block unchanged. After j steps, exact Gram-rank and weight calculations give

$$h_s(C_{\mathrm{mix},j}^{(r)}) = 2(r - j),$$
$$W_{C_{\mathrm{mix},j}^{(r)}}(Y) = (1 + 3Y^3)^{j+1}(1 + 3Y^2)^{r-j}.$$

The minimum distance remains two while an unextended non-LCD constituent remains and becomes three after the final step. Independent enumeration of every nonzero codeword and all intermediate Gram matrices gives the following mixed-hull results. The corresponding results are reported in Table 4.

Table 4. Exact iterative validation for mixed-hull binary mother codes.

r	k	h_0	Steps	$n_0 \to n_r$	h_r	$d_0 \to d_r$
2	6	4	2	$9 \to 11$	0	$2 \to 3$
3	8	6	3	$12 \to 15$	0	$2 \to 3$
4	10	8	4	$15 \to 19$	0	$2 \to 3$
6	14	12	6	$21 \to 27$	0	$2 \to 3$
8	18	16	8	$27 \to 35$	0	$2 \to 3$

Note: The arrows indicate the change from the initial code to the code after all extension steps.

A further mixed mother code couples the first non-LCD constituent and the LCD constituent through one additional coordinate. Starting from $r = 2$, define

$$\mathbf{X}_\star = [\mathbf{X}_{\mathrm{mix}}^{(2)} \mid \mathbf{e}_1 + \mathbf{e}_5], \qquad \mathbf{Z}_\star = [\mathbf{Z}_{\mathrm{mix}}^{(2)} \mid \mathbf{0}].$$

The added coordinate contributes zero to the Gram matrix. Applying the two original extension pairs to this length-ten mother code produces the independently verified sequences $h_s = 4, 2, 0$ and $d_s = 2, 2, 3$, with fixed dimension $k = 6$.

The mixed mother code with $r = 2$ also permits a higher-dimensional orbit calculation. Let $\mathbf{P}$ exchange its first two blocks of three coordinate positions, fixing the last block. Exchanging the corresponding two pairs of generator rows gives a matrix $\mathbf{A}$ with $\mathbf{A}\mathbf{X}^{(2)}_{\mathrm{mix}} = \mathbf{X}^{(2)}_{\mathrm{mix}}\mathbf{P}$ and $\mathbf{A}\mathbf{Z}^{(2)}_{\mathrm{mix}} = \mathbf{Z}^{(2)}_{\mathrm{mix}}\mathbf{P}$. Thus, $H = \{\mathbf{I}_9, \mathbf{P}\}$ is a subgroup of $\Gamma_s(C^{(2)}_{\mathrm{mix},0})$. The parameter space has 2^{18} pairs, of which 2^{12} are fixed by the block exchange. Simultaneous one-step hull reduction and distance increase occur exactly when

$$u_3 = u_6 = 1, \qquad v_1 + v_2 = v_4 + v_5 = 1.$$

There are 2^{14} such successful pairs and 2^{10} are fixed by $\mathbf{P}$. The Burnside formula of Theorem 13 applied to this subgroup gives

$$|(\mathbb{F}_2^9 \times \mathbb{F}_2^9)/H| = \frac{2^{18} + 2^{12}}{2} = 133120,$$
$$|\Omega_{\mathrm{red},3}/H| = \frac{2^{14} + 2^{10}}{2} = 8704.$$

The ranks of the two generator blocks are 3 and 5, respectively. Consequently, the 2^{18} parameter pairs determine $2^8 = 256$ distinct appended-column pairs, with 2^{10} parameter preimages per column pair. The subgroup reduces the 256 effective column pairs to 144 orbits and the 16 successful effective pairs to 12 orbits. These are exact orbit counts under a specified subgroup of the mother-code automorphism group.

A parameter-matched comparison can be made with the symplectic Plotkin-sum construction [16]. Let $\mathbf{H}_r \in \mathbb{F}_2^{r\times 4r}$ have disjoint rows, each containing three consecutive ones followed by a zero. Its Euclidean Gram matrix is the identity. The Plotkin-sum generator

$$\mathbf{G}_{\mathrm{P}} = \begin{pmatrix} \mathbf{H}_r & \mathbf{H}_r \\ 0 & \mathbf{H}_r \end{pmatrix}$$

therefore has a nonsingular symplectic Gram matrix and minimum symplectic distance three by the Plotkin-sum LCD criterion. It gives the same parameters $[8r, 2r, 3]_2^{\mathrm{s}}$ as the completed extension above. Viewing $\mathbf{H}_r$ over $\mathbb{F}_4$ also yields a Hermitian LCD code because $\mathbf{H}_r\overline{\mathbf{H}}_r^{\top} = \mathbf{I}_r$. This quaternary linear code has parameters $[4r, r, 3]_4$, cardinality 4^r and Hamming distance three, matching the quaternary additive image of the proposed extension in length, cardinality and distance [18].

The matching minimum distances distinguish parameter performance from the effect of the construction. The Plotkin and Hermitian benchmarks directly generate LCD codes, whereas the proposed sequence starts with a specified code of hull dimension $2r$ and minimum distance two, preserves its dimension, and reaches the same LCD parameters through r controlled extensions. These direct-sum tests establish multi-step feasibility and distance improvement without asserting an asymptotic distance gain.

6.2. Binary LCD Mother Code: LCD Preservation, Distance Increase, and Exact Counting

We next consider the binary symplectic LCD code $C_2 \subseteq \mathbb{F}_2^8$ generated by

$$\mathbf{G}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} = [\mathbf{X}_2 \mid \mathbf{Z}_2].$$

Its Gram matrix is

$$\mathbf{M}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has rank four. Hence,

$$h_s(C_2) = 0.$$

Direct enumeration gives

$$d_s(C_2) = 2$$

and

$$W_{C_2}(Y) = 1 + 3Y^2 + 6Y^3 + 6Y^4.$$

The projective minimum-weight coefficient set is

$$\mathbb{P}A_2(C_2) = \{(0,0,1,1), (1,0,0,1), (1,1,0,1)\}.$$

Take the one-sided extension

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then,

$$\mathbf{X}_2\mathbf{u} = \mathbf{0}, \qquad \mathbf{Z}_2\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The Gram matrix is unchanged:

$$\overline{\mathbf{M}}_2 = \mathbf{M}_2,$$

so the extended code remains symplectic LCD. For each of the three projective minimum-weight coefficient vectors,

$$\mathbf{a}\mathbf{Z}_2\mathbf{v} = 1.$$

Hence, no minimum-weight codeword remains unchanged in symplectic weight, and Corollary 4 gives

$$d_s(\overline{C}_2) = 3.$$

The extended generator matrix is

$$\overline{\mathbf{G}}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The kernel subcode defined by the appended pair has weight enumerator

$$W_{C^0_{\mathbf{u},\mathbf{v}}}(Y) = 1 + 4Y^3 + 3Y^4.$$

Consequently, Theorem 5 predicts

$$\begin{aligned} W_{\overline{C}_2}(Y) &= YW_{C_2}(Y) + (1 - Y)W_{C^0_{\mathbf{u},\mathbf{v}}}(Y) \\ &= 1 + 7Y^3 + 5Y^4 + 3Y^5. \end{aligned}$$

Direct enumeration of all 16 codewords of the extended code gives exactly the same weight enumerator.

We next enumerate all

$$2^4 \cdot 2^4 = 256$$

extension pairs. The resulting counts are summarized in Table 5.

Table 5. Exhaustive enumeration for the binary symplectic LCD mother code C_2.

Property of the Extension	Number of Pairs
All extension pairs	256
Distance 3	108
Distance 2	148
Symplectic LCD	136
Non-LCD	120
Symplectic LCD and distance 3	48
Symplectic LCD and distance 2	88

The exact inclusion-exclusion formulas independently reproduce these counts. In particular,

$$N_{\mathbf{X},3} = 2, \qquad N_{\mathbf{Z},3} = 2, \qquad N_3 = 108, \qquad N_{\mathrm{LCD},3} = 48.$$

Furthermore,

$$\mathbf{T} = \mathbf{X}_2^{\top} \mathbf{M}_2^{-1} \mathbf{Z}_2$$

has rank four. Hence, Lemma 3 gives

$$\Lambda_2(4,4,4) = (2^4 - 1)2^3 = 120,$$

which is exactly the number of extension pairs failing the LCD condition in the exhaustive enumeration. Thus, the LCD criterion and the exact distance-counting formula agree independently with direct enumeration.

For this mother code, the projective minimum-weight set contains $t = 3$ elements, so each inclusion–exclusion formula has only $2^3 = 8$ intersection terms, compared with $2^8 = 256$ effective appended-column pairs in a direct scan. The computed counts $N_3 = 108$ and $N_{\mathrm{LCD},3} = 48$ agree with both procedures. Each intersection term requires rank calculations, so the term count alone is not a wall-clock speed comparison.

6.3. Ternary Validation

To verify that the preceding mechanism is not restricted to characteristic two, consider the ternary symplectic code $C_3 \subseteq \mathbb{F}_3^4$ generated by

$$\mathbf{G}_3 = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 2 & 0 & 0 \end{pmatrix} = [\mathbf{X}_3 \mid \mathbf{Z}_3].$$

Its Gram matrix is zero:

$$\mathbf{M}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and therefore

$$h_s(C_3) = 2.$$

Direct enumeration gives

$$d_s(C_3) = 2, \qquad W_{C_3}(Y) = 1 + 8Y^2.$$

Again, $\mathbf{K} = \mathbf{I}_2$. We have

$$\mathrm{Im}(\mathbf{K}\mathbf{X}_3) = \mathbb{F}_3^2, \qquad \mathrm{Im}(\mathbf{K}\mathbf{Z}_3) = \mathrm{span}\left\{\begin{pmatrix}2\\0\end{pmatrix}\right\}.$$

Hence,

$$\alpha = 2, \qquad \beta = 1, \qquad \gamma = 1.$$

The exact image-space count is

$$N_3(2,1,1) = 3^3 - 3^2 - 3 - (3-1)3 + 3 = 12.$$

Each pair of image vectors has $3^{2-2} = 1$ and $3^{2-1} = 3$ preimages, respectively. Thus, Proposition 1 predicts

$$3^{4-2-1}N_3(2,1,1) = 3 \cdot 12 = 36$$

hull-reducing extension pairs.

Choose

$$\mathbf{u} = \mathbf{v} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Then,

$$\mathbf{X}_3\mathbf{u} = \begin{pmatrix}2\\2\end{pmatrix}, \qquad \mathbf{Z}_3\mathbf{v} = \begin{pmatrix}2\\0\end{pmatrix},$$

which are linearly independent over $\mathbb{F}_3$. The extended generator matrix is

$$\overline{\mathbf{G}}_3 = \begin{pmatrix}0 & 2 & 2 & 1 & 2 & 2\\ 2 & 2 & 2 & 0 & 0 & 0\end{pmatrix},$$

with Gram matrix

$$\overline{\mathbf{M}}_3 = \begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}.$$

This matrix has rank two over $\mathbb{F}_3$, so

$$h_s(\overline{C}_3) = 0.$$

Moreover,

$$d_s(\overline{C}_3) = 3, \qquad W_{\overline{C}_3}(Y) = 1 + 8Y^3.$$

Thus,

$$\boxed{[4,2,2]_3 \longrightarrow [6,2,3]_3}$$

is obtained.

Finally, exhaustive enumeration of all

$$3^2 \cdot 3^2 = 81$$

extension pairs gives

rank$[\mathbf{p}\ \mathbf{f}]$	Number of pairs	$h_s(\overline{C}_3)$	$d_s(\overline{C}_3)$
0	3	2	2
1	42	2	2
2	36	0	3

which agrees exactly with the predicted hull-reducing count of 36.

The selection cost can be measured without enumerating codewords or distance-forbidden pairs. For the binary experiments below, take $n = k$ and use the full-row-rank family

$$\mathbf{X}_0 = \mathbf{I}_k, \qquad \mathbf{Z}_0 = \mathbf{I}_k + \sum_{j=1}^{(k-h)/2} \mathbf{e}_{2j-1}\mathbf{e}_{2j}^{\top}.$$

Here $k - h$ is even and the sum is empty when $h = k$. Direct Gram-rank calculation gives $h_s(C_0) = h$ and $\alpha_0 = \beta_0 = h$, so the full reduction algorithm succeeds in each case. The measurements include the initial hull and image computations, the selection of all preimages and construction of the successive generator matrices. They exclude the independent rank checks used to verify the output. The measured running times are listed in Table 6.

Table 6. Measured running times for deterministic binary iterative LCD construction.

$k = n$	h	r	$\lvert C_0 \rvert$	Median Time in ms
8	4	2	2^8	1.3
16	4	2	2^{16}	1.8
32	4	2	2^{32}	3.7
64	4	2	2^{64}	13.3
128	4	2	2^{128}	90.5
64	2	1	2^{64}	14.7
64	8	4	2^{64}	15.9
64	16	8	2^{64}	17.7
64	32	16	2^{64}	35.3
64	64	32	2^{64}	87.5

Each entry is a median over repeated runs using Python 3.13.5 and NumPy 2.3.5 on an Intel Xeon Platinum 8573C processor. The runs were repeated 35 times for $k \leq 32$, 21 times for $k = 64$ and 13 times for $k = 128$. The corresponding first-step exhaustive search spaces range from 2^{16} to 2^{256}. For fixed $k = 64$, increasing the initial hull dimension from 2 to 64 raises the required number of extensions from 1 to 32. Exact arithmetic also verified the intermediate Gram ranks on 18 feasible structured instances and 426 random full-row-rank instances over $\mathbb{F}_2$ and $\mathbb{F}_3$.

6.4. Summary of the Computational Validation

The binary and ternary enumerations agree with the hull, distance and counting formulas. The larger-hull and mixed-hull families verify repeated reduction through initial hull dimension sixteen, while the mixed instance also gives exact parameter and effective-column orbit counts. Parameter-matched Plotkin and Hermitian constructions attain the same final distance on the direct-sum family. The experiments therefore establish the controlled transformation of prescribed non-LCD mother codes into LCD codes with unchanged dimension, including cases with a nondegenerate constituent.

7. Conclusions and Further Directions

This paper developed a systematic framework for the nonstandard two-coordinate symplectic extension $\overline{\mathbf{G}}(\mathbf{u}, \mathbf{v}) = [\mathbf{X} \mid \mathbf{Xu} \mid \mathbf{Z} \mid \mathbf{Zv}]$. The central structural feature is that the associated symplectic Gram matrix is modified by an alternating perturbation of rank at most two. This implies that the symplectic hull dimension can change only by -2, 0, or $+2$. The exact alternatives are determined first by the rank of $[\mathbf{KXu}\ \mathbf{KZv}]$; in the remaining rank-zero case, the scalar λ on a nondegenerate complement distinguishes hull preservation from a two-dimensional hull increase. In particular, the condition $\operatorname{rank}[\mathbf{KXu}\ \mathbf{KZv}] = 2$ characterizes a one-step reduction of the hull dimension by two. For an initial hull of dimension $2r$, a complete sequence of r successive reductions exists precisely when $\operatorname{rank}(\mathbf{K}_0\mathbf{X}_0) \geq r$ and $\operatorname{rank}(\mathbf{K}_0\mathbf{Z}_0) \geq r$. This criterion does not by itself guarantee simultaneous distance increases. For an initial symplectic LCD code, we also obtained an exact criterion for preserving the LCD property under the extension.

The metric behavior of the construction is similarly rigid. A single extension either preserves the minimum symplectic distance or increases it by exactly one, according to

whether a minimum-weight coefficient vector lies in the corresponding extension kernel. This leads to forbidden-pair descriptions, a weight-enumerator identity, and criteria for simultaneous control of the hull and the minimum distance. Beyond existence results, exact counting formulas were derived for hull-reducing and distance-improving extension pairs. Inclusion–exclusion gives exact counts under simultaneous constraints, while orbit reduction removes equivalent parameter pairs whenever suitable split symplectic automorphisms of the mother code are available.

In the binary case, the standard correspondence between binary symplectic codes and quaternary additive codes transfers these results to the trace-Hermitian setting. The proposed two-coordinate binary extension becomes a one-symbol extension over $\mathbb{F}_4$, and the hull, LCD, distance, and weight-enumerator criteria yield corresponding statements for quaternary trace-Hermitian additive complementary dual codes. Since $\mathbb{F}_4$-linearity is not required, the construction applies to additive codes beyond the Hermitian LCD subclass. Comparisons with Hermitian LCD codes should therefore be interpreted through the conditional benchmark criterion established in this paper and require reliable bounds or optimum values for the corresponding parameters.

The explicit length-four additive example matches the optimal Hermitian LCD minimum distance, and the present results do not establish a strict distance advantage over this subclass.

The exhaustive experiments over $\mathbb{F}_2$ and $\mathbb{F}_3$ provide finite-instance validation of the theoretical results. They reproduce the predicted hull transitions, LCD-preserving behavior, distance changes, weight enumerators, and exact counting formulas. In particular, the ternary example confirms that the hull-reduction mechanism is not restricted to characteristic two. These computations are intended to validate the theory rather than to assert new best-known code parameters.

Several directions remain open. It would be useful to develop more efficient algorithms that combine the exact counting formulas with automorphism orbits for large extension spaces. Another natural problem is to extend the present framework of alternating perturbations of rank at most two to higher-dimensional structured extensions and determine whether similarly precise hull-variation and distance criteria can be obtained. For cyclic, constacyclic, quasi-cyclic, and related structured families, polynomial or module representations may lead to more explicit algebraic versions of the present conditions. Finally, a systematic study of the resulting quaternary trace-Hermitian ACD codes, including their $\mathbb{F}_4$-linearity, equivalence classes, and comparison with rigorously established Hermitian LCD benchmarks, may further clarify the additional possibilities provided by the additive setting.

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