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A Mixed-Adjoint Framework for 2 × 2 Operator Matrices and Their Numerical Radii

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Abstract

We develop a semi-inner product framework for 2 × 2 block operator matrices acting on the direct sum $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$, generated by the diagonal operator matrix $\mathbb{A} = \mathrm{diag}(A, B)$, where A and B are distinct positive operators acting on $\mathcal{H}$, neither assumed to be injective. This extends the setting of Bhunia, Paul, and Nayak [Electron. J. Linear Algebra 36 (2020), 143–157], who treated the case $\mathbb{A} = \mathrm{diag}(A, A)$ with A injective. Using Douglas' majorization theorem, we employ and develop the mixed (A, B)- and (B, A)-adjoint formalism in the block operator matrix setting; these adjoints relate the two distinct seminorms induced by A and B and take the place of the ordinary adjoint when these seminorms differ. Within this framework, we establish upper and lower bounds for the $\mathbb{A}$-numerical radius of 2 × 2 operator matrices belonging to the appropriate $\mathbb{A}$-bounded or $\mathbb{A}$-adjointable classes, extending and refining the bounds of the above-mentioned paper to the case of two distinct, not necessarily injective, positive generators.

Keywords: mixed adjoint operator; semi-Hilbertian space; $\mathbb{A}$-numerical radius; block operator matrix; Douglas theorem; operator inequalities

MSC: 47A12; 47A30; 47A63

1. Introduction and Preliminaries

Inequalities serve as fundamental tools in functional analysis, with numerical radius inequalities playing a central role in operator theory (see [1–3] and the references therein). In this paper, we aim to establish generalized numerical radius inequalities for 2 × 2 block operator matrices acting on asymmetric semi-Hilbertian spaces—that is, spaces induced by a diagonal positive operator matrix on a direct sum Hilbert space, where the diagonal entries are distinct positive operators acting on the same complex Hilbert space. The primary goal of this section is to outline the foundational context of semi-Hilbertian spaces and recall the essential properties of generalized numerical radii. This establishes the necessary theoretical framework and clarifies the notation that will be utilized throughout the manuscript.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $\mathbf{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $X \in \mathbf{L}(\mathcal{H})$, the operator norm is $\|X\| = \sup\{\|Xu\| : u \in \mathcal{H},\ \|u\| = 1\}$ and the numerical radius is

$$\omega(X) = \sup\{|\langle Xu, u \rangle| : u \in \mathcal{H},\ \|u\| = 1\}.$$

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The numerical radius defines a norm equivalent to the standard operator norm, specifically satisfying $\frac{1}{2}\|X\| \leq \omega(X) \leq \|X\|$. It is widely used to study the numerical range and estimate the location of polynomial zeros (see, for example, [4,5]). Improving these bounds has inspired an extensive body of literature [2,6,7], including significant work on operator matrices [8–10].

The theory of bounded operators extends naturally to semi-Hilbertian spaces, a framework initiated by Krein [11] and extensively developed in recent years [12–15]. A positive operator $A \in \mathbf{L}(\mathscr{H})$, meaning that $\langle Au, u\rangle \geq 0$ for all $u \in \mathscr{H}$, induces the semi-inner product $\langle u, v\rangle_A = \langle Au, v\rangle$ and the seminorm $\|u\|_A = \sqrt{\langle Au, u\rangle}$, which becomes a norm if and only if A is injective. The set of all non-zero positive operators on $\mathscr{H}$ (explicitly excluding the zero operator $\mathbf{0}$) will be denoted by $\mathbf{L}^+(\mathscr{H})$. Note that $(\mathscr{H}, \langle\cdot,\cdot\rangle_A)$ is in general neither a normed space nor a complete space (see [12]).

For $X \in \mathbf{L}(\mathscr{H})$, we denote by $\mathrm{Ran}(X)$ its range, by $\mathrm{Ker}(X)$ its kernel, and by X^* its adjoint. For a subspace $\mathscr{M} \subseteq \mathscr{H}$, $\overline{\mathscr{M}}$ stands for its closure with respect to the norm topology of $\mathscr{H}$. For a closed subspace $\mathscr{M} \subseteq \mathscr{H}$, we write $\Pi_{\mathscr{M}}$ for the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. We denote by $\mathbf{0}$ the zero operator on $\mathscr{H}$, and by $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$ (we simply write I when the underlying space is clear from the context). The unit sphere with respect to the A-seminorm is denoted by $S_A^1(\mathscr{H}) = \{u \in \mathscr{H} : \|u\|_A = 1\}$. Throughout, for $n \in \mathbb{N}$, $\mathbb{M}_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries. Before stating the fundamental range inclusion theorem, we recall the Löwner partial order for self-adjoint operators. For self-adjoint operators $C, D \in \mathbf{L}(\mathscr{H})$, the operator inequality $C \leq D$ means that $\langle Cu, u\rangle \leq \langle Du, u\rangle$ for all $u \in \mathscr{H}$. The following celebrated assertion is known as the Douglas theorem or Douglas majorization theorem (for its proof see [16]).

Theorem 1. *If $X, Y \in \mathbf{L}(\mathscr{H})$, then the following statements are equivalent:*

(i) $\mathrm{Ran}(Y) \subseteq \mathrm{Ran}(X)$;

(ii) *There exists a positive constant λ such that $YY^* \leq \lambda^2 XX^*$, which is equivalent to the norm inequality $\|Y^*u\| \leq \lambda\|X^*u\|$ for all $u \in \mathscr{H}$;*

(iii) $XZ = Y$ *for some* $Z \in \mathbf{L}(\mathscr{H})$.

Moreover, if one of these conditions holds, then there exists a unique solution $Q \in \mathbf{L}(\mathscr{H})$ of the equation $XZ = Y$ (known as the Douglas solution) such that $\mathrm{Ran}(Q) \subseteq \overline{\mathrm{Ran}(X^*)}$.

Let $X \in \mathbf{L}(\mathscr{H})$. An operator $Y \in \mathbf{L}(\mathscr{H})$ is said to be an A-adjoint of X if for all $u, v \in \mathscr{H}$, the identity $\langle Xu, v\rangle_A = \langle u, Yv\rangle_A$ holds [12]. Thus, the existence of an A-adjoint of X is equivalent to the existence of a solution of the equation $AY = X^*A$. By Douglas' theorem, this is equivalent to the range inclusion $\mathrm{Ran}(X^*A) \subseteq \mathrm{Ran}(A)$. Clearly, the existence of an A-adjoint operator for all $X \in \mathbf{L}(\mathscr{H})$ is not guaranteed unless the generator is invertible. We denote by $\mathbf{L}_A(\mathscr{H})$ the subspace of all operators admitting A-adjoints:

$$\mathbf{L}_A(\mathscr{H}) = \{X \in \mathbf{L}(\mathscr{H}) : \mathrm{Ran}(X^*A) \subseteq \mathrm{Ran}(A)\}.$$

If $X \in \mathbf{L}_A(\mathscr{H})$, the reduced solution of the equation $AY = X^*A$ is a distinguished A-adjoint operator of X, which is denoted by $X^{\sharp_A}$. Note that $X^{\sharp_A} = A^\dagger X^* A$, where $A^\dagger$ is the Moore–Penrose inverse of A [13]. Recall that given an operator $T \in \mathbf{L}(\mathscr{H})$, its Moore–Penrose inverse $T^\dagger$ is defined as the unique linear extension of the inverse of the restricted isomorphism $T|_{\mathrm{Ker}(T)^\perp} : \mathrm{Ker}(T)^\perp \to \mathrm{Ran}(T)$ to the domain $\mathrm{Dom}(T^\dagger) = \mathrm{Ran}(T) \oplus \mathrm{Ran}(T)^\perp$, satisfying $\mathrm{Ker}(T^\dagger) = \mathrm{Ran}(T)^\perp$. Furthermore, $T^\dagger$ represents the unique solution to the four Moore–Penrose equations: $TT^\dagger T = T$, $T^\dagger TT^\dagger = T^\dagger$, $T^\dagger T = \Pi_{\mathrm{Ker}(T)^\perp}$, and $TT^\dagger = \Pi_{\overline{\mathrm{Ran}(T)}}|_{\mathrm{Dom}(T^\dagger)}$. The operator $T^\dagger$ has a closed graph, and it is bounded if and only if $\mathrm{Ran}(T)$ is closed. Additionally, if T is a self-adjoint operator, then $(T^\dagger)^* = T^\dagger$. Consequently, if T is a positive operator, its Moore–Penrose inverse satisfies $T^\dagger = (\sqrt{T})^\dagger(\sqrt{T})^\dagger$ [13]. Notice that if $X \in$

$\mathbf{L}_A(\mathscr{H})$, then $X^{\sharp_A} \in \mathbf{L}_A(\mathscr{H})$, $(X^{\sharp_A})^{\sharp_A} = \Pi_{\overline{\mathrm{Ran}(A)}} X \Pi_{\overline{\mathrm{Ran}(A)}}$, and $((X^{\sharp_A})^{\sharp_A})^{\sharp_A} = X^{\sharp_A}$. Moreover, if $Y \in \mathbf{L}_A(\mathscr{H})$ then $XY \in \mathbf{L}_A(\mathscr{H})$ and $(XY)^{\sharp_A} = Y^{\sharp_A} X^{\sharp_A}$. For more results concerning $X^{\sharp_A}$, we refer the reader to [12,14].

An operator X is called A-bounded if there exists $\lambda > 0$ such that $\|Xu\|_A \leq \lambda \|u\|_A$ for all $u \in \mathscr{H}$. An application of Douglas' theorem shows that the subspace of all operators admitting $\sqrt{A}$-adjoints, denoted by $\mathbf{L}_{\sqrt{A}}(\mathscr{H})$, is equal to the collection of all A-bounded operators:

$$\mathbf{L}_{\sqrt{A}}(\mathscr{H}) = \{X \in \mathbf{L}(\mathscr{H}) : \exists\, \lambda > 0 \text{ such that } \|Xu\|_A \leq \lambda \|u\|_A, \forall\, u \in \mathscr{H}\}.$$

Notice that $\mathbf{L}_A(\mathscr{H})$ and $\mathbf{L}_{\sqrt{A}}(\mathscr{H})$ are two subalgebras of $\mathbf{L}(\mathscr{H})$ which are, in general, neither closed nor dense in $\mathbf{L}(\mathscr{H})$. Moreover, we have the inclusion $\mathbf{L}_A(\mathscr{H}) \subseteq \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ [12,14].

Example 1. *We show that the inclusion* $\mathbf{L}_A(\mathscr{H}) \subseteq \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ *may be strict when* $\mathrm{Ran}(A)$ *is not closed. Let* $\mathscr{H} = \ell^2(\mathbb{N})$ *with canonical orthonormal basis* $\{e_n\}_{n\geq 1}$. *Let* $A = \mathrm{diag}(1, 2^{-2}, 3^{-2}, \ldots)$. *Then* $A \in \mathbf{L}^+(\mathscr{H})$ *and* $\sqrt{A} = \mathrm{diag}(1, 2^{-1}, 3^{-1}, \ldots)$. *Consider the vector* $v = (n^{-2})_{n\geq 1} \in \mathscr{H}$.

Let $u = (n^{-1})_{n\geq 1}$. *Since* $\sum_{n=1}^{\infty} n^{-2} < \infty$, *we have* $u \in \mathscr{H}$. *Moreover,* $\sqrt{A}\, u = v$. *Hence* $v \in \mathrm{Ran}(\sqrt{A})$. *On the other hand, suppose that* $Aw = v$ *for some* $w \in \mathscr{H}$. *Then* $w_n = n^2 \cdot n^{-2} = 1$ *for all* n. *Since* $\sum_{n=1}^{\infty} 1 = \infty$, *this is impossible. Thus* $v \notin \mathrm{Ran}(A)$.

Now define the rank-one operator $X \in \mathbf{L}(\mathscr{H})$ *by*

$$Xx = \langle x, v\rangle e_1, \quad x \in \mathscr{H}.$$

Its adjoint is given by $X^* y = \langle y, e_1\rangle v$ *for all* $y \in \mathscr{H}$. *Since* $\sqrt{A}$ *and* A *are diagonal, we have*

$$X^* \sqrt{A}\, y = X^* A\, y = y_1 v, \quad y \in \mathscr{H}.$$

Therefore $\mathrm{Ran}(X^*\sqrt{A}) = \mathrm{Ran}(X^*A) = \mathrm{span}\{v\}$.

Since $\mathrm{span}\{v\} \subseteq \mathrm{Ran}(\sqrt{A})$, *Douglas' theorem implies that* $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$. *Indeed, for all* $x \in \mathscr{H}$, *we have*

$$\|Xx\|_A = |\langle x, v\rangle|\, \|e_1\|_A = |\langle \sqrt{A}\, x, u\rangle| \leq \|u\|\, \|x\|_A.$$

On the other hand, since $\mathrm{span}\{v\} \not\subseteq \mathrm{Ran}(A)$, *we get* $\mathrm{Ran}(X^*A) \not\subseteq \mathrm{Ran}(A)$. *By Douglas' theorem, the equation* $AY = X^*A$ *has no solution in* $\mathbf{L}(\mathscr{H})$. *Thus* X *admits no* A*-adjoint, that is,* $X \notin \mathbf{L}_A(\mathscr{H})$. *Consequently,* $\mathbf{L}_A(\mathscr{H}) \subsetneq \mathbf{L}_{\sqrt{A}}(\mathscr{H})$.

This explains why the formulas involving $X^{\sharp_A}$ *and the mixed adjoints require the assumption* $X \in \mathbf{L}_A(\mathscr{H})$, *and not merely* $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$.

Clearly, $\langle \cdot, \cdot \rangle_A$ induces a seminorm on $\mathbf{L}_{\sqrt{A}}(\mathscr{H})$. Indeed, if $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$, then it holds that

$$\|X\|_A = \sup_{\substack{u \in \overline{\mathrm{Ran}(A)}, \\ u \neq 0}} \frac{\|Xu\|_A}{\|u\|_A} = \sup_{u \in S_A^1(\mathscr{H})} \|Xu\|_A < \infty.$$

Furthermore, the A-numerical radius of an operator $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ was firstly defined by Saddi in [17] as

$$\omega_A(X) = \sup_{u \in S_A^1(\mathscr{H})} |\langle Xu, u\rangle_A|.$$

It should be emphasized that it may happen that $\|X\|_A$ and $\omega_A(X)$ are equal to $+\infty$ for some $X \in \mathbf{L}(\mathscr{H}) \setminus \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ [18]. However, these quantities are equivalent seminorms

on $\mathbf{L}_{\sqrt{A}}(\mathscr{H})$. More precisely, it was shown in [15] that for every $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$, we have $\frac{1}{2}\|X\|_A \leq \omega_A(X) \leq \|X\|_A$. Notice that if $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ and satisfies $AX^2 = \mathbf{0}$, then by (Corollary 2, [19]) we have

$$\omega_A(X) = \frac{1}{2}\|X\|_A. \tag{1}$$

For the sequel, for any arbitrary operator $X \in \mathbf{L}_A(\mathscr{H})$, we denote the real and imaginary parts by

$$\Re_A(X) = \frac{X + X^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(X) = \frac{X - X^{\sharp_A}}{2i}.$$

It has been proved in [20] that for $X \in \mathbf{L}_A(\mathscr{H})$, it holds

$$\omega_A(X) = \sup_{\theta\in\mathbb{R}} \left\|\Re_A(e^{i\theta}X)\right\|_A = \sup_{\theta\in\mathbb{R}} \left\|\Im_A(e^{i\theta}X)\right\|_A. \tag{2}$$

Recently, the concept of the A-spectral radius of A-bounded operators has been introduced in [19] as follows:

$$r_A(X) := \inf_{n\in\mathbb{N}} \|X^n\|_A^{\frac{1}{n}} = \lim_{n\to\infty} \|X^n\|_A^{\frac{1}{n}}, \quad X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H}).$$

Moreover, like the classical spectral radius of Hilbert space operators, it was shown in [19] that $r_A(\cdot)$ satisfies the commutativity property, which asserts that

$$r_A(XY) = r_A(YX),$$

for all $X, Y \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$. In addition, the following relation between the A-spectral and A-numerical radii of A-bounded operators was also proved in [19]:

$$r_A(X) \leq \omega_A(X), \quad \text{for all } X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H}).$$

Recall that an operator X is said to be A-positive if $AX \geq \mathbf{0}$, in which case we write $X \geq_A \mathbf{0}$. Clearly, an A-positive operator is always A-selfadjoint (that is, AX is selfadjoint), since $\mathscr{H}$ is a complex Hilbert space. It is also clear that A-selfadjoint operators belong to $\mathbf{L}_A(\mathscr{H})$. Notice that if X is A-selfadjoint, then by [19],

$$\omega_A(X) = r_A(X) = \|X\|_A. \tag{3}$$

Moreover, if $X \in \mathbf{L}_A(\mathscr{H})$, then $XX^{\sharp_A} \geq_A \mathbf{0}$, $X^{\sharp_A}X \geq_A \mathbf{0}$, and

$$\|X^{\sharp_A}X\|_A = \|XX^{\sharp_A}\|_A = \|X\|_A^2 = \|X^{\sharp_A}\|_A^2. \tag{4}$$

Finally, if $X, Y \in \mathbf{L}(\mathscr{H})$ satisfy $X - Y \geq_A \mathbf{0}$, then we write $X \geq_A Y$. In recent years, several results covering some classes of operators on a complex Hilbert space $\mathscr{H}$ were extended to semi-Hilbertian spaces. Of course, the extension is not trivial since many difficulties arise. For instance, as it is mentioned above, it may happen that $\|X\|_A = \infty$ for some $X \in \mathbf{L}(\mathscr{H})$. Moreover, not any operator admits an adjoint operator for the semi-inner product $\langle\cdot,\cdot\rangle_A$. In addition, as mentioned earlier, even if $X \in \mathbf{L}_A(\mathscr{H})$, we have $(X^{\sharp_A})^{\sharp_A} \neq X$.

An operator $X \in \mathbf{L}(\mathscr{H})$ is said to be an A-isometry if $\|Xu\|_A = \|u\|_A$ for all $u \in \mathscr{H}$. An operator $U \in \mathbf{L}(\mathscr{H})$ is said to be A-unitary if $U \in \mathbf{L}_A(\mathscr{H})$ and $\|Uu\|_A = \|U^{\sharp_A}u\|_A = \|u\|_A$ for all $u \in \mathscr{H}$. We denote the set of all A-unitary operators by $\mathcal{U}_A(\mathscr{H})$. Baklouti and Namouri (Lemma 2.4, [21]) connected these notions with the fundamental properties of the usual isometries and unitary operators, showing that $X \in \mathbf{L}_A(\mathscr{H})$ is an A-isometry if and only if $AX^{\sharp_A}X = A$, and that $U \in \mathbf{L}_A(\mathscr{H})$ is A-unitary if and only if $AU^{\sharp_A}U = AUU^{\sharp_A} = A$. The reader is referred to [12–14,22] and the references therein for more details on this theory.

The exploration of numerical radii in semi-Hilbertian spaces has recently stimulated a wealth of significant contributions. Various authors have focused on generalizing and

improving fundamental estimations for the A-numerical radius of single operators [23–25]. Parallel investigations have successfully utilized the Moore–Penrose inverse and extended Buzano inequalities to refine A-spectral bounds [26–28]. Furthermore, advanced refinements and exact bounds for the A-numerical radius of semi-Hilbertian operators continue to be extensively explored [29–31]. In the context of block matrices and structural configurations, sophisticated convex-block approaches have yielded highly accurate numerical radius inequalities [32–34]. Related concepts, including A-partial isometries and A-Berezin number inequalities, have also attracted considerable attention [35,36].

The concept of the numerical radius has also been extended to block operator matrices. For 2×2 matrices, the investigation was initiated by Bhunia, Paul, and Nayak [37], who investigated the operator matrix $\mathbb{A} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$ acting on $\mathscr{K}$, under the restrictive assumption that A is strictly non-zero positive operator. Subsequently, Rout et al. [38,39] and Feki [40] removed this assumption, establishing $\mathbb{A}$-numerical radius inequalities for 2×2 matrices where A is simply a non-zero positive operator. Further results under the same conditions were established by Bhunia and Paul in [41]. Several authors then generalized the framework to $q \times q$ operator matrices [42,43], retaining the homogeneous generator $\mathbb{A} = \mathrm{diag}(A, \dots, A)$ defined on the uniform direct sum $\mathscr{H} \oplus \cdots \oplus \mathscr{H}$.

A common feature of these results is that the same operator A occupies every diagonal entry of $\mathbb{A}$. While convenient, this uniformity assumption sidesteps a genuine structural difficulty: in general, the two direct summands of $\mathscr{K}$ carry different semi-inner products, so the off-diagonal blocks act as transformations between two distinct seminorms—a situation the uniform theory simply does not capture. Closing this gap is the main goal of the present manuscript. Throughout, we work with

$$\mathbb{A} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix},$$

where $A, B \in \mathbf{L}^+(\mathscr{H})$ are two, possibly distinct, non-zero positive operators, neither of which is assumed to be injective.

The notion of a mixed adjoint relative to two positive operators goes back to Arias, Corach, and Gonzalez ([14], [Section 3]), who studied the equation $AY = X^*B$ in the context of lifting properties and described its reduced solutions. Using Douglas' theorem [16], we define the mixed (A, B)- and (B, A)-adjoints in the block matrix setting, where they play the role of the usual adjoint once the two seminorms no longer coincide. When $A = B$, they reduce to the ordinary A-adjoint, and all of our results recover the known ones for the uniform generator $\mathrm{diag}(A, A)$. These adjoints are our main tool for deriving $\mathbb{A}$-numerical radius inequalities for arbitrary 2×2 operator matrices. In particular, our bounds extend the quadratic numerical radius inequalities of Kittaneh [2], as well as their block versions due to Bhunia, Paul, and Nayak [37], to distinct and possibly singular positive generators.

The paper is organized as follows. Section 2 introduces mixed adjoint operators and mixed seminorms, and establishes their basic metric and submultiplicative properties. Section 3 gives necessary and sufficient conditions for a 2×2 block operator matrix to admit a mixed adjoint, together with upper bounds for its spectral radius. Section 4 then uses these tools to derive bounds for the generalized $\mathbb{A}$-numerical radius, treating the diagonal and anti-diagonal block structures separately across the two seminorms.

2. Mixed Adjoint Operators and Mixed Seminorms

In this section, we define the foundational concepts of mixed adjoints and mixed seminorms bridging different semi-Hilbertian spaces, and establish their core metric and algebraic properties.

The next definition formalizes the notion of a mixed adjoint operator.

Definition 1. *Let* $X \in \mathbf{L}(\mathscr{H})$. *An operator* $Y \in \mathbf{L}(\mathscr{H})$ *is called a mixed* (A, B)*-adjoint of* X *if it satisfies*

$$\langle Xu, v\rangle_B = \langle u, Yv\rangle_A, \quad \textit{for all } u, v \in \mathscr{H}.$$

This defining relation is algebraically equivalent to the operator equation $AY = X^*B$. *We denote by* $\mathbf{L}_{A,B}(\mathscr{H})$ *the set of all operators in* $\mathbf{L}(\mathscr{H})$ *that admit a mixed* (A, B)*-adjoint*.

By applying Douglas' theorem (Theorem 1), we deduce that the operator equation $AY = X^*B$ possesses a solution if and only if the range inclusion $\mathrm{Ran}(X^*B) \subseteq \mathrm{Ran}(A)$ holds. If this condition holds, then Douglas' theorem guarantees the existence of a unique reduced solution, denoted by $X^{\sharp_{A,B}}$, that satisfies $\mathrm{Ran}(X^{\sharp_{A,B}}) \subseteq \overline{\mathrm{Ran}(A)}$ and $\mathrm{Ker}(X^{\sharp_{A,B}}) = \mathrm{Ker}(X^*B)$. Specifically, by using the Moore–Penrose inverse of A, this distinguished solution can be explicitly represented as $X^{\sharp_{A,B}} = A^\dagger X^* B$.

The next result generalizes a profound property established by Arias et al. [12] for uniform generators. It establishes an explicit formula for the double mixed adjoint in terms of orthogonal projections.

Lemma 1. *Let* $X \in \mathbf{L}_{A,B}(\mathscr{H})$. *Then,* $X^{\sharp_{A,B}} \in \mathbf{L}_{B,A}(\mathscr{H})$ *and we have*

$$(X^{\sharp_{A,B}})^{\sharp_{B,A}} = \Pi_{\overline{\mathrm{Ran}(B)}} X \Pi_{\overline{\mathrm{Ran}(A)}} \tag{5}$$

and

$$((X^{\sharp_{A,B}})^{\sharp_{B,A}})^{\sharp_{A,B}} = X^{\sharp_{A,B}}. \tag{6}$$

Proof. Let $Y = X^{\sharp_{A,B}}$. By definition, Y is the unique reduced solution satisfying the equation $AY = X^*B$ such that $\mathrm{Ran}(Y) \subseteq \overline{\mathrm{Ran}(A)}$. Using the properties of the Moore–Penrose pseudoinverse, this solution is explicitly given by $Y = A^\dagger X^* B$.

Since $A^\dagger$ is typically an unbounded operator (unless $\mathrm{Ran}(A)$ is closed), we must carefully apply the adjoint extension relations for densely defined operators. Specifically, the adjoint of a product satisfies $(\mathcal{S}\mathcal{T})^* \supset \mathcal{T}^*\mathcal{S}^*$. Applying this to Y, we obtain:

$$Y^* = (A^\dagger X^* B)^* \supset B^* (X^*)^* (A^\dagger)^* = B X A^\dagger.$$

Since Y is a bounded operator on $\mathscr{H}$, its adjoint Y^* is everywhere defined. The operator $BXA^\dagger$ is defined on the dense domain $\mathscr{D}(A^\dagger) = \mathrm{Ran}(A) \oplus \mathrm{Ker}(A)$. Therefore, for any vector $u \in \mathscr{D}(A^\dagger)$, we have the exact equality $Y^*u = BXA^\dagger u$.

Let $Z = \Pi_{\overline{\mathrm{Ran}(B)}} X \Pi_{\overline{\mathrm{Ran}(A)}}$. We aim to show that $BZ = Y^*A$. Both BZ and Y^*A are everywhere defined bounded operators. Let $v \in \mathscr{H}$. Clearly, $Av \in \mathrm{Ran}(A) \subseteq \mathscr{D}(A^\dagger)$. By evaluating Y^* at Av, we get:

$$Y^*(Av) = BXA^\dagger(Av) = BX\Pi_{\overline{\mathrm{Ran}(A)}} v.$$

Since B is self-adjoint, we have $B = B\Pi_{\overline{\mathrm{Ran}(B)}}$. Substituting this yields:

$$Y^*Av = B\Pi_{\overline{\mathrm{Ran}(B)}} X \Pi_{\overline{\mathrm{Ran}(A)}} v = BZv.$$

This establishes the bounded operator identity $BZ = Y^*A$. Since $\mathrm{Ran}(Z) \subseteq \overline{\mathrm{Ran}(B)}$ by construction, the uniqueness of the Douglas reduced solution guarantees that $Y^{\sharp_{B,A}} = Z$, which proves (5).

To prove (6), let $W = Z^{\sharp_{A,B}}$. We seek to show $W = Y$. By definition, W is the unique reduced solution to $AW = Z^*B$ with $\mathrm{Ran}(W) \subseteq \overline{\mathrm{Ran}(A)}$. Taking the classical adjoint of the bounded operator Z, we get $Z^* = \Pi_{\overline{\mathrm{Ran}(A)}} X^* \Pi_{\overline{\mathrm{Ran}(B)}}$. Multiplying by B, we obtain:

$$Z^*B = \Pi_{\overline{\mathrm{Ran}(A)}} X^* \Pi_{\overline{\mathrm{Ran}(B)}} B = \Pi_{\overline{\mathrm{Ran}(A)}} X^* B.$$

Since Y solves $AY = X^*B$ and A commutes with its range projection, substituting X^*B yields:

$$Z^*B = \Pi_{\overline{\mathrm{Ran}(A)}}(AY) = AY.$$

Since Y already satisfies $\mathrm{Ran}(Y) \subseteq \overline{\mathrm{Ran}(A)}$, the uniqueness of the reduced solution ensures $W = Y$. This proves (6) and completes the proof. □

By using similar reasoning, we apply Douglas' theorem to the square roots of the generators to establish a broader class of bounded operators.

The next definition introduces boundedness across mixed generators.

Definition 2. *An operator $X \in \mathbf{L}(\mathscr{H})$ admits a mixed $(\sqrt{A}, \sqrt{B})$-adjoint if it satisfies the range inclusion condition:*

$$\mathrm{Ran}(X^*\sqrt{B}) \subseteq \mathrm{Ran}(\sqrt{A}).$$

By Douglas' theorem, this inclusion is equivalent to the existence of a constant $c > 0$ such that $X^*BX \leq c^2A$. This operator inequality translates precisely to the mixed boundedness condition:

$$\|Xu\|_B \leq c\|u\|_A, \quad \textit{for all } u \in \mathscr{H}.$$

We denote the set of all such $(\sqrt{A}, \sqrt{B})$-bounded operators by $\mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$.

The next result establishes the inclusion between these mixed operator spaces. The proof utilizes operator convexity properties for contractive maps [44,45] and extends a brilliant technique due to Antezana, first presented by Arias, Corach, and Gonzalez in [14].

Lemma 2. *We have $\mathbf{L}_{A,B}(\mathscr{H}) \subseteq \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$.*

Proof. Let $X \in \mathbf{L}_{A,B}(\mathscr{H})$. Assume that X is a contraction, meaning that $\|X\| \leq 1$; otherwise, we scale it. In this case, the map $\Phi : \mathbf{L}(\mathscr{H}) \to \mathbf{L}(\mathscr{H})$ defined by $\Phi(E) = X^*EX$ is a contractive positive map. By the definition of the mixed space, there exists an operator $Y \in \mathbf{L}(\mathscr{H})$ such that $AY = X^*B$. By substituting this, we obtain:

$$X^*B^2X = (X^*B)(BX) = (AY)(Y^*A) = AYY^*A \leq \|Y\|^2A^2.$$

By leveraging the operator convexity properties established for contractive maps [44,45], we obtain $X^*BX \leq \sqrt{X^*B^2X}$. On the other hand, since the function $f(t) = \sqrt{t}$ is operator monotone, we deduce that $\sqrt{X^*B^2X} \leq \sqrt{\|Y\|^2A^2} = \|Y\|A$. By combining these bounds, we get:

$$(X^*\sqrt{B})(X^*\sqrt{B})^* = X^*BX \leq \|Y\|A = \|Y\|(\sqrt{A})(\sqrt{A})^*.$$

By using the equivalence between assertion (ii) and assertion (i) of Douglas' theorem (Theorem 1), this operator inequality guarantees the range inclusion $\mathrm{Ran}(X^*\sqrt{B}) \subseteq \mathrm{Ran}(\sqrt{A})$. Thus, $X \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$. This completes the proof. □

The following definition establishes the core metric topology across mixed generator spaces.

Definition 3. *For an operator $X \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$, we define the mixed (A, B)-operator seminorm by*

$$\|X\|_{A,B} = \sup\{\|Xu\|_B : u \in S^1_A(\mathscr{H})\}.$$

Remark 1. *Let $X \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$. Then $\|X\|_{A,B} = 0$ if and only if $X\big(\overline{\mathrm{Ran}(A)}\big) \subseteq \mathrm{Ker}(B)$. In general, therefore, $\|\cdot\|_{A,B}$ is only a seminorm on $\mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$. It is a norm precisely when B is injective, and no injectivity assumption on A is needed.*

Analogously, the spaces $\mathbf{L}_{B,A}(\mathscr{H})$ and $\mathbf{L}_{\sqrt{B},\sqrt{A}}(\mathscr{H})$, the distinguished mixed adjoint $Y^{\sharp_{B,A}}$, and the mixed seminorm $\|Y\|_{B,A}$ are defined symmetrically for an operator $Y \in \mathbf{L}(\mathscr{H})$ by reversing the roles of A and B. When considering operators under a uniform generator A, we naturally recover the standard notations $X^{\sharp_A}$ and $\mathbf{L}_A(\mathscr{H})$.

The next result expresses the mixed seminorm in terms of the semi-inner product, establishes the fundamental isometric property of the mixed adjoint mapping, and provides a generalized submultiplicativity property for operators mapping across different semi-inner product spaces.

Proposition 1. *Let* $A, B, C \in \mathbf{L}^+(\mathscr{H})$. *Then, the following assertions hold:*

(i) *If* $X \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$, *then the mixed* (A,B)*-operator seminorm satisfies*

$$\|X\|_{A,B} = \sup\{|\langle Xu, v\rangle_B| : u \in S^1_A(\mathscr{H}), v \in S^1_B(\mathscr{H})\}.$$

Consequently, if $X \in \mathbf{L}_{A,B}(\mathscr{H})$, *we obtain the equality*

$$\|X^{\sharp_{A,B}}\|_{B,A} = \|X\|_{A,B}. \tag{7}$$

(ii) *If* $Y \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$ *and* $X \in \mathbf{L}_{\sqrt{B},\sqrt{C}}(\mathscr{H})$, *then* $XY \in \mathbf{L}_{\sqrt{A},\sqrt{C}}(\mathscr{H})$ *and we have*

$$\|XY\|_{A,C} \le \|X\|_{B,C}\|Y\|_{A,B}.$$

Proof. (i) Suppose that $X \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$. By taking the supremum over all $v \in S^1_B(\mathscr{H})$, we obtain $\sup_{v\in S^1_B(\mathscr{H})} |\langle Xu, v\rangle_B| \le \|Xu\|_B$. On the other hand, if $\|Xu\|_B \neq 0$, by choosing $v = \frac{Xu}{\|Xu\|_B} \in S^1_B(\mathscr{H})$, we deduce $\langle Xu, v\rangle_B = \|Xu\|_B$. If $\|Xu\|_B = 0$, the equality holds trivially. Thus, we deduce

$$\sup_{v\in S^1_B(\mathscr{H})} |\langle Xu, v\rangle_B| = \|Xu\|_B. \tag{8}$$

By taking the supremum over all $u \in S^1_A(\mathscr{H})$ in (8), we obtain the first identity.

To establish the norm equality, we utilize the definition of the mixed adjoint. Assume that $X \in \mathbf{L}_{A,B}(\mathscr{H})$. By definition, we have $\langle Xu, v\rangle_B = \langle u, X^{\sharp_{A,B}}v\rangle_A$. By substituting this into the established supremum formula, we obtain:

$$\begin{aligned}\|X\|_{A,B} &= \sup_{u\in S^1_A(\mathscr{H}), v\in S^1_B(\mathscr{H})} |\langle Xu, v\rangle_B| \\ &= \sup_{u\in S^1_A(\mathscr{H}), v\in S^1_B(\mathscr{H})} |\langle u, X^{\sharp_{A,B}}v\rangle_A| \\ &= \|X^{\sharp_{A,B}}\|_{B,A}.\end{aligned}$$

This implies the desired equality.

(ii) Let $u \in S^1_A(\mathscr{H})$. By using the definition of the mixed (B,C)-operator seminorm for X, we obtain for any vector $v \in \mathscr{H}$ that $\|Xv\|_C \le \|X\|_{B,C}\|v\|_B$. By substituting $v = Yu$, we deduce:

$$\|X(Yu)\|_C \le \|X\|_{B,C}\|Yu\|_B.$$

Since $Y \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$, we further bound the right side using the mixed (A,B)-seminorm of Y:

$$\|XYu\|_C \le \|X\|_{B,C}\|Y\|_{A,B}\|u\|_A.$$

Since this bound holds for all $u \in \mathscr{H}$, it follows that $XY \in \mathbf{L}_{\sqrt{A},\sqrt{C}}(\mathscr{H})$. By taking the supremum over all $u \in S^1_A(\mathscr{H})$, we obtain $\|XY\|_{A,C} \le \|X\|_{B,C}\|Y\|_{A,B}$. This completes the proof. □

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3. Generalized Semi-Inner Products for 2 × 2 Operator Matrices

The goal of this section is to determine necessary and sufficient conditions for block operator matrices to admit mixed adjoints. Building upon these conditions, we develop new mixed-generator upper bounds for the spectral radii and seminorms of 2 × 2 matrices within the direct sum space.

In this section, we introduce the foundational spaces and operators utilized throughout the manuscript. Let $\mathscr{H}$ be a complex Hilbert space. Throughout this paper, $\mathcal{K}$ exclusively denotes the direct sum Hilbert space $\mathscr{H} \oplus \mathscr{H}$, and $\mathbb{A}$ denotes the 2 × 2 diagonal operator matrix $\mathrm{diag}(A, B)$, where $A, B \in \mathbf{L}^+(\mathscr{H})$. We equip $\mathcal{K}$ with the canonical inner product:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle u_1, v_1 \rangle_{\mathscr{H}} + \langle u_2, v_2 \rangle_{\mathscr{H}},$$

for all column vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{K}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{K}$. It is evident that $\mathbb{A} \in \mathbf{L}^+(\mathcal{K})$.

The matrix $\mathbb{A}$ induces a positive semi-definite sesquilinear form on $\mathcal{K}$:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{A}} = \langle \mathbb{A}\mathbf{u}, \mathbf{v} \rangle = \langle Au_1, v_1 \rangle_{\mathscr{H}} + \langle Bu_2, v_2 \rangle_{\mathscr{H}} = \langle u_1, v_1 \rangle_A + \langle u_2, v_2 \rangle_B.$$

The following theorem provides necessary and sufficient conditions for an operator matrix to admit an $\mathbb{A}$-adjoint.

Theorem 2. *An operator matrix* $\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ *admits an* $\mathbb{A}$*-adjoint (i.e.,* $\mathbb{X} \in \mathbf{L}_{\mathbb{A}}(\mathcal{K})$*) if and only if its components admit their respective mixed adjoints, specifically:* $X_{11} \in \mathbf{L}_A(\mathscr{H})$, $X_{22} \in \mathbf{L}_B(\mathscr{H})$, $X_{12} \in \mathbf{L}_{B,A}(\mathscr{H})$, *and* $X_{21} \in \mathbf{L}_{A,B}(\mathscr{H})$. *In this case, we have*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{\sharp_{\mathbb{A}}} = \begin{bmatrix} X_{11}^{\sharp_A} & X_{21}^{\sharp_{A,B}} \\ X_{12}^{\sharp_{B,A}} & X_{22}^{\sharp_B} \end{bmatrix}. \tag{9}$$

Proof. By invoking Douglas' theorem, an operator matrix $\mathbb{X}$ belongs to $\mathbf{L}_{\mathbb{A}}(\mathcal{K})$ if and only if the range inclusion $\mathrm{Ran}(\mathbb{X}^*\mathbb{A}) \subseteq \mathrm{Ran}(\mathbb{A})$ holds. Let us explicitly evaluate the product $\mathbb{X}^*\mathbb{A}$:

$$\mathbb{X}^*\mathbb{A} = \begin{bmatrix} X_{11}^* & X_{21}^* \\ X_{12}^* & X_{22}^* \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} = \begin{bmatrix} X_{11}^*A & X_{21}^*B \\ X_{12}^*A & X_{22}^*B \end{bmatrix}.$$

Since $\mathbb{A} = \mathrm{diag}(A, B)$, its range is strictly the direct sum of the ranges of its diagonal components, meaning $\mathrm{Ran}(\mathbb{A}) = \mathrm{Ran}(A) \oplus \mathrm{Ran}(B)$. Therefore, the global condition $\mathrm{Ran}(\mathbb{X}^*\mathbb{A}) \subseteq \mathrm{Ran}(\mathbb{A})$ is mathematically equivalent to stating that for any column vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{K}$, the following holds:

$$\begin{bmatrix} X_{11}^*Au_1 + X_{21}^*Bu_2 \\ X_{12}^*Au_1 + X_{22}^*Bu_2 \end{bmatrix} \in \mathrm{Ran}(A) \oplus \mathrm{Ran}(B). \tag{10}$$

Assume that $\mathbb{X} \in \mathbf{L}_{\mathbb{A}}(\mathcal{K})$. By setting $u_2 = 0$ in (10), we obtain $X_{11}^*Au_1 \in \mathrm{Ran}(A)$ and $X_{12}^*Au_1 \in \mathrm{Ran}(B)$ for all $u_1 \in \mathscr{H}$. This implies $\mathrm{Ran}(X_{11}^*A) \subseteq \mathrm{Ran}(A)$ and $\mathrm{Ran}(X_{12}^*A) \subseteq \mathrm{Ran}(B)$. By Douglas' theorem, this guarantees $X_{11} \in \mathbf{L}_A(\mathscr{H})$ and $X_{12} \in \mathbf{L}_{B,A}(\mathscr{H})$. Similarly, by setting $u_1 = 0$, we obtain $X_{21}^*Bu_2 \in \mathrm{Ran}(A)$ and $X_{22}^*Bu_2 \in \mathrm{Ran}(B)$ for all $u_2 \in \mathscr{H}$. This implies $\mathrm{Ran}(X_{21}^*B) \subseteq \mathrm{Ran}(A)$ and $\mathrm{Ran}(X_{22}^*B) \subseteq \mathrm{Ran}(B)$. Hence, $X_{21} \in \mathbf{L}_{A,B}(\mathscr{H})$ and $X_{22} \in \mathbf{L}_B(\mathscr{H})$.

Conversely, assume that the components belong to their respective mixed spaces. By definition, we have $\mathrm{Ran}(X_{11}^{*}A) \subseteq \mathrm{Ran}(A)$, $\mathrm{Ran}(X_{12}^{*}A) \subseteq \mathrm{Ran}(B)$, $\mathrm{Ran}(X_{21}^{*}B) \subseteq \mathrm{Ran}(A)$, and $\mathrm{Ran}(X_{22}^{*}B) \subseteq \mathrm{Ran}(B)$. Since $\mathrm{Ran}(A)$ and $\mathrm{Ran}(B)$ are linear subspaces, they are closed under addition. Thus, for any vectors $u_1, u_2 \in \mathscr{H}$, we obtain:

$$X_{11}^{*}Au_1 + X_{21}^{*}Bu_2 \in \mathrm{Ran}(A),$$

$$X_{12}^{*}Au_1 + X_{22}^{*}Bu_2 \in \mathrm{Ran}(B).$$

It follows that the resulting vector resides in $\mathrm{Ran}(A) \oplus \mathrm{Ran}(B) = \mathrm{Ran}(\mathbb{A})$. Therefore, $\mathrm{Ran}(\mathbb{X}^{*}\mathbb{A}) \subseteq \mathrm{Ran}(\mathbb{A})$, which implies $\mathbb{X} \in \mathbf{L}_{\mathbb{A}}(\mathscr{K})$.

Let $\mathbb{S}$ be the matrix on the right-hand side of (9). By hypothesis, we have the identities $AX_{11}^{\sharp_A} = X_{11}^{*}A$, $BX_{22}^{\sharp_B} = X_{22}^{*}B$, $BX_{12}^{\sharp_{B,A}} = X_{12}^{*}A$, and $AX_{21}^{\sharp_{A,B}} = X_{21}^{*}B$. By computing the matrix product $\mathbb{A}\mathbb{S}$, we obtain:

$$\begin{aligned}
\mathbb{A}\mathbb{S} &= \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} \begin{bmatrix} X_{11}^{\sharp_A} & X_{21}^{\sharp_{A,B}} \\ X_{12}^{\sharp_{B,A}} & X_{22}^{\sharp_B} \end{bmatrix} \\
&= \begin{bmatrix} AX_{11}^{\sharp_A} & AX_{21}^{\sharp_{A,B}} \\ BX_{12}^{\sharp_{B,A}} & BX_{22}^{\sharp_B} \end{bmatrix} \\
&= \begin{bmatrix} X_{11}^{*}A & X_{21}^{*}B \\ X_{12}^{*}A & X_{22}^{*}B \end{bmatrix} \\
&= \begin{bmatrix} X_{11}^{*} & X_{21}^{*} \\ X_{12}^{*} & X_{22}^{*} \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} = \mathbb{X}^{*}\mathbb{A}.
\end{aligned}$$

Since $\mathbb{X}^{\sharp_{\mathbb{A}}}$ is the unique operator satisfying $\mathbb{A}\mathbb{X}^{\sharp_{\mathbb{A}}} = \mathbb{X}^{*}\mathbb{A}$ such that $\mathrm{Ran}(\mathbb{X}^{\sharp_{\mathbb{A}}}) \subseteq \overline{\mathrm{Ran}(\mathbb{A})}$, it remains to show that $\mathrm{Ran}(\mathbb{S}) \subseteq \overline{\mathrm{Ran}(\mathbb{A})}$. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathscr{K}$. Since $\mathrm{Ran}(X_{11}^{\sharp_A}) \subseteq \overline{\mathrm{Ran}(A)}$ and $\mathrm{Ran}(X_{21}^{\sharp_{A,B}}) \subseteq \overline{\mathrm{Ran}(A)}$, the first component of $\mathbb{S}\mathbf{u}$ resides in $\overline{\mathrm{Ran}(A)}$. Similarly, the second component resides in $\overline{\mathrm{Ran}(B)}$. Thus, we deduce that $\mathbb{S}\mathbf{u} \in \overline{\mathrm{Ran}(A)} \oplus \overline{\mathrm{Ran}(B)} = \overline{\mathrm{Ran}(\mathbb{A})}$. This implies that $\mathrm{Ran}(\mathbb{S}) \subseteq \overline{\mathrm{Ran}(\mathbb{A})}$. This completes the proof. □

Remark 2. *It should be noted here that the first summand of $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$ carries the seminorm $\|\cdot\|_A$ and the second carries $\|\cdot\|_B$. Throughout the paper, the block X_{ij} of an operator matrix acts from the j-th summand to the i-th summand, so its domain and codomain are in general equipped with different seminorms. For instance, X_{12} maps the* B*-summand into the* A*-summand, which is why $X_{12} \in \mathbf{L}_{B,A}(\mathscr{H})$ and why its seminorm is $\|X_{12}\|_{B,A}$; likewise, X_{21} maps the* A*-summand into the* B*-summand and belongs to $\mathbf{L}_{A,B}(\mathscr{H})$. In every mixed notion attached to a block, the generator of the domain is written first and that of the codomain second. The mixed adjoint reverses the direction of the block. Thus $X_{21}^{\sharp_{A,B}}$ maps the* B*-summand into the* A*-summand and occupies the $(1,2)$ position in (9), while $X_{12}^{\sharp_{B,A}}$ occupies the $(2,1)$ position. This convention is adopted solely to avoid confusion between the indices of the blocks and those of the generators.*

In the following corollary, we can directly apply Theorem 2 by replacing the positive generators A and B with their respective square roots $\sqrt{A}$ and $\sqrt{B}$.

Corollary 1. *An operator matrix $\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbf{L}_{\sqrt{\mathbb{A}}}(\mathscr{K})$ if and only if its components satisfy $X_{11} \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$, $X_{22} \in \mathbf{L}_{\sqrt{B}}(\mathscr{H})$, $X_{12} \in \mathbf{L}_{\sqrt{B},\sqrt{A}}(\mathscr{H})$, and $X_{21} \in \mathbf{L}_{\sqrt{A},\sqrt{B}}(\mathscr{H})$.*

The next result provides an upper bound for the $\mathbb{A}$-operator seminorm.

Lemma 3. *Let* $\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbf{L}_{\sqrt{\mathbb{A}}}(\mathscr{K})$. *Then, we have*

$$\|\mathbb{X}\|_{\mathbb{A}} \leq \|\widehat{\mathbb{X}}^{\mathbb{A}}\|, \tag{11}$$

where $\widehat{\mathbb{X}}^{\mathbb{A}} = \begin{bmatrix} \|X_{11}\|_A & \|X_{12}\|_{B,A} \\ \|X_{21}\|_{A,B} & \|X_{22}\|_B \end{bmatrix}$.

Proof. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathscr{K}$. By using the definition, we obtain:

$$\begin{aligned} \|\mathbb{X}\mathbf{u}\|_{\mathbb{A}}^2 &= \|\sqrt{\mathbb{A}}\mathbb{X}\mathbf{u}\|^2 \\ &= \|X_{11}u_1 + X_{12}u_2\|_A^2 + \|X_{21}u_1 + X_{22}u_2\|_B^2 \\ &\leq (\|X_{11}u_1\|_A + \|X_{12}u_2\|_A)^2 + (\|X_{21}u_1\|_B + \|X_{22}u_2\|_B)^2. \end{aligned} \tag{12}$$

Let $\hat{\mathbf{u}}^{\mathbb{A}} = \begin{bmatrix} \|u_1\|_A \\ \|u_2\|_B \end{bmatrix} \in \mathbb{R}^2$. Since $\|\hat{\mathbf{u}}^{\mathbb{A}}\| = \sqrt{\|u_1\|_A^2 + \|u_2\|_B^2} = \|\mathbf{u}\|_{\mathbb{A}}$, by expanding (12), we obtain:

$$\begin{aligned} \|\mathbb{X}\mathbf{u}\|_{\mathbb{A}}^2 &\leq (\|X_{11}\|_A\|u_1\|_A + \|X_{12}\|_{B,A}\|u_2\|_B)^2 \\ &\quad + (\|X_{21}\|_{A,B}\|u_1\|_A + \|X_{22}\|_B\|u_2\|_B)^2 \\ &= \left\|\widehat{\mathbb{X}}^{\mathbb{A}}\hat{\mathbf{u}}^{\mathbb{A}}\right\|^2 \\ &\leq \|\widehat{\mathbb{X}}^{\mathbb{A}}\|^2\|\hat{\mathbf{u}}^{\mathbb{A}}\|^2 = \|\widehat{\mathbb{X}}^{\mathbb{A}}\|^2\|\mathbf{u}\|_{\mathbb{A}}^2. \end{aligned}$$

By taking the supremum over all $\mathbf{u} \in \mathrm{S}_{\mathbb{A}}^1(\mathscr{K})$ in the last inequality, we get (11) as requested. This completes the proof. □

We are now in a position to state our next main result, which bounds the $\mathbb{A}$-spectral radius of an arbitrary 2×2 block operator matrix by the ordinary spectral radius of the 2×2 scalar matrix formed from the mixed seminorms of its entries.

Theorem 3. *Let* $\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbf{L}_{\sqrt{\mathbb{A}}}(\mathscr{K})$. *Then,* $r_{\mathbb{A}}(\mathbb{X}) \leq r(\widehat{\mathbb{X}}^{\mathbb{A}})$. *That is*

$$r_{\mathbb{A}}\left(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\right) \leq r\left(\begin{bmatrix} \|X_{11}\|_A & \|X_{12}\|_{B,A} \\ \|X_{21}\|_{A,B} & \|X_{22}\|_B \end{bmatrix}\right). \tag{13}$$

Proof. Let $\mathbb{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \in \mathbf{L}_{\sqrt{\mathbb{A}}}(\mathscr{K})$. To prove (13), we first establish a general entrywise inequality between the 2×2 real matrices $\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}$ and $\widehat{\mathbb{X}}^{\mathbb{A}}\widehat{\mathbb{Y}}^{\mathbb{A}}$ for appropriately bounded block matrices $\mathbb{X}$ and $\mathbb{Y}$. To formalize the mappings, let $E_1 = A$ and $E_2 = B$. The (i, j) block component of an operator matrix on $\mathscr{K}$ maps from the space equipped with the generator E_j to the space equipped with the generator E_i. Thus, its bounding mixed seminorm is evaluated as $\|\cdot\|_{E_j,E_i}$.

By evaluating the (i,j) entry of $\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}$ using matrix multiplication and the triangle inequality, we obtain:

$$\left(\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}\right)_{ij} = \left\|\sum_{k=1}^{2} \mathrm{X}_{ik}\mathrm{Y}_{kj}\right\|_{\mathrm{E}_j,\mathrm{E}_i} \leq \sum_{k=1}^{2} \left\|\mathrm{X}_{ik}\mathrm{Y}_{kj}\right\|_{\mathrm{E}_j,\mathrm{E}_i}.$$

By employing the generalized mixed submultiplicativity established in Proposition 1(ii), we systematically bound each term:

$$\left(\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}\right)_{ij} \leq \sum_{k=1}^{2} \|\mathrm{X}_{ik}\|_{\mathrm{E}_k,\mathrm{E}_i} \|\mathrm{Y}_{kj}\|_{\mathrm{E}_j,\mathrm{E}_k}.$$

We observe that this upper bound is exactly the algebraic formula for the (i,j) entry of the matrix product $\widehat{\mathbb{X}}^{\mathbb{A}}\widehat{\mathbb{Y}}^{\mathbb{A}}$. Since this holds for all indices $i,j \in \{1,2\}$, we deduce the global entrywise matrix inequality:

$$0 \leq \widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}} \leq \widehat{\mathbb{X}}^{\mathbb{A}}\widehat{\mathbb{Y}}^{\mathbb{A}}.$$

It is a standard property of the spectral norm for matrices with non-negative entries that if $0 \leq \mathrm{M} \leq \mathrm{N}$ entrywise, then $\|\mathrm{M}\| \leq \|\mathrm{N}\|$. By applying this monotonicity property, we infer:

$$\|\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}\| \leq \|\widehat{\mathbb{X}}^{\mathbb{A}}\widehat{\mathbb{Y}}^{\mathbb{A}}\|.$$

Finally, since the standard matrix norm is submultiplicative, we obtain $\|\widehat{\mathbb{X}}^{\mathbb{A}}\widehat{\mathbb{Y}}^{\mathbb{A}}\| \leq \|\widehat{\mathbb{X}}^{\mathbb{A}}\|\|\widehat{\mathbb{Y}}^{\mathbb{A}}\|$, which yields the fundamental matrix bound:

$$\|\widehat{\mathbb{X}\mathbb{Y}}^{\mathbb{A}}\| \leq \|\widehat{\mathbb{X}}^{\mathbb{A}}\|\|\widehat{\mathbb{Y}}^{\mathbb{A}}\|. \tag{14}$$

By employing (11) alongside (14) and an induction argument, we deduce:

$$\|\mathbb{X}^n\|_{\mathbb{A}} \leq \left\|\widehat{\mathbb{X}^n}^{\mathbb{A}}\right\| \leq \left\|\left(\widehat{\mathbb{X}}^{\mathbb{A}}\right)^n\right\|,$$

for all $n \in \mathbb{N}$. By utilizing the formula for the $\mathbb{A}$-spectral radius, we obtain:

$$r_{\mathbb{A}}(\mathbb{X}) = \lim_{n\to\infty} \|\mathbb{X}^n\|_{\mathbb{A}}^{\frac{1}{n}} \leq \lim_{n\to\infty} \left\|\left(\widehat{\mathbb{X}}^{\mathbb{A}}\right)^n\right\|^{\frac{1}{n}} = r(\widehat{\mathbb{X}}^{\mathbb{A}}).$$

By using this, we obtain (13). □

To establish our subsequent result, we evaluate the seminorms and numerical radii by isolating the diagonal and anti-diagonal components.

Proposition 2. *Let* $\mathbb{X} = \begin{bmatrix} \mathrm{X}_{11} & \mathbf{0} \\ \mathbf{0} & \mathrm{X}_{22} \end{bmatrix}, \mathbb{Y} = \begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix} \in \mathbf{L}_{\sqrt{\mathbb{A}}}(\mathscr{K})$. *Then, the following assertions hold:*

(a) $\|\mathbb{X}\|_{\mathbb{A}} = \max(\|\mathrm{X}_{11}\|_{\mathrm{A}}, \|\mathrm{X}_{22}\|_{\mathrm{B}})$.
(b) $\|\mathbb{Y}\|_{\mathbb{A}} = \max\left(\|\mathrm{X}_{12}\|_{\mathrm{B,A}}, \|\mathrm{X}_{21}\|_{\mathrm{A,B}}\right)$.

Proof. (a) Let $\mathbf{u} = \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} \in \mathscr{K}$. Applying the Cauchy–Schwarz and arithmetic-geometric mean inequalities, we obtain:

$$\begin{aligned}|\langle \mathbb{X}\mathbf{u}, \mathbf{v}\rangle_{\mathbb{A}}| &\leq |\langle X_{11}u_1, v_1\rangle_A| + |\langle X_{22}u_2, v_2\rangle_B| \\ &\leq \|X_{11}\|_A\|u_1\|_A\|v_1\|_A + \|X_{22}\|_B\|u_2\|_B\|v_2\|_B \\ &\leq \max(\|X_{11}\|_A, \|X_{22}\|_B)\frac{1}{2}\left(\|u_1\|_A^2 + \|v_1\|_A^2 + \|u_2\|_B^2 + \|v_2\|_B^2\right) \\ &= \frac{1}{2}\left(\|\mathbf{u}\|_{\mathbb{A}}^2 + \|\mathbf{v}\|_{\mathbb{A}}^2\right)\max(\|X_{11}\|_A, \|X_{22}\|_B). \end{aligned} \tag{15}$$

By taking the supremum over independent vectors $\mathbf{u}, \mathbf{v} \in S^1_{\mathbb{A}}(\mathcal{K})$ in (15), we deduce:

$$\|\mathbb{X}\|_{\mathbb{A}} \leq \max(\|X_{11}\|_A, \|X_{22}\|_B).$$

Let $x, z \in S^1_A(\mathscr{H})$. By letting $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} z \\ 0 \end{bmatrix}$, we have $\mathbf{u}, \mathbf{v} \in S^1_{\mathbb{A}}(\mathcal{K})$. By evaluating the inner product, we obtain:

$$\|\mathbb{X}\|_{\mathbb{A}} \geq |\langle \mathbb{X}\mathbf{u}, \mathbf{v}\rangle_{\mathbb{A}}| = |\langle X_{11}x, z\rangle_A|.$$

By taking the supremum over all such x and z, we deduce that $\|\mathbb{X}\|_{\mathbb{A}} \geq \|X_{11}\|_A$. Symmetrically, by isolating the second component, we infer that $\|\mathbb{X}\|_{\mathbb{A}} \geq \|X_{22}\|_B$. This implies the desired equality.

(b) Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{K}$. By expanding the squared seminorm, we obtain:

$$\begin{aligned}\|\mathbb{Y}\mathbf{u}\|_{\mathbb{A}}^2 &= \|X_{12}u_2\|_A^2 + \|X_{21}u_1\|_B^2 \\ &\leq \|X_{12}\|_{B,A}^2\|u_2\|_B^2 + \|X_{21}\|_{A,B}^2\|u_1\|_A^2 \\ &\leq \max\left(\|X_{12}\|_{B,A}^2, \|X_{21}\|_{A,B}^2\right)\left(\|u_1\|_A^2 + \|u_2\|_B^2\right) \\ &= \max\left(\|X_{12}\|_{B,A}^2, \|X_{21}\|_{A,B}^2\right)\|\mathbf{u}\|_{\mathbb{A}}^2.\end{aligned}$$

This implies that $\|\mathbb{Y}\|_{\mathbb{A}} \leq \max(\|X_{12}\|_{B,A}, \|X_{21}\|_{A,B})$. Let $x \in S^1_B(\mathscr{H})$ and define $\mathbf{u} = \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathcal{K}$. Since $\mathbf{u} \in S^1_{\mathbb{A}}(\mathcal{K})$, we obtain:

$$\|\mathbb{Y}\|_{\mathbb{A}} \geq \|\mathbb{Y}\mathbf{u}\|_{\mathbb{A}} = \|X_{12}x\|_A.$$

By taking the supremum over all $x \in S^1_B(\mathscr{H})$, we deduce $\|\mathbb{Y}\|_{\mathbb{A}} \geq \|X_{12}\|_{B,A}$. Similarly, by isolating the first component, we infer $\|\mathbb{Y}\|_{\mathbb{A}} \geq \|X_{21}\|_{A,B}$. This implies the equality, which completes the proof. □

The next definition introduces the concepts of mixed isometries and mixed unitary operators mapping between two distinct semi-inner product spaces, extending the standard A-unitary framework [12].

Definition 4. *An operator $X \in \mathbf{L}(\mathscr{H})$ is called an (A, B)-isometry if it satisfies $\|Xu\|_B = \|u\|_A$ for all $u \in \mathscr{H}$. Furthermore, an operator $U \in \mathbf{L}_{A,B}(\mathscr{H})$ is called an (A, B)-unitary operator if U is an (A, B)-isometry and $U^{\sharp_{A,B}}$ is a (B, A)-isometry. We denote the set of all (A, B)-unitary operators by $\mathcal{U}_{A,B}(\mathscr{H})$.*

Remark 3. *It should be noted here that if $A = B$, then $\mathcal{U}_{A,B}(\mathscr{H})$ reduces to $\mathcal{U}_A(\mathscr{H})$, the set of all A-unitary operators, originally introduced in [12].*

The next result establishes a generalized unitary invariance property for the numerical radius across different semi-inner product spaces.

Lemma 4. *Suppose that* $U \in \mathcal{U}_{A,B}(\mathscr{H})$ *is an* (A,B)*-unitary operator. Let* $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ *and* $Y \in \mathbf{L}_{\sqrt{B}}(\mathscr{H})$*. Then, the following hold:*

$$\omega_B(UXU^{\sharp_{A,B}}) = \omega_A(X), \tag{16}$$

and

$$\omega_A(U^{\sharp_{A,B}}YU) = \omega_B(Y). \tag{17}$$

Proof. Let $u \in S_B^1(\mathscr{H})$. By definition, since U is an (A,B)-unitary operator, we have $\|U^{\sharp_{A,B}}u\|_A = \|u\|_B = 1$. Let $v = U^{\sharp_{A,B}}u \in S_A^1(\mathscr{H})$. By using the definition of the mixed adjoint, we obtain:

$$\langle UXU^{\sharp_{A,B}}u, u\rangle_B = \langle XU^{\sharp_{A,B}}u, U^{\sharp_{A,B}}u\rangle_A = \langle Xv, v\rangle_A.$$

By taking the supremum over all $u \in S_B^1(\mathscr{H})$, we deduce that $\omega_B(UXU^{\sharp_{A,B}}) \leq \omega_A(X)$.

Conversely, suppose that $y \in S_A^1(\mathscr{H})$. Let $z = Uy$. Since $\|Uy\|_B = \|y\|_A = 1$, we get $z \in S_B^1(\mathscr{H})$. Since $\|Uy\|_B^2 = \|y\|_A^2$, we deduce that $\langle U^*BUy, y\rangle = \langle Ay, y\rangle$ for all y, which implies $U^*BU = A$. By definition of the mixed adjoint, we have $AU^{\sharp_{A,B}} = U^*B$. Thus, we obtain $AU^{\sharp_{A,B}}U = U^*BU = A$. This implies that $U^{\sharp_{A,B}}Uy - y \in \mathrm{Ker}(A)$. Since $X \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$, it leaves $\mathrm{Ker}(A)$ invariant, so $AXU^{\sharp_{A,B}}Uy = AXy$. Therefore, we obtain:

$$\begin{aligned}\langle UXU^{\sharp_{A,B}}z, z\rangle_B &= \langle XU^{\sharp_{A,B}}Uy, U^{\sharp_{A,B}}Uy\rangle_A\\ &= \langle AXU^{\sharp_{A,B}}Uy, U^{\sharp_{A,B}}Uy\rangle\\ &= \langle AXy, U^{\sharp_{A,B}}Uy\rangle\\ &= \langle Xy, AU^{\sharp_{A,B}}Uy\rangle\\ &= \langle Xy, Ay\rangle = \langle Xy, y\rangle_A.\end{aligned}$$

By taking the supremum over all such $z \in S_B^1(\mathscr{H})$, we deduce $\omega_A(X) \leq \omega_B(UXU^{\sharp_{A,B}})$. Consequently, we obtain (16) as desired.

Let $x \in S_A^1(\mathscr{H})$. Let $w = Ux$. Because U is an (A,B)-unitary operator, we have $\|w\|_B = \|Ux\|_B = \|x\|_A = 1$. Thus, we obtain $w \in S_B^1(\mathscr{H})$. By applying the definition of the mixed adjoint, we obtain:

$$\langle U^{\sharp_{A,B}}YUx, x\rangle_A = \langle YUx, Ux\rangle_B = \langle Yw, w\rangle_B.$$

By taking the supremum over all $x \in S_A^1(\mathscr{H})$, we deduce $\omega_A(U^{\sharp_{A,B}}YU) \leq \omega_B(Y)$.

Conversely, suppose that $t \in S_B^1(\mathscr{H})$. Let $s = U^{\sharp_{A,B}}t$. Because U is an (A,B)-unitary operator, we have $\|s\|_A = \|U^{\sharp_{A,B}}t\|_A = \|t\|_B = 1$, which implies $s \in S_A^1(\mathscr{H})$. Since $\|U^{\sharp_{A,B}}t\|_A^2 = \|t\|_B^2$, we deduce that $\langle AU^{\sharp_{A,B}}t, U^{\sharp_{A,B}}t\rangle = \langle Bt, t\rangle$ for all t. By definition of the mixed adjoint, we have $AU^{\sharp_{A,B}} = U^*B$. This implies $\langle U^*Bt, U^{\sharp_{A,B}}t\rangle = \langle Bt, t\rangle$, and therefore $\langle Bt, UU^{\sharp_{A,B}}t\rangle = \langle Bt, t\rangle$. Thus, we obtain $BUU^{\sharp_{A,B}} = B$. This implies that $UU^{\sharp_{A,B}}t - t \in \mathrm{Ker}(B)$. Since $Y \in \mathbf{L}_{\sqrt{B}}(\mathscr{H})$, it leaves $\mathrm{Ker}(B)$ invariant. This implies $BYUU^{\sharp_{A,B}}t = BYt$. Therefore, we deduce:

$$\begin{aligned}\langle U^{\sharp_{A,B}}YUs, s\rangle_A &= \langle YUU^{\sharp_{A,B}}t, UU^{\sharp_{A,B}}t\rangle_B\\ &= \langle BYUU^{\sharp_{A,B}}t, UU^{\sharp_{A,B}}t\rangle\\ &= \langle BYt, UU^{\sharp_{A,B}}t\rangle\\ &= \langle Yt, BUU^{\sharp_{A,B}}t\rangle\\ &= \langle Yt, Bt\rangle = \langle Yt, t\rangle_B.\end{aligned}$$

By taking the supremum over all $t \in S_B^1(\mathscr{H})$, we deduce $\omega_B(Y) \leq \omega_A(U^{\sharp_{A,B}}YU)$. Consequently, we obtain (17) as claimed. This completes the proof. □

Remark 4. *We point out that if* $\mathrm{A} = \mathrm{B}$, *then* $\mathcal{U}_{\mathrm{A},\mathrm{B}}(\mathscr{H})$ *reduces to* $\mathcal{U}_{\mathrm{A}}(\mathscr{H})$, *the class of* A*-unitary operators, and Lemma 4 recovers the known unitary invariance of the* A*-numerical radius,*

$$\omega_{\mathrm{A}}(\mathrm{UXU}^{\sharp_{\mathrm{A}}}) = \omega_{\mathrm{A}}(\mathrm{U}^{\sharp_{\mathrm{A}}}\mathrm{XU}) = \omega_{\mathrm{A}}(\mathrm{X}), \tag{18}$$

for all $\mathrm{U} \in \mathcal{U}_{\mathrm{A}}(\mathscr{H})$ *and* $\mathrm{X} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$; *see [46].*

The next theorem establishes several structural assertions for appropriately bounded operators.

Theorem 4. *The following assertions hold for appropriately bounded operators:*

(i) Let $\mathrm{X}_{11} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$ *and* $\mathrm{X}_{22} \in \mathbf{L}_{\sqrt{\mathrm{B}}}(\mathscr{H})$. *Then*

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathrm{X}_{11} & \mathbf{0} \\ \mathbf{0} & \mathrm{X}_{22} \end{bmatrix}\right) = \max\{\omega_{\mathrm{A}}(\mathrm{X}_{11}), \omega_{\mathrm{B}}(\mathrm{X}_{22})\}.$$

(ii) Let $\begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix} \in \mathbf{L}_{\mathbb{A}}(\mathscr{K})$ *and* $\widetilde{\mathbb{A}} = \mathrm{diag}(\mathrm{B}, \mathrm{A})$. *Then*

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix}\right) = \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \mathrm{X}_{22} & \mathrm{X}_{21} \\ \mathrm{X}_{12} & \mathrm{X}_{11} \end{bmatrix}\right). \tag{19}$$

In particular, we have

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right) = \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{21} \\ \mathrm{X}_{12} & \mathbf{0} \end{bmatrix}\right). \tag{20}$$

(iii) Let $\mathrm{X}_{12} \in \mathbf{L}_{\mathrm{B},\mathrm{A}}(\mathscr{H})$ *and* $\mathrm{X}_{21} \in \mathbf{L}_{\mathrm{A},\mathrm{B}}(\mathscr{H})$. *Then for any* $\theta \in \mathbb{R}$,

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ e^{i\theta}\mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right) = \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right).$$

Proof. (i) Let $\mathbb{T} = \begin{bmatrix} \mathrm{X}_{11} & \mathbf{0} \\ \mathbf{0} & \mathrm{X}_{22} \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix} \in \mathrm{S}^1_{\mathbb{A}}(\mathscr{K})$, which implies $\|\mathrm{u}_1\|^2_{\mathrm{A}} + \|\mathrm{u}_2\|^2_{\mathrm{B}} = 1$. By using the Cauchy–Schwarz inequality for the mixed components, we deduce:

$$\begin{aligned} |\langle \mathbb{T}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{A}}| &\leq |\langle \mathrm{X}_{11}\mathrm{u}_1, \mathrm{u}_1\rangle_{\mathrm{A}}| + |\langle \mathrm{X}_{22}\mathrm{u}_2, \mathrm{u}_2\rangle_{\mathrm{B}}| \\ &\leq \omega_{\mathrm{A}}(\mathrm{X}_{11})\|\mathrm{u}_1\|^2_{\mathrm{A}} + \omega_{\mathrm{B}}(\mathrm{X}_{22})\|\mathrm{u}_2\|^2_{\mathrm{B}} \\ &\leq \max\{\omega_{\mathrm{A}}(\mathrm{X}_{11}), \omega_{\mathrm{B}}(\mathrm{X}_{22})\}. \end{aligned}$$

By taking the supremum over all $\mathbf{u} \in \mathrm{S}^1_{\mathbb{A}}(\mathscr{K})$, we infer that

$$\omega_{\mathbb{A}}(\mathbb{T}) \leq \max\{\omega_{\mathrm{A}}(\mathrm{X}_{11}), \omega_{\mathrm{B}}(\mathrm{X}_{22})\}.$$

Suppose $\mathrm{x} \in \mathrm{S}^1_{\mathrm{A}}(\mathscr{H})$ and define $\mathbf{u} = \begin{bmatrix} \mathrm{x} \\ 0 \end{bmatrix} \in \mathrm{S}^1_{\mathbb{A}}(\mathscr{K})$. Then, we obtain

$$|\langle \mathbb{T}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{A}}| = |\langle \mathrm{X}_{11}\mathrm{x}, \mathrm{x}\rangle_{\mathrm{A}}|.$$

By taking the supremum over all $\mathrm{x} \in \mathrm{S}^1_{\mathrm{A}}(\mathscr{H})$, we deduce

$$\sup_{\mathrm{x} \in \mathrm{S}^1_{\mathrm{A}}(\mathscr{H})} |\langle \mathbb{T}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{A}}| = \omega_{\mathrm{A}}(\mathrm{X}_{11}),$$

and thus we infer $\omega_{\mathbb{A}}(\mathbb{T}) \geq \omega_{A}(X_{11})$. Similarly, for any $y \in S_{B}^{1}(\mathscr{H})$, by defining $\mathbf{v} = \begin{bmatrix} 0 \\ y \end{bmatrix} \in S_{\mathbb{A}}^{1}(\mathcal{K})$, we deduce $\omega_{\mathbb{A}}(\mathbb{T}) \geq \omega_{B}(X_{22})$. Therefore, we obtain $\omega_{\mathbb{A}}(\mathbb{T}) = \max\{\omega_{A}(X_{11}), \omega_{B}(X_{22})\}$. This completes the proof of Theorem 4(i).

(ii) Let $\widetilde{\mathbb{A}} = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$ be a diagonal operator matrix on $\mathcal{K}$. Let $\mathbb{U} = \begin{bmatrix} \mathbf{0} & I_{\mathscr{H}} \\ I_{\mathscr{H}} & \mathbf{0} \end{bmatrix}$. By computing the matrix multiplication, we obtain:

$$\mathbb{U}^{*}\mathbb{A} = \begin{bmatrix} \mathbf{0} & I_{\mathscr{H}} \\ I_{\mathscr{H}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} = \begin{bmatrix} \mathbf{0} & B \\ A & \mathbf{0} \end{bmatrix}.$$

Since $\mathrm{Ran}(\mathbb{U}^{*}\mathbb{A}) \subseteq \mathrm{Ran}(\widetilde{\mathbb{A}})$ holds trivially, we deduce that $\mathbb{U} \in \mathbf{L}_{\widetilde{\mathbb{A}},\mathbb{A}}(\mathcal{K})$. By solving the equation $\widetilde{\mathbb{A}}\mathbb{Y} = \mathbb{U}^{*}\mathbb{A}$ blockwise, we compute the explicit mixed adjoint using the Moore–Penrose pseudoinverse. Specifically, we multiply the diagonal pseudoinverse $\widetilde{\mathbb{A}}^{\dagger} = \mathrm{diag}(B^{\dagger}, A^{\dagger})$ by the matrix $\mathbb{U}^{*}\mathbb{A}$. Applying the fundamental property $T^{\dagger}T = \Pi_{\overline{\mathrm{Ran}(T)}}$ to the positive generators A and B, we obtain:

$$\mathbb{U}^{\sharp_{\widetilde{\mathbb{A}},\mathbb{A}}} = \widetilde{\mathbb{A}}^{\dagger}\mathbb{U}^{*}\mathbb{A} = \begin{bmatrix} B^{\dagger} & \mathbf{0} \\ \mathbf{0} & A^{\dagger} \end{bmatrix} \begin{bmatrix} \mathbf{0} & B \\ A & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & B^{\dagger}B \\ A^{\dagger}A & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \Pi_{\overline{\mathrm{Ran}(B)}} \\ \Pi_{\overline{\mathrm{Ran}(A)}} & \mathbf{0} \end{bmatrix}.$$

By evaluating the respective mixed seminorms, we verify the isometric properties $\|\mathbb{U}\mathbf{u}\|_{\mathbb{A}} = \|\mathbf{u}\|_{\widetilde{\mathbb{A}}}$ for all $\mathbf{u} \in \mathcal{K}$ and $\|\mathbb{U}^{\sharp_{\widetilde{\mathbb{A}},\mathbb{A}}}\mathbf{v}\|_{\widetilde{\mathbb{A}}} = \|\mathbf{v}\|_{\mathbb{A}}$ for all $\mathbf{v} \in \mathcal{K}$. Thus, we deduce that $\mathbb{U}$ is a $(\widetilde{\mathbb{A}}, \mathbb{A})$-unitary operator mapping between the two semi-inner product spaces. Let $\mathbb{S} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$. Applying unitary invariance across these isometrically isomorphic spaces presented in (17), we obtain:

$$\begin{aligned}
\omega_{\mathbb{A}}(\mathbb{S}) &= \omega_{\widetilde{\mathbb{A}}}\left(\mathbb{U}^{\sharp_{\widetilde{\mathbb{A}},\mathbb{A}}}\mathbb{S}\mathbb{U}\right) \\
&= \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \Pi_{\overline{\mathrm{Ran}(B)}}X_{22} & \Pi_{\overline{\mathrm{Ran}(B)}}X_{21} \\ \Pi_{\overline{\mathrm{Ran}(A)}}X_{12} & \Pi_{\overline{\mathrm{Ran}(A)}}X_{11} \end{bmatrix}\right) \\
&= \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \Pi_{\overline{\mathrm{Ran}(B)}} & \mathbf{0} \\ \mathbf{0} & \Pi_{\overline{\mathrm{Ran}(A)}} \end{bmatrix}\begin{bmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{bmatrix}\right) \\
&= \omega_{\widetilde{\mathbb{A}}}\left(\Pi_{\overline{\mathrm{Ran}(\widetilde{\mathbb{A}})}}\begin{bmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{bmatrix}\right) = \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} X_{22} & X_{21} \\ X_{12} & X_{11} \end{bmatrix}\right).
\end{aligned}$$

The last equality follows from [38]. This proves (19). Suppose that $X_{11} = \mathbf{0}$ and $X_{22} = \mathbf{0}$. By substituting these into (19), we obtain the desired equality (20).

(iii) Suppose that $\theta \in \mathbb{R}$. Let $\mathbb{T} = \begin{bmatrix} \mathbf{0} & X_{12} \\ X_{21} & \mathbf{0} \end{bmatrix}$ and $\mathbb{U} = \begin{bmatrix} I_{\mathscr{H}} & \mathbf{0} \\ \mathbf{0} & e^{-i\frac{\theta}{2}}I_{\mathscr{H}} \end{bmatrix}$. It can be seen that $\mathbb{U}\mathbb{U}^{*} = I_{\mathcal{K}}$ and $\mathbb{A}\mathbb{U} = \mathbb{U}\mathbb{A}$. By definition, we have $\mathbb{A}\mathbb{U}^{\sharp_{\mathbb{A}}} = \mathbb{U}^{*}\mathbb{A}$. Then, we obtain $\mathbb{U}\mathbb{A}\mathbb{U}^{\sharp_{\mathbb{A}}} = \mathbb{U}\mathbb{U}^{*}\mathbb{A} = \mathbb{A}$. Since $\mathbb{A}\mathbb{U} = \mathbb{U}\mathbb{A}$, it follows that $\mathbb{A}\mathbb{U}\mathbb{U}^{\sharp_{\mathbb{A}}} = \mathbb{A}$. Similarly, we get $\mathbb{A}\mathbb{U}^{\sharp_{\mathbb{A}}}\mathbb{U} = \mathbb{A}$. Thus, we deduce that $\mathbb{U} \in \mathcal{U}_{\mathbb{A}}(\mathcal{K})$. Applying (18), we obtain $\omega_{\mathbb{A}}(\mathbb{U}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbb{U}) = \omega_{\mathbb{A}}(\mathbb{T})$. Since $\mathbb{A}\mathbb{U} = \mathbb{U}\mathbb{A}$, we have $\mathbb{U}^{\sharp_{\mathbb{A}}} = \Pi_{\overline{\mathrm{Ran}(\mathbb{A})}}\mathbb{U}^{*}$ [12]. Since $\Pi_{\overline{\mathrm{Ran}(\mathbb{A})}}$ does not affect the $\mathbb{A}$-numerical

radius, it follows that $\omega_{\mathbb{A}}(\mathbb{U}^{\sharp_{\mathbb{A}}}\mathbb{T}\mathbb{U}) = \omega_{\mathbb{A}}(\mathbb{U}^*\mathbb{T}\mathbb{U})$. By computing the matrix multiplication, we get:

$$\mathbb{U}^*\mathbb{T}\mathbb{U} = \begin{bmatrix} \mathrm{I}_{\mathscr{H}} & \mathbf{0} \\ \mathbf{0} & e^{i\frac{\theta}{2}}\mathrm{I}_{\mathscr{H}} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathrm{I}_{\mathscr{H}} & \mathbf{0} \\ \mathbf{0} & e^{-i\frac{\theta}{2}}\mathrm{I}_{\mathscr{H}} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & e^{-i\frac{\theta}{2}}\mathrm{X}_{12} \\ e^{i\frac{\theta}{2}}\mathrm{X}_{21} & \mathbf{0} \end{bmatrix} = e^{-i\frac{\theta}{2}}\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ e^{i\theta}\mathrm{X}_{21} & \mathbf{0} \end{bmatrix}.$$

We have $\omega_{\mathbb{A}}(e^{-i\frac{\theta}{2}}\mathbb{S}) = \omega_{\mathbb{A}}(\mathbb{S})$ for any block operator matrix $\mathbb{S}$. We deduce:

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ e^{i\theta}\mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right) = \omega_{\mathbb{A}}(\mathbb{T}).$$

This completes the proof. □

Remark 5. *Alternatively, as pointed out by the referee, we can prove Theorem 4(ii) directly without explicitly using generalized mixed-unitary operators. Let* $\mathbb{U} = \begin{bmatrix} \mathbf{0} & \mathrm{I}_{\mathscr{H}} \\ \mathrm{I}_{\mathscr{H}} & \mathbf{0} \end{bmatrix}$. *We observe that* $\mathbb{U}^*\widetilde{\mathbb{A}}\mathbb{U} = \mathbb{A}$. *Let* $\mathbf{u} \in \mathcal{K}$. *Since* $\mathbb{U}\mathbf{u} = \begin{bmatrix} \mathrm{u}_2 \\ \mathrm{u}_1 \end{bmatrix}$, *we have*

$$\|\mathbb{U}\mathbf{u}\|^2_{\widetilde{\mathbb{A}}} = \langle\widetilde{\mathbb{A}}\mathbb{U}\mathbf{u}, \mathbb{U}\mathbf{u}\rangle = \langle\mathbb{U}^*\widetilde{\mathbb{A}}\mathbb{U}\mathbf{u}, \mathbf{u}\rangle = \langle\mathbb{A}\mathbf{u}, \mathbf{u}\rangle = \|\mathbf{u}\|^2_{\mathbb{A}}.$$

Furthermore, for any block operator matrix $\mathbb{S}$, *the numerical radius formulation yields*

$$\omega_{\widetilde{\mathbb{A}}}(\mathbb{U}\mathbb{S}\mathbb{U}) = \sup_{\mathbf{v}\in\mathbb{S}^1_{\widetilde{\mathbb{A}}}(\mathcal{K})} |\langle\mathbb{U}\mathbb{S}\mathbb{U}\mathbf{v}, \mathbf{v}\rangle_{\widetilde{\mathbb{A}}}|.$$

By performing the change in variables $\mathbf{v} = \mathbb{U}\mathbf{u}$ *and utilizing the identity* $\mathbb{U}^*\widetilde{\mathbb{A}}\mathbb{U} = \mathbb{A}$, *we obtain* $\langle\mathbb{U}\mathbb{S}\mathbb{U}\mathbf{v}, \mathbf{v}\rangle_{\widetilde{\mathbb{A}}} = \langle\widetilde{\mathbb{A}}\mathbb{U}\mathbb{S}\mathbf{u}, \mathbb{U}\mathbf{u}\rangle = \langle\mathbb{U}^*\widetilde{\mathbb{A}}\mathbb{U}\mathbb{S}\mathbf{u}, \mathbf{u}\rangle = \langle\mathbb{A}\mathbb{S}\mathbf{u}, \mathbf{u}\rangle = \langle\mathbb{S}\mathbf{u}, \mathbf{u}\rangle_{\mathbb{A}}$. *Taking the supremum over all unit vectors gives*

$$\omega_{\widetilde{\mathbb{A}}}(\mathbb{U}\mathbb{S}\mathbb{U}) = \omega_{\mathbb{A}}(\mathbb{S}).$$

By applying this to $\mathbb{S} = \begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix}$, *we immediately deduce* (19).

The following result is a consequence of Theorem 4(i) by setting A = B and was already proved in [38]. For the sake of completeness, we include its proof.

Corollary 2. *Let* $\mathrm{X}, \mathrm{Y} \in \mathbf{L}_{\mathrm{A}}(\mathscr{H})$.

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathrm{X} & \mathrm{Y} \\ \mathrm{Y} & \mathrm{X} \end{bmatrix}\right) = \max\{\omega_{\mathrm{A}}(\mathrm{X}+\mathrm{Y}), \omega_{\mathrm{A}}(\mathrm{X}-\mathrm{Y})\}.$$

In particular, by taking $\mathrm{X} = \mathbf{0}$, *we obtain:*

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{Y} \\ \mathrm{Y} & \mathbf{0} \end{bmatrix}\right) = \omega_{\mathrm{A}}(\mathrm{Y}).$$

Proof. Assume that A = B. Then $\mathbb{A} = \mathrm{diag}(\mathrm{A}, \mathrm{A})$. Let $\mathbb{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{I}_{\mathscr{H}} & \mathrm{I}_{\mathscr{H}} \\ \mathrm{I}_{\mathscr{H}} & -\mathrm{I}_{\mathscr{H}} \end{bmatrix}$. Since $\mathbb{U}\mathbb{U}^* = \mathrm{I}_{\mathcal{K}}$ and $\mathbb{U}\mathbb{A} = \mathbb{A}\mathbb{U}$, we deduce that $\mathbb{U}$ is an $\mathbb{A}$-unitary operator. Let $\mathbb{S} = \begin{bmatrix} \mathrm{X} & \mathrm{Y} \\ \mathrm{Y} & \mathrm{X} \end{bmatrix}$. By computing

the matrix multiplication, we obtain $\mathbb{U}^*\mathbb{S}\mathbb{U} = \begin{bmatrix} X+Y & \mathbf{0} \\ \mathbf{0} & X-Y \end{bmatrix}$. By applying the unitary invariance stated in (18), we obtain $\omega_{\mathbb{A}}(\mathbb{S}) = \omega_{\mathbb{A}}(\mathbb{U}^{\sharp_{\mathbb{A}}}\mathbb{S}\mathbb{U})$. Since $\mathbb{U}^{\sharp_{\mathbb{A}}} = \Pi_{\overline{\mathrm{Ran}(\mathbb{A})}}\mathbb{U}^*$ and $\Pi_{\overline{\mathrm{Ran}(\mathbb{A})}}$ does not affect the $\mathbb{A}$-numerical radius, we have $\omega_{\mathbb{A}}(\mathbb{S}) = \omega_{\mathbb{A}}(\mathbb{U}^*\mathbb{S}\mathbb{U})$. By using Theorem 4(i) for the diagonal block matrix, we obtain

$$\omega_{\mathbb{A}}(\mathbb{U}^*\mathbb{S}\mathbb{U}) = \max\{\omega_A(X+Y), \omega_A(X-Y)\}.$$

This yields the first identity. Suppose that $X = \mathbf{0}$. We obtain $\max\{\omega_A(Y), \omega_A(-Y)\}$. Since $\omega_A(-Y) = \omega_A(Y)$, we deduce the second identity. This completes the proof. □

4. Applications to Generalized Numerical Radius for 2 × 2 Operator Matrices

In this section, we aim to demonstrate the invariance of the generalized numerical radius under distinct isometric and unitary transformations, and calculate sharp estimations for anti-diagonal operator blocks across separate seminorms.

The next result presents a computational lemma for the anti-diagonal component of a 2 × 2 operator matrix across distinct generators.

Lemma 5. *Let $X_{12} \in \mathbf{L}_{B,A}(\mathscr{H})$ and $X_{21} \in \mathbf{L}_{A,B}(\mathscr{H})$. Then, we have*

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & X_{12} \\ X_{21} & \mathbf{0} \end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}X_{12} + e^{-i\theta}X_{21}^{\sharp_{A,B}}\right\|_{B,A}.$$

Proof. Let $\mathbb{Y} = \begin{bmatrix} \mathbf{0} & X_{12} \\ X_{21} & \mathbf{0} \end{bmatrix}$. By using Proposition 2(b) together with (7), we obtain:

$$\left\|\begin{bmatrix} \mathbf{0} & Z \\ Z^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}} = \|Z\|_{B,A}, \tag{21}$$

for every $Z \in \mathbf{L}_{B,A}(\mathscr{H})$. By utilizing the properties of the numerical radius for the generalized adjoint (since $\omega_{\mathbb{A}}(\mathbb{Y}) = \omega_{\mathbb{A}}(\mathbb{Y}^{\sharp_{\mathbb{A}}})$) and applying the first identity in (2) directly to $\mathbb{Y}^{\sharp_{\mathbb{A}}}$ together with (9), we deduce:

$$\begin{aligned}
\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & X_{12} \\ X_{21} & \mathbf{0} \end{bmatrix}\right) &= \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & X_{12} \\ X_{21} & \mathbf{0} \end{bmatrix}^{\sharp_{\mathbb{A}}}\right) \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}\begin{bmatrix} \mathbf{0} & X_{21}^{\sharp_{A,B}} \\ X_{12}^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix} + e^{-i\theta}\begin{bmatrix} \mathbf{0} & (X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}} \\ (X_{21}^{\sharp_{A,B}})^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \mathbf{0} & e^{i\theta}X_{21}^{\sharp_{A,B}} + e^{-i\theta}(X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}} \\ e^{i\theta}X_{12}^{\sharp_{B,A}} + e^{-i\theta}(X_{21}^{\sharp_{A,B}})^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} \mathbf{0} & e^{i\theta}X_{21}^{\sharp_{A,B}} + e^{-i\theta}(X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}} \\ \left(e^{i\theta}X_{21}^{\sharp_{A,B}} + e^{-i\theta}(X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}}\right)^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}},
\end{aligned}$$

where the last equality follows from (6). Since $e^{i\theta}X_{21}^{\sharp_{A,B}} + e^{-i\theta}(X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}} \in \mathbf{L}_{B,A}(\mathscr{H})$ and its (B, A)-adjoint constitutes the (2, 1) block component, then by evaluating the mixed norms through (21), the maximum simplifies to a single term. By using this, we deduce:

$$\begin{aligned}\omega_{\mathbb{A}}(\mathbb{Y}) &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}X_{21}^{\sharp_{A,B}} + e^{-i\theta}(X_{12}^{\sharp_{B,A}})^{\sharp_{A,B}}\right\|_{B,A}\\ &= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{i\theta}X_{12}^{\sharp_{B,A}} + e^{-i\theta}X_{21}\right\|_{A,B}.\end{aligned}$$

The last equality holds by directly applying (7) to the composite operator $Z = e^{i\theta}X_{12}^{\sharp_{B,A}} + e^{-i\theta}X_{21}$, since the expression inside the (B, A)-seminorm is precisely $Z^{\sharp_{A,B}}$. By replacing θ with $-\theta$ in the last equality, we obtain:

$$\omega_{\mathbb{A}}\left(\begin{bmatrix}\mathbf{0} & X_{12}\\ X_{21} & \mathbf{0}\end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{-i\theta}X_{12}^{\sharp_{B,A}} + e^{i\theta}X_{21}\right\|_{A,B}. \tag{22}$$

Let $\widetilde{\mathbb{A}} = \begin{bmatrix}B & \mathbf{0}\\ \mathbf{0} & A\end{bmatrix}$ be an operator matrix on $\mathscr{K}$. By Theorem 4(ii), we infer that

$$\omega_{\mathbb{A}}\left(\begin{bmatrix}\mathbf{0} & X_{12}\\ X_{21} & \mathbf{0}\end{bmatrix}\right) = \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix}\mathbf{0} & X_{21}\\ X_{12} & \mathbf{0}\end{bmatrix}\right). \tag{23}$$

By applying the established identity (22) to the permuted matrix $\begin{bmatrix}\mathbf{0} & X_{21}\\ X_{12} & \mathbf{0}\end{bmatrix}$ operating under the flipped seminorm system generated by $\widetilde{\mathbb{A}}$, we explicitly interchange the roles of X_{12} and X_{21}, along with the generators A and B. From this substitution, we obtain:

$$\omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix}\mathbf{0} & X_{21}\\ X_{12} & \mathbf{0}\end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|e^{-i\theta}X_{21}^{\sharp_{A,B}} + e^{i\theta}X_{12}\right\|_{B,A}. \tag{24}$$

By substituting (24) into the unitary equivalence (23), we get the desired result. This completes the proof. □

The next result provides the main generalized upper bound utilizing a 2×2 scalar matrix structure.

Corollary 3. *Let* $\mathbb{X} = \begin{bmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix} \in \mathbf{L}_{\mathbb{A}}(\mathscr{K})$. *Then,*

$$\omega_{\mathbb{A}}(\mathbb{X}) \le \omega(\mathbb{S}),$$

where $\mathbb{S} = [S_{ij}] \in \mathbb{M}_2(\mathbb{R})$ *is given by*

$$\mathbb{S} = \begin{bmatrix}\omega_A(X_{11}) & \omega_{\mathbb{A}}\left(\begin{bmatrix}\mathbf{0} & X_{12}\\ X_{21} & \mathbf{0}\end{bmatrix}\right)\\ \omega_{\mathbb{A}}\left(\begin{bmatrix}\mathbf{0} & X_{12}\\ X_{21} & \mathbf{0}\end{bmatrix}\right) & \omega_B(X_{22})\end{bmatrix}. \tag{25}$$

Proof. By using the definition of the real part with respect to $\mathbb{A}$ for $\theta \in \mathbb{R}$, we obtain:

$$\begin{aligned}\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}) &= \frac{1}{2}\left(e^{i\theta}\mathbb{X} + e^{-i\theta}\mathbb{X}^{\sharp_{\mathbb{A}}}\right)\\ &= \begin{bmatrix}\mathfrak{R}_A(e^{i\theta}X_{11}) & \frac{1}{2}\left(e^{i\theta}X_{12} + e^{-i\theta}X_{21}^{\sharp_{A,B}}\right)\\ \frac{1}{2}\left(e^{i\theta}X_{21} + e^{-i\theta}X_{12}^{\sharp_{B,A}}\right) & \mathfrak{R}_B(e^{i\theta}X_{22})\end{bmatrix}.\end{aligned}$$

By applying Lemma 3, specifically the block seminorm estimate (11), alongside the norm monotonicity of matrices with non-negative entries, and subsequently utilizing Lemma 5 and the properties of the numerical radius, we deduce:

$$\left\|\Re_{\mathbb{A}}(e^{i\theta}\mathbb{X})\right\|_{\mathbb{A}} \leq \left\|\begin{bmatrix} \omega_{\mathrm{A}}(\mathrm{X}_{11}) & \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right) \\ \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{21} \\ \mathrm{X}_{12} & \mathbf{0} \end{bmatrix}\right) & \omega_{\mathrm{B}}(\mathrm{X}_{22}) \end{bmatrix}\right\|.$$

By taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality, we obtain $\omega_{\mathbb{A}}(\mathbb{X}) \leq \|\mathbb{S}\|$, where $\mathbb{S}$ is defined in (25). Finally, by using (23), we deduce $\omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{21} \\ \mathrm{X}_{12} & \mathbf{0} \end{bmatrix}\right) = \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right)$. Since $\mathbb{S}$ is a real symmetric matrix, then $\omega(\mathbb{S}) = \|\mathbb{S}\|$. By using this, we infer the required inequality. This completes the proof. □

Recall that the notation $\mathrm{Y} \leq_{\mathrm{A}} \mathrm{X}$ means that $\mathrm{X} - \mathrm{Y} \geq_{\mathrm{A}} \mathbf{0}$, which is equivalent to the operator inequality $\mathrm{A}(\mathrm{X} - \mathrm{Y}) \geq \mathbf{0}$. The next lemma establishes an essential monotonicity property and was already stated in [37].

Lemma 6. *Let* $\mathrm{X}, \mathrm{Y} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$ *be* A*-bounded operators. If* $\mathbf{0} \leq_{\mathrm{A}} \mathrm{Y} \leq_{\mathrm{A}} \mathrm{X}$, *then* $\|\mathrm{Y}\|_{\mathrm{A}} \leq \|\mathrm{X}\|_{\mathrm{A}}$.

The next result provides the main bounds for the mixed anti-diagonal matrix.

Theorem 5. *Let* $\mathrm{X}_{12} \in \mathbf{L}_{\mathrm{B,A}}(\mathscr{H})$ *and* $\mathrm{X}_{21} \in \mathbf{L}_{\mathrm{A,B}}(\mathscr{H})$. *Then, for* $\mathbb{T} = \begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}$, *the following bounds hold:*

$$\omega_{\mathbb{A}}^2(\mathbb{T}) \geq \frac{1}{4}\max\left\{\|\mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} + \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\mathrm{X}_{21}\|_{\mathrm{A}}, \|\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\mathrm{X}_{12} + \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\|_{\mathrm{B}}\right\},$$

and

$$\omega_{\mathbb{A}}^2(\mathbb{T}) \leq \frac{1}{2}\max\left\{\|\mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} + \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\mathrm{X}_{21}\|_{\mathrm{A}}, \|\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\mathrm{X}_{12} + \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\|_{\mathrm{B}}\right\}.$$

Proof. Let $\mathbb{H}_\theta = \Re_{\mathbb{A}}(e^{i\theta}\mathbb{T}) = \frac{1}{2}(e^{i\theta}\mathbb{T} + e^{-i\theta}\mathbb{T}^{\sharp_{\mathbb{A}}})$ and

$$\mathbb{K}_\theta = \Im_{\mathbb{A}}(e^{i\theta}\mathbb{T}) = \frac{1}{2i}(e^{i\theta}\mathbb{T} - e^{-i\theta}\mathbb{T}^{\sharp_{\mathbb{A}}}).$$

By expanding the squares of these components, we deduce:

$$\begin{aligned} \mathbb{H}_\theta^2 + \mathbb{K}_\theta^2 &= \frac{1}{4}\left(e^{2i\theta}\mathbb{T}^2 + \mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} + \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + e^{-2i\theta}(\mathbb{T}^{\sharp_{\mathbb{A}}})^2\right) \\ &\quad - \frac{1}{4}\left(e^{2i\theta}\mathbb{T}^2 - \mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} - \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} + e^{-2i\theta}(\mathbb{T}^{\sharp_{\mathbb{A}}})^2\right) \\ &= \frac{1}{2}\left(\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} + \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\right). \end{aligned}$$

Next, by explicitly computing the products $\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}}$ and $\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}$ based on Theorem 2, we obtain:

$$\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} = \begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} \\ \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \\ \mathbf{0} & \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} \end{bmatrix},$$

$$\mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} = \begin{bmatrix} \mathbf{0} & \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} \\ \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\mathrm{X}_{21} & \mathbf{0} \\ \mathbf{0} & \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\mathrm{X}_{12} \end{bmatrix}.$$

By summing these two block diagonal matrices, we obtain:

$$\mathbb{H}_\theta^2 + \mathbb{K}_\theta^2 = \frac{1}{2}\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}, \tag{26}$$

where we define $\mathrm{M} = \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} + \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\mathrm{X}_{21} \in \mathbf{L}_{\mathrm{A}}(\mathscr{H})$ and $\mathrm{N} = \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\mathrm{X}_{12} + \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} \in \mathbf{L}_{\mathrm{B}}(\mathscr{H})$.

By taking the norm on both sides and utilizing the triangle inequality and submultiplicativity of the seminorm, we deduce

$$\frac{1}{2}\left\|\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}\right\|_{\mathbb{A}} = \|\mathbb{H}_\theta^2 + \mathbb{K}_\theta^2\|_{\mathbb{A}} \leq \|\mathbb{H}_\theta\|_{\mathbb{A}}^2 + \|\mathbb{K}_\theta\|_{\mathbb{A}}^2 \leq 2\omega_{\mathbb{A}}^2(\mathbb{T}).$$

By evaluating the norm of the diagonal operator matrix, we obtain

$$\frac{1}{2}\max\{\|\mathrm{M}\|_{\mathrm{A}}, \|\mathrm{N}\|_{\mathrm{B}}\} \leq 2\omega_{\mathbb{A}}^2(\mathbb{T}).$$

This completes the proof of the first inequality.

Conversely, by rearranging the equality $\mathbb{H}_\theta^2 + \mathbb{K}_\theta^2 = \frac{1}{2}\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}$, we obtain:

$$\mathbb{H}_\theta^2 - \frac{1}{2}\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix} = -\mathbb{K}_\theta^2 \leq_{\mathbb{A}} \mathbf{0}.$$

Consequently, we infer that $\mathbb{H}_\theta^2 \leq_{\mathbb{A}} \frac{1}{2}\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}$. Since $\mathbb{H}_\theta$ is $\mathbb{A}$-selfadjoint, its square satisfies $\mathbb{H}_\theta^2 \geq_{\mathbb{A}} \mathbf{0}$. Moreover, the block diagonal matrix $\frac{1}{2}\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}$ is clearly $\mathbb{A}$-positive by taking (26) into consideration. Thus, the hypotheses of Lemma 6 are satisfied. Applying this lemma, together with the identity $\|\mathbb{H}_\theta^2\|_{\mathbb{A}} = \|\mathbb{H}_\theta\|_{\mathbb{A}}^2$, which holds because $\mathbb{H}_\theta$ is $\mathbb{A}$-selfadjoint (see [19]), we deduce that

$$\|\mathbb{H}_\theta\|_{\mathbb{A}}^2 \leq \frac{1}{2}\left\|\begin{bmatrix} \mathrm{M} & \mathbf{0} \\ \mathbf{0} & \mathrm{N} \end{bmatrix}\right\|_{\mathbb{A}} = \frac{1}{2}\max\{\|\mathrm{M}\|_{\mathrm{A}}, \|\mathrm{N}\|_{\mathrm{B}}\}.$$

By taking the supremum over all $\theta \in \mathbb{R}$, we infer

$$\omega_{\mathbb{A}}^2(\mathbb{T}) \leq \frac{1}{2}\max\{\|\mathrm{M}\|_{\mathrm{A}}, \|\mathrm{N}\|_{\mathrm{B}}\}.$$

This finishes the proof of the second inequality. □

The next result is a direct corollary for the uniform single-space framework. It follows by letting $\mathrm{X}_{12} = \mathrm{X}_{21} = \mathrm{T}$ in Theorem 5, and it was originally obtained in [20]. For $\mathrm{A} = \mathrm{I}_{\mathscr{H}}$, it recovers the well-known inequalities of Kittaneh [2].

Corollary 4. *Let* $\mathrm{T} \in \mathbf{L}_{\mathrm{A}}(\mathscr{H})$. *Then*

$$\frac{1}{4}\|\mathrm{T}\mathrm{T}^{\sharp_{\mathrm{A}}} + \mathrm{T}^{\sharp_{\mathrm{A}}}\mathrm{T}\|_{\mathrm{A}} \leq \omega_{\mathrm{A}}^2(\mathrm{T}) \leq \frac{1}{2}\|\mathrm{T}\mathrm{T}^{\sharp_{\mathrm{A}}} + \mathrm{T}^{\sharp_{\mathrm{A}}}\mathrm{T}\|_{\mathrm{A}}.$$

The next result provides the commutativity property of the mixed spectral radius and will be required in order to prove our next result.

Lemma 7. *If* $\mathrm{X} \in \mathbf{L}_{\sqrt{\mathrm{A}},\sqrt{\mathrm{B}}}(\mathscr{H})$ *and* $\mathrm{Y} \in \mathbf{L}_{\sqrt{\mathrm{B}},\sqrt{\mathrm{A}}}(\mathscr{H})$, *then* $\mathrm{YX} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$, $\mathrm{XY} \in \mathbf{L}_{\sqrt{\mathrm{B}}}(\mathscr{H})$, *and*

$$r_{\mathrm{A}}(\mathrm{YX}) = r_{\mathrm{B}}(\mathrm{XY}).$$

Proof. By using Proposition 1(ii), we obtain $YX \in \mathbf{L}_{\sqrt{A}}(\mathscr{H})$ and $XY \in \mathbf{L}_{\sqrt{B}}(\mathscr{H})$. Let $n \geq 2$ be an integer. We have

$$(YX)^n = Y(XY)^{n-1}X.$$

By applying the submultiplicativity property for mixed seminorms, we get

$$\|(YX)^n\|_A \leq \|Y\|_{B,A}\|(XY)^{n-1}\|_B\|X\|_{A,B}.$$

It follows that

$$\|(YX)^n\|_A^{\frac{1}{n}} \leq \left(\|Y\|_{B,A}\|X\|_{A,B}\right)^{\frac{1}{n}}\|(XY)^{n-1}\|_B^{\frac{1}{n}}.$$

By taking the limit as $n \to \infty$, we obtain

$$r_A(YX) \leq \lim_{n\to\infty}\left(\|(XY)^{n-1}\|_B^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}} = r_B(XY).$$

Similarly, we get $r_B(XY) \leq r_A(YX)$. Hence, $r_A(YX) = r_B(XY)$. This completes the proof. □

The next result provides an upper bound for the $\mathbb{A}$-numerical radius through a decomposition of the matrix into row blocks.

Theorem 6. *Let* $\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbf{L}_{\mathbb{A}}(\mathcal{K})$. *Then*

$$\omega_{\mathbb{A}}(\mathbb{X}) \leq \frac{1}{2}\left(\|X_{11}\|_A + \sqrt{\|X_{11}X_{11}^{\sharp_A} + X_{12}X_{12}^{\sharp_{B,A}}\|_A}\right) + \frac{1}{2}\left(\|X_{22}\|_B + \sqrt{\|X_{21}X_{21}^{\sharp_{A,B}} + X_{22}X_{22}^{\sharp_B}\|_B}\right).$$

Proof. Step 1: We first establish a key auxiliary bound for the upper triangular block $\mathbb{X}_1 = \begin{bmatrix} X_{11} & X_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. We claim that:

$$\omega_{\mathbb{A}}(\mathbb{X}_1) \leq \frac{1}{2}\left(\|X_{11}\|_A + \sqrt{\|X_{11}X_{11}^{\sharp_A} + X_{12}X_{12}^{\sharp_{B,A}}\|_A}\right). \tag{27}$$

Since the operator $\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)$ is $\mathbb{A}$-selfadjoint for $\theta \in \mathbb{R}$, we apply (3) to obtain

$$r_{\mathbb{A}}\left(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\right) = \|\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\|_{\mathbb{A}}.$$

On the other hand, by using the spectral radius formulation, we get

$$r_{\mathbb{A}}\left(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\right) = \frac{1}{2}r_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1 + e^{-i\theta}\mathbb{X}_1^{\sharp_{\mathbb{A}}}) = \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} e^{i\theta}X_{11} + e^{-i\theta}X_{11}^{\sharp_A} & e^{i\theta}X_{12} \\ e^{-i\theta}X_{12}^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\right).$$

By factoring the block matrix, we obtain

$$r_{\mathbb{A}}\left(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\right) = \frac{1}{2}r_{\mathbb{A}}\left(\begin{bmatrix} X_{11}^{\sharp_A} & e^{i\theta}I_{\mathscr{H}} \\ X_{12}^{\sharp_{B,A}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} e^{-i\theta}I_{\mathscr{H}} & \mathbf{0} \\ X_{11} & X_{12} \end{bmatrix}\right).$$

Let $\mathbb{C} = \begin{bmatrix} e^{-i\theta}\mathrm{I}_{\mathscr{H}} & \mathbf{0} \\ \mathrm{X}_{11} & \mathrm{X}_{12} \end{bmatrix}$ and $\mathbb{D} = \begin{bmatrix} \mathrm{X}_{11}^{\sharp_{\mathrm{A}}} & e^{i\theta}\mathrm{I}_{\mathscr{H}} \\ \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \end{bmatrix}$. Let $\mathbb{A}' = \mathrm{diag}(\mathrm{A}, \mathrm{A})$ acting on $\mathscr{K}$. Since $\mathrm{X}_{11} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$ and $\mathrm{X}_{12} \in \mathbf{L}_{\sqrt{\mathrm{B}},\sqrt{\mathrm{A}}}(\mathscr{H})$, we deduce $\mathbb{C} \in \mathbf{L}_{\sqrt{\mathbb{A}},\sqrt{\mathbb{A}'}}(\mathscr{K})$. Similarly, since $\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} \in \mathbf{L}_{\sqrt{\mathrm{A}}}(\mathscr{H})$ and $\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} \in \mathbf{L}_{\sqrt{\mathrm{A}},\sqrt{\mathrm{B}}}(\mathscr{H})$, we obtain $\mathbb{D} \in \mathbf{L}_{\sqrt{\mathbb{A}'},\sqrt{\mathbb{A}}}(\mathscr{K})$. By applying Lemma 7, we get

$$\begin{aligned} r_{\mathbb{A}}\big(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\big) &= \frac{1}{2} r_{\mathbb{A}'}(\mathbb{C}\mathbb{D}) \\ &= \frac{1}{2} r_{\mathbb{A}'}\left(\begin{bmatrix} e^{-i\theta}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} & \mathrm{I}_{\mathscr{H}} \\ \mathrm{X}_{11}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} + \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & e^{i\theta}\mathrm{X}_{11} \end{bmatrix}\right). \end{aligned}$$

By using (13) together with the identity $\|\mathrm{X}_{11}^{\sharp_{\mathrm{A}}}\|_{\mathrm{A}} = \|\mathrm{X}_{11}\|_{\mathrm{A}}$, we get

$$\begin{aligned} r_{\mathbb{A}}\big(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\big) &\leq \frac{1}{2} r\left(\begin{bmatrix} \|e^{-i\theta}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}}\|_{\mathrm{A}} & \|\mathrm{I}_{\mathscr{H}}\|_{\mathrm{A}} \\ \|\mathrm{X}_{11}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} + \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\|_{\mathrm{A}} & \|e^{i\theta}\mathrm{X}_{11}\|_{\mathrm{A}} \end{bmatrix}\right) \\ &\leq \frac{1}{2} r\left(\begin{bmatrix} \|\mathrm{X}_{11}\|_{\mathrm{A}} & 1 \\ \|\mathrm{X}_{11}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} + \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\|_{\mathrm{A}} & \|\mathrm{X}_{11}\|_{\mathrm{A}} \end{bmatrix}\right). \end{aligned}$$

It follows that

$$r_{\mathbb{A}}\big(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\big) \leq \frac{1}{2}\left(\|\mathrm{X}_{11}\|_{\mathrm{A}} + \sqrt{\|\mathrm{X}_{11}\mathrm{X}_{11}^{\sharp_{\mathrm{A}}} + \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\|_{\mathrm{A}}}\right).$$

Since

$$\omega_{\mathbb{A}}(\mathbb{X}_1) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\|_{\mathbb{A}} = \sup_{\theta\in\mathbb{R}} r_{\mathbb{A}}\big(\mathfrak{R}_{\mathbb{A}}(e^{i\theta}\mathbb{X}_1)\big),$$

we obtain (27).

Step 2: We estimate the lower triangular block $\mathbb{X}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix}$ and combine the bounds. Let $\widetilde{\mathbb{A}} = \begin{bmatrix} \mathrm{B} & \mathbf{0} \\ \mathbf{0} & \mathrm{A} \end{bmatrix}$ be a diagonal operator matrix on $\mathscr{K}$, and let $\mathbb{U} = \begin{bmatrix} \mathbf{0} & \mathrm{I}_{\mathscr{H}} \\ \mathrm{I}_{\mathscr{H}} & \mathbf{0} \end{bmatrix}$. By (19), we infer that

$$\omega_{\mathbb{A}}(\mathbb{X}_2) = \omega_{\widetilde{\mathbb{A}}}\left(\begin{bmatrix} \mathrm{X}_{22} & \mathrm{X}_{21} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right).$$

By applying (27) to this matrix under the permuted seminorm system generated by $\widetilde{\mathbb{A}}$, we obtain

$$\omega_{\mathbb{A}}(\mathbb{X}_2) \leq \frac{1}{2}\left(\|\mathrm{X}_{22}\|_{\mathrm{B}} + \sqrt{\|\mathrm{X}_{22}\mathrm{X}_{22}^{\sharp_{\mathrm{B}}} + \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\|_{\mathrm{B}}}\right).$$

Since $\mathbb{X} = \mathbb{X}_1 + \mathbb{X}_2$, the triangle inequality for the $\mathbb{A}$-numerical radius gives $\omega_{\mathbb{A}}(\mathbb{X}) \leq \omega_{\mathbb{A}}(\mathbb{X}_1) + \omega_{\mathbb{A}}(\mathbb{X}_2)$. Hence, we obtain the required inequality. This completes the proof. □

The next result presents our final generalized upper bound based on isolating the diagonal and anti-diagonal elements.

Theorem 7. *Let* $\mathbb{X} = \begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix} \in \mathbf{L}_{\mathbb{A}}(\mathscr{K})$. *Then, we have*

$$\omega_{\mathbb{A}}(\mathbb{X}) \leq \max\{\omega_{\mathrm{A}}(\mathrm{X}_{11}), \omega_{\mathrm{B}}(\mathrm{X}_{22})\} + \frac{1}{2}\sqrt{\|\mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\|_{\mathrm{A}}} + \frac{1}{2}\sqrt{\|\mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\|_{\mathrm{B}}}.$$

Proof. By employing the triangle inequality and Theorem 4(i), we obtain:

$$\omega_{\mathbb{A}}(\mathbb{X}) \leq \max\{\omega_{\mathrm{A}}(\mathrm{X}_{11}), \omega_{\mathrm{B}}(\mathrm{X}_{22})\} + \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) + \omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}\right).$$

Since strictly off-diagonal block matrices are nilpotent of degree 2, then by applying (1), we deduce that:

$$\omega_{\mathbb{A}}\left(\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) = \frac{1}{2}\left\|\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}}.$$

Moreover, by (4), namely $\|\mathbb{Z}\|_{\mathbb{A}}^2 = \|\mathbb{Z}\mathbb{Z}^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}}$, along with Proposition 2(a), we obtain:

$$\begin{aligned} \left\|\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}}^2 &= \left\|\begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}} \\ &= \left\|\begin{bmatrix} \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right\|_{\mathbb{A}} \\ &= \|\mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\|_{\mathrm{A}}. \end{aligned}$$

Since scaling by the half coefficient evaluates the first off-diagonal term, we apply analogous arguments to the transposed term corresponding to X_{21} to obtain $\frac{1}{2}\sqrt{\|\mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\|_{\mathrm{B}}}$. Finally, by adding these expressions, we complete the proof. □

The following example illustrates the utility of the mixed-adjoint framework for distinct singular generators, a scenario not covered by the uniform setting $\mathrm{diag}(\mathrm{A}, \mathrm{A})$ as stated.

Example 2. *Let* $\mathscr{H} = \mathbb{C}^2$ *with the standard inner product. Let* $\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\mathrm{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *Then* A *and* B *are distinct singular positive operators. Let* $\mathbb{X} = \begin{bmatrix} \mathbf{0} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathbf{0} \end{bmatrix}$ *act on* $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$, *where* $\mathrm{X}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $\mathrm{X}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *A direct computation gives* $\mathrm{X}_{12}^*\mathrm{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *Since* $\mathrm{Ran}(\mathrm{B}) = \mathrm{span}\{\mathrm{e}_2\}$, *we have* $\mathrm{Ran}(\mathrm{X}_{12}^*\mathrm{A}) \subseteq \mathrm{Ran}(\mathrm{B})$. *Thus* $\mathrm{X}_{12} \in \mathbf{L}_{\mathrm{B,A}}(\mathscr{H})$. *The reduced solution of* $\mathrm{BY} = \mathrm{X}_{12}^*\mathrm{A}$ *is* $\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Similarly, $\mathrm{X}_{21}^*\mathrm{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $\mathrm{Ran}(\mathrm{A}) = \mathrm{span}\{\mathrm{e}_1\}$. *Hence* $\mathrm{Ran}(\mathrm{X}_{21}^*\mathrm{B}) \subseteq \mathrm{Ran}(\mathrm{A})$, *so* $\mathrm{X}_{21} \in \mathbf{L}_{\mathrm{A,B}}(\mathscr{H})$ *and* $\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. *By Theorem 2,* $\mathbb{X} \in \mathbf{L}_{\mathbb{A}}(\mathscr{K})$. *Now, by applying Theorem 5, wee have*

$$\begin{aligned} \mathrm{M} &= \mathrm{X}_{12}\mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}} + \mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}}\mathrm{X}_{21} = 2\mathrm{A}, \\ \mathrm{N} &= \mathrm{X}_{12}^{\sharp_{\mathrm{B,A}}}\mathrm{X}_{12} + \mathrm{X}_{21}\mathrm{X}_{21}^{\sharp_{\mathrm{A,B}}} = 2\mathrm{B}. \end{aligned}$$

Thus $\mathrm{diag}(\mathrm{M}, \mathrm{N}) = 2\mathbb{A}$ *and* $\|\mathrm{M}\|_{\mathrm{A}} = \|\mathrm{N}\|_{\mathrm{B}} = 2$. *Therefore, Theorem 5 gives*

$$\frac{1}{2} \leq \omega_{\mathbb{A}}(\mathbb{X})^2 \leq 1.$$

In fact, $\omega_{\mathbb{A}}(\mathbb{X}) = 1$. *Let* $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in S^1_{\mathbb{A}}(\mathcal{K})$ *with* $u_1 = (a, b)$ *and* $u_2 = (c, d)$. *Then* $|a|^2 + |d|^2 = 1$. *Since* $X_{12}u_2 = (d, 0)$ *and* $X_{21}u_1 = (0, a)$, *we obtain*

$$\langle \mathbb{X}\mathbf{u}, \mathbf{u} \rangle_{\mathbb{A}} = d\bar{a} + a\bar{d} = 2\,\mathrm{Re}(a\bar{d}).$$

Hence $|\langle \mathbb{X}\mathbf{u}, \mathbf{u} \rangle_{\mathbb{A}}| \leq 2|a||d| \leq |a|^2 + |d|^2 = 1$, *with equality for* $a = d = \frac{1}{\sqrt{2}}$. *Thus* $\omega_{\mathbb{A}}(\mathbb{X}) = 1$, *and the upper bound of Theorem 5 is attained.*

We note that on $\overline{\mathrm{Ran}(A)} \oplus \overline{\mathrm{Ran}(B)}$ *the matrix* $\mathbb{X}$ *acts as the ordinary* 2×2 *flip matrix. Hence the example does not produce new numerical values. It only shows that the mixed-adjoint framework applies directly to distinct singular generators* $A \neq B$, *where the uniform setting* $\mathrm{diag}(A, A)$ *does not apply as stated.*

5. Conclusions

In this paper, we have ventured beyond the standard uniform semi-inner product framework to address a natural yet technically demanding question: how do 2×2 block operator matrices behave when their off-diagonal components map between entirely distinct, non-injective semi-Hilbertian spaces? By deploying the generalized two-generator mixed adjoint via Douglas' theorem, we successfully bridged this gap, extracting a cohesive theory of numerical radii that elegantly captures the asymmetry of distinct positive generators. The structural upper and lower bounds established here not only encapsulate previous uniform results as special cases but also open the door to analyzing highly singular, asymmetric physical and operator systems. We hope this distinct-generator framework will serve as a robust and inspiring mathematical tool for researchers exploring the next generation of operator theoretic bounds.

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References

1. Gustafson, K.E.; Rao, D.K.M. *Numerical Range*; Springer: New York, NY, USA, 1997.
2. Kittaneh, F. Numerical radius inequalities for Hilbert spaces operators. *Stud. Math.* **2005**, *168*, 73–80. [CrossRef]
3. Paul, K.; Bag, S. On the numerical radius of a matrix and estimation of bounds for zeros of a polynomial. *Int. J. Math. Math. Sci.* **2012**, *2012*, 129132. [CrossRef]
4. Bhunia, P.; Bag, S.; Paul, K. Numerical radius inequalities and its applications in estimation of zeros of polynomials. *Linear Algebra Appl.* **2019**, *573*, 166–177. [CrossRef]

5. Paul, K.; Bag, S. Estimation of bounds for the zeros of a polynomial using numerical radius. *Appl. Math. Comput.* **2013**, *222*, 231–243. [CrossRef]
6. Yamazaki, T. On upper and lower bounds of the numerical radius and an equality condition. *Stud. Math.* **2007**, *178*, 83–89. [CrossRef]
7. Kittaneh, F.; Moslehian, M.S.; Yamazaki, T. Cartesian decomposition and numerical radius inequalities. *Linear Algebra Appl.* **2015**, *471*, 46–53. [CrossRef]
8. Hirzallah, O.; Kittaneh, F.; Shebrawi, K. Numerical radius inequalities for 2 × 2 operator matrices. *Stud. Math.* **2012**, *210*, 99–115. [CrossRef]
9. Shebrawi, K. Numerical radius inequalities for certain 2 × 2 operator matrices II. *Linear Algebra Appl.* **2017**, *523*, 1–12. [CrossRef]
10. Bhunia, P.; Bag, S.; Paul, K. Numerical radius inequalities of operator matrices with applications. *Linear Multilinear Algebra* **2021**, *69*, 1635–1644. [CrossRef]
11. Krein, M.G. Compact linear operators on functional spaces with two norms. *Integral Equ. Oper. Theory* **1998**, *30*, 140–162. [CrossRef]
12. Arias, M.L.; Corach, G.; Gonzalez, M.C. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **2008**, *428*, 1460–1475. [CrossRef]
13. Arias, M.L.; Corach, G.; Gonzalez, M.C. Metric properties of projections in semi-Hilbertian spaces. *Integral Equ. Oper. Theory* **2008**, *62*, 11–28. [CrossRef]
14. Arias, M.L.; Corach, G.; Gonzalez, M.C. Lifting properties in operator ranges. *Acta Sci. Math.* **2009**, *75*, 635–653.
15. Baklouti, H.; Feki, K.; Sid Ahmed, O.A.M. Joint numerical ranges of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2018**, *555*, 266–284. [CrossRef]
16. Douglas, R.G. On majorization, factorization and range inclusion of operators in Hilbert space. *Proc. Am. Math. Soc.* **1966**, *17*, 413–416. [CrossRef]
17. Saddi, A. *A*-Normal operators in Semi-Hilbertian spaces. *Aust. J. Math. Anal. Appl.* **2012**, *9*, 1–12.
18. Feki, K. A note on the *A*-numerical radius of operators in semi-Hilbertian spaces. *Arch. Math.* **2020**, *115*, 535–544. [CrossRef]
19. Feki, K. Spectral radius of semi-Hilbertian space operators and its applications. *Ann. Funct. Anal.* **2020**, *11*, 929–946. [CrossRef]
20. Zamani, A. *A*-numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.* **2019**, *578*, 159–183. [CrossRef]
21. Baklouti, H.; Namouri, S. Spectral analysis of bounded operators on semi-Hilbertian spaces. *Banach J. Math. Anal.* **2022**, *16*, 12. [CrossRef]
22. Tam, T.-Y.; Zhang, P. Spectral decomposition of selfadjoint matrices in positive semidefinite inner product spaces and its applications. *Linear Multilinear Algebra* **2019**, *67*, 1829–1838. [CrossRef]
23. Bhunia, P.; Nayak, R.K.; Paul, K. Improvement of *A*-numerical radius inequalities of semi-Hilbertian space operators. *Results Math.* **2021**, *76*, 120. [CrossRef]
24. Moslehian, M.S.; Xu, Q.; Zamani, A. Seminorm and numerical radius inequalities of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2020**, *591*, 299–321. [CrossRef]
25. Sen, A.; Birbonshi, R.; Paul, K. A note on the *A*-numerical range of semi-Hilbertian operators. *Linear Algebra Appl.* **2024**, *703*, 268–288. [CrossRef]
26. Bhunia, P.; Kittaneh, F.; Paul, K.; Sen, A. Anderson's theorem and *A*-spectral radius bounds for semi-Hilbertian space operators. *Linear Algebra Appl.* **2023**, *657*, 147–162. [CrossRef]
27. Kittaneh, F.; Zamani, A. Bounds for *A*-numerical radius based on an extension of *A*-Buzano inequality. *J. Comput. Appl. Math.* **2023**, *426*, 115070. [CrossRef]
28. Sababheh, M.; Djordjević, D.S.; Moradi, H.R. Numerical radius and norm bounds via the Moore-Penrose inverse. *Complex Anal. Oper. Theory* **2024**, *18*, 117. [CrossRef]
29. Gourty, A.; Ighachane, M.A.; Kittaneh, F. Further *A*-numerical radius inequalities for semi-Hilbertian space operators. *J. Anal.* **2025**, *33*, 2023–2045. [CrossRef]
30. Guesba, M.; Kittaneh, F. New bounds for the *A*-numerical radius of the product of semi-Hilbert space operators. *J. Anal.* **2026**, *early access*. [CrossRef]
31. Guesba, M.; Tapdigoglu, R. Refined inequalities for the *A*-numerical radius. *J. Anal.* **2026**, *early access*. [CrossRef]
32. Sababheh, M.; Conde, C.; Moradi, H.R. A convex-block approach for numerical radius inequalities. *Funct. Anal. Appl.* **2023**, *57*, 26–30. [CrossRef]
33. Sababheh, M.; Conde, C.; Moradi, H.R. On the numerical radius of an operator matrix. *Math. Inequal. Appl.* **2024**, *27*, 417–434. [CrossRef]
34. Sababheh, M.; Moradi, H.R. More accurate numerical radius inequalities (I). *Linear Multilinear Algebra* **2021**, *69*, 1964–1973. [CrossRef]
35. Arias, M.L.; Mbekhta, M. A-partial isometries and generalized inverses. *Linear Algebra Appl.* **2013**, *439*, 1286–1293. [CrossRef]

36. Zamani, A.; Sahoo, S.; Tapdigoglu, R.; Garaev, M. A-Berezin number inequalities for 2 × 2 operator matrices. *Bull. Malays. Math. Sci. Soc.* **2024**, *47*, 114.
37. Bhunia, P.; Paul, K.; Nayak, R.K. On inequalities for *A*-numerical radius of operators. *Electron. J. Linear Algebra* **2020**, *36*, 143–157.
38. Rout, N.C.; Sahoo, S.; Mishra, D. On $\mathbb{A}$-numerical radius inequalities for 2 × 2 operator matrices. *Linear Multilinear Algebra* **2022**, *70*, 2672–2692.
39. Rout, N.C.; Mishra, D. Further results on $\mathbb{A}$-numerical radius inequalities. *Ann. Funct. Anal.* **2022**, *13*, 13. [CrossRef]
40. Feki, K. Some bounds for the $\mathbb{A}$-numerical radius of certain 2 × 2 operator matrices. *Hacet. J. Math. Stat.* **2021**, *50*, 795–810. [CrossRef]
41. Bhunia, P.; Paul, K. Some improvements of numerical radius inequalities of operators and operator matrices. *Linear Multilinear Algebra* **2022**, *70*, 1995–2013. [CrossRef]
42. Feki, K. Some $\mathbb{A}$-numerical radius inequalities for $d \times d$ operator matrices. *Rend. Circ. Mat. Palermo II. Ser.* **2022**, *71*, 85–103.
43. Daptari, S.; Kittaneh, F.; Sahoo, S. New *A*-numerical radius equalities and inequalities for certain operator matrices and applications. *Results Math.* **2025**, *80*, 22. [CrossRef]
44. Hansen, F.; Pedersen, G.K. Jensen's inequality for operators and Löwner's theorem. *Math. Ann.* **1982**, *258*, 229–241.
45. Hansen, F.; Pedersen, G.K. Jensen's operator inequality. *Bull. Lond. Math. Soc.* **2003**, *35*, 553–564. [CrossRef]
46. Bhunia, P.; Feki, K.; Paul, K. *A*-Numerical radius orthogonality and parallelism of semi-Hilbertian space operators and their applications. *Bull. Iran. Math. Soc.* **2021**, *47*, 435–457. [CrossRef]

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