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Application of the Boundary Control Method to the Dynamical Inverse Problem for the Goupillaud Model

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Abstract

We apply Belishev's Boundary Control method to the dynamical inverse problem for the Goupillaud model—a one-dimensional discrete layered medium with equal two-way wave travel times—providing a rigorous operator setting for this problem. The inverse problem consists of determining the layer reflection coefficients from the system response to an impulsive excitation. Since the model is classically specified as an input–output system without an explicit state space, we construct its operator realization as a controlled system, which is necessary for the applicability of this method. To achieve a unified description, we introduce a boundary parameter $\epsilon \in [-1, 1]$ that unifies the free ($\epsilon = 1$) and non-free ($\epsilon = 0$) surface cases, while accommodating intermediate impedance loads. Our central result is an explicit formula, valid for any ϵ, that expresses the connecting operator through the response operator. We show that this representation leads to the Krein approach to the inverse problem, whereas its factorized form yields the Marchenko approach. Following the Boundary Control method, the inverse problem equations are formulated as a special direct control problem, providing a unified derivation of the discrete Krein and Marchenko equations along with layer-by-layer reconstruction algorithms. This work links control theory and inverse problems for layered media, offering a novel operator interpretation of classical algorithms.

Keywords: boundary control method; Goupillaud model; dynamic inverse problem

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1. Introduction

Dynamical inverse problems for inhomogeneous media arise in wave dynamics, geophysics, and acoustics [1–4]. Their goal is to determine the internal characteristics of a medium from wavefield measurements recorded at its boundaries or at accessible points. One of the basic models is the Goupillaud model, a one-dimensional layered medium in which the two-way travel time of a wave in each internal layer is the same. This model, proposed by Goupillaud in 1961 [5], reduces the inverse problem to recovering reflection coefficients from the impulse response of the medium. Classical approaches to its solution, namely the discrete Krein and Marchenko equations and dynamic deconvolution, are well studied and described, for instance, by Bardan and Robinson [6].

On the other hand, the boundary control method (BC method), proposed by M. I. Belishev [7], is a powerful approach to inverse problems based on its connection with control

theory. In the continuous case, it has been successfully applied to wave problems and various other equations of mathematical physics [8–10]. The key advantage of this approach lies in its applicability to multidimensional dynamical inverse problems [11–13]. In the discrete case, the BC method was developed by A. S. Mikhailov and V. S. Mikhailov for dynamical systems associated with Jacobi matrices [14]; in particular, the authors extended the applicability of the method to systems with an infinite wave propagation speed [15].

However, to the best of our knowledge, the BC method has not yet been applied directly to the Goupillaud model. The present paper fills this gap. Our main motivation is to study this classical discrete wave problem from the perspective of the BC method, with the aim of subsequently using its key techniques in the multidimensional case. To this end, we formulate the Goupillaud model as a controlled dynamical system in the state space and apply the BC method to it. Note that the Goupillaud model is originally specified as an input–output system: its description reduces to the response operator, and the state space is not explicitly present. Since the BC method relies on a control operator, its application is impossible without a state-space realization. Therefore, constructing such a realization is a necessary preliminary step.

In broad terms, this paper develops a unified operator framework for the classical Goupillaud inverse problem by adapting the BC method to this setting. Its key novelty is that the Krein and Marchenko approaches are not treated as separate techniques but are derived—across the entire range of boundary scenarios—from a single connecting operator identity and its factorization. Realized through the Robinson polynomials, this identity and the resulting algebra relating the response and connecting operators place the problem at the interface of inverse problems, control theory, and the theory of orthogonal polynomials on the unit circle. Compared with classical approaches that cover the entire boundary range under consideration using the Toeplitz structure of the autocorrelation matrix [16], and with existing unifying frameworks where the inversion equations are synthesized by analyzing the transfer matrices of cascading layers [17], the proposed state-space framework provides a common operator language for all boundary conditions, offering both an explicit state-space realization of the Goupillaud model and a unified derivation of the classical equations as special cases of a single underlying control-theoretic scheme.

To specify our contribution more precisely, the main results are detailed as follows. First, we introduce a single parameter $\epsilon \in [-1, 1]$ that describes the condition on the upper boundary. The value $\epsilon = 1$ corresponds to a free surface (underwater acoustics), $\epsilon = 0$ to a non-free surface (seismoacoustics), and the remaining values within the interval represent an impedance load. This parametrization allows us to incorporate the full range of boundary conditions into a single operator framework, enabling a unified derivation of the corresponding Krein and Marchenko equations.

Second, we construct a state-space realization of the Goupillaud model [18,19] along with its operator formulation. We introduce the control and state spaces, as well as the control operator relating them. The dynamics operator turns out to be an upper Hessenberg matrix, which is built from the Robinson polynomials and inherits their fundamental properties. In the theory of orthogonal polynomials on the unit circle [20], this matrix plays a role analogous to that of the Jacobi matrix in the theory of orthogonal polynomials on the real line.

Third, the BC method itself is constructed. We introduce the connecting operator and prove a theorem on its representation in terms of the response operator [7]. This representation, a discrete analogue of Belishev's formula for the continuous case, plays a key role in solving the inverse problem [7]. The equations of the inverse problem are then formulated as equations of a special direct control problem.

Fourth, the Krein and Marchenko equations are derived within the framework of the BC method. The Krein equation explicitly links the coefficient vector of the Robinson polynomial to the connecting operator and the cumulative transmission coefficient, whereas the Marchenko equation is formulated via a factorized representation of the connecting operator combined with the coefficient vector of a special polynomial. We demonstrate that for $\epsilon = 1$, the general Krein equation reduces to its classical counterpart from the work of Bardan and Robinson [6], while for $\epsilon = 0$, the Marchenko equation yields the corresponding classical expression from the same study. Thus, the proposed approach not only unifies and generalizes these established results but also provides them with a novel operator-theoretic interpretation.

The paper is organized as follows. Section 2 describes the Goupillaud model, the transfer matrix, and the Robinson polynomials. In Section 3, we construct the dynamical system and introduce the response and control operators. Section 4 formulates the BC method, introduces the connecting operator, and proves the representation theorem. The Krein and Marchenko equations and the corresponding algorithms are also derived in this section. A brief discussion of the results is given in Section 5. Finally, a short Appendix provides supplementary material.

2. Goupillaud Model

The Goupillaud model describes wave dynamics in a semi-infinite layered medium with an arbitrary number of layers, where wave excitation and registration occur at the upper boundary. Since within a fixed observation time the signal can reflect only from a finite number of layers, the original semi-infinite medium can be represented as a filtration structure: a chain of nested finite-layer models.

With the number of layers fixed for now, the present section outlines the key aspects of the Goupillaud model description in the time domain, the z-domain, and the domain of Robinson polynomials. Our approach is largely based on the paper [6], with minor modifications regarding the layer numbering, the specification of boundary conditions, and the definition of the z-transform (which is expressed in terms of positive powers of z). Consequently, slight adjustments have been made to the Robinson polynomials as well.

2.1. Model Description in the Time Domain

Let the layer index range from $l = 0, \dots, L$ (corresponding to a model with $L+1$ layers), and let the boundary index coincide with the index of the underlying layer. Each l-th interface is characterized by reflection coefficients γ_l ($|\gamma_l| < 1$) and transmission coefficients σ_l, which are related by the local energy conservation law $\sigma_l^2 + \gamma_l^2 = 1$. The unit of time is chosen such that the two-way wave travel time in each internal layer equals unity.

The state of the medium is described at discrete times $t = k/2$, $k \in \mathbb{Z}$, by downgoing $u_l^+[t]$ and upgoing $u_l^-[t]$ waves at each interface l. At the internal interfaces ($l = 1, \dots, L$), the continuity relations hold [6]:

$$\begin{aligned} u_l^+[t] &= \sigma_l\, u_{l-1}^+[t - \tfrac{1}{2}] - \gamma_l\, u_l^-[t], \\ u_{l-1}^-[t + \tfrac{1}{2}] &= \gamma_l\, u_{l-1}^+[t - \tfrac{1}{2}] + \sigma_l\, u_l^-[t], \end{aligned} \tag{1}$$

whereas at the external boundaries, the excitation and absorption conditions are given by

$$u_0^+[t] + \epsilon u_0^-[t] = f[t], \quad |\epsilon| \le 1, \tag{2}$$

$$u_L^-[t] = 0. \tag{3}$$

The source $f[t]$ is specified by a finite sequence of pulses f_k arriving at integer times. The model is assumed to be causal: $u_l^{\pm}[t] = 0$ for $t < 0$.

It is worth noting the following points regarding the proposed Goupillaud model description:

1. For a given source, the wavefield vanishes at even-numbered interfaces for half-integer t, and at odd-numbered interfaces for integer t.
2. The upper boundary condition (2) generalizes the models with and without a free surface ($\epsilon = 1$ and $\epsilon = 0$, respectively), which were considered separately in [6]. Here $\epsilon = \gamma_0$ is the reflection coefficient at the upper boundary. The corresponding relation follows from the wave balance at $l = 0$: the downgoing wave $u_0^+[t]$ is the sum of the incident pulse $f[t]$ and the upgoing wave $u_0^-[t]$ reflected with coefficient $-\gamma_0$, so that $u_0^+[t] = f[t] - \gamma_0 u_0^-[t]$.
3. The lower boundary condition (3) does not restrict generality: if the index L is chosen to exceed the observation time by one, the response of the semi-infinite medium coincides with that of the $(L+1)$-layer model.

2.2. Model Description in the z-Domain

To analyze the system algebraically, the time-domain interface conditions can be naturally formulated using the transfer matrix method, which reduces the equations to recurrence relations between the fields at neighboring interfaces. This transition employs the generating function (the formal z-transform)

$$u_l^{\pm}(z) = \sum_{k \in \mathbb{Z}} u_l^{\pm}[k/2] z^{k/2}, \tag{4}$$

and yields the matrix equality

$$\begin{bmatrix} u_l^+(z) \\ u_l^-(z) \end{bmatrix} = T_l(z) \begin{bmatrix} u_{l-1}^+(z) \\ u_{l-1}^-(z) \end{bmatrix}, \quad l = 1, \dots, L, \tag{5}$$

where

$$T_l(z) = \frac{z^{-1/2}}{\sigma_l} \begin{bmatrix} z & -\gamma_l \\ -\gamma_l z & 1 \end{bmatrix}.$$

The matrix $T_l(z)$, which relates the wavefields at neighboring interfaces, is called the *transfer matrix*.

In the z-domain, the conditions on the external boundaries (2) and (3) take the form:

$$u_0^+(z) + \epsilon u_0^-(z) = f(z), \tag{6}$$

$$u_L^-(z) = 0, \tag{7}$$

where the source function $f(z) = \sum_{k=0}^{\infty} f_k z^k$ has a finite number of nonzero coefficients, i.e., it is a polynomial.

The condition $\sigma_l^2 + \gamma_l^2 = 1$ is written as $\det T_l(z) = 1$. Then, relation (5) implies that the quantity $u_l^+(z)u_l^+(z^{-1}) - u_l^-(z)u_l^-(z^{-1})$ is invariant with respect to the index l, which reflects the global energy conservation law in the Goupillaud model.

Remark 1. *As a consequence of Point 1 above, the exponents of z in (4) are, in fact, non-negative integers. This differs from the convention adopted in digital signal processing, where the z-transform is constructed in terms of negative powers of z.*

2.3. Robinson Polynomials

The wave propagation through the first l layers is described by the cumulative transfer matrix $T_{l,1}(z) := T_l(z) \cdots T_1(z)$, which relates the wave fields at the boundaries of this segment:

$$\begin{bmatrix} u_l^+(z) \\ u_l^-(z) \end{bmatrix} = T_{l,1}(z) \begin{bmatrix} u_0^+(z) \\ u_0^-(z) \end{bmatrix}. \tag{8}$$

This matrix can be represented as:

$$T_{l,1}(z) = \frac{z^{-l/2}}{\sigma_{l,1}} \begin{bmatrix} p_l(z) & -q_l(z) \\ -q_l^*(z) & p_l^*(z) \end{bmatrix}, \tag{9}$$

where $\sigma_{l,1} := \prod_{k=1}^{l} \sigma_k$ is the cumulative transmission coefficient. The recurrence relations $T_{l,1}(z) = T_l(z)T_{l-1,1}(z)$ and $\sigma_{l,1} = \sigma_l \sigma_{l-1,1}$ hold under the initial conditions $T_{0,1}(z) = I$ and $\sigma_{0,1} = 1$.

The functions $p_l(z)$ and $q_l(z)$ in (9), as well as their reciprocal forms $p_l^*(z) := z^l p_l(z^{-1})$ and $q_l^*(z) := z^l q_l(z^{-1})$, are polynomials in z and are called the *Robinson polynomials*. They satisfy the recurrence relations:

$$\begin{aligned} p_l(z) &= z p_{l-1}(z) + \gamma_l q_{l-1}^*(z), \\ q_l(z) &= z q_{l-1}(z) + \gamma_l p_{l-1}^*(z), \quad l = 1, \dots, L, \end{aligned} \tag{10}$$

with the initial conditions $p_0(z) = 1$, $q_0(z) = 0$. In explicit form, these polynomials are written as:

$$\begin{aligned} p_l(z) &= z^l + \cdots + \gamma_1 \gamma_l z, \\ q_l(z) &= \gamma_1 z^{l-1} + \cdots + \gamma_l, \quad l = 2, \dots, L. \end{aligned} \tag{11}$$

The unimodularity of the matrices $T_l(z)$ implies the identity:

$$p_l(z)p_l(z^{-1}) - q_l(z)q_l(z^{-1}) = \sigma_{l,1}^2, \quad l = 0, \dots, L, \tag{12}$$

which expresses the global energy conservation law in terms of the Robinson polynomials.

This paper also employs the functions $\phi_l(\epsilon, z) := p_l(z) + \epsilon q_l(z)$, hereafter referred to as the *Robinson ϵ-polynomials*. Relation (11) shows that these are monic polynomials of degree l. Consequently, their ordered set $\phi_0(\epsilon, z), \dots, \phi_{L-1}(\epsilon, z)$ forms a basis in the space $\mathbb{R}_{L-1}[z]$ of polynomials of degree at most $L-1$.

Setting $l = L$ in relation (8) and taking into account the boundary conditions (6) and (7), we find the expression for the response $u_0^-(z)$ to the source signal $f(z)$:

$$u_0^-(z) = \frac{q_L^*(z)}{\phi_L^*(\epsilon, z)} f(z). \tag{13}$$

This formula completely determines the dynamical properties of the model, which are discussed in detail in the next section.

Remark 2. *The Robinson polynomials defined here differ from those introduced in [6]. From a mathematical viewpoint, they represent the same polynomials, but expressed in terms of positive powers of the variable z.*

For $|\epsilon| = 1$, the Robinson ϵ-polynomials reduce to the classical Szegő polynomials orthogonal on the unit circle [20], with the Verblunsky coefficients equal, up to a sign, to the reflection coefficients. This follows because, for this parameter value, they satisfy the Szegő recurrence with the initial condition $\phi_0(1, z) = 1$.

3. Dynamical System

The BC method is based on the representation of a dynamical model as a controlled system in state space [7]. Its basic element is the control operator, which relates input signals (controls) to the internal state of the medium. Since the Goupillaud model is conventionally described in terms of an input–output map (the response operator in the terminology of the BC method), the realization problem arises [18,19,21]—namely, constructing a system of equations that explicitly specifies the relation between the control space and the state space of the model.

3.1. Response Operator

In the Goupillaud model, the role of the input is played by the source signal $f(z)$, and that of the output by the reflected wave signal $u_0^-(z)$. The dynamical system defined by this input–output relation will be referred to as the *Goupillaud system*. According to (13), the transfer function of this system is given by

$$h_L(\epsilon, z) := \frac{q_L^*(z)}{\phi_L^*(\epsilon, z)}. \tag{14}$$

By Corollary A1 in Appendix A, this function is analytic in the disk $|z| < 1$ and therefore expands into a power series in a neighborhood of the origin, thereby determining the impulse response of the system. Since $h_L(\epsilon, 0) = 0$, this series vanishes at the origin and begins directly with the power z.

As the *control space* $\mathcal{F}_L$ of the Goupillaud system, we choose the vector space $\mathbb{R}_{L-1}[z]$ endowed with the standard inner product $\langle \cdot, \cdot \rangle$ on the space of coefficients. The corresponding *response space* is taken to be the quotient ring $\mathbb{R}[[z]]/\langle z^L \rangle$, which is naturally identified with the control space $\mathcal{F}_L$. Under this identification, the *response operator* $R_L(\epsilon) \colon \mathcal{F}_L \to \mathcal{F}_L$ is then defined simply as the multiplication operator by the transfer function in the indicated quotient ring.

When expressed in the standard monomial basis $1, z, \dots, z^{L-1}$ shared by both the control and response spaces, this operator is given by the strictly lower triangular Toeplitz matrix

$$R_L(\epsilon) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ r_1 & 0 & \cdots & 0 & 0 \\ r_2 & r_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{L-1} & r_{L-2} & \cdots & r_1 & 0 \end{bmatrix}, \tag{15}$$

which is composed of the elements of the system's impulse response.

Throughout, the matrices of operators in monomial bases are denoted by the same symbols as the operators themselves.

The Goupillaud model for an arbitrary parameter value $\epsilon \in [-1, 1]$ can be reduced to the canonical cases $\epsilon = 0$ or $\epsilon = 1$ by virtue of the following lemma.

Lemma 1. *The following relation holds:*

$$\begin{aligned} R_L(\epsilon)[I - \epsilon R_L(\epsilon)]^{-1} &= R_L(0), \\ [I - \epsilon R_L(\epsilon)]^{-1} R_L(\epsilon) &= R_L(0). \end{aligned} \tag{16}$$

Proof. We begin with the relation $[1 + \epsilon h_L(0, z)] h_L(\epsilon, z) = h_L(0, z)$, which follows from the definitions of the transfer function (14) and the polynomial $\phi_L(\epsilon, z)$. We expand both sides into power series in z in a neighborhood of the origin. If we restrict ourselves to the L-term truncations of the series, their multiplication in the quotient ring corresponds exactly to the product of lower triangular Toeplitz matrices. Passing to these matrices and isolating $R_L(\epsilon)$ yields (16). □

3.2. Control Operator

Let $\mathbb{R}[z]/\langle \phi_L(\epsilon, z) \rangle$ be the quotient ring of $\mathbb{R}[z]$ modulo the ideal generated by $\phi_L(\epsilon, z)$, and let S_z be the operator of multiplication by z in this ring. We denote by $V_L(\epsilon) \colon \mathbb{R}[z]/\langle \phi_L(\epsilon, z) \rangle \to \mathbb{R}_{L-1}[z]$ the linear isomorphism mapping the basis of normalized Robinson ϵ-polynomials $\{\phi_k(\epsilon, z)/\sigma_{k,1}\}_{k=0}^{L-1}$ onto the monomial basis. This isomorphism induces the action of the operator $A_L(\epsilon) := V_L(\epsilon) S_z V_L^{-1}(\epsilon)$ on the space $\mathbb{R}_{L-1}[z]$.

We define the *state space* $\mathcal{X}_L(\epsilon)$ of the Goupillaud system as the space $\mathbb{R}_{L-1}[z]$ endowed with the inner product $\langle \cdot, \cdot \rangle$ and equipped with the operator $A_L(\epsilon)$. The *control operator* $W_L(\epsilon)\colon \mathcal{F}_L \to \mathcal{X}_L(\epsilon)$ is defined as the composition $W_L(\epsilon) := V_L(\epsilon) \circ \pi|_{\mathcal{F}_L}$, where $\pi|_{\mathcal{F}_L}$ is the restriction of the canonical projection onto the subspace $\mathcal{F}_L$.

The relationship between these operators is illustrated by the commutative diagram in Figure 1. Specifically, the control operator $W_L(\epsilon) = V_L(\epsilon) \circ \pi|_{\mathcal{F}_L}$ sends a control polynomial to the state space $\mathcal{X}_L(\epsilon)$ by first identifying it with an element of the quotient ring via $\pi|_{\mathcal{F}_L}$ and then mapping it into $\mathcal{X}_L(\epsilon)$ via $V_L(\epsilon)$. Under this construction, the mapping $V_L(\epsilon)$ intertwines the shift S_z (multiplication by z) in the quotient ring with the dynamics $A_L(\epsilon)$. Equivalently, the state evolution is directly obtained from the shift in the quotient representation by the same mapping $V_L(\epsilon)$ that defines the control operator.

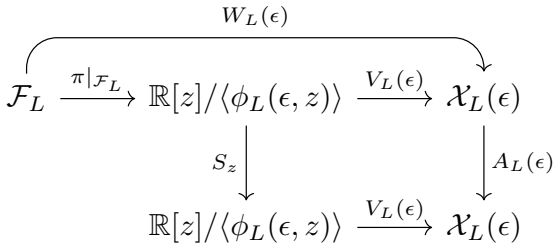


Figure 1. Relationship between the control and dynamics operators. The control operator $W_L(\epsilon)$ maps the control space $\mathcal{F}_L$ into the state space $\mathcal{X}_L(\epsilon)$, while the dynamics operator $A_L(\epsilon)$ corresponds to multiplication by z in $\mathbb{R}[z]/\langle \phi_L(\epsilon, z) \rangle$. The commutativity of the diagram shows that the dynamics operator $A_L(\epsilon)$ corresponds exactly to the shift S_z in the quotient ring transferred to the state space via the isomorphism $V_L(\epsilon)$.

In the monomial basis, the operator S_z is given by the companion matrix of the polynomial $\phi_L(\epsilon, z)$, while the operator $V_L(\epsilon)$ (and, consequently, the control operator $W_L(\epsilon)$) is given by the matrix product

$$V_L(\epsilon) = W_L(\epsilon) = \Sigma_L \Phi_L^{-1}(\epsilon), \tag{17}$$

where $\Sigma_L := \operatorname{diag}(\sigma_{0,1}, \ldots, \sigma_{L-1,1})$ is the diagonal matrix of cumulative transmission coefficients, and $\Phi_L(\epsilon)$ is the transition matrix from the monomial basis to the basis of Robinson ϵ-polynomials (ordinary, non-normalized ones). The columns of the matrix $\Phi_L(\epsilon)$ are formed by the coefficient vectors $\phi_l(\epsilon)$, $l = 0, \ldots, L-1$, of the polynomials $\phi_l(\epsilon, z)$. In explicit form,

$$S_z = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_{L,0} \\ 1 & 0 & \cdots & 0 & -\alpha_{L,1} \\ 0 & 1 & \cdots & 0 & -\alpha_{L,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{L,L-1} \end{bmatrix}, \quad \Phi_L(\epsilon) = \begin{bmatrix} 1 & \alpha_{1,0} & \alpha_{2,0} & \cdots & \alpha_{L-1,0} \\ 0 & 1 & \alpha_{2,1} & \cdots & \alpha_{L-1,1} \\ 0 & 0 & 1 & \cdots & \alpha_{L-1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

3.3. Realization of the Goupillaud System

Theorem 1. *In the state space, the Goupillaud system is realized by the system of equations*

$$\begin{aligned} u[t+1] &= Au[t] + Bf[t], \quad t \in \mathbb{N}_0, \\ r[t] &= Cu[t], \\ u[0] &= 0, \end{aligned} \tag{18}$$

where $A := V_L(\epsilon) S_z V_L^{-1}(\epsilon)$, $B := V_L(\epsilon)[1, 0, \ldots, 0]^\top$, *and* $C := [r_1, \ldots, r_L] V_L^{-1}(\epsilon)$.

Proof. System (18) represents a realization in the controllability canonical form ([19] p. 319), expressed in the basis of normalized Robinson ϵ-polynomials. In this form, with the input vector $\tilde{B} := [1, 0, \dots, 0]^\top$ and the output vector $\tilde{C} := [r_1, \dots, r_L]$, the controllability matrix is the identity matrix, and the impulse response elements satisfy the equality $\tilde{C}S_z^k\tilde{B} = r_k$. The isomorphism $V_L(\epsilon)$ defines a change in basis $u[t] = V_L(\epsilon)\tilde{u}[t]$, which brings the system to the form (18) with the matrices A, B, and C, while preserving its impulse response. □

Remark 3. *The matrix A is an upper Hessenberg matrix. In the theory of orthogonal polynomials on the unit circle, it plays the same role as the tridiagonal Jacobi matrix for polynomials orthogonal on the real line [20].*

In this realization, the final basis state $e_{L-1} := [0, \dots, 0, 1]^\top$ plays a special role; it generates the weighted L-th reflection coefficient.

Lemma 2. *The following relation holds:*

$$Ce_{L-1} = \sigma_{L-1,1}\gamma_L. \tag{19}$$

Proof. The recurrence relations (10) and the energy conservation law (12) imply the identity

$$q_L^*(z)\phi_{L-1}^*(z) - q_{L-1}^*(z)\phi_L^*(z) = \gamma_L\sigma_{L-1,1}^2 z^L,$$

which, taking into account (14), can be rewritten as

$$h_L(\epsilon, z)\phi_{L-1}^*(\epsilon, z) = q_{L-1}^*(z) + \frac{\gamma_L\sigma_{L-1,1}^2 z^L}{\phi_L^*(\epsilon, z)}.$$

Comparing the coefficients of z^L on both sides of this equality, we obtain $[r_1, \dots, r_L]\phi_{L-1}(\epsilon) = \gamma_L\sigma_{L-1,1}^2$, which is equivalent to (19). □

4. Boundary Control Method

In this section, we apply the BC method to the inverse problem for the Goupillaud model. Using the system realization constructed in the previous section, we introduce the connecting operator, which is the central concept of this approach. Our main result is a representation of this operator in terms of the response operator, as well as its factorization. Following the BC method, we formulate a special control problem and utilize this representation to derive the governing equations. These equations coincide with the classical equations of the inverse problem in the Krein ($\epsilon = 1$) and Marchenko ($\epsilon = 0$) approaches [6,17]. We present algorithms to reconstruct the reflection coefficients layer by layer, treating the medium at each step as a finite system.

4.1. Connecting Operator

We define the *connecting operator* $G_L(\epsilon) : \mathcal{F}_L \to \mathcal{F}_L$ by the equality

$$G_L(\epsilon) := W_L(\epsilon)^\top W_L(\epsilon),$$

where $W_L(\epsilon)^\top : \mathcal{X}_L(\epsilon) \to \mathcal{F}_L$ is the adjoint operator to $W_L(\epsilon)$ with respect to the inner products in the spaces $\mathcal{F}_L$ and $\mathcal{X}_L(\epsilon)$. In the monomial basis, according to (17), the matrix of this operator takes the form

$$G_L(\epsilon) = [\Phi_L^{-1}(\epsilon)]^\top\Sigma_L^2\Phi_L^{-1}(\epsilon). \tag{20}$$

In the BC method, the connecting operator plays a key role since it is expressed directly in terms of the response operator.

Lemma 3. *The following relation holds:*

$$R_L(\epsilon) = [\Phi_L^{-1}(\epsilon)]^\top Q_L^\top, \tag{21}$$

where $R_L(\epsilon)$ is the matrix of the response operator in the monomial basis and Q_L is the coefficient matrix of the polynomials $q_0(z), \dots, q_{L-1}(z)$.

Proof. Using relation (16), we reduce the verification of equality (21) to proving the formula

$$R_L(0) = (P_L^{-1})^\top Q_L^\top, \tag{22}$$

where $P_L := \Phi_L(0)$. Taking into account the structure of the matrices under consideration, to prove (22) it suffices to verify the coincidence of the first $l = 0, \dots, L-1$ terms of the power series expansion at the origin of the functions $h_l(z) := q_l^*(z)/p_l^*(z)4$ and $h_L(z)$. For this purpose, we use Cauchy's integral formula for derivatives.

According to this formula, the indicated coincidence implies that the difference $h_L(z) - h_l(z)$ has a zero of order at least l at the origin, that is,

$$\frac{q_L^*(z)}{p_L^*(z)} - \frac{q_l^*(z)}{p_l^*(z)} = z^l \cdot \delta(z),$$

where the function $\delta(z)$ is analytic in a neighborhood of the origin. Since the denominators of the fractions do not vanish at the origin, the problem reduces to verifying the divisibility of the polynomial $q_L^*(z)p_l^*(z) - q_l^*(z)p_L^*(z)$ by z^l. By induction, it suffices to verify the divisibility of the polynomial $q_l^*(z)p_{l-1}^*(z) - q_{l-1}^*(z)p_l^*(z)$ by z^{l-1}. Substituting expressions (10) into this relation and applying (12), we find

$$\begin{aligned} &q_l^*(z)p_{l-1}^*(z) - q_{l-1}^*(z)p_l^*(z) \\ &\qquad = \gamma_l z[p_{l-1}(z)p_{l-1}^*(z) - q_{l-1}(z)q_{l-1}^*(z)] = \gamma_l z^l \sigma_{l-1,1}^2, \end{aligned}$$

which proves the required property. □

Remark 4. *The statement of the lemma follows naturally from the nesting property of the $(L-1)$-layer Goupillaud model within the L-layer one. From an algebraic viewpoint, this property guarantees the invariance of the impulse response coefficients when the recording window is extended, which makes the presented matrix identity intuitively obvious. Nevertheless, the provided analytical proof yields an independent verification of this fact.*

Theorem 2. *For the matrix of the connecting operator, the representations*

$$G_L(\epsilon) = I - \epsilon R_L(\epsilon) - \epsilon R_L(\epsilon)^\top - (1-\epsilon^2) R_L(\epsilon) R_L(\epsilon)^\top \tag{23a}$$

$$= [I - \epsilon R_L(\epsilon) - R_L(\epsilon)J][I - \epsilon R_L(\epsilon)^\top + J R_L(\epsilon)^\top], \tag{23b}$$

hold, where J is the anti-diagonal identity matrix.

Proof. The second equality in (23) is obtained by expanding the product using the identity $R_L(\epsilon)^\top = J R_L(\epsilon) J$, which holds for a triangular Toeplitz matrix. To prove the first equality, we substitute expressions (20), (21), and $\Phi_L(\epsilon) = P_L + \epsilon Q_L$ into it. Subsequent algebraic transformations reduce the problem to verifying the equality

$$\Sigma_L^2 = P_L^\top P_L - Q_L^\top Q_L,$$

which is the matrix form of the energy conservation law (12). □

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Remark 5. *Representations* (23) *are related to the natural energy measure of the Goupillaud model, which is a real positive measure on the unit circle* $|z| = 1$,

$$d\mu_L(\epsilon, z) := \frac{\sigma_{L,1}^2}{2\pi i |\phi_L(\epsilon, z)|^2} \frac{dz}{z}. \tag{24}$$

For any polynomial $\pi(z)$, *we set* $\overline{\pi(z)} := \pi(z^{-1})$ *and* $|\pi(z)|^2 := \pi(z)\pi(z^{-1})$. *The substitution* $p_l(z) = \phi_l(\epsilon, z) - \epsilon q_l(z)$ *into the energy conservation law* (12) *brings it to the form*

$$1 - \epsilon \frac{q_l(z)}{\phi_l(\epsilon, z)} - \epsilon \frac{\overline{q_l(z)}}{\overline{\phi_l(\epsilon, z)}} - (1 - \epsilon^2) \frac{|q_l(z)|^2}{|\phi_l(\epsilon, z)|^2} = \frac{\sigma_{l,1}^2}{|\phi_l(\epsilon, z)|^2}, \tag{25}$$

which, upon passing to matrices in the quotient ring, corresponds to identity (23). *For* $\epsilon = 1$, *measure* (24) *is the classical Bernstein–Szegő measure for polynomials on the unit circle [20].*

4.2. Krein Algorithm

The central idea of the BC method is to interpret inverse problem equations as equations in the external space of a system, which ensure the solution of special control problems ([7] p. 108). For the Goupillaud system, the L-th special control problem is formulated as the problem of finding an input signal that drives the system in $L-1$ steps to the state $\sigma_{L-1,1} z^{L-1}$, the weighted final basis element of the state space. According to the definition of the control operator, the required signal is the Robinson ϵ-polynomial $\phi_{L-1}(\epsilon, z)$.

The connection between this control problem and the inverse problem for the Goupillaud model stems from the fact that the coefficient vector of this polynomial satisfies the following fundamental equation.

Lemma 4. *The coefficient vector* $\phi_{L-1}(\epsilon)$ *of the polynomial* $\phi_{L-1}(\epsilon, z)$ *is the unique solution to the equation*

$$G_L(\epsilon)\phi_{L-1}(\epsilon) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_{L-1,1}^2 \end{bmatrix}. \tag{26}$$

Proof. According to (20), the complete system of coefficient vectors of the Robinson ϵ-polynomials satisfies the matrix equality

$$G_L(\epsilon)\Phi_L(\epsilon) = [\Phi_L^{-1}(\epsilon)]^\top \Sigma_L^2. \tag{27}$$

Since the matrix $\Phi_L(\epsilon)$ is upper triangular with ones on the main diagonal, its inverse transpose matrix $[\Phi_L^{-1}(\epsilon)]^\top$ is also lower triangular with the same unit diagonal. Isolating the last column in (27), we arrive at the required relation (26). The non-singularity of the matrix of the connecting operator follows from its definition. □

Lemma 5. *The relation*

$$[0, r_1, \dots, r_{L-1}]\phi_{L-1}(\epsilon) = \gamma_{L-1} \tag{28}$$

holds.

Proof. Formula (28) is a coefficient-wise representation of the identity $h_L(\epsilon, z)\phi_L^*(\epsilon, z) = q_L^*(z)$ in the quotient ring $\mathbb{R}[[z]]/\langle z^L \rangle$. Its validity follows from the structure of the matrix of the response operator (15) and the explicit form of the Robinson polynomial $q_{L-1}(z)$ in formula (11), whose constant term equals γ_{L-1}. □

Theorem 3. *Let $G_L(\epsilon)$ be the matrix of the connecting operator given by representation (23a). Then the solutions to Equation (26), found sequentially for all L, provide a solution to the inverse problem for the Goupillaud model. The reflection coefficients are recovered via Formula (28).*

Proof. The theorem is a direct consequence of Lemmas 4 and 5. □

Remark 6.

1. *It follows from the explicit representation (11) that for nonzero ϵ, the coefficient γ_{L-1} is recovered by the formula $\gamma_{L-1} = \phi_{L-1}(\epsilon, 0)/\epsilon$. However, for $\epsilon = 0$, this relation becomes ill-defined, whereas the application of Formula (28) allows one to solve the inverse problem in a unified manner for any values of $\epsilon \in [-1, 1]$.*
2. *To identify the reflection coefficients, one can employ the equality $[r_1, \dots, r_L]\phi_{L-1}(\epsilon) = \gamma_L \sigma^2_{L-1,1}$ resulting from Lemma 2. For $\epsilon = 1$, it underlies the fast inversion algorithms that utilize the Levinson recursion.*
3. *For $\epsilon = 1$, the system of Equations (26) and (23a) reduces to equation (47) from [6], which is a discrete analogue of the integral equation for the continuous inverse scattering problem associated with the name of Krein [17].*

The Krein algorithm for the layer-by-layer reconstruction of the reflection coefficients γ_l within the BC method is implemented as the following step-by-step procedure:

0. **Initialization.** The initial value for the system order is set to $L = 2$.
1. **Computation of the connecting operator.** For the current L, the lower triangular Toeplitz matrix of the response operator $R_L(\epsilon)$ (15) is formed from the impulse response. The matrix of the connecting operator $G_L(\epsilon)$ is computed via formula (23a).
2. **Solution of the Krein equation.** The vector $\phi_{L-1}(\epsilon)$ and the cumulative parameter $\sigma^2_{L-1,1}$ are found by solving the linear system (26). In practice, the components of the vector $\phi_{L-1}(\epsilon)$ (whose leading coefficient equals unity) are determined from the first $L-1$ equations of the system, after which the value $\sigma^2_{L-1,1}$ is found from the last equation.
3. **Extraction of the reflection coefficient.** Based on the found vector $\phi_{L-1}(\epsilon)$, the reflection coefficient γ_{L-1} is computed via formula (28).
   Steps 1–3 are repeated cyclically in the transition $L \to L + 1$.

4.3. Marchenko Algorithm

The Marchenko algorithm is based on representation (23b). While the Krein algorithm computes the Robinson ϵ-polynomial $\phi_{L-1}(\epsilon, z)$ for each L, the Marchenko scheme seeks a polynomial combination $g_{L-1}(z) := p_{L-1}(z) + q^*_{L-1}(z)$, whose matrix form is obtained as a result of the action of the second factor of expression (23b) on the control $\phi_{L-1}(\epsilon)$.

Lemma 6. *The expression*

$$[I - \epsilon R_L(\epsilon)^\top + J R_L(\epsilon)^\top]\phi_{L-1}(\epsilon) = g_{L-1} \tag{29}$$

holds, where $g_{L-1} := p_{L-1} + J q_{L-1}$ is the coefficient vector of the polynomial $g_{L-1}(z)$.

Proof. We verify this relation directly using the matrix identity (21). Substituting it into the left-hand side of (29) and taking into account that $\Phi_L(\epsilon) = P_L + \epsilon Q_L$ and $J^2 = I$, we obtain the required structure of the vector g_{L-1} in terms of the coefficients of the Robinson polynomials. □

Theorem 4. *The solutions to the systems of equations*

$$[I - \epsilon R_L(\epsilon) - R_L(\epsilon)J]g_{L-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma^2_{L-1,1} \end{bmatrix}, \tag{30}$$

which are closed by the relation $\sigma^2_{L-1,1} = \sigma^2_{L-2,1}\left(1 - (g_{L-1,L-1} - 1)^2\right)$ *and are found sequentially for all L, provide a solution to the inverse problem for the Goupillaud model. The reflection coefficients are recovered via formula*

$$\gamma_{L-1} = g_{L-1,L-1} - 1. \tag{31}$$

Proof. Equation (30) follows directly from equation (26) upon substituting the factorization (23b) and applying Lemma 6. Although system (30) contains L equations for $L+1$ unknowns (the components of the vector g_{L-1} and the parameter $\sigma^2_{L-1,1}$), its first equation yields $g_{L-1,0} = 0$. Taking into account the nonlinear closure via the parameter $\sigma^2_{L-2,1}$ known from the previous step, the system becomes closed with respect to L unknowns and uniquely determines the vector g_{L-1}.

The extraction formula (31) follows from the definition $g_{L-1}(z) := p_{L-1}(z) + q^*_{L-1}(z)$ and the explicit form of the Robinson polynomials (11), according to which the leading coefficients of the polynomials $p_{L-1}(z)$ and $q^*_{L-1}(z)$ are equal to 1 and γ_{L-1}, respectively. □

Remark 7. *For* $\epsilon = 0$*, the system of Equation* (30) *reduces to Equation (59) from [6], which is a discrete analogue of the integral equation for the continuous inverse scattering problem associated with the name of Marchenko and characterized by a Hankel matrix [17].*

The Marchenko algorithm for the layer-by-layer reconstruction of the reflection coefficients γ_l within the BC method is implemented as the following step-by-step procedure:

0. **Initialization.** The initial value for the system order is set to $L = 2$.
1. **Computation of the factorization operator.** For the current L, the lower triangular Toeplitz matrix of the response operator $R_L(\epsilon)$ of the form (15) is formed from the impulse response, and the system operator matrix $[I - \epsilon R_L(\epsilon) - R_L(\epsilon)J]$ is computed.
2. **Solution of the Marchenko equation.** The vector g_{L-1} and the cumulative parameter $\sigma^2_{L-1,1}$ are found by solving the closed system (30).
3. **Extraction of the reflection coefficient.** The required reflection coefficient γ_{L-1} is extracted from the last component of the vector g_{L-1} via Formula (31).

Steps 1–3 are repeated cyclically in the transition from L to $L+1$.

5. Discussion

This paper develops a new approach to the dynamic inverse problem for the Goupillaud model by adapting Belishev's BC method to this discrete setting. By constructing an operator realization of the model as a controlled system, we bridge the gap between the abstract state-space framework required by the BC method and the model's original input–output description. We introduce a unified boundary parameter $\epsilon \in [-1, 1]$ to describe free surfaces, non-free surfaces, and intermediate impedance loads. Under this framework, we derive an explicit formula that expresses the connecting operator through the response operator for any ϵ. This representation directly implies Krein's approach, while its factorized form leads to Marchenko's approach. Furthermore, formulating the inverse problem equations as a specific forward control problem provides a unified derivation of both discrete Krein and Marchenko equations, as well as the corresponding layer-stripping algorithms for reflection coefficient recovery.

To explicitly clarify the innovations of the proposed operator framework and avoid any misconception, it is important to distinguish our formal unified derivations from brand-new physical or computational conclusions. The purpose of this paper is not to introduce a new physical model or a new numerical inversion algorithm. The physical model under consideration remains the classical Goupillaud layered-earth model, and the final Krein and Marchenko inversion procedures recovered here are the classical ones. We stress that the proposed operator framework should be viewed as a unifying language rather than a new physical theory or a novel fast inversion algorithm. Consequently, the present work does not claim novelty in the field of numerical algorithms. Nevertheless, the contribution of this work lies in establishing an explicit operator correspondence, which clarifies the relationship between classical polynomial recursions and the Krein/Marchenko equations through the lens of the system's underlying metric and energy properties. This rigorous mathematical foundation serves as a necessary prerequisite for subsequent developments: a multichannel generalization of the Goupillaud model is planned, and it is within this extension that the authors expect to obtain new algorithms and draw new physical conclusions.

This work builds upon the insight that the BC method provides a unified framework for deriving the classical integral equations of inverse scattering (Gelfand–Levitan, Krein, and Marchenko) in the continuous case [22,23]. We extend this unified, operator-based approach to the discrete Goupillaud system, demonstrating that the well-known Krein and Marchenko equations arise naturally from two different representations—namely, the direct and the factorized forms—of the same connecting operator. Formulating the inverse problem equations as a specific forward control problem thus yields a unified derivation of both discrete equations, as well as the corresponding layer-stripping algorithms for reflection coefficient recovery.

Owing to the strictly theoretical and methodological character of this work, a numerical simulation section is deliberately omitted. Consequently, highly important practical issues—such as operator conditioning, targeted regularization strategies, computational complexity scaling, and the algorithms' resilience to noise—are not quantified herein. Furthermore, physical perturbations that typically challenge layer-stripping methods, such as mild dissipation and absorption effects, frequency-dependent attenuation with dispersion, or modeling departures arising from non-normal wave incidence and slightly unequal round-trip travel times per layer, fall outside the scope of our current analysis. We note that for the classical formulations of the Goupillaud inverse problem and its extensions, these computational and stability limitations have been extensively investigated; in particular, the critical dependency of inversion success on low-frequency data accuracy, the mechanisms of error accumulation with depth, and various stabilization techniques (such as thresholding, adaptive windowing, or data smoothing) have been rigorously analyzed through matrix and differential methods [24–26]. Concurrently, advanced forward modeling and inversion methodologies in the time and frequency domains have been developed to explicitly incorporate absorption, dispersion pairs, and non-normal incidence wavefields within the Goupillaud earth model [27,28]. Translating these well-known computational and physical problems from the classical algebraic and wave-propagational languages to our control-theoretic state-space framework remains a separate, compelling task for subsequent applied research.

While this work is purely theoretical, the valuable inquiries raised during the review process have highlighted several important directions for subsequent numerical and applied research:

- It is promising to investigate the existence and properties of fast numerical solvers for intermediate values of the boundary parameter $\epsilon \neq 0, 1$. Although the strict Toeplitz and Hankel structures of the corresponding matrices are lost in this general case, this

problem appears to be tractable, making the development of such solvers a subject for future research.

- Extending the boundary parameter ϵ to the complex plane represents a natural generalization to reactive or dissipative boundaries. However, in the Goupillaud setting, incorporating such boundary dissipation inherently requires a consistent treatment of wave propagation within the medium. By the principle of causality, absorption is accompanied by dispersion [28]; thus, if the medium or boundary is dissipative, the phase velocity becomes frequency-dependent, and the two-way travel time in each layer is no longer constant. As noted in [28], this frequency dependence invalidates the direct use of standard z-transforms, making it a promising direction for future research.
- Another interesting question is to establish the relationship between the derived Hessenberg dynamics matrix A and the five-diagonal CMV matrix. Since A is non-unitary for a general boundary parameter ϵ, transforming it into a CMV-like form faces nontrivial algebraic obstacles, and the existence of a definitive CMV analogue for this system remains open.

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Appendix A. Analyticity of the Transfer Function

Theorem A1. *For $|\epsilon| \le 1$, all zeros of the polynomial $\phi_l(\epsilon, z)$ lie in the disk $|z| < 1$. Consequently, all zeros of the polynomial $\phi_l^*(\epsilon, z)$ lie outside this disk.*

Proof. On the boundary $|z| = 1$, according to (12), we have $|p_l(z)| - |q_l(z)| > 0$ and, therefore, $|\epsilon q_l(z)| \le |q_l(z)| < |p_l(z)|$. By Rouché's theorem, the polynomials $p_l(z)$ and $\phi_l(\epsilon, z)$ have the same number of zeros in the disk $|z| < 1$.

We prove by induction on l that the polynomial $p_l(z)$ has exactly l zeros in the disk $|z| < 1$. For $p_0(z) = 1$, there are no zeros. Suppose that the number of zeros of the polynomial $p_{l-1}(z)$ in the disk $|z| < 1$ equals $l - 1$. On the boundary $|z| = 1$, from the first relation in (10) and from (12), we find $|p_l - zp_{l-1}| < |q_{l-1}| < |zp_{l-1}|$. By Rouché's theorem, the polynomials $zpper_{l-1}(z)$ (or more precisely, $zp_{l-1}(z)$) and $p_l(z)$ have the same number of zeros in the disk $|z| < 1$, that is, l. □

Corollary A1. *Since all zeros of $\phi_L^*(\epsilon, z)$ lie outside the unit disk, the denominator of the transfer function $h_L(\epsilon, z) = q_L^*(z)/\phi_L^*(\epsilon, z)$ does not vanish for $|z| < 1$. Hence $h_L(\epsilon, z)$ is analytic in the disk $|z| < 1$.*

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