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Shape Differentiation and Anisotropy Onset of Quadratic Invariant-Manifold Jets in Surface Navier–Stokes Flow

Amr M. Y. Abdelaty *, Hesham A. Abdelghafar * and Ibrahim S. Elshazly

Department of Basic Sciences, Common First Year Deanship, King Saud University, Riyadh 12373, Saudi Arabia; iali2.c@ksu.edu.sa

* Correspondence: ayoussef@ksu.edu.sa (A.M.Y.A.); habdelghafar@ksu.edu.sa (H.A.A.)

Abstract

For a C^1 family of smooth closed genus-zero surfaces, we study the quadratic normal jet of unforced surface Navier–Stokes flow near the zero equilibrium. Under an isolated rank-three Riesz slow block and a three-weight slow–fast gap, local C^2 invariant graphs share this quotient-valued second derivative after tangent-input identification, without requiring uniqueness of the full graph. A fixed-complement representative is C^1 in shape in a bilinear Sobolev-operator norm with infinite-dimensional output. An exact covariance law handles admissible comparisons, including spatial transports with derivative loss. At the unit sphere, the jet vanishes. For symmetric affine deformations, its first variation is a degree-three spherical-harmonic tensor with coefficient $-1/(45\nu)$, vanishing exactly for scalar deformation matrices. For each fixed nonscalar D, the jet is therefore nonzero for all sufficiently small $0 < |\delta| < \delta_0(D)$. The onset is local and non-quantitative; no explicit value of $\delta_0(D)$ is obtained.

Keywords: surface Navier–Stokes equations; invariant manifold; quadratic normal jet; shape differentiation; spectral projection; anisotropy

MSC: 37L10; 58J37; 35Q30; 76D05; 58J50

1. Introduction and Relation to Earlier Results

We consider a C^1 family of smooth closed fluid surfaces, each stationary during its own Navier–Stokes evolution, and ask how surface deformation changes the leading nonlinear geometry near the zero equilibrium. Shape changes both the viscous spectrum and the quadratic coupling of the modes, so the slow spectral subspace alone does not determine how a nearby invariant graph bends in phase space.

1.1. Terminology

The *quadratic normal jet*, abbreviated *normal jet*, is the second derivative of a local invariant graph modulo its slow tangent space; *quotient-valued jet* refers to this same tensor. A *fixed-complement representative* $\overline{W}_\varepsilon$, with ε the shape parameter, expresses it in a chosen reference complement after identifying the tangent inputs. The *leading normal correction* is the Taylor term $\frac{1}{2}\overline{W}_\varepsilon(z,z)$, with z denoting the slow coordinates. Here, "normal" means transverse to the slow space in phase space, not normal to the physical surface. The quotient removes quadratic changes of internal graph coordinates.

Under the quantitative three-weight slow–fast gap used below, each sufficiently small shape admits a normalized local C^2 invariant graph, and every admissible graph in the

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specified class has the same jet after tangent-input identification. A fixed-complement representative of this common jet is C^1 in shape in a bilinear Sobolev-operator norm with infinite-dimensional output, while admissible state and spectral comparisons satisfy an exact covariance law. At the unit sphere, the base jet is zero, so its first shape variation is independent of first-order admissible comparison choices that agree at the base point. For the symmetric affine family $R_\delta(x) = (I + \delta D)x$, that variation is a degree-three spherical-harmonic tensor with coefficient $-1/(45\nu)$, vanishing exactly when the symmetric matrix D is scalar. For every fixed nonscalar D, the actual quadratic normal jet is nonzero for all sufficiently small $0 < |\delta| < \delta_0(D)$. These are the main shape-sensitivity and local onset conclusions. The uniqueness assertion concerns the quadratic quotient tensor: admissible full invariant graphs need not be unique, and their higher-order geometry is not identified by this result.

Parameter dependence of invariant manifolds is available in several related settings. The parameterization method constructs invariant manifolds associated with nonresonant subspaces [1], and Cabré–Fontich–de la Llave prove parameter regularity for families of maps on Banach spaces under spectral and nonresonance hypotheses [2] (Theorem 1.1). Spectral-submanifold theory provides a nonresonant model-reduction framework [3]. For a finite-dimensional system, King et al. obtain the C^r parameter dependence of low-order SSM coefficients across a Hopf bifurcation, with coefficient uniqueness under their stated assumptions [4] (Lemma III.4 and Theorem III.7). Jain–Li develop coefficient recursions for parameter-dependent invariant manifolds of polynomial ODEs, including systems obtained by spatial discretization [5] (Section 2); related SSM sensitivity and optimization methods are developed in [6,7].

In infinite-dimensional problems, rigorous invariant-manifold results include the nonlinear beam analysis of Kogelbauer–Haller [8], continuum-vibration SSMs [9], time-delay SSMs [10], and stable and unstable manifolds for capillary waves and other quasilinear PDEs [11]. For Navier–Stokes flow on fixed domains, Buza constructs spectral submanifolds and complementary invariant foliations near equilibria [12] (Theorem 4.10), with smooth-semiflow foliation theory developed further in [13]; equation-driven invariant-manifold reduction for Navier–Stokes Hopf bifurcations is developed in [14]. On fixed surfaces, Prüss–Simonett–Wilke provide Navier–Stokes well-posedness and surface–Stokes spectral foundations [15] (Theorem 3.5 and Proposition 4.1), and Simonett–Wilke establish a bounded H^∞ calculus for the shifted surface–Stokes operator [16] (Theorem 3.1).

Domain and shape dependence enter the literature through a different set of results. Varchon proves convergence of local invariant manifolds under domain perturbation [17] (Theorem 4.1), while Sa Ngiamsunthorn proves the upper and lower semicontinuity of local stable and unstable manifolds under the respective domain-convergence and hyperbolicity hypotheses [18] (Theorems 2.5–2.6). Chitour–Kateb–Long develop the shape differentiation of Stokes spectral data for variable Dirichlet domains in three dimensions [19] (Theorem 2.6 and Section 2.4). These works provide the relevant comparison points for parameter dependence, infinite-dimensional fluid dynamics, and domain/shape variation.

1.2. Contribution and Novelty

The principal advancement is the shape differentiation of a graph-independent quadratic quotient tensor for a moving-surface PDE whose nonlinearity and spatial comparisons lose derivatives. The Banach-map parameter theory above supplies a broader parameter-regularity context; the cited finite-dimensional SSM results treat coefficients of ODE invariant manifolds. The fixed-domain fluid results establish invariant manifolds for an unchanged equation, while the domain-perturbation and Stokes shape results concern manifold convergence or linear spectral data. Here, the fixed-flux common domain, finite-

time generator-domain gain and raised coefficient estimates permit a C^1 derivative of the common quadratic normal jet in an infinite-dimensional output norm. The exact covariance law records its dependence on the physical and spectral comparisons. The spherical calculation then identifies an explicit cubic-harmonic first variation at a zero jet.

Proposition 1 provides the bridge to that calculation: on a genus-zero surface, stream functions modulo constants represent divergence-free velocities, and the fixed-flux stream map conjugates the generators, Riesz projections and quadratic convection. Thus, the scalar spherical tensor represents the same velocity-space quotient jet. The proposition is proved after the surface–Stokes realization and sign conventions have been established.

In fixed-complement coordinates, $\frac{1}{2}\overline{W}_\varepsilon(z,z)$ reconstructs the leading complementary part of a state from its slow amplitudes. Its shape derivative measures how geometry changes that nonlinear reconstruction. At the sphere, the selected rotational modes have zero quadratic normal correction; a nonscalar deformation activates a quadratic coupling into complementary cubic-harmonic stream modes. This is information about the local invariant geometry. It does not by itself imply a change of stability or a transition to turbulence, and the reduced slow dynamics are a separate quantity, computed in Proposition 7.

1.3. Scope and Limitations

The analysis concerns a C^1 family in C^{s+7} of smooth closed genus-zero surfaces, the unforced zero equilibrium, constant viscosity, deformation-tensor surface stress and a quantitatively separated rank-three slow block. The three-weight gap is sufficient rather than optimal. When the base jet is nonzero, the raw shape derivative is generally comparison dependent. The onset result is local and non-quantitative: for each fixed nonscalar D the argument yields some $\delta_0(D) > 0$, but no explicit or numerical value. The exact ellipsoid certificate is a separate spectral-separation application; it does not establish a nonzero jet on that ellipsoid.

2. Model, Hypotheses and Main Conclusions

2.1. The Equation and Its Fixed Scale

Let M be a compact, connected, oriented smooth surface without boundary, diffeomorphic to S^2. For a metric g, write ∇_g for the Levi–Civita connection, k_g for Gaussian curvature, and J_g for positive rotation by $\pi/2$ in the tangent plane. Our scalar Laplacian is $\Delta_g = \operatorname{div}_g \operatorname{grad}_g$, with nonpositive spectrum. The Bochner Laplacian has the same sign. With the surface deformation tensor

$$\operatorname{Def}_g u = \tfrac{1}{2}\mathcal{L}_u g, \qquad 2\operatorname{div}_g \operatorname{Def}_g u = \Delta_{B,g} u + k_g u + \operatorname{grad}_g \operatorname{div}_g u, \tag{1}$$

the unforced, unit-density equation is

$$\partial_t u + \nabla_u u - 2\nu \operatorname{div}_g \operatorname{Def}_g u + \operatorname{grad}_g p = 0, \qquad \operatorname{div}_g u = 0, \qquad \nu > 0. \tag{2}$$

The viscosity ν is constant in space and in the shape parameter. All surfaces are stationary during the fluid evolution; the parameter labels different equations.

For scalar functions, define the constant-valued mean and augmented Laplacian

$$m_g f = \frac{\int_M f \, \mathrm{d}\mu_g}{\int_M 1 \, \mathrm{d}\mu_g}, \qquad \mathscr{D}_g = \Delta_g - m_g, \qquad \Pi_g = I - \operatorname{grad}_g \mathscr{D}_g^{-1} \operatorname{div}_g . \tag{3}$$

The inverse in this formula acts on the full scalar space: $\mathscr{D}_g = -I$ on constants and equals Δ_g on mean-zero functions. The construction therefore remains on full scalar spaces, with the constant/mean-zero splitting used only to describe the inverse.

Let $R_\varepsilon : M \to \mathbb{R}^3$ be embeddings, $g_\varepsilon = R_\varepsilon^* g_{\mathrm{Eucl}}$, and $j_\varepsilon = \mathrm{d}\mu_{g_\varepsilon} / \mathrm{d}\mu_{g_0}$. The physical Piola transport of a reference flux field is

$$\tau_\varepsilon q = \mathrm{d}R_\varepsilon(j_\varepsilon^{-1} q), \qquad \mathrm{div}_{g_\varepsilon}(j_\varepsilon^{-1} q) = j_\varepsilon^{-1} \mathrm{div}_{g_0} q. \tag{4}$$

Thus, the divergence-free constraint is fixed. For an integer $s \geq 3$, set

$$H^r_\sigma = \{q \in H^r(TM) : \mathrm{div}_{g_0} q = 0\}, \qquad X_0 = H^s_\sigma, \quad Y = H^{s+1}_\sigma, \quad X_1 = H^{s+2}_\sigma. \tag{5}$$

All norms on M refer to a fixed finite smooth atlas and g_0; uniformly equivalent geometric norms will be used when convenient. The transported equation is

$$\dot{q} = A_\varepsilon q + B_\varepsilon(q, q), \tag{6}$$

where, with $g = g_\varepsilon$ and $j = j_\varepsilon$,

$$A_\varepsilon q = j \Pi_g \mathcal{V}_g(j^{-1} q), \qquad \mathcal{V}_g = \nu(\Delta_{B,g} + k_g), \tag{7}$$

$$B_\varepsilon(q, r) = -j \Pi_g \mathcal{C}_g(j^{-1} q, j^{-1} r), \qquad \mathcal{C}_g(u, v) = \tfrac{1}{2}(\nabla_u v + \nabla_v u). \tag{8}$$

Here, $A_\varepsilon : X_1 \to X_0$ has a common domain, while $B_\varepsilon : Y \times Y \to X_0$ loses one spatial derivative. The zero solution is an equilibrium for every shape.

2.2. Uniform Assumptions

Assumption 1 (Surface family and slow–fast separation)**.** *The following hypotheses are imposed on a sufficiently small parameter interval about* 0*:*

(H1) *$M \simeq S^2$ and $s \geq 3$ is an integer. Each R_ε is smooth, and $\varepsilon \mapsto R_\varepsilon$ is C^1 in $C^{s+7}(M; \mathbb{R}^3)$. The metrics are uniformly positive definite and their indicated coefficient norms are locally bounded.*

(H2) *A_ε is the realization of (7) on (5). A single contour Γ in its resolvent set encloses the three eigenvalues closest to zero, counted with multiplicity, and no others. Let P_ε be its Riesz projection, $E_\varepsilon = \mathrm{ran}\, P_\varepsilon$, and $Q_\varepsilon = I - P_\varepsilon$.*

(H3) *For constants C, α, β, η independent of ε,*

$$\left\| e^{tA_\varepsilon} P_\varepsilon \right\|_{X_0 \to Y} \leq C e^{-\alpha t} \qquad (t \leq 0), \tag{9}$$

$$\left\| e^{tA_\varepsilon} Q_\varepsilon \right\|_{X_0 \to Y} \leq C t^{-1/2} e^{-\beta t} \qquad (t > 0), \tag{10}$$

$$0 \leq \alpha < \eta < \beta/3. \tag{11}$$

The group in (9) is only the finite-dimensional group on E_ε; no backward full semiflow is assumed.

(H4) *The primary comparison of the changing spectral spaces is*

$$T_\varepsilon = P_\varepsilon P_0 + Q_\varepsilon Q_0, \qquad T_0 = I. \tag{12}$$

Alternative physical and spectral comparisons are restricted to Definition 2; the regularity of the comparison and of its inverse form an independent requirement.

The common-domain and regularity conclusions needed in (H2)–(H4) follow from Sections 3 and 4. The spectral and gap conditions are hypotheses of the general theory: Theorem 5 verifies them for one ellipsoid, and Lemma 8 gives their local persistence. This application does not place that ellipsoid in the spherical onset neighborhood. Repeated eigenvalues inside the rank-three block are allowed.

Role and sharpness of the assumptions

In (H1), genus zero makes stream functions modulo constants a complete representation of divergence-free fields, without a harmonic-field component. The index $s \geq 3$ supplies the Sobolev product bounds used for the derivative-losing convection and the smoothed mild flow. The $C^1(C^{s+7})$ shape assumption is a sufficient finite derivative budget: it supports the common-domain operator rows, the raised H^{s+4} quadratic coefficients and the spatial comparison estimates in Sections 4 and 9. These indices are not asserted to be minimal; lowering them would require rechecking those multiplication, elliptic and composition estimates.

The choice of rank three in (H2) is motivated by the three rotational modes at the sphere and their continuation under small deformations. A Riesz projection tracks their whole subspace through possible internal eigenvalue crossings, so no differentiable choice of individual eigenvectors is needed. Other ranks would require a corresponding spectral selection and gap analysis. In (H3), the three history weights $\eta, 2\eta, 3\eta$ control the graph construction, its first and second derivatives, and the difference estimates in Section 6. The stronger condition $\beta > 3\eta > 3\alpha$ also implies the quadratic homological separation $\beta - 2\alpha > 0$ used in Lemma 12. This is a sufficient construction hypothesis, not an optimal nonresonance criterion.

In (H4), the regularity of both comparison maps and their inverses is needed to differentiate the same physical jet in different coordinates. This does not follow from the regularity of the conjugated equation alone; Remark 2 gives a counterexample. Spatial Piola families are differentiated in a two-derivative-loss operator norm, even though their infinitesimal generator is first order. The raised coefficient estimates supply the required output regularity.

The verified geometries are the sphere and its sufficiently small deformations, and the certified triaxial ellipsoid and its local neighborhood. These examples and persistence results do not establish how frequently the gap holds among arbitrary surface geometries.

At the unit sphere, the three selected rotational Stokes rates are zero, so the rank-three block is central. On the strictly triaxial ellipsoid used below, the corresponding selected slow rates are small but positive decay rates. Hypothesis (11) permits $\alpha = 0$; hence, the spherical endpoint is not excluded by a strict-stability requirement.

Mere spectral isolation cannot replace (11). For example, the system $\dot{x} = -x$, $\dot{y} = -2y + x^2$ has an isolated slow line. An invariant graph $y = h(x)$ would satisfy $xh' - 2h = -x^2$; hence, $h(x) = x^2(C - \log|x|)$ on either side of zero. It is not C^2 at zero. Here, the fast decay rate 2 equals twice the slow rate 1, so the quadratic homological denominator vanishes despite spectral isolation. In Theorem 1, (H3) both provides the C^2 graph needed to define $D^2K_\varepsilon(0)$ and separates the six quadratic homological resolvents. The example explains why an isolated slow block alone cannot support that theorem; it does not show that the chosen three-weight gap is necessary.

2.3. The Normal Jet and Its Derivative

For Banach spaces U, V, $\mathcal{L}(U, V)$ denotes bounded linear maps, and $\mathcal{B}^2_{\mathrm{sym}}(U, V)$ denotes bounded symmetric bilinear maps $U \times U \to V$, with norm

$$\|W\|_{\mathcal{B}^2_{\mathrm{sym}}(U,V)} = \sup_{\|a\|_U \leq 1,\, \|b\|_U \leq 1} \|W(a,b)\|_V.$$

All bilinear operator norms below use this convention.

We use E_0 itself as the fixed parameter space, and denote its inclusion into X_1 by i. A normalized local graph at ε has a parameterization K_ε satisfying

$$K_\varepsilon(0) = 0, \qquad DK_\varepsilon(0) = T_\varepsilon i, \qquad P_\varepsilon K_\varepsilon(z) = T_\varepsilon i z. \tag{13}$$

Its quadratic normal object is the quotient-valued tensor

$$\mathcal{J}_2(\varepsilon) = [D^2 K_\varepsilon(0)] \in \mathcal{B}^2_{\mathrm{sym}}(E_0, Y/E_\varepsilon). \tag{14}$$

The inputs in this display are identified with E_ε by $T_\varepsilon i$. Without that identification, the natural tensor is in $\mathcal{B}^2_{\mathrm{sym}}(E_\varepsilon, Y/E_\varepsilon)$. The representative in a fixed complement is

$$\overline{W}_\varepsilon = T_\varepsilon^{-1} Q_\varepsilon D^2 K_\varepsilon(0) \in \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 X_1). \tag{15}$$

Its membership in X_1 is established below as a coefficient-level regularity conclusion.

At each fixed shape, the Taylor expansion in Y gives

$$Q_0 T_\varepsilon^{-1} K_\varepsilon(z) = \tfrac{1}{2}\overline{W}_\varepsilon(z,z) + o_Y(|z|^2).$$

Thus, $\overline{W}_\varepsilon$ describes the leading quadratic displacement normal to the slow space after the stated comparisons. The factor $1/2$ appears because $\overline{W}_\varepsilon$ represents a second derivative. The shape derivative below concerns this coefficient. The Taylor expansion is taken separately at each fixed shape, and shape differentiability is established directly at the coefficient level.

Theorem 1 (Existence, unique quadratic jet and shape differentiation)**.** *Under Assumption 1, each sufficiently small ε admits a normalized locally invariant C^2 graph in Y satisfying (13) in the precise mild sense of Definition 1. Every graph in that class has the same Tensor (14) after its linear tangent-input identification is fixed. No uniqueness of the full graph is required; an arbitrary locally invariant C^2 parameterization tangent to E_ε may first be normalized by the finite-dimensional coordinate change described in Section 7.1.*

Define

$$\overline{A}_\varepsilon = T_\varepsilon^{-1} A_\varepsilon T_\varepsilon, \qquad \overline{B}_\varepsilon(a,b) = T_\varepsilon^{-1} B_\varepsilon(T_\varepsilon a, T_\varepsilon b), \qquad L_\varepsilon = i^{-1}\overline{A}_\varepsilon i. \tag{16}$$

On $\mathcal{Z}_1 = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 X_1)$ with values in $\mathcal{Z}_0 = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 X_0)$, let

$$(\mathscr{H}_\varepsilon W)(a,b) = \overline{A}_{\varepsilon,Q} W(a,b) - W(L_\varepsilon a, b) - W(a, L_\varepsilon b), \tag{17}$$

$$b_\varepsilon(a,b) = -2Q_0 \overline{B}_\varepsilon(ia, ib). \tag{18}$$

Then, $\mathscr{H}_\varepsilon : \mathcal{Z}_1 \to \mathcal{Z}_0$ is invertible and

$$\mathscr{H}_\varepsilon \overline{W}_\varepsilon = b_\varepsilon. \tag{19}$$

The map $\varepsilon \mapsto \overline{W}_\varepsilon$ is C^1 in $\mathcal{Z}_1$, with

$$\overline{W}_0' = \mathscr{H}_0^{-1}(b_0' - \mathscr{H}_0' \overline{W}_0), \qquad \left\| \overline{W}_\varepsilon - \overline{W}_0 - \varepsilon \overline{W}_0' \right\|_{\mathcal{Z}_1} = o(|\varepsilon|). \tag{20}$$

Proposition 8 establishes the stronger C^1 regularity of the six quadratic coefficients with values in $\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma)$, including the corresponding $o(|\varepsilon|)$ remainder. This coefficient-level result supplies the two derivatives needed by spatial comparisons; full-graph regularity lies outside the statement. The graph proof is in Section 6, the generator-

domain argument in Section 7, and the differentiability proof, including norm remainders, in Section 8. Thus, (20) has six bilinear input components with infinite-dimensional outputs.

Theorem 2 (Comparison law and the zero-jet case)**.** *Let two admissible physical transports describe the same equation, with relative state map G_ε, so that $\widehat{F}_\varepsilon = G_\varepsilon F_\varepsilon G_\varepsilon^{-1}$ in the linear-state conjugacy sense. Let $T_\varepsilon, \widehat{T}_\varepsilon$ be admissible spectral comparisons, all equal to I at 0. Set*

$$D_\varepsilon = \widehat{T}_\varepsilon^{-1} G_\varepsilon T_\varepsilon, \qquad D_{\varepsilon,E} = D_\varepsilon|_{E_0}, \quad D_{\varepsilon,Q} = D_\varepsilon|_{Q_0 Y}. \tag{21}$$

Then, D_ε is block diagonal, and the representatives obey

$$\widehat{\overline{W}}_\varepsilon = D_{\varepsilon,Q}\overline{W}_\varepsilon(D_{\varepsilon,E}^{-1}\cdot, D_{\varepsilon,E}^{-1}\cdot), \tag{22}$$

$$\widehat{\overline{W}}_0' = \overline{W}_0' + d_Q\overline{W}_0 - \overline{W}_0(d_E\cdot,\cdot) - \overline{W}_0(\cdot, d_E\cdot), \quad d = D_0'. \tag{23}$$

The derivative formula holds in $\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0X_1)$ with the losses specified in Definition 2. In particular, when $\overline{W}_0 = 0$, the first variation is unchanged by these first-order comparisons.

The proof is in Section 9: the correction terms in (23) generally do not vanish, while for a purely tangential surface reparameterization the jet is constant after the exact physical and input comparisons and its physically compared derivative is zero.

Theorem 3 (Anisotropy onset at the sphere)**.** *Let $M = S^2$, $R_\delta(x) = (I + \delta D)x$ with a real symmetric 3×3 matrix D, and use the rank-three block continuing the rotational modes. Put $f(x) = x^T Dx$, $\psi_p(x) = p \cdot x$, and let Π_3 denote scalar projection onto degree-three spherical harmonics. With the outward orientation and bracket $\{a, b\}_0 = (x \times \nabla a) \cdot \nabla b$, the stream-function representative of the first normal-jet variation is*

$$W_0'(p,q) = -\frac{1}{45\nu}\Pi_3\big[\psi_p\{\psi_q, f\}_0 + \psi_q\{\psi_p, f\}_0\big]. \tag{24}$$

The tensor is nonzero if and only if D is not a scalar matrix. For any fixed nonscalar D, the quadratic jet is nonzero for all sufficiently small $0 < |\delta| < \delta_0(D)$; the proof gives no explicit $\delta_0(D)$. Formula (24) is independent of first-order admissible comparisons agreeing at the sphere.

The corresponding velocity representative is $J_0 \operatorname{grad}_0 W_0'(p,q)$. Section 10 gives the complete affine derivation together with a purely normal consistency calculation; the result applies to the central spherical endpoint and its nearby slow blocks.

Proof architecture

The proof first fixes the divergence constraint and the Stokes domain, then differentiates the operator and Riesz projection on that fixed scale (Sections 3 and 4). The isolated block provides the finite-dimensional slow coordinates. The three-weight gap then yields a C^2 mild invariant graph in Y (Section 6). Differentiating its finite-time invariance identity twice gives an identity for the quadratic coefficient. The generator difference quotient in Proposition 5 places its six symmetric components in $D(A_\varepsilon) = X_1$; this is a domain gain for the coefficient, not a claim that the whole graph takes values in X_1. The separated homological inverse makes that coefficient independent of the graph (Section 7). Raised elliptic estimates then give the shape derivative and its norm remainders (Section 8), with enough regularity for the derivative-losing comparison law (Section 9). Finally, the scalar–vector bridge reduces the spherical calculation to harmonic polynomials (Section 10). The spectral certificate in Section 5 is independent of this onset calculation; onset uses the general near-sphere gap estimates.

3. Uniform Surface–Stokes Realization

3.1. A Fixed Divergence Constraint and a Full-Space Pressure Inverse

For distributions the divergence is defined by $\langle \operatorname{div}_g v, \varphi \rangle_g = -\int g(v, \operatorname{grad}_g \varphi)\, \mathrm{d}\mu_g$. Scalar distribution spaces below are fixed using the reference density; multiplication by the smooth positive density identifies this definition with those spaces. In particular, for a scalar distribution $f \in H^{-1}$ in reference-density notation,

$$m_g f = \frac{\langle f, j \rangle_0}{\int_M j \, \mathrm{d}\mu_0}, \qquad \langle f, a \rangle_g = \langle f, ja \rangle_0.$$

Thus, the geometric mean pairs with 1 in the g-density convention, equivalently with j in the fixed one. All negative-order spaces in this section are scalar spaces, not additional fluid phase spaces.

Lemma 1 (Scalar inverse and pressure projection)**.** *Fix a finite atlas on M and a positive lower metric bound. Suppose that the metric coefficients and their inverses have locally bounded C^{s+5} norms. For every integer $-1 \le \ell \le s+3$, the operator*

$$\mathscr{D}_g : H^{\ell+2}(M) \longrightarrow H^{\ell}(M)$$

is an isomorphism with locally uniform inverse bound. For every integer $0 \le r \le s+4$, the compositions

$$H^r(TM) \xrightarrow{\operatorname{div}_g} H^{r-1} \xrightarrow{\mathscr{D}_g^{-1}} H^{r+1} \xrightarrow{\operatorname{grad}_g} H^r(TM) \tag{25}$$

are bounded, and Π_g is a projection onto $\ker \operatorname{div}_g$*. On $L^2(TM, g)$, it is the orthogonal projection. These definitions are compatible when an input belongs to two of the indicated Sobolev spaces.*

Proof. Write $f = f_\perp + m_g f$, where $m_g f_\perp = 0$. The weak Laplacian on $H^1/\mathbb{R}$ is invertible by the coercivity of $\int |\operatorname{grad}_g v|^2 \, \mathrm{d}\mu_g$ and the Poincaré inequality. Equivalently, there is a unique v with $m_g v = 0$ and $\Delta_g v = f_\perp$. The scalar regularity estimate proven in Appendix A is

$$\|v\|_{H^{\ell+2}} \le C_\ell \big(\|\Delta_g v\|_{H^\ell} + \|v\|_{L^2} \big), \quad 0 \le \ell \le s+3. \tag{26}$$

Its weak $H^{-1} \to H^1$ counterpart follows from the same coercivity. It follows that

$$\mathscr{D}_g^{-1} f = (\Delta_g|_{m_g=0})^{-1}(f - m_g f) - m_g f. \tag{27}$$

This formula proves both surjectivity and injectivity: $\mathscr{D}_g$ is $-I$ on constants and Δ_g on the mean-zero summand. The summands describe the inversion, while the parameter-dependent operator itself retains the full fixed domain and range stated above. Appendix A also proves the uniformity of the bounds under the specified coefficient controls.

The divergence of any vector distribution has g-mean zero. Thus, $v = \mathscr{D}_g^{-1} \operatorname{div}_g w$ has mean zero and $\Delta_g v = \operatorname{div}_g w$. Consequently, $\operatorname{div}_g \Pi_g w = 0$, while $\Pi_g w = w$ if $\operatorname{div}_g w = 0$. This proves the projection assertion, including idempotence. The three factors have exactly the spaces in (25), including $r = 0$. Integration by parts shows that gradients are L^2_g orthogonal to divergence-free fields, which proves orthogonality. Uniqueness of the weak scalar solution proves consistency of all the realizations. □

The coordinate divergence formula $\operatorname{div}_g v = |g|^{-1/2} \partial_i (|g|^{1/2} v^i)$ proves

$$\operatorname{div}_{g_\varepsilon}(j_\varepsilon^{-1} q) = j_\varepsilon^{-1} |g_0|^{-1/2} \partial_i (|g_0|^{1/2} q^i) = j_\varepsilon^{-1} \operatorname{div}_{g_0} q.$$

This holds for distributions by testing against smooth functions. Multiplication by $j_\varepsilon^{\pm 1}$ and the tangent map $\mathrm{d}R_\varepsilon$ are bounded isomorphisms on the finite levels under consideration. Therefore, (4) identifies the physical solenoidal spaces with the fixed closed spaces H^r_σ. Smooth solenoidal fields are dense in H^r_σ: approximate a field by smooth fields and apply the bounded, fixed projection Π_{g_0}.

3.2. Stream Functions, the Viscous Identity and Weak Forms

Lemma 2 (The scalar quotient realization)**.** *For a smooth oriented genus-zero surface, the map*

$$\psi \longmapsto J_g \operatorname{grad}_g \psi$$

after quotienting constants is an isomorphism from $H^{r+1}/\mathbb{R}$ to the divergence-free subspace of $H^r(TM)$ for every integer $r \geq 0$. On any of the finite levels used here, its norm and inverse norm are locally uniform in g. Piola transport gives

$$j^{-1} J_0 \operatorname{grad}_0 \psi = J_g \operatorname{grad}_g \psi. \tag{28}$$

With $\{a, b\}_g = (J_g \operatorname{grad}_g a) b$ and $\omega = \Delta_g \psi$, Equation (2) is equivalent, i.e., modulo pressure, to

$$\partial_t \Delta_g \psi = \nu \left[\Delta_g^2 \psi + 2 \operatorname{div}_g (k_g \operatorname{grad}_g \psi) \right] - \{\psi, \Delta_g \psi\}_g. \tag{29}$$

Proof. The convention for J gives $\iota_{J_g \operatorname{grad}_g \psi} \mu_g = -\mathrm{d}\psi$. Conversely, $\operatorname{div}_g u = 0$ says that the one-form $\iota_u \mu_g$ is closed. Since $M \simeq S^2$, a smooth closed one-form is exact by path integration on this simply connected surface. The weak conclusion follows by the approximation argument below. The potential is unique modulo constants. The identity $\mathrm{d}\psi = -\iota_u \mu_g$, chartwise differentiation, and Poincaré give

$$\|\psi\|_{H^{r+1}/\mathbb{R}} \leq C_r \|u\|_{H^r}, \qquad \|J_g \operatorname{grad}_g \psi\|_{H^r} \leq C_r \|\psi\|_{H^{r+1}/\mathbb{R}}.$$

For the weak exactness assertion, one can first approximate a solenoidal field by smooth solenoidal fields using Lemma 1; the potentials, normalized to reference mean zero, are Cauchy by these inequalities. The smooth exactness assertion follows from path integration on the simply connected sphere. The contraction identity with the area form also proves (28), since $\mu_g = j\mu_0$.

For completeness, contracting the covariant commutator in an orthonormal frame gives

$$\nabla^i (\nabla_i u_j + \nabla_j u_i) = (\Delta_B u)_j + \nabla_j \operatorname{div}_g u + k_g u_j.$$

This is (1). On one-forms, the same commutator gives $\Delta_B = \Delta_H + k$, where $\Delta_H = -(\mathrm{d}\delta + \delta\,\mathrm{d})$ has nonpositive sign. The exterior derivative and the Hodge star commute with Δ_H. For $u = J \operatorname{grad} \psi$, this yields

$$\operatorname{curl}(\Delta_B u + k u) = \Delta_g^2 \psi + 2 \operatorname{curl}(k J \operatorname{grad} \psi) = \Delta_g^2 \psi + 2 \operatorname{div}_g (k \operatorname{grad}_g \psi).$$

Here, curl means $\mathrm{d}(u^\flat) = (\operatorname{curl} u)\mu_g$; in positively oriented flat coordinates, $J \operatorname{grad} \psi = (-\psi_y, \psi_x)$, so its curl is $\Delta\psi$. To verify the nonlinear term, use $(\nabla_u u)^\flat = \iota_u \mathrm{d}(u^\flat) + \frac{1}{2}\mathrm{d}|u|^2$. Taking d and using $\operatorname{div}_g u = 0$ gives $\operatorname{curl} \nabla_u u = u(\operatorname{curl} u) = \{\psi, \Delta\psi\}_g$. Pressure has zero curl, which proves (29). Conversely, a curl-free difference of the two velocity equations is a gradient on this genus-zero surface and is absorbed in the normalized pressure. Hence, the vector and scalar formulations are equivalent. □

On the fixed scalar quotient, the linear and symmetric bilinear maps corresponding to (7) and (8) are

$$A_g^{\psi}\psi = \nu\left[\Delta_g\psi + 2\Delta_g^{-1}\operatorname{div}_g(k_g \operatorname{grad}_g \psi)\right], \tag{30}$$

$$B_g^{\psi}(\psi,\phi) = -\tfrac{1}{2}\Delta_g^{-1}\left[\{\psi,\Delta_g\phi\}_g + \{\phi,\Delta_g\psi\}_g\right]. \tag{31}$$

In these formulas, Δ_g^{-1} is the inverse between scalar quotient spaces. The bracket right-hand sides have zero g-mean, because Hamiltonian vector fields are divergence free. This is distinct from the full-space pressure inverse $\mathscr{D}_g^{-1}$. The velocity spaces X_0, Y, X_1 correspond respectively to $H^{s+1}/\mathbb{R}, H^{s+2}/\mathbb{R}, H^{s+3}/\mathbb{R}$.

For scalar functions a, b, c, integration by parts in (30) and (31) gives

$$\mathfrak{m}_g^{\psi}(a,b) = \int_M \operatorname{grad}_g a \cdot_g \operatorname{grad}_g b \,\mathrm{d}\mu_g, \tag{32}$$

$$\mathfrak{a}_g^{\psi}(a,b) = \nu\int_M [\Delta_g a\,\Delta_g b - 2k_g \operatorname{grad}_g a \cdot_g \operatorname{grad}_g b]\,\mathrm{d}\mu_g, \tag{33}$$

$$\mathfrak{b}_g^{\psi}(a,b;c) = -\tfrac{1}{2}\int_M [\Delta_g a\{b,c\}_g + \Delta_g b\{a,c\}_g]\,\mathrm{d}\mu_g. \tag{34}$$

The last equality uses $\int c\{a,d\}_g = -\int d\{a,c\}_g$; This fixes the sign of the weak convective form. In particular, the scalar stiffness is nonnegative, because it is the deformation energy of the corresponding velocity, even though its displayed integrand need not be pointwise nonnegative.

3.3. Sign Conventions and Stream-Function Reduction

The onset calculation uses the following sign and inverse conventions from Lemma 2 and (30) and (31).

The surface orientation is the chosen positive orientation and J_g is positive rotation by $\pi/2$, characterized for a stream function by

$$\iota_{J_g \operatorname{grad}_g \psi}\mu_g = -\,\mathrm{d}\psi, \qquad u = J_g \operatorname{grad}_g \psi.$$

We use

$$\mathrm{d}(u^{\flat}) = (\operatorname{curl}_g u)\mu_g, \qquad \Delta_g = \operatorname{div}_g \operatorname{grad}_g,$$

with Δ_g nonpositive. Hence, $\operatorname{curl}_g u = \Delta_g\psi$. With $\{a,b\}_g = (J_g \operatorname{grad}_g a)b$, the two curl identities are

$$\operatorname{curl}_g\left((\Delta_{B,g} + k_g)u\right) = \Delta_g^2\psi + 2\operatorname{div}_g(k_g \operatorname{grad}_g \psi), \qquad \operatorname{curl}_g(\nabla_u u) = \{\psi,\Delta_g\psi\}_g.$$

Accordingly, the scalar evolution is exactly

$$\partial_t\Delta_g\psi = \nu\left[\Delta_g^2\psi + 2\operatorname{div}_g(k_g \operatorname{grad}_g \psi)\right] - \{\psi,\Delta_g\psi\}_g,$$

and, on scalar quotient space modulo constants,

$$A_g^{\psi}\psi = \nu\left[\Delta_g\psi + 2\Delta_g^{-1}\operatorname{div}_g(k_g \operatorname{grad}_g \psi)\right],$$
$$B_g^{\psi}(\psi,\phi) = -\tfrac{1}{2}\Delta_g^{-1}\left[\{\psi,\Delta_g\phi\}_g + \{\phi,\Delta_g\psi\}_g\right].$$

The factor $-1/2$ is fixed by the symmetric bilinearization $\mathcal{C}_g(u,v) = \frac{1}{2}(\nabla_u v + \nabla_v u)$ together with the minus sign in the velocity nonlinearity. Finally, Δ_g^{-1} above is the inverse on quotient spaces $H^r(M)/\mathbb{R}$, whereas $\mathscr{D}_g^{-1}$ in Lemma 1 is the inverse of the augmented Laplacian on the full scalar pressure space; these are distinct inverses with the stated domains.

3.4. Realization Theorem

Simonett–Wilke [16] (§2, (2.2)–(2.3), Theorem 3.1) show that, on a smooth compact connected embedded hypersurface without boundary, the operator $-\nu\Pi_g(\Delta_{B,g} + \mathrm{Ric}_g)$ on $L^2_{g,\sigma}$, with domain $H^2_{g,\sigma}$, is closed and densely defined, its negative generates an analytic semigroup, and sufficiently large positive real shifts are invertible. We apply this L^2 statement at $q = 2$ in dimension two, where $\mathrm{Ric}_g = k_g$. Each $R_\varepsilon(M)$ satisfies the geometric hypotheses, $\nu > 0$, and Lemma 1 supplies its Helmholtz projection. Piola transport is a bounded isomorphism on both L^2 and H^2, so the result applies to A_ε with the opposite, generator sign. The shape-uniform higher-Sobolev estimates and parameter regularity used below are established separately in this section.

Theorem 4 (Self-adjoint transported surface–Stokes realization and spectral form)**.** *Under (H1), let*

$$\mathcal{H}_\varepsilon = (H^0_\sigma, \mathfrak{m}_\varepsilon),$$

where

$$\mathfrak{m}_\varepsilon(q, v) = \int_M g_\varepsilon(q, v) j_\varepsilon^{-1} \, \mathrm{d}\mu_0. \tag{35}$$

Define the transported energy form

$$\mathfrak{a}_\varepsilon(q, v) = 2\nu \int_M \mathrm{Def}_g(j^{-1}q) :_g \mathrm{Def}_g(j^{-1}v) \, \mathrm{d}\mu_g. \tag{36}$$

Its form domain is the fixed closed space H^1_σ, and it is a densely defined nonnegative closed form on $\mathcal{H}_\varepsilon$. The associated nonnegative operator is $-A_{\varepsilon,0}$.

For every integer $0 \le r \le s + 2$, the compatible realization $A_{\varepsilon,r}$ on H^r_σ is closed and densely defined with

$$D(A_{\varepsilon,r}) = H^{r+2}_\sigma, \qquad A_{\varepsilon,r} q = A_\varepsilon q. \tag{37}$$

For q in that domain, with $f = -A_\varepsilon q$, the physical velocity $u = j_\varepsilon^{-1} q$ and mean-zero pressure solve

$$-\nu(\Delta_{B,g} + k_g)u + \mathrm{grad}_g \, p = j^{-1} f, \qquad \mathrm{div}_g \, u = 0, \qquad m_g p = 0, \tag{38}$$

and satisfy the locally uniform estimate

$$\|q\|_{H^{r+2}} + \|p\|_{H^{r+1}} \le C_r\big(\|f\|_{H^r} + \|q\|_{L^2}\big). \tag{39}$$

The L^2 realization is nonpositive and self-adjoint relative to the named mass inner product $\mathfrak{m}_\varepsilon$, has compact resolvent, and

$$\ker A_{\varepsilon,0} = \{q \in H^2_\sigma : \ \mathrm{Def}_{g_\varepsilon}(j_\varepsilon^{-1} q) = 0\},$$

the transported Killing fields. All higher Sobolev realizations have the same eigenvalues with the same multiplicities; their resolvents and Riesz projections are compatible on common inputs, and the Riesz ranges lie in every raised level used below.

If $0 \le \lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots$ are the decay rates of $-A_{\varepsilon,0}$, repeated according to multiplicity and including any zero Killing rates, then the generalized min–max formula on the mass/form pair is

$$\lambda_n(\varepsilon) = \min_{\substack{V \subset H^1_\sigma \\ \dim V = n}} \max_{0 \ne q \in V} \frac{\mathfrak{a}_\varepsilon(q, q)}{\mathfrak{m}_\varepsilon(q, q)}, \qquad n \ge 1. \tag{40}$$

Proof. The imported L^2 domain statement and integration by parts imply $\mathfrak{m}_\varepsilon(A_\varepsilon q, v) = -\mathfrak{a}_\varepsilon(q, v)$ for $q, v \in H^2_\sigma$. For divergence-free physical fields, the integration-by-parts identity

$$2\| \operatorname{Def}_g u\|^2_{L^2_g} = \|\nabla_g u\|^2_{L^2_g} - \int_M k_g |u|^2_g \, \mathrm{d}\mu_g$$

shows that the norm $\big(\mathfrak{m}_\varepsilon(q,q) + \mathfrak{a}_\varepsilon(q,q)\big)^{1/2}$ is equivalent to the H^1 norm on H^1_σ. Thus, the energy form is densely defined, nonnegative and closed with form domain H^1_σ, and its associated operator is $-A_{\varepsilon,0}$. Hence, $A_{\varepsilon,0}$ is symmetric and nonpositive. For a sufficiently large real ω, the bounded everywhere-defined inverse of $\omega - A_{\varepsilon,0}$ is symmetric and, hence, self-adjoint. Taking its inverse proves the self-adjointness of $A_{\varepsilon,0}$. Its kernel is exactly the transported Killing space, since $\mathfrak{a}_\varepsilon(q,q) = 0$ is equivalent to $\operatorname{Def}_g(j^{-1}q) = 0$. The compactness of $H^1_\sigma \hookrightarrow H^0_\sigma$ at the form level, and equivalently of $H^2_\sigma \hookrightarrow H^0_\sigma$ at the operator level, gives compact resolvent. The spectral theorem for this closed mass/form pair gives (40), including multiplicity and any zero Killing rates.

The higher estimate follows from stream-function elliptic regularity. For $u = J_g \operatorname{grad}_g \psi$, write $\mathscr{L}_g \psi = \Delta_g^2 \psi + 2 \operatorname{div}_g(k_g \operatorname{grad}_g \psi)$. The scalar estimate in (26), applied twice, and the multiplier bounds from the fixed atlas give, for $0 \le r \le s+2$,

$$\begin{aligned} \|\psi\|_{H^{r+3}/\mathbb{R}} &\le C_r\big(\|\Delta_g \psi\|_{H^{r+1}} + \|\psi\|_{H^1/\mathbb{R}}\big) \\ &\le C_r\big(\|\Delta_g^2 \psi\|_{H^{r-1}} + \|\psi\|_{H^{r+1}/\mathbb{R}}\big) \\ &\le C_r\big(\|\mathscr{L}_g \psi\|_{H^{r-1}} + \|\psi\|_{H^{r+1}/\mathbb{R}}\big). \end{aligned} \tag{41}$$

For $r = 0$, the second application is the weak $H^{-1} \to H^1$ inverse estimate; $\Delta_g \psi$ has mean zero. The lower-order term is controlled because $\operatorname{div}_g(k_g \operatorname{grad}_g)$ has order two. The inequality

$$\|\psi\|_{H^{r+1}/\mathbb{R}} \le a\|\psi\|_{H^{r+3}/\mathbb{R}} + C_{r,a}\|\psi\|_{H^1/\mathbb{R}}$$

follows by splitting local Fourier frequencies at a fixed radius. Choose a to absorb this term in (41). Taking the curl in (38), as justified in Lemma 2, gives $-\nu \mathscr{L}_g \psi = \operatorname{curl}_g(j^{-1} f)$ in H^{r-1}. The stream isomorphism therefore proves the velocity part of (39).

This argument also upgrades weak solutions: start with $q \in H^2_\sigma$, then $\psi \in H^3/\mathbb{R}$. If $f \in H^r$ and the preceding induction step gives $\psi \in H^{r+2}/\mathbb{R}$, then

$$\Delta_g(\Delta_g \psi) = -\nu^{-1} \operatorname{curl}_g(j^{-1} f) - 2 \operatorname{div}_g(k_g \operatorname{grad}_g \psi) \in H^{r-1}.$$

The scalar weak-to-strong estimate gives $\Delta_g \psi \in H^{r+1}$ and then $\psi \in H^{r+3}/\mathbb{R}$. This induction reaches $r = s+2$. At a fixed smooth shape, it continues through any prescribed integer level, although its constants above $s+2$ are not asserted uniformly in the given family.

The pressure in (38) is explicitly

$$p = \mathscr{D}_g^{-1} \operatorname{div}_g\big(\nu(\Delta_{B,g} + k_g)u\big). \tag{42}$$

Indeed, $\Pi_g \nu(\Delta_B + k)u = -j^{-1} f$; subtracting that projection gives the gradient in (38). In (42), the maps are $H^{r+2}(TM) \to H^r(TM) \to H^{r-1} \to H^{r+1}$. The output has g-mean zero, so this supplies the pressure estimate with a fixed normalization.

For $f \in H^r_\sigma$, the L^2 solution of $(\omega - A)q = f$ first lies in H^2_σ. The same induction, with the lower-order term ωq included, gives $q \in H^{r+2}_\sigma$. Its a priori estimate is (39) with that zeroth-order term absorbed and the bounded L^2 inverse used. Thus, $(\omega - A) : H^{r+2}_\sigma \to H^r_\sigma$ is onto and has a bounded inverse. It is one-to-one by its L^2 realization. This identifies the closed realization and its domain; density follows from smooth solenoidal approximation.

The compactness of the domain embedding again gives compact resolvent. Compatibility and the full spectral assertion are proven separately in Lemma 3 below.

Here is the uniformity in (39). Choose a finite atlas with relatively compact inner patches covering M, and fixed smooth cutoff functions supported in the outer patches. The same patches and cutoffs serve every ε. A common lower bound for g gives common ellipticity and area bounds. The C^{s+7} bound on R_ε gives common C^{s+6} metric/density bounds and C^{s+4} curvature and zeroth-order viscous-coefficient bounds. Appendix A uses only these finitely many derivatives, so its Young inequalities can be chosen uniformly over the family. Products by $j^{\pm 1}$ and the stream norm equivalences use the same bounds. A single sufficiently small neighborhood preserves these bounds and the finitely many inverse estimates. □

Lemma 3 (Compatibility of spectra, resolvents and projections)**.** *For the realizations in Theorem 4, a complex number z is a resolvent point on H^r_σ if and only if it is a resolvent point on H^0_σ. At such a point, the resolvent on H^r_σ is the restriction of the L^2 resolvent and maps H^r_σ boundedly to H^{r+2}_σ. Riesz projections defined on different levels agree on their common inputs.*

Proof. Let z be in the L^2 resolvent. For $f \in H^r_\sigma$, its unique L^2 resolvent solution initially belongs to H^2_σ. In the bootstrap in the preceding proof, replace f by $f - zq$, treating zq as a lower-order term. This yields $q \in H^{r+2}_\sigma$, and the elliptic estimate with interpolation gives

$$\|q\|_{H^{r+2}} \le C_{r,z}\big(\|f\|_{H^r} + \|q\|_{L^2}\big) \le C_{r,z}(1 + \|\mathscr{R}_{\varepsilon,0}(z)\|_{L^2\to L^2})\|f\|_{H^r}. \tag{43}$$

Here, the second subscript 0 denotes the L^2 level at the fixed shape ε. Thus, z remains a resolvent point at level r.

Conversely, the compact-resolvent, self-adjoint L^2 operator has only real eigenvalues of finite multiplicity, tending to $-\infty$. If $Aq = zq$ with $q \neq 0$ initially in H^2_σ, the repeated application of the same bootstrap puts q in every finite Sobolev space at that fixed smooth shape. Hence, it is smooth and remains an eigenfield of each raised realization; no L^2 spectral point is lost. Nor can the raised realization introduce another spectral point, because every L^2 resolvent point has just been shown to remain a resolvent point. The kernels agree. There are no additional generalized eigenfields: such a vector on a raised level would also be a generalized eigenvector of the self-adjoint L^2 operator. This proves equality with multiplicity.

Resolvent uniqueness proves agreement on common inputs. A Riesz projection is the norm integral of these resolvents around a common compact contour, so the projections agree there as well. Its range is a finite sum of the smooth eigenspaces just identified; in particular, it is invariant on each raised level. Uniform bounds on a contour follow from (43) and compactness, or directly from the inverse estimate in Section 4.1. □

Proposition 1 (Unified scalar–vector conjugacy and quotient-jet comparison)**.** *Let $g = g_\varepsilon$ and, for every integer level used below, define the stream-to-reference-flux isomorphism*

$$\mathscr{S}_{\varepsilon,r} : H^{r+1}(M)/\mathbb{R} \longrightarrow H^r_\sigma, \qquad \mathscr{S}_{\varepsilon,r}\psi := j_\varepsilon J_{g_\varepsilon} \operatorname{grad}_{g_\varepsilon} \psi = J_0 \operatorname{grad}_0 \psi. \tag{44}$$

The last equality is exactly (28); in the fixed reference coordinates, the map is therefore independent of ε, although the scalar and velocity realizations it identifies depend on g_ε. When the Sobolev level is clear from the surrounding domain and codomain, we suppress the subscript r on $\mathscr{S}_{\varepsilon,r}$.

Write $A_\varepsilon^\psi = A_{g_\varepsilon}^\psi$ and $B_\varepsilon^\psi = B_{g_\varepsilon}^\psi$. On the compatible levels,

$$D(A_{\varepsilon,r}^\psi) = H^{r+3}(M)/\mathbb{R}, \qquad A_{\varepsilon,r}\mathscr{S}_{\varepsilon,r+2} = \mathscr{S}_{\varepsilon,r}A_{\varepsilon,r}^\psi, \tag{45}$$

$$(z - A_{\varepsilon,r})^{-1}\mathscr{S}_{\varepsilon,r} = \mathscr{S}_{\varepsilon,r+2}(z - A_{\varepsilon,r}^\psi)^{-1}, \qquad P_\varepsilon\mathscr{S}_\varepsilon = \mathscr{S}_\varepsilon P_\varepsilon^\psi, \tag{46}$$

$$B_\varepsilon(\mathscr{S}_\varepsilon\psi, \mathscr{S}_\varepsilon\phi) = \mathscr{S}_\varepsilon B_\varepsilon^\psi(\psi, \phi). \tag{47}$$

Here, $P_\varepsilon^\psi := \mathscr{S}_\varepsilon^{-1}P_\varepsilon\mathscr{S}_\varepsilon$ and $Q_\varepsilon^\psi = I - P_\varepsilon^\psi$ are the scalar Riesz projections. Thus, the rank-three scalar slow space $E_\varepsilon^\psi := \operatorname{ran} P_\varepsilon^\psi$ is carried isomorphically onto the rank-three velocity slow space $E_\varepsilon = \operatorname{ran} P_\varepsilon$. The canonical scalar comparison from the base scalar block to the moving scalar block

$$T_\varepsilon^\psi = P_\varepsilon^\psi P_0^\psi + Q_\varepsilon^\psi Q_0^\psi$$

satisfies

$$T_\varepsilon\mathscr{S}_0 = \mathscr{S}_\varepsilon T_\varepsilon^\psi. \tag{48}$$

No individual eigenvector, and in particular no moving eigenbasis inside a repeated rank-three block, is used in these identities.

The same isomorphism acts on quotient outputs: the induced map

$$\widetilde{\mathscr{S}}_\varepsilon : (H^{s+2}/\mathbb{R})/E_\varepsilon^\psi \longrightarrow Y/E_\varepsilon, \qquad [\psi] \longmapsto [\mathscr{S}_\varepsilon\psi] \tag{49}$$

is well-defined. If K_ε^ψ is the corresponding scalar graph, define its fixed-complement representative by

$$\overline{W}_\varepsilon^\psi = (T_\varepsilon^\psi)^{-1}Q_\varepsilon^\psi D^2K_\varepsilon^\psi(0) \in \mathcal{B}^2_{\mathrm{sym}}(E_0^\psi, Q_0^\psi(H^{s+3}(M)/\mathbb{R})),$$

which is the scalar counterpart of (15). Then, for $a, b \in E_0^\psi$, the fixed-complement representatives obey

$$\overline{W}_\varepsilon^{\mathrm{v}}(\mathscr{S}_0 a, \mathscr{S}_0 b) = \mathscr{S}_0\,\overline{W}_\varepsilon^\psi(a, b). \tag{50}$$

Thus, scalar quotient representatives and velocity quotient representatives are compared by an explicit induced quotient map, not identified as the same vector. In particular, at the spherical base point, the scalar first variation W_0' in Section 10 represents the velocity first variation $J_0\,\mathrm{grad}_0\,W_0'$.

Proof. Lemma 2 gives the isomorphism and (44). The weak mass and energy identities in (32) and (33), together with Theorem 4, identify the scalar and velocity operators and their domains, proving (45). Resolvent uniqueness then gives the first identity in (46); contour integration gives the Riesz identity, and the canonical comparison identity (48) follows algebraically from its definition. Parameter regularity used later is supplied separately by Proposition 3; the conjugacy identities here are algebraic on the compatible levels.

The equivalence of the velocity equation and (29), with the symmetric scalar operator (31), proves (47). Therefore, applying $\mathscr{S}_\varepsilon$ to a scalar normalized mild graph produces the corresponding velocity graph (and conversely). Differentiation with respect to the finite-dimensional graph variables gives the induced quotient law (49); applying Q_ε, then the commuting canonical comparisons, gives (50). The shape differentiation of the spatial map $\mathscr{S}_\varepsilon$ is treated separately from this algebraic conjugacy. □

Lemma 4 (Uniform semigroup bounds on the finite Sobolev scale)**.** *The L^2 semigroup restricts to a strongly continuous analytic semigroup on each H^r_σ, $0 \le r \le s+4$. On H^r_σ, $0 \le r \le s+2$, its generator is precisely (37). If the rates in a complementary Riesz subspace are at least $\beta_* > 0$, then, for $0 \le r \le s+2$ and $0 < \beta < \beta_*$,*

$$\|e^{tA_\varepsilon}Q_\varepsilon\|_{H^r_\sigma \to H^{r+1}_\sigma} \le C_{r,\beta}t^{-1/2}e^{-\beta t}, \qquad t > 0. \tag{51}$$

The constants are locally uniform when the finite coefficient bounds and $\beta_* - \beta$ are fixed. On a rank-three slow space with rates in $[0, \alpha]$, the backward bound in (9) holds uniformly.

Proof. Put $\Lambda = I - A_{\varepsilon,0}$ on the Hilbert space with Mass (35). It is self-adjoint and $\Lambda \geq I$. We first identify its spectral power norms with the Sobolev norms, including the odd levels. For divergence-free physical fields, the integration of (1) gives

$$2\| \operatorname{Def}_g u\|^2_{L^2_g} = \|\nabla_g u\|^2_{L^2_g} - \int_M k_g |u|^2_g \, \mathrm{d}\mu_g.$$

Consequently, $\mathfrak{m}_\varepsilon(q,q) + \mathfrak{a}_\varepsilon(q,q)$ is uniformly equivalent to $\|q\|^2_{H^1}$. The positive closed form associated with Λ therefore has domain H^1_σ, and the spectral theorem for a positive operator identifies this form domain with $D(\Lambda^{1/2})$. This use of the closed-form theorem requires density, symmetry, closedness and positivity; density was proven above, the displayed norm equivalence proves closedness, and the other two properties follow from (35) and (36). Its associated operator is $I - A$, by integration by parts and the already established surjectivity.

Theorem 4 gives $\|q\|_{H^{k+2}} \asymp \|\Lambda q\|_{H^k}$ for $0 \leq k \leq s+2$; the lower L^2 term is controlled by $\|\Lambda q\|_{L^2}$ because $\Lambda \geq I$. Starting at $k = 0$ and at the form level $k = 1$, induction and $D(\Lambda^{(k+2)/2}) = \{q \in D(\Lambda) : \Lambda q \in D(\Lambda^{k/2})\}$ give

$$D(\Lambda^{r/2}) = H^r_\sigma, \qquad \|\Lambda^{r/2} q\|_{\mathfrak{m}_\varepsilon} \asymp \|q\|_{H^r}, \quad 0 \leq r \leq s+4. \tag{52}$$

All constants use only the finite uniform bounds already obtained, and the same scale through $s+4$ identifies the odd endpoint.

In a mass-orthonormal eigenbasis with rates $\lambda_n \geq 0$, the square of the norm in (52) is $\sum_n (1+\lambda_n)^r |q_n|^2$. Multiplication by $e^{-\lambda_n t}$ is strongly continuous in that weighted sequence space by dominated convergence and is analytic for $\Re t > 0$. Its generator domain adds the factor λ_n^2, giving H^{r+2}_σ for $r \leq s+2$. For fast coefficients, its one-derivative norm is bounded by

$$\sup_{\lambda \geq \beta_*} (1+\lambda)^{1/2} e^{-\lambda t} \leq C_{\beta,\beta_*} t^{-1/2} e^{-\beta t}.$$

For $0 < t \leq 1$, this follows by maximizing $\sqrt{t(1+\lambda)} e^{-(\lambda-\beta)t}$; for $t \geq 1$, the positive gap $\beta_* - \beta$ absorbs the factor $\sqrt{t}$. Transfer by (52) proves (51). On the finite slow space, the factors for $t \leq 0$ are bounded by $e^{-\alpha t}$; the Sobolev-to-mass norm equivalences of its smooth eigenfields are uniform. This proves the remaining assertion and permits zero modes inside the excluded slow block. □

4. Operator-Norm Shape Calculus

4.1. Products, Inverses and a Norm Convention

If $\mathfrak{X}_F$ is the space assigned to a family F in the tables below, define

$$E_F(\varepsilon) = F_\varepsilon - F_0 - \varepsilon F_0', \qquad \|E_F(\varepsilon)\|_{\mathfrak{X}_F} \leq |\varepsilon| \omega_F(\varepsilon), \qquad \omega_F(\varepsilon) \longrightarrow 0. \tag{53}$$

For a bilinear map, the norm is the supremum over the product of the two input unit balls. Coefficient norms are taken in the fixed atlas. Every little-o assertion below uses (53) in its displayed space; it does not mean pointwise convergence on individual inputs.

Lemma 5 (Bounded inverse and product remainders). *Let $D_\varepsilon : V \to U$ be C^1 between fixed Banach spaces, with D_0 invertible. Put $J_0 = D_0^{-1}$ and $\Delta D = D_\varepsilon - D_0$. If $\|J_0 \Delta D\|_{\mathcal{L}(V,V)} \leq 1/2$, then*

$$\|D_\varepsilon^{-1}\|_{U\to V} \le 2\|J_0\|_{U\to V}, \tag{54}$$

$$(D^{-1})_0' = -J_0 D_0' J_0, \tag{55}$$

$$\|D_\varepsilon^{-1} - J_0 + \varepsilon J_0 D_0' J_0\|_{U\to V} \le \|J_0\|^2 \|E_D(\varepsilon)\|_{V\to U} + 2\|J_0\|^3 \|\Delta D\|_{V\to U}^2. \tag{56}$$

Finite sums and matching compositions preserve C^1 regularity and (53), including composition on either input of a bilinear map.

Proof. The series $\sum_{n\ge 0}(-J_0\Delta D)^n J_0$ converges in $\mathcal{L}(U,V)$ and is a two-sided inverse, with the stated bound. The exact identity

$$D_\varepsilon^{-1} = J_0 - J_0\Delta D J_0 + J_0\Delta D J_0 \Delta D D_\varepsilon^{-1}$$

proves (55) and (56). For matching factors $F_\varepsilon, G_\varepsilon$, their remainder is exactly $F_0E_G + E_F G_0 + (F_\varepsilon - F_0)(G_\varepsilon - G_0)$. Its norm is bounded by the sum of the three corresponding products of norms. Repeating this identity for finitely many factors proves the composition assertion. For a bilinear factor, expand its two input operators separately; products containing two increments are $O(\varepsilon^2)$ on the input unit balls. All estimates apply at any base parameter. The inverse formula and the product identities there imply the continuity of the derivatives. □

For an integer $r \ge 0$, local Leibniz rules give

$$\|c\,\partial^a v\|_{H^r} \le C_r \|c\|_{W^{r,\infty}} \|v\|_{H^{r+|a|}}. \tag{57}$$

For the H^{-1} scalar row, use the divergence-form weak pairing, so the same estimate is obtained by duality from H^1. For $r > 1$ on a surface, H^r is an algebra. One direct local proof uses $\|\widehat{u}\|_{L^1} \le C_r\|u\|_{H^r}$ and the convolution formula for $\widehat{uv}$, distributing the weight of order r to either factor and applying Young's inequality. Localization preserves the estimate. In particular, for each of the three indices $r = s, s+1, s+2$,

$$\|B_\varepsilon(u,v)\|_{H^r} \le C_r \|u\|_{H^{r+1}} \|v\|_{H^{r+1}}. \tag{58}$$

A convective term differentiates one input once. The Helmholtz projection has order zero in the explicitly proven sense of (25).

4.2. Geometry and Full First Variations

Put $V = \partial_\varepsilon R_\varepsilon|_0 = V_T + V_n n$, using the tangent identification at R_0. Let $b(X,Y) = \langle D_X n, Y\rangle$ and $H = \operatorname{tr}_g b$, so the unit outward sphere has $b = g$. The differentiation of $g_{ij} = \partial_i R \cdot \partial_j R$ and of its determinant gives

$$h = g_0' = \mathcal{L}_{V_T} g_0 + 2V_n b, \qquad \kappa = j_0' = \tfrac{1}{2}\operatorname{tr}_g h = \operatorname{div}_g V_T + HV_n. \tag{59}$$

All quantities on the right of the following formulas are evaluated at zero, where $j_0 = 1$:

$$(g^{-1})' = -g^{-1}hg^{-1}, \qquad (j^{-1})' = -\kappa, \tag{60}$$

$$(\operatorname{grad} f)' = -h^\sharp \operatorname{grad} f, \qquad (\operatorname{div} w)' = w(\kappa), \tag{61}$$

$$\Delta' f = -h^{ij}\nabla_i\nabla_j f - (\nabla^i h_i{}^k - \nabla^k\kappa)\nabla_k f, \tag{62}$$

$$\dot\Gamma_{ij}^k = \tfrac{1}{2}g^{k\ell}(\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}), \tag{63}$$

$$k' = -\tfrac{1}{2}k\operatorname{tr}_g h + \tfrac{1}{2}\nabla^i\nabla^j h_{ij} - \tfrac{1}{2}\Delta\operatorname{tr}_g h. \tag{64}$$

To justify the differential identities, differentiate the coordinate formulas for inverse metric, density divergence and Christoffel symbols. Substituting $\dot{\Gamma}$ in the contraction of the covariant Hessian gives (62). For curvature, the variation of Ricci is $\dot{\mathrm{Ric}}_{ij} = \nabla_k \dot{\Gamma}^k_{ij} - \nabla_j \dot{\Gamma}^k_{ik}$. Differentiate $2k = g^{ij}\,\mathrm{Ric}_{ij}$, use $\mathrm{Ric} = kg$, and substitute (63); the two contracted second-derivative terms are $\nabla^i \nabla^j h_{ij}$ and $-\Delta\,\mathrm{tr}\,h$, yielding (64) with the displayed sign and factor $1/2$.

The moving mean and the augmented inverse have derivatives

$$m_g' f = m_g(\kappa f) - m_g(\kappa) m_g f, \qquad \mathscr{D}_g' = \Delta_g' - m_g', \tag{65}$$

$$(\mathscr{D}^{-1})' = -\mathscr{D}^{-1}\mathscr{D}'\mathscr{D}^{-1}. \tag{66}$$

Consequently,

$$\Pi' = -\,\mathrm{grad}'\,\mathscr{D}^{-1}\,\mathrm{div} + \mathrm{grad}\,\mathscr{D}^{-1}\mathscr{D}'\mathscr{D}^{-1}\,\mathrm{div} - \mathrm{grad}\,\mathscr{D}^{-1}\,\mathrm{div}'\,. \tag{67}$$

For each $0 \le r \le s+4$, the first and third terms follow the chain in (25); in the middle term, the internal chain is $H^{r-1} \xrightarrow{\mathscr{D}^{-1}} H^{r+1} \xrightarrow{\mathscr{D}'} H^{r-1} \xrightarrow{\mathscr{D}^{-1}} H^{r+1}$. Thus, this is a bounded derivative on $H^r(TM)$, including the pressure contribution at the lowest level.

For the covariant vector Hessian, differentiating both its vector index and its covariant input index yields

$$\begin{aligned}((\Delta_B)'w)^k = {} & -h^{ij}(\nabla_i \nabla_j w)^k \\ & + g^{ij}\big[(\nabla_i \dot{\Gamma})^k_{j\ell} w^\ell + 2\dot{\Gamma}^k_{i\ell} \nabla_j w^\ell - \dot{\Gamma}^\ell_{ij} \nabla_\ell w^k\big].\end{aligned} \tag{68}$$

The first two differentiated vector-connection terms combine into the factor 2 because g^{ij} is symmetric. The last term comes from differentiating the connection on the index j of $\nabla_j w$. Applying the product rule to (7) and (8) now gives

$$A_0' q = \kappa A_0 q + \Pi' \mathcal{V}_0 q + \nu \Pi_0\big((\Delta_B)' q + k' q\big) - \Pi_0 \mathcal{V}_0(\kappa q), \tag{69}$$

$$\begin{aligned}B_0'(q,r) = {} & \kappa B_0(q,r) - \Pi' \mathcal{C}_0(q,r) - \Pi_0 \mathcal{C}^{\mathrm{met}}(q,r) \\ & + \Pi_0 \mathcal{C}_0(\kappa q, r) + \Pi_0 \mathcal{C}_0(q, \kappa r),\end{aligned} \tag{70}$$

$$(\mathcal{C}^{\mathrm{met}}(q,r))^k = \dot{\Gamma}^k_{ij} q^i r^j. \tag{71}$$

Here, $\mathcal{V}_0 = \nu(\Delta_{B,0} + k_0)$, as in (7). The density terms cannot be reduced to multiplication alone, since

$$\mathcal{C}_0(\kappa q, r) + \mathcal{C}_0(q, \kappa r) = 2\kappa \mathcal{C}_0(q,r) + \tfrac{1}{2}[(q\kappa)r + (r\kappa)q]. \tag{72}$$

Because the unforced equilibrium is identically zero throughout the shape family, these formulas contain no equilibrium-derivative contribution.

4.3. Regularity Summary and Operator Mappings

Tables 1 and 2 summarize the finite regularity and operator topologies used below. Throughout Table 1, all rows follow from the common assumption $R \in C^1(C^{s+7})$; this regularity budget is sufficient rather than minimal. Each row includes the remainder (53) in the listed topology and a continuous derivative in that topology. All bounds are locally uniform for the fixed atlas, s, ν, a positive lower metric bound, and the stated finite coefficient bounds. Inverse rows additionally depend on the base inverse bound; Riesz rows depend on the compact contour and its resolvent bound; homological rows depend on the positive separation in Lemma 12. The constants are uniform over the corresponding input unit balls.

Table 1. Coefficient families, their derivative and remainder topologies.

Family	Space for Family, Derivative and Remainder	First Derivative
g	C^{s+6} covariant tensors	h, (59)
g^{-1}	C^{s+6} contravariant tensors	$-g^{-1}hg^{-1}$
j, j^{-1}	C^{s+6} scalar functions	$\kappa, -\kappa$ at zero
Γ	C^{s+5} coordinate coefficient arrays	(63)
k	C^{s+4} scalar functions	(64)
Δ_B coefficients of orders $2, 1, 0$	$C^{s+6}, C^{s+5}, C^{s+4}$, respectively	(68)

Table 2. Operator families. The middle column gives the norm space for the derivative and its $o(|\varepsilon|)$ remainder. Each primary row follows from $R \in C^1(C^{s+7})$.

Family	Source, Target and Derivative Topology	Derivative and Inverse Justification
$\mathscr{D}_\varepsilon$	$\mathcal{L}(H^{\ell+2}, H^\ell)$, every integer $-1 \le \ell \le s+3$	$\Delta' - m'$, (65); weak pairing for $\ell = -1$
$\mathscr{D}_\varepsilon^{-1}$	$\mathcal{L}(H^\ell, H^{\ell+2})$, every integer $-1 \le \ell \le s+3$	(66); Lemmas 1 and 5
Π_ε	$\mathcal{L}(H^r(TM), H^r(TM))$, every integer $0 \le r \le s+4$	(67) with (25)
A_ε	$\mathcal{L}(H^{r+2}_\sigma, H^r_\sigma)$, every integer $0 \le r \le s+2$	(69); common domain from (37)
B_ε	$\mathcal{B}^2(H^{r+1}_\sigma \times H^{r+1}_\sigma, H^r_\sigma)$, $r = s, s+1, s+2$	(70); one-derivative estimate (58)
$\mathscr{R}_\varepsilon(z)$	$\mathcal{L}(H^r_\sigma, H^{r+2}_\sigma)$, $0 \le r \le s+2$, uniformly for $z \in \Gamma$	$\mathscr{R}A'\mathscr{R}$, Proposition 3
$P_\varepsilon, Q_\varepsilon$	$\mathcal{L}(H^r_\sigma, H^r_\sigma)$ for $r = s, s+1, s+2, s+3, s+4$; also P_ε in $\mathcal{L}(H^s_\sigma, H^{s+4}_\sigma)$	P' in (79), $Q' = -P'$; two-resolvent identity in (80)
$T_\varepsilon, T_\varepsilon^{-1}$	$\mathcal{L}(H^r_\sigma, H^r_\sigma)$, $r = s, s+1, s+2, s+3, s+4$	$T_0' = [P_0', P_0]$, $(T^{-1})_0' = -T_0'$
$\mathscr{H}_\varepsilon$	$\mathcal{L}(\mathcal{Z}_{1,r}, \mathcal{Z}_{0,r})$, $r = s, s+1, s+2$	(141), Lemma 12
$\mathscr{H}_\varepsilon^{-1}$	$\mathcal{L}(\mathcal{Z}_{0,r}, \mathcal{Z}_{1,r})$, $r = s, s+1, s+2$	$-\mathscr{H}^{-1}\mathscr{H}'\mathscr{H}^{-1}$, (143)

Here, $\mathcal{Z}_{1,r} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{r+2}_\sigma)$ and $\mathcal{Z}_{0,r} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^r_\sigma)$; these fixed spaces are used in Section 8. The last two rows are established there.

Proposition 2 (Norm differentiability of the transported equation)**.** *Under (H1), the coefficient rows and the first five operator rows of Table 2 satisfy* (53)*. In particular, with* $E_A = A_\varepsilon - A_0 - \varepsilon A_0'$ *and* $E_B = B_\varepsilon - B_0 - \varepsilon B_0'$,

$$\|E_A\|_{H^{s+2}_\sigma \to H^s_\sigma} = o(|\varepsilon|), \qquad \|E_B\|_{\mathcal{B}^2(H^{s+1}_\sigma \times H^{s+1}_\sigma, H^s_\sigma)} = o(|\varepsilon|), \tag{73}$$

$$\|E_A\|_{H^{s+3}_\sigma \to H^{s+1}_\sigma} = o(|\varepsilon|), \qquad \|E_B\|_{\mathcal{B}^2(H^{s+2}_\sigma \times H^{s+2}_\sigma, H^{s+1}_\sigma)} = o(|\varepsilon|), \tag{74}$$

$$\|E_A\|_{H^{s+4}_\sigma \to H^{s+2}_\sigma} = o(|\varepsilon|), \qquad \|E_B\|_{\mathcal{B}^2(H^{s+3}_\sigma \times H^{s+3}_\sigma, H^{s+2}_\sigma)} = o(|\varepsilon|). \tag{75}$$

All six norms have locally uniform bounds and continuous first derivatives.

Proof. The matrix inverse and determinant are smooth on a set of uniformly positive definite matrices. Their finite Taylor expansions, followed by chartwise product rules through order $s+6$, give the first three coefficient rows. Differentiating the Christoffel and curvature formulas uses respectively one and two additional derivatives of the metric, giving the remaining coefficient rows. Their remainders are bounded by a constant multiplied by the C^{s+7} remainder of R_ε, plus a constant multiplied by $\|R_\varepsilon - R_0\|^2_{C^{s+7}}$. Thus, every coefficient remainder has its asserted small norm.

For each $\ell = -1, 0, \dots, s+3$, the Laplacian and mean have the displayed operator remainder by (57) or the weak pairing at -1. Lemma 1 supplies the base inverse on those fixed spaces. Applying (56) at each of these finitely many levels proves the full inverse row with a common small interval. Composing along (25) proves the pressure-projection row for each $r = 0, \dots, s+4$. Every inverse occurs between the divergence output and the gradient input.

For A, its unprojected differential expression has order two; the order-zero coefficient includes two derivatives of g. Multiplication by j^{-1} takes H^{r+2}_σ to $H^{r+2}(TM)$, the viscous expression takes that to $H^r(TM)$, the pressure projection stays in $H^r(TM)$, and multiplication by j returns to H^r_σ. Apply the finite product remainder identity at every $0 \le r \le s+2$. For B and each of $r = s, s+1, s+2$, the two inputs are in H^{r+1}, the covariant convection is in H^r by (58), the projection is in H^r, and the output multiplication fixes the divergence constraint. The bilinear product remainder is bounded on the two unit balls by the sum of coefficient and projection remainders and products of two increments. This gives all six explicitly listed norms. Their derivatives are (69) and (70). The output remains solenoidal because each family has that property and the solenoidal subspace is closed. Repeating the estimates at an arbitrary parameter and using the assumed C^1 coefficient dependence proves continuity of the derivatives. □

4.4. Riesz Projection and the Canonical Spectral Comparison

Proposition 3 (Riesz calculus on all required levels)**.** *On a fixed contour Γ enclosing the rank-three block, put $\mathscr{R}_\varepsilon(z) = (z - A_\varepsilon)^{-1}$. Then, for every $0 \le r \le s+2$, uniformly in $z \in \Gamma$,*

$$\mathscr{R}_0'(z) = \mathscr{R}_0(z) A_0' \mathscr{R}_0(z), \tag{76}$$

$$\sup_{z\in\Gamma} \|\mathscr{R}_\varepsilon(z) - \mathscr{R}_0(z) - \varepsilon \mathscr{R}_0(z) A_0' \mathscr{R}_0(z)\|_{H^r_\sigma \to H^{r+2}_\sigma} = o(|\varepsilon|). \tag{77}$$

The projection and comparison rows in Table 2 hold. In particular, P_ε is C^1 as a map $H^s_\sigma \to H^{s+4}_\sigma$, and $T_\varepsilon, T_\varepsilon^{-1}$ are C^1 endomorphisms on each of the five indicated levels.

Proof. Lemma 3 supplies the base inverse of $z - A_0$ from H^r_σ to H^{r+2}_σ. Apply Lemma 5 to $D_\varepsilon = z - A_\varepsilon$. On the compact contour, all base inverse norms have a finite maximum. The bound in (56) is therefore uniform in z, with $E_D = -E_A$, and proves (76) and (77). There are finitely many r, so one parameter interval works for all of them. This also proves the persistence of the contour under a small shape perturbation.

Contour integration gives

$$P_\varepsilon = \frac{1}{2\pi i} \int_\Gamma \mathscr{R}_\varepsilon(z)\, \mathrm{d}z, \tag{78}$$

$$P_0' = \frac{1}{2\pi i} \int_\Gamma \mathscr{R}_0(z) A_0' \mathscr{R}_0(z)\, \mathrm{d}z. \tag{79}$$

Its remainder norm on any available level is bounded by the contour length divided by 2π times the supremum in (77). For the highest output level, use

$$P_\varepsilon = \frac{1}{2\pi i}\int_\Gamma z\,\mathscr{R}_\varepsilon(z)^2\,\mathrm{d}z : H^s_\sigma \longrightarrow H^{s+4}_\sigma. \tag{80}$$

Indeed, $\partial_z\mathscr{R} = -\mathscr{R}^2$, and the integral of $\partial_z(z\mathscr{R})$ is zero; equivalently, its residues vanish. In the squared resolvent, the right factor maps $H^s_\sigma \to H^{s+2}_\sigma$ and the left one maps $H^{s+2}_\sigma \to H^{s+4}_\sigma$. Their product remainders prove (80) in norm, including its C^1 dependence. Restricting its input to each of $H^{s+1}_\sigma, \dots, H^{s+4}_\sigma$ and including the output in that space gives all five endomorphism rows. Thus, P is controlled on the raised level using operator domains through H^{s+4}_σ.

The contour can be chosen invariant under complex conjugation, so P_ε preserves real fields. Differentiating $P_\varepsilon^2 = P_\varepsilon$ yields

$$P_0P_0'P_0 = 0, \qquad Q_0P_0'Q_0 = 0. \tag{81}$$

The map $T_\varepsilon = P_\varepsilon P_0 + Q_\varepsilon Q_0$ satisfies

$$P_\varepsilon T_\varepsilon = T_\varepsilon P_0, \qquad T_\varepsilon = I + (P_\varepsilon - P_0)(P_0 - Q_0), \qquad T_0' = [P_0', P_0]. \tag{82}$$

It is close to I in each of the five endomorphism norms. Apply the inverse lemma there and take the minimum of the five small parameter intervals. This gives T_ε^{-1}, its derivative and its remainder in exactly those norms. Rank is preserved because T_ε intertwines the two projections invertibly. All these maps agree on common inputs by Lemma 3. □

5. An Analytic Spectral Certificate and Its Persistence

5.1. Generator Signs and the Spherical Endpoint

Write the Stokes decay rates as $0 \le \lambda_1 \le \lambda_2 \le \cdots$, counting multiplicities and including any zero Killing rates. The corresponding generator eigenvalues are $-\lambda_j$. At the unit sphere, (30) is

$$A_0^\psi = \nu(\Delta_0 + 2) \quad \text{on scalar functions modulo constants.}$$

The spherical harmonics of degree $\ell \ge 1$ have rates $\nu(\ell(\ell+1) - 2)$: the three degree-one rotations have rate zero, degree two has rate 4ν, and degree three has rate 10ν. Thus, $|z| = \nu$ isolates the central rank-three block. The absence of simplicity within this block causes no difficulty for the Riesz construction.

Lemma 6 (No Killing fields on a strictly triaxial ellipsoid)**.** *An ellipsoid with three distinct positive semiaxes has no nonzero Killing field. All its Stokes decay rates are strictly positive.*

Proof. Order the semiaxes as $a > b > c$, set $Q = \mathrm{diag}(a^{-2}, b^{-2}, c^{-2})$, and write the ellipsoid as $X^TQX = 1$. Its outward normal is $QX/|QX|$. The restriction of $Q/|QX|$ to the tangent plane is its second fundamental form. The determinant on that plane is $\det Q\,(n^TQ^{-1}n)/|QX|^2$, where $n = QX/|QX|$. Since $X^TQX = 1$, this gives

$$k(X) = \frac{\det Q}{(X^TQ^2X)^2}.$$

The maximum occurs only at $p_\pm = \pm(a, 0, 0)$, since $X^T Q^2 X$ on $X^T Q X = 1$ has its unique smallest eigendirection along the longest axis. A connected isometry flow must fix each of these two points. In the tangent coordinates $(y, z) = (X_2, X_3)$ at p_+,

$$X^T Q^2 X = a^{-2} + \frac{a^2 - b^2}{a^2 b^4} y^2 + \frac{a^2 - c^2}{a^2 c^4} z^2.$$

The induced metric at the origin of this chart is Euclidean. The two displayed quadratic coefficients differ: $(a^2 - t)/(a^2 t^2)$ is strictly decreasing for $0 < t < a^2$. The Hessian of k therefore has two distinct eigendirections there. The derivative of the connected isometry flow belongs to $SO(2)$ and preserves this Hessian; its only connected possibility is the identity. An isometry fixing a point and its derivative fixes every geodesic from that point, and hence the whole connected complete surface. Thus, the flow is the identity and its Killing generator is zero. The kernel characterization in Theorem 4 and compact resolvent now imply $\lambda_1 > 0$. □

5.2. Min–Max on the Correct Scalar Quotient

Let $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots$ be the eigenvalues of $-\Delta_g$. The subscript 4 below always means the fourth *nonconstant* scalar eigenvalue, not the fourth entry of a list beginning with σ_0.

Lemma 7 (Scalar lower bound for the fourth Stokes rate)**.** *If $k_g \le k_{\max}$ on a closed genus-zero surface, then*

$$\lambda_4 \ge \nu(\sigma_4 - 2k_{\max}). \tag{83}$$

Proof. On $H^2(M)/\mathbb{R}$, with mass space $H^1(M)/\mathbb{R}$, use

$$c_g(\psi) = \int (\Delta_g \psi)^2 \, \mathrm{d}\mu_g, \qquad m_g^\psi(\psi) = \int |\operatorname{grad}_g \psi|_g^2 \, \mathrm{d}\mu_g.$$

In an L_g^2 orthonormal scalar eigenbasis, a function with coefficients c_j, $j \ge 1$, has $c_g = \sum_{j\ge1} \sigma_j^2 |c_j|^2$ and $m_g^\psi = \sum_{j\ge1} \sigma_j |c_j|^2$. The generalized eigenvalues of these two forms are exactly σ_j, including multiplicity. The domain of the numerator is $H^2/\mathbb{R}$, by scalar elliptic regularity, and the mass completion is $H^1/\mathbb{R}$. Formula (33) gives the pointwise form inequality

$$\mathfrak{a}_g^\psi(\psi, \psi) \ge \nu\big(c_g(\psi) - 2k_{\max} m_g^\psi(\psi)\big).$$

For clarity, the min–max statement used here is that the nth eigenvalue of a lower-bounded closed form with compact form-domain inclusion into its mass space equals the infimum, over n-dimensional subspaces of its form domain, of the maximum of its Rayleigh quotient. It follows by expansion in the orthonormal eigenbasis: the span of the first n eigenvectors gives the upper bound, and every n-dimensional subspace contains a vector orthogonal to the first $n - 1$ eigenvectors, giving the lower bound. The shifted forms above are closed and have compact inclusion by the elliptic regularity and compactness of $H^2 \hookrightarrow H^1$. Applying this characterization to the same domain and mass on both sides proves (83). Lemma 2 identifies the entire velocity space with this quotient, so no harmonic vector fields are omitted. □

Theorem 5 (Exact ellipsoid certificate). *For the ellipsoid with semiaxes* $(24/25, 1, 26/25)$,

$$\frac{\lambda_3}{\nu} < \alpha_{\mathrm{rat}} = \frac{2905746390625}{16181071183872} < \frac{9}{50}, \tag{84}$$

$$\frac{\lambda_4}{\nu} \geq \beta_{\mathrm{bd}} = \frac{438707}{158184} > \frac{277}{100}. \tag{85}$$

In particular,

$$\lambda_4 - 2\lambda_3 > \nu(\beta_{\mathrm{bd}} - 2\alpha_{\mathrm{rat}}) > \frac{12}{5}\nu. \tag{86}$$

The contour $|z| = \nu$ *selects exactly the first three generator eigenvalues. In a sufficiently small shape neighborhood, Assumption 1 holds with* $\alpha = \nu/5$, $\eta = \nu/2$ *and* $\beta = 5\nu/2$.

Proof. For this certificate, use the auxiliary round metric $g_\circ$ and area $\mu_\circ$ on S^2, irrespective of the parameter center chosen elsewhere. This does not change any Stokes rate: if $\mu_{g_0} = j_* \mu_\circ$, the fixed flux isomorphism is $q_\circ = j_* q_0$ and the two area factors satisfy $j_\circ = j_* j_0$. Hence, $j_\circ^{-1} q_\circ = j_0^{-1} q_0$ is the same physical velocity. The Piola identity transfers the divergence constraint, and this fixed isomorphism conjugates the two realizations. All fluxes and densities in the present proof refer to $\mu_\circ$.

Take general semiaxes $0 < m \leq b \leq M$, $R(x) = Sx$, $S = \mathrm{diag}(m, b, M)$ and $H = S^2$. The area transformation formula for a linear map gives

$$j(x) = \det S \, |S^{-T} x| = mbM\sqrt{x^T H^{-1} x}.$$

It follows, including the endpoint extrema, that

$$m^2 g_\circ \leq g \leq M^2 g_\circ, \qquad j_{\min} = mb \leq j \leq bM = j_{\max}, \qquad k_g \leq k_{\max} = \frac{M^2}{b^2 m^2}. \tag{87}$$

The curvature bound follows from the curvature formula in Lemma 6; it remains valid when two axes agree. The scalar Rayleigh quotient of any nonzero $\psi \in H^1$ satisfies

$$\frac{\int |\operatorname{grad}_g \psi|_g^2 \, \mathrm{d}\mu_g}{\int |\psi|^2 \, \mathrm{d}\mu_g} \geq \frac{j_{\min}}{M^2 j_{\max}} \frac{\int |\operatorname{grad}_\circ \psi|^2 \, \mathrm{d}\mu_\circ}{\int |\psi|^2 \, \mathrm{d}\mu_\circ}.$$

Apply min–max to the full scalar space, *including* the constant eigenfunction before numbering the nonconstant modes. This avoids equating two different mean-zero subspaces. Since $\sigma_4(S^2) = 6$, Lemma 7 yields

$$\sigma_4 \geq \frac{6 j_{\min}}{M^2 j_{\max}}, \qquad \lambda_4 \geq \nu\left(\frac{6 j_{\min}}{M^2 j_{\max}} - 2k_{\max}\right). \tag{88}$$

For the upper bound, use the *reference fluxes* $q_p = x \times p = \Omega_p x$, where Ω_p is the skew matrix of this map. Their pulled-back *physical velocities* are $v_p = j^{-1} q_p$, and their actual tangent velocities are $\mathrm{d}R \, v_p$. They satisfy $\operatorname{div}_g v_p = 0$ by (4). The mass and energy notation is fixed as follows:

$$\mathsf{M}_g(v) = \int g(v, v) \, \mathrm{d}\mu_g, \qquad \mathfrak{m}(q, q) = \mathsf{M}_g(j^{-1} q) = \int g(q, q) j^{-1} \, \mathrm{d}\mu_\circ, \tag{89}$$

$$\mathsf{E}_g(v) = 2\nu \int |\operatorname{Def}_g v|_g^2 \, \mathrm{d}\mu_g, \qquad \mathfrak{a}(q, q) = \mathsf{E}_g(j^{-1} q). \tag{90}$$

Thus, $\mathfrak{m}(q_p, q_p) = \mathsf{M}_g(v_p)$ and $\mathfrak{a}(q_p, q_p) = \mathsf{E}_g(v_p)$, with q_p retained as the reference-flux argument of the transported forms.

For tangent vectors, $\mathcal{L}_{q_p} g$ is the restriction of $H\Omega_p - \Omega_p H$. In diagonal coordinates, its off-diagonal entries are $(H_{ii} - H_{jj})(\Omega_p)_{ij}$, and

$$|H\Omega_p - \Omega_p H|_F^2 = 2\sum_{i<j}(H_{ii} - H_{jj})^2(\Omega_p)_{ij}^2 \le 2(M^2 - m^2)^2|p|^2.$$

A covariant two-tensor's g-norm is at most m^{-2} times its ambient Frobenius norm. Since $\mathrm{Def}_g q = \frac{1}{2}\mathcal{L}_q g$,

$$|\mathrm{Def}_g q_p|_g \le \frac{\sqrt{2}}{2}\chi|p|, \qquad \chi = \frac{M^2}{m^2} - 1. \tag{91}$$

Differentiating the exact density on the sphere gives

$$\mathrm{grad}_\circ \log j = \frac{(I - x\otimes x)H^{-1}x}{x^T H^{-1}x}.$$

Here is the variance estimate used to bound this expression. For a symmetric matrix with eigenvalues in $[a_-, a_+]$ and weights x_i^2 summing to one, put $\bar{a} = \sum x_i^2 a_i$. From $(a_i - a_-)(a_+ - a_i) \ge 0$, one obtains

$$\sum x_i^2(a_i - \bar{a})^2 \le (a_+ - \bar{a})(\bar{a} - a_-) \le \tfrac{1}{4}(a_+ - a_-)^2.$$

Use $a_- = M^{-2}$, $a_+ = m^{-2}$, $x^T H^{-1}x \ge M^{-2}$, and the dual metric bound $|\cdot|_g \le m^{-1}|\cdot|_0$ to conclude

$$|\,\mathrm{d}\log j|_g \le \frac{\chi}{2m}. \tag{92}$$

The product identity

$$\mathrm{Def}_g(j^{-1}q) = j^{-1}\big(\mathrm{Def}_g q - \mathrm{sym}(q^\flat \otimes \mathrm{d}\log j)\big)$$

and $|\,\mathrm{sym}(a\otimes b)| \le |a|\,|b|$, $|q_p|_g \le M|p|$, now give

$$|\,\mathrm{Def}_g v_p|_g \le C_{\mathrm{def}}|p|, \qquad C_{\mathrm{def}} = \frac{\chi}{2j_{\min}}\left(\sqrt{2} + \frac{M}{m}\right). \tag{93}$$

The pointwise bound and (89) and (90) imply

$$\mathfrak{a}(q_p, q_p) = \mathsf{E}_g(v_p) \le 8\pi\nu j_{\max} C_{\mathrm{def}}^2|p|^2, \tag{94}$$

$$\mathfrak{m}(q_p, q_p) = \mathsf{M}_g(v_p) \ge \frac{m^2}{j_{\max}}\int_{S^2}|x\times p|^2\,\mathrm{d}\mu_\circ = \frac{8\pi}{3}\frac{m^2}{j_{\max}}|p|^2. \tag{95}$$

The last integral follows from $\int x_i x_j\,\mathrm{d}\mu_\circ = (4\pi/3)\delta_{ij}$. If q_p vanishes everywhere, $x\times p = 0$ for every unit x; hence, $p = 0$. Multiplication by positive j^{-1} and the injective tangent map $\mathrm{d}R$ preserve this independence. The trial space is therefore exactly three-dimensional. By the generalized mass/form min–max Formula (40) in Theorem 4, its maximum Rayleigh quotient bounds the third decay rate:

$$\lambda_3 \le \nu\frac{3j_{\max}^2}{m^2}C_{\mathrm{def}}^2. \tag{96}$$

All constants for the specified ellipsoid are given in Table 3. The strict inequality $\sqrt{2} < 99/70$ follows from $2 \cdot 70^2 = 9800 < 9801 = 99^2$. Therefore,

$$C_{\mathrm{def}} < \frac{25/144}{2(24/25)}\left(\frac{99}{70} + \frac{13}{12}\right) = \frac{131125}{580608} =: C_{\mathrm{rat}}, \tag{97}$$

$$\frac{3j_{\max}^2}{m^2}C_{\mathrm{rat}}^2 = \frac{169}{48}\left(\frac{131125}{580608}\right)^2 = \frac{2905746390625}{16181071183872}, \tag{98}$$

$$\frac{6j_{\min}}{M^2 j_{\max}} - 2k_{\max} = \frac{11250}{2197} - \frac{169}{72} = \frac{438707}{158184}. \tag{99}$$

The following positive rational differences establish all numerical comparisons in the theorem exactly:

$$\frac{9}{50} - \alpha_{\mathrm{rat}} = \frac{171160561799}{404526779596800} > 0, \tag{100}$$

$$\beta_{\mathrm{bd}} - \frac{277}{100} = \frac{13433}{3954600} > 0, \tag{101}$$

$$\beta_{\mathrm{bd}} - 2\alpha_{\mathrm{rat}} - \frac{12}{5} = \frac{1265842631757751}{88874533477416960} > 0. \tag{102}$$

This proves (84)–(86). Lemma 6 gives the strict positivity of all rates. The first three negative generator eigenvalues lie in $(-9\nu/50, 0)$, inside $|z| = \nu$, whereas the fourth and all higher ones lie at or below $-277\nu/100$, outside. Persistence and the semigroup constants follow from Lemma 8 below and Lemma 4. Their proofs use only the already established realization and coefficient estimates. □

Table 3. Exact constants and bounds used in the ellipsoid spectral certificate.

Quantity	Exact Value or Bound
Metric comparison $m^2 g_\circ \le g \le M^2 g_\circ$	$m^2 = 576/625$, $M^2 = 676/625$
Area-density extrema	$j_{\min} = 24/25$, $j_{\max} = 26/25$
Curvature upper bound	$k_{\max} = 169/144$
Scalar Rayleigh comparison factor	$j_{\min}/(M^2 j_{\max}) = 1875/2197$
Fourth nonconstant scalar rate	$\sigma_4 \ge 11250/2197$
Metric anisotropy χ	$25/144$
Reference-flux deformation	$\lvert \mathrm{Def}_g q_p \rvert_g \le (25\sqrt{2}/288)\lvert p \rvert$
Density gradient	$\lvert \mathrm{d} \log j \rvert_g \le 625/6912$
Physical-velocity deformation	$\lvert \mathrm{Def}_g v_p \rvert_g < C_{\mathrm{rat}}\lvert p \rvert$, $C_{\mathrm{rat}} = 131125/580608$ for $p \neq 0$
Mass lower coefficient	$m^2/j_{\max} = 288/325$, multiplying $(8\pi/3)\lvert p \rvert^2$
Energy upper coefficient	$j_{\max} C_{\mathrm{rat}}^2 = 8940758125/168552824832$, multiplying $8\pi\nu\lvert p \rvert^2$
Trial Rayleigh prefactor	$3j_{\max}^2/m^2 = 169/48$
Third Stokes-rate upper bound	$\alpha_{\mathrm{rat}} = 2905746390625/16181071183872$
Fourth Stokes-rate lower bound	$\beta_{\mathrm{bd}} = 438707/158184$

For orientation, the exact fractions correspond to $\alpha_{\mathrm{rat}} \approx 0.179577$, $\beta_{\mathrm{bd}} \approx 2.773397$, and $\beta_{\mathrm{bd}} - 2\alpha_{\mathrm{rat}} \approx 2.414243$; the proof uses the exact rational values above.

5.3. Local Persistence with a Common Contour

Lemma 8 (Continuity of the ordered rates)**.** *For a shape family satisfying (H1), each ordered rate $\lambda_k(\varepsilon)$ is continuous. Around the certified ellipsoid, the first three rates are less than $\nu/5$ and the fourth exceeds $27\nu/10$ after shrinking the parameter interval. The first rate remains positive there. These bounds give the contour and constants asserted in Theorem 5.*

Proof. Use the fixed form domain H^1_σ of fluxes, the mass from (35) and energy from (36). Coefficient continuity and the product formula for $\mathrm{Def}_g(j^{-1}q)$ give, for a function $d(\varepsilon) \to 0$,

$$|\mathfrak{m}_\varepsilon(q,q) - \mathfrak{m}_0(q,q)| \le d(\varepsilon)\mathfrak{m}_0(q,q), \qquad |\mathfrak{a}_\varepsilon(q,q) - \mathfrak{a}_0(q,q)| \le d(\varepsilon)\|q\|^2_{H^1}.$$

Indeed, the energy is a quadratic expression in first derivatives and values of q, whose coefficients depend continuously on g, j and their first derivatives; Cauchy–Schwarz bounds the difference by the displayed norm. The form estimate proved in Lemma 4 gives $\|q\|^2_{H^1} \le C(\mathfrak{a}_0(q,q) + \mathfrak{m}_0(q,q))$. Absorb this C in d. For $d < 1$, the two Rayleigh quotients therefore satisfy

$$\frac{(1-d)\mathfrak{a}_0/\mathfrak{m}_0 - d}{1+d} \le \frac{\mathfrak{a}_\varepsilon}{\mathfrak{m}_\varepsilon} \le \frac{(1+d)\mathfrak{a}_0/\mathfrak{m}_0 + d}{1-d}.$$

All subspaces in the min–max characterization are subspaces of the same H^1_σ. Taking their maxima and then infima preserves these inequalities, with $\lambda_k(0)$ in place of $\mathfrak{a}_0/\mathfrak{m}_0$ and $\lambda_k(\varepsilon)$ in the middle. Hence, $\lambda_k(\varepsilon) \to \lambda_k(0)$ for every fixed k. This also proves continuity when an eigenvalue is multiple, without choosing differentiable individual eigenvectors.

At the ellipsoid, apply continuity to $k = 1, 3, 4$, using the strict bounds and $\lambda_1(0) > 0$. This gives $0 < \lambda_1 \le \lambda_3 < \nu/5$ and $\lambda_4 > 27\nu/10$. The spectrum is real and nonpositive, so $|z| = \nu$ encloses exactly three generator eigenvalues throughout this neighborhood. The contour-resolvent bounds also follow directly from Proposition 3. Lemma 4 with $\alpha = \nu/5$ and any $\beta < 27\nu/10$, in particular $\beta = 5\nu/2$, supplies (9) and (10). Choosing $\eta = \nu/2$ gives $\alpha < \eta$ and $3\eta < \beta$, as required. □

For affine families approaching the sphere, the metric spread χ in (93) is $O(|\delta|)$ and the remaining factors stay bounded. Nonnegativity and (96) consequently give

$$0 \le \lambda_3(\delta) = O(\delta^2), \qquad \lambda_4(\delta) \longrightarrow 4\nu. \tag{103}$$

The second conclusion follows from Lemma 8. Thus, a common rank-three block and the three-weight gap hold on both sides of the spherical endpoint. For a scalar affine matrix, all three rates remain zero; the same sufficient gap still holds.

6. A Local C^2 Invariant Graph on the Derivative-Losing Scale

6.1. The Local Mild Semiflow

At a fixed shape, let $X_0 = H^s_\sigma$, $Y = H^{s+1}_\sigma$ and $X_1 = H^{s+2}_\sigma$. Suppress the shape subscript. A mild solution satisfies

$$q(t) = e^{tA}q_0 + \int_0^t e^{(t-a)A}B(q(a), q(a))\,\mathrm{d}a. \tag{104}$$

The smoothing estimate for the full semigroup on $0 < t \le T_0$ has the form $Ct^{-1/2}$ from X_0 to Y; the finite slow part is bounded and can be absorbed in that estimate on this finite interval.

Lemma 9 (Finite-time differentiability of the mild flow)**.** *Near zero,* (104) *has a unique local mild semiflow on Y, which is C^2 in its initial value as a map to $C([0,T], Y)$ for a common sufficiently small T. At the zero solution,*

$$D\Phi^t(0)a = e^{tA}a, \tag{105}$$

$$D^2\Phi^t(0)[a,b] = 2\int_0^t e^{(t-v)A}B(e^{vA}a, e^{vA}b)\,\mathrm{d}v. \tag{106}$$

The integral is a Y-valued Bochner integral for $t > 0$ and is continuous at $t = 0$ with value zero.

Proof. On $\mathcal{X}_T = C([0,T], Y)$, the integral of a forcing in $C([0,T], X_0)$ has norm at most $2C\sqrt{T}$ times its supremum norm. On a ball of radius R, the quadratic map has Lipschitz constant at most $2C_B R$, by (58). Choose R larger than twice the semigroup bound multiplied by the allowed initial radius, and T so that the integral map stays in this ball and its Lipschitz constant is less than $1/2$. Geometric iteration gives the unique fixed point. Subtracting two fixed-point equations shows Lipschitz dependence on initial values.

At a solution q, the first-variation equation is

$$v_a(t) = e^{tA}a + 2\int_0^t e^{(t-v)A} B(q(v), v_a(v))\,\mathrm{d}v.$$

The integral operator on its right has norm less than $1/2$, so its inverse has norm at most 2. For $d = q(q_0 + a) - q(q_0)$, subtracting the variation equation leaves the source $B(d,d)$; hence, $\|d - v_a\|_{\mathcal{X}_T} \le C\|a\|_Y^2$. The second variation solves

$$w_{a,b}(t) = 2\int_0^t e^{(t-v)A}[B(q(v), w_{a,b}(v)) + B(v_a(v), v_b(v))]\,\mathrm{d}v.$$

Again, invert the same contraction. Subtract the first-variation equations at $q_0 + a$ and q_0, then subtract this second-variation equation. The remaining sources are $B(d - v_a, v_b)$ and $B(d, v_b(q_0 + a) - v_b(q_0))$, each bounded by $C\|a\|_Y^2\|b\|_Y$. Their integrals have the same bound in $\mathcal{X}_T$. Difference equations for $w_{a,b}$ at two base points give a bound $C\|q_0 - \widetilde{q}_0\|_Y\|a\|_Y\|b\|_Y$, since all coefficients and first variations are Lipschitz there. This proves the Fréchet C^2 assertion, not just directional variations. At $q = 0$, the coefficient $B(q,\cdot)$ vanishes, giving (105) and (106). The integrable kernel and continuous integrands justify all Bochner integrals and continuity at zero. Time uniqueness also gives the semiflow composition law on its domain. □

6.2. The Cutoff and Its Derivative Bounds

Let $E = \operatorname{ran} P$ be the slow space at the fixed shape, $i : E \hookrightarrow Y$ its inclusion and $L = i^{-1}Ai$. A spectral comparison returns this notation to E_0 later. Choose a smooth scalar function χ equal to one on $[0,1]$ and zero on $[4,\infty)$, and define

$$N_\rho(u) = \chi(\|u\|_Y^2/\rho^2)B(u,u). \tag{107}$$

The cutoff uses the Hilbert norm on Y at the fixed shape; shape differentiation enters only after the fixed-shape graph construction.

Lemma 10 (Global bounds for a fixed cutoff radius)**.** *For every fixed $\rho > 0$, $N_\rho : Y \to X_0$ is C^3 and globally Lipschitz. Its derivatives satisfy*

$$\begin{aligned} N_\rho(0) = DN_\rho(0) = 0, \qquad & \sup\|DN_\rho\| \le C\rho =: L_\rho, \\ \sup\|D^2N_\rho\| \le C =: M_2, \qquad & \sup\|D^3N_\rho\| \le C\rho^{-1} =: M_3. \end{aligned} \tag{108}$$

The suprema range over all $u \in Y$; the norms are multilinear norms into X_0.

Proof. Write $Q(u) = B(u,u)$, $r(u) = \|u\|_Y^2/\rho^2$, and $r_a = 2\langle u,a\rangle_Y/\rho^2$, $r_{ab} = 2\langle a,b\rangle_Y/\rho^2$. Then, $Q_a = 2B(u,a)$, $Q_{ab} = 2B(a,b)$, and third derivatives of r, Q vanish. The full product derivatives are

$$\begin{aligned} DN(u)a &= \chi Q_a + \chi' r_a Q, \\ D^2N(u)[a,b] &= \chi Q_{ab} + \chi'(r_a Q_b + r_b Q_a + r_{ab} Q) + \chi'' r_a r_b Q, \\ D^3N(u)[a,b,c] &= \chi'(r_a Q_{bc} + r_b Q_{ac} + r_c Q_{ab}) \\ &\quad + \chi'(r_{ab} Q_c + r_{ac} Q_b + r_{bc} Q_a) \\ &\quad + \chi''(r_a r_b Q_c + r_a r_c Q_b + r_b r_c Q_a) \\ &\quad + \chi''(r_{ab} r_c + r_{ac} r_b + r_{bc} r_a) Q + \chi''' r_a r_b r_c Q. \end{aligned}$$

All scalar derivatives of χ are evaluated at $r(u)$. Where a nonzero term occurs, $\|u\|_Y \le 2\rho$; outside this ball, all the terms vanish. Therefore, $\|Q\| \le C\rho^2$, $\|Q_a\| \le C\rho\|a\|$, $|r_a| \le C\rho^{-1}\|a\|$, and $|r_{ab}| \le C\rho^{-2}\|a\|\|b\|$. Each displayed derivative has exactly the bound in (108). At zero, the asserted values follow from the quadratic factor. The integral identity $N(u) - N(v) = \int_0^1 DN(v + t(u-v))(u-v)\,\mathrm{d}t$ proves global Lipschitz continuity. We first fix a sufficiently small positive ρ and then differentiate; the argument uses this fixed cutoff radius throughout the parameter calculation. □

6.3. History Spaces and the Two Integral Estimates

For $\theta > 0$, define the Banach spaces

$$Z_\theta = \{u \in C((-\infty,0], Y) : \|u\|_\theta := \sup_{t\le 0} e^{\theta t}\|u(t)\|_Y < \infty\}, \tag{109}$$

$$Z_\theta^0 = \{f \in C((-\infty,0], X_0) : \|f\|_\theta^0 := \sup_{t\le 0} e^{\theta t}\|f(t)\|_{X_0} < \infty\}. \tag{110}$$

Completeness follows by multiplication with $e^{\theta t}$ and the completeness of bounded continuous functions. If $\theta_1 \le \theta_2$, the inclusions $Z_{\theta_1} \subset Z_{\theta_2}$ and $Z_{\theta_1}^0 \subset Z_{\theta_2}^0$ have norm at most one, since $t \le 0$. Evaluation $u \mapsto u(0)$ has norm at most one to Y from every Z_θ.

For $\alpha < \theta < \beta$, define

$$(\mathscr{K}f)(t) = -\int_t^0 e^{(t-a)A_E} P f(a)\,\mathrm{d}a + \int_{-\infty}^t e^{(t-a)A_Q} Q f(a)\,\mathrm{d}a. \tag{111}$$

Lemma 11 (Weighted integral bound)**.** *The operator in* (111) *maps Z_θ^0 boundedly to Z_θ, with*

$$\|\mathscr{K}\|_{Z_\theta^0 \to Z_\theta} \le M_\theta := C\Big[(\theta-\alpha)^{-1} + \sqrt{\pi}(\beta-\theta)^{-1/2}\Big]. \tag{112}$$

The homogeneous slow history $U\xi(t) = ie^{tL}\xi$ satisfies $\|U\xi\|_\theta \le C|\xi|$.

Proof. For the slow integral, set $b = a - t \ge 0$ and use the backward finite-dimensional group bound. Its weighted norm at t is at most

$$C\|f\|_\theta^0 \int_t^0 e^{\theta t - \alpha(t-a) - \theta a}\,\mathrm{d}a = C\|f\|_\theta^0 \int_0^{-t} e^{-(\theta-\alpha)b}\,\mathrm{d}b \le \frac{C}{\theta-\alpha}\|f\|_\theta^0.$$

For the fast integral, set $b = t - a > 0$. Its weighted norm is at most

$$C\|f\|_\theta^0 \int_0^\infty b^{-1/2} e^{-(\beta-\theta)b}\,\mathrm{d}b = \frac{C\sqrt{\pi}}{\sqrt{\beta-\theta}}\|f\|_\theta^0.$$

The last identity follows from $y = \sqrt{(\beta-\theta)b}$ and $2\int_0^\infty e^{-y^2}\,dy = \sqrt{\pi}$. Both are Bochner integrals in Y. Their continuity in t follows by splitting the fast integral into $0 < b < a_0$, a finite middle interval and a tail: the first part is uniformly small by integrability of $b^{-1/2}$, the tail by exponential decay, and the middle part is continuous by strong continuity and continuity of f. The slow term is a continuous finite-dimensional integral. Finally, $e^{\theta t}\|ie^{tL}\xi\|_Y \le Ce^{(\theta-\alpha)t}|\xi| \le C|\xi|$ for $t \le 0$. □

Proposition 4 (Central spherical endpoint)**.** *The three-weight construction remains valid when the selected slow block is central, $\alpha = 0$. For each $\theta \in \{\eta, 2\eta, 3\eta\}$,*

$$M_\theta = C\left[\theta^{-1} + \sqrt{\pi}(\beta-\theta)^{-1/2}\right] < \infty,$$

because $\eta > 0$ and $3\eta < \beta$. Thus, the slow group may be bounded rather than decaying: no positive slow decay rate is required for the contraction. The cutoff radius is fixed sufficiently small so that (113) *holds and is not sent to zero. The finite-time generator-domain argument of Section 7 is unchanged, and the homological separation becomes $\gamma = \beta - 2\alpha = \beta > 0$.*

Proof. The slow contribution in Lemma 11 is $C(\theta-\alpha)^{-1}$; at $\alpha = 0$, this is C/θ. The fast contribution is finite because each listed weight is below β. Hence, all three contraction constants used in the proof of Theorem 6 are finite, and the same fixed positive cutoff radius can be chosen to make their products with L_ρ smaller than $1/2$. The later domain-gain proof uses only finite-time mild invariance and strong continuity, while the shifted fast resolvents are separated from the sums of slow rates by β. No additional decay assumption is introduced. □

6.4. Contraction and Full Three-Weight Differentiability

Definition 1 (Normalized locally invariant C^2 graph)**.** *At a fixed shape, let $E = \operatorname{ran} P$, let $i : E \hookrightarrow Y$ be the inclusion, and let $U \subset E$ be an open neighborhood of 0. A map $K : U \to Y$ is a* normalized locally invariant C^2 graph in the mild sense *if K is C^2,*

$$K(0) = 0, \qquad DK(0) = i, \qquad PK(\xi) = i\xi,$$

and there is a neighborhood $\mathcal{N} \subset Y$ of 0 such that, whenever $\xi \in U$ and the positive-time mild orbit $\Phi^\tau(K(\xi))$ exists and remains in $\mathcal{N}$ for $0 \le \tau \le t$, one has

$$\Phi^t(K(\xi)) = K(\xi_t), \qquad \xi_t = i^{-1}P\Phi^t(K(\xi)) \in U.$$

For a moving shape, we use the fixed parameter space E_0: a normalized graph is $K_\varepsilon : U_\varepsilon \subset E_0 \to Y$ with $K_\varepsilon(0) = 0$, $DK_\varepsilon(0) = T_\varepsilon i$ and $P_\varepsilon K_\varepsilon(z) = T_\varepsilon iz$, and with the same positive-time mild local-invariance condition for the ε-semiflow while the orbit remains in its chosen local neighborhood. This is the exact class used below for the normalized uniqueness assertion.

Proposition 6 proves that every graph in this class has the same quotient-valued quadratic jet once the tangent inputs are fixed.

Theorem 6 (Local invariant graph)**.** *Under* (9)–(11) *and* (58)*, there exists a local C^2 map $h : E \to QY$, with $h(0) = Dh(0) = 0$, such that $K(\xi) = i\xi + h(\xi)$ is a normalized locally invariant graph in the sense of Definition 1. The construction does not assert uniqueness among all full local C^2 graphs.*

Proof. Fix $\rho > 0$ so small that

$$L_\rho \max\{M_\eta, M_{2\eta}, M_{3\eta}\} < \tfrac{1}{2}. \tag{113}$$

All three Ms are finite by $\alpha < \eta$ and $3\eta < \beta$. For each $\xi \in E$, solve

$$u_\xi = U\xi + \mathscr{K}N_\rho(u_\xi). \tag{114}$$

The substitution is globally Lipschitz from Z_θ to Z^0_θ, with constant L_ρ, because $N_\rho(0) = 0$ and its derivative is globally bounded. The right side is therefore a contraction on each of $Z_\eta, Z_{2\eta}, Z_{3\eta}$. Completeness and the contraction bound give a unique fixed point. The Z_η solution belongs to the other two spaces by inclusion and solves their equations, so the three fixed points coincide. Subtracting the equations for $\xi + a$ and ξ gives

$$\|u_{\xi+a} - u_\xi\|_\eta \le 2\|Ua\|_\eta \le C|a|. \tag{115}$$

The constants below are independent of the base history because the derivatives of the fixed cutoff are globally bounded.

Put $u = u_\xi$, $N = N_\rho$ and $\mathcal{A}_u v = \mathscr{K}(DN(u)v)$. On each of the three weights, $\|\mathcal{A}_u\| < 1/2$, so $I - \mathcal{A}_u$ has inverse norm at most 2. Define, in order,

$$p_a = Ua + \mathcal{A}_u p_a, \tag{116}$$

$$r_{a,b} = \mathcal{A}_u r_{a,b} + \mathscr{K}(D^2N(u)[p_a, p_b]). \tag{117}$$

The first equation is solved on Z_η and gives $\|p_a\|_\eta \le C|a|$. A pointwise bilinear product of histories of weights θ_1, θ_2 belongs to the forcing space of weight $\theta_1 + \theta_2$, with norm at most the bilinear bound times the two history norms. Thus, the forcing of the second equation belongs to $Z^0_{2\eta}$, and $\|r_{a,b}\|_{2\eta} \le C|a||b|$. The solutions are linear and symmetric bilinear in their direction arguments, respectively, by the uniqueness of the linear equations.

Let $d = u_{\xi+a} - u_\xi$. Its exact increment equation is

$$(I - \mathcal{A}_u)(d - p_a) = \mathscr{K}\big(N(u + d) - N(u) - DN(u)d\big).$$

The pointwise remainder is bounded by $(M_2/2)\|d(t)\|^2_Y$. Hence, using the weight assignment $\eta + \eta = 2\eta$ and (115),

$$\|d - p_a\|_{2\eta} \le M_{2\eta} M_2 \|d\|^2_\eta \le C|a|^2. \tag{118}$$

To estimate the base-point dependence of p_b, put $\Delta p_b = p_b(\xi + a) - p_b(\xi)$. The subtraction of its two equations gives

$$(I - \mathcal{A}_u)\Delta p_b = \mathscr{K}\big([DN(u + d) - DN(u)]p_b(\xi + a)\big).$$

Since the bracket is bounded pointwise by $M_2\|d(t)\|_Y$, its forcing has weight $\eta + \eta = 2\eta$. Therefore,

$$\|\Delta p_b\|_{2\eta} \le C|a||b|. \tag{119}$$

After evaluation at zero, (118) proves the Fréchet derivative $DK(\xi)a = p_a(0)$ for $K(\xi) = u_\xi(0)$, and (119) proves its continuity.

For the second derivative, subtract (117) from the equation for Δp_b. With $e_{a,b} = \Delta p_b - r_{a,b}$, the exact equation is

$$(I - \mathcal{A}_u)e_{a,b} = \mathscr{K}(S_1 + S_2 + S_3), \tag{120}$$

$$S_1 = [DN(u+d) - DN(u)]\Delta p_b, \tag{121}$$

$$S_2 = [DN(u+d) - DN(u) - D^2N(u)[d,\cdot]]p_b, \tag{122}$$

$$S_3 = D^2N(u)[d - p_a, p_b]. \tag{123}$$

The complete forcing estimates are

$$\begin{aligned} \|S_1\|^0_{3\eta} &\le M_2\|d\|_\eta\|\Delta p_b\|_{2\eta} \le C|a|^2|b| && (\eta + 2\eta),\\ \|S_2\|^0_{3\eta} &\le \tfrac{1}{2}M_3\|d\|^2_\eta\|p_b\|_\eta \le C_\rho|a|^2|b| && (\eta+\eta+\eta),\\ \|S_3\|^0_{3\eta} &\le M_2\|d - p_a\|_{2\eta}\|p_b\|_\eta \le C|a|^2|b| && (2\eta+\eta). \end{aligned}$$

The second estimate is the integral Taylor remainder for DN, using the bounded third derivative of N. Inverting on $Z_{3\eta}$ yields

$$\|p_b(\xi + a) - p_b(\xi) - r_{a,b}(\xi)\|_{3\eta} \le C_\rho|a|^2|b|. \tag{124}$$

Taking the supremum over $|b| \le 1$ and then evaluating at zero proves the Fréchet derivative of DK, with $D^2K(\xi)[a,b] = r_{a,b}(0)$. No twice-differentiable substitution map on a single history weight has been assumed.

The continuity of this second derivative also has an explicit three-weight equation. Let $v = u_{\xi+c}$ and write $p_a(v), r_{a,b}(v)$ for the variations at this base. Put $R = r_{a,b}(v) - r_{a,b}(u)$. Then,

$$\begin{aligned} (I - \mathcal{A}_u)R = \mathscr{K}\big([DN(v) - DN(u)]r_{a,b}(v)\\ + [D^2N(v) - D^2N(u)][p_a(v), p_b(v)]\\ + D^2N(u)[p_a(v) - p_a(u), p_b(v)]\\ + D^2N(u)[p_a(u), p_b(v) - p_b(u)]\big). \end{aligned} \tag{125}$$

The four $Z^0_{3\eta}$ source bounds, in their displayed order, are

$$\begin{aligned} &M_2\|v - u\|_\eta\|r_{a,b}(v)\|_{2\eta} \le C|c||a||b|,\\ &M_3\|v - u\|_\eta\|p_a(v)\|_\eta\|p_b(v)\|_\eta \le C_\rho|c||a||b|,\\ &M_2\|p_a(v) - p_a(u)\|_{2\eta}\|p_b(v)\|_\eta \le C|c||a||b|,\\ &M_2\|p_a(u)\|_\eta\|p_b(v) - p_b(u)\|_{2\eta} \le C|c||a||b|. \end{aligned}$$

They use respectively $\eta + 2\eta$, 3η, $2\eta + \eta$ and $\eta + 2\eta$. The inverse on $Z_{3\eta}$ and evaluation at zero give continuity in the bilinear norm of D^2K. Since $P\mathscr{K}f(0) = 0$, Equation (114) gives $PK(\xi) = i\xi$ and $h = QK$. The zero history solves the fixed-point equation at $\xi = 0$. There, $DN(0) = 0$, so $p_a = Ua$, proving $h(0) = Dh(0) = 0$.

Directly splitting the two integrals in (111) shows that u_ξ satisfies the mild equation between any two finite times $t_1 \le t_2 \le 0$. The slow term uses only its finite-dimensional group; the fast convolution uses its semigroup. The cutoff equation has a global forward mild solution from $K(\xi)$: on each interval of a fixed sufficiently small length, its integral map is a contraction on the whole space $C([0,T],Y)$, since N is globally Lipschitz, and the finite-interval semigroup bounds are independent of the initial size. Repeated intervals cover any finite positive time and give finite continuous norms.

Concatenate this forward solution with u_ξ to obtain w on $(-\infty, T]$, and set $v(t) = w(t+T)$ for $t \le 0$. For $t + T \le 0$, $e^{\eta t}\|v(t)\|_Y \le e^{-\eta T}\|u_\xi\|_\eta$; for $-T \le t \le 0$, the norm is bounded by continuity of the forward segment. Thus, $v \in Z_\eta$. It obeys the cutoff mild equation on every finite subinterval, including those crossing the concatenation time. For its fast component, send a lower time $a \to -\infty$ in the variation-of-constants formula. The homogeneous term tends to zero because $\|e^{(t-a)A_Q}Qv(a)\|_Y \le C_t e^{(\beta-\eta)a}\|v\|_\eta$ for sufficiently negative a, and $\beta > \eta$. The slow component is obtained by solving its finite-dimensional equation backward from zero. These two formulas are exactly (114) with slow input $i^{-1}Pv(0)$. The uniqueness of its history fixed point gives $v(0) = K(i^{-1}Pv(0))$. Thus, the cutoff graph is forward invariant. On the neighborhood in which $\|K(\xi)\|_Y < \rho$, the cutoff equals the original quadratic nonlinearity. For every orbit segment staying in that neighborhood, uniqueness identifies the cutoff and original solutions. This is the asserted local invariance for (104). □

7. Finite-Time Domain Gain and the Unique Quotient Jet

7.1. The Quotient Object and Its Fixed Representatives

Let $\pi_\varepsilon : Y \to Y/E_\varepsilon$ denote the quotient map, where the quotient is equipped with its Banach quotient norm. Since E_ε is finite dimensional and closed, the quotient is a Banach space. The map

$$\eta_\varepsilon : Y/E_\varepsilon \longrightarrow Q_\varepsilon Y, \qquad \eta_\varepsilon([y]) = Q_\varepsilon y \tag{126}$$

is well-defined and is a bounded isomorphism. Its inverse is $\pi_\varepsilon|_{Q_\varepsilon Y}$; its norm is at most $\|Q_\varepsilon\|$, since one may take the infimum over representatives $y + e$, $e \in E_\varepsilon$.

The distinction between inputs, quotient outputs and complement representatives is displayed by

$$\begin{array}{ccc} & & E_0 \xrightarrow{\iota_\varepsilon = T_\varepsilon i} E_\varepsilon, \\ Y/E_0 & \xrightarrow{\widetilde{T}_\varepsilon} & Y/E_\varepsilon \\ {\scriptstyle \eta_0}\downarrow & & \downarrow{\scriptstyle \eta_\varepsilon} \qquad \widetilde{T}_\varepsilon([y]) = [T_\varepsilon y]. \\ Q_0 Y & \xrightarrow[T_\varepsilon|_{Q_0 Y}]{} & Q_\varepsilon Y, \end{array} \tag{127}$$

The upper map is well-defined because $T_\varepsilon E_0 = E_\varepsilon$, and the square commutes because $Q_\varepsilon T_\varepsilon = T_\varepsilon Q_0$.

For a C^2 parameterization K_ε satisfying (13), the natural tensor with tangent inputs is

$$\mathcal{J}_2^{\mathrm{nat}}(\varepsilon)(v, w) = \pi_\varepsilon D^2 K_\varepsilon(0)[\iota_\varepsilon^{-1}v, \iota_\varepsilon^{-1}w], \quad v, w \in E_\varepsilon. \tag{128}$$

Thus, $\mathcal{J}_2^{\mathrm{nat}}(\varepsilon) \in \mathcal{B}^2_{\mathrm{sym}}(E_\varepsilon, Y/E_\varepsilon)$. The tensor $\mathcal{J}_2(\varepsilon)$ in (14) has its two inputs pulled to E_0 by ι_ε; its values are still in the moving quotient. Pulling its output back by $\widetilde{T}_\varepsilon^{-1}$ and then applying η_0 gives exactly

$$\eta_0 \widetilde{T}_\varepsilon^{-1} \mathcal{J}_2(\varepsilon) = T_\varepsilon^{-1} Q_\varepsilon D^2 K_\varepsilon(0) = \overline{W}_\varepsilon.$$

This is first an equality in $\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 Y)$. Its improved output regularity will follow from the finite-time argument, not from the definition of the quotient.

If the internal coordinates change by $\chi(0) = 0$, $D\chi(0) = I$, then the chain rule gives

$$D^2(K \circ \chi)(0)[a, b] = D^2 K(0)[a, b] + DK(0) D^2\chi(0)[a, b]. \tag{129}$$

The additional term is tangent and is annihilated by π. With a general invertible $D\chi(0)$, the two inputs also transform by that linear map; Formula (128) compensates for it. Any

local C^2 embedding tangent to E can be normalized to $PK(z) = iz$: the derivative of $i^{-1}PK$ at zero is invertible, so its finite-dimensional local inverse supplies the coordinate change. Here, the inverse function theorem is used only for a map between three-dimensional spaces with an invertible derivative; it is not applied to a parabolic time map.

7.2. Domain Gain from an Identity at Positive Time

Proposition 5 (Finite-time identity and generator-domain gain)**.** *At a fixed shape, let $K : E \to Y$ be any normalized locally invariant C^2 graph in the mild sense of Definition 1. Then, $W = QD^2K(0)$ belongs to $\mathcal{B}^2_{\mathrm{sym}}(E, QX_1)$ and obeys*

$$A_Q W(a,b) - W(La,b) - W(a,Lb) = -2QB(ia,ib). \tag{130}$$

No assertion that $K(z)$ or its higher-order remainder belongs to $D(A)$ is needed.

Proof. For a fixed sufficiently small $t \geq 0$, local invariance and the normalization give $\Phi^t(K(z)) = K(z_t)$, with $z_t = i^{-1}P\Phi^t(K(z))$ a C^2 finite-dimensional map near zero. Lemma 9 gives a C^2 finite-time mild solution map with values in Y, so the invariance identity may be differentiated twice in Y before the generator is invoked. Its linear part is $Dz_t(0) = e^{tL}$. Apply Q to the second derivative: the term $DK(0)D^2z_t(0)$ disappears because $QDK(0) = 0$. The bounded finite-time maps in Lemma 9 are differentiated first; the generator is applied only after the resulting Y-identity has been formed. That identity is

$$e^{tA_Q}W(a,b) + 2\int_0^t e^{(t-v)A_Q}QB(ie^{vL}a, ie^{vL}b)\,\mathrm{d}v = W(e^{tL}a, e^{tL}b). \tag{131}$$

To obtain the domain gain, we next view (131) in X_0 and take a right difference quotient. The norm limit gives QX_1-valued quadratic coefficients by the semigroup generator characterization, without requiring $AK(z)$ or a domain-valued Taylor remainder.

Choose a mass-orthonormal eigenbasis e_1, e_2, e_3 for the finite-dimensional L, with eigenvalues $a_i \leq 0$. Repetition is allowed because self-adjointness, not simplicity, supplies such a basis. Write $\zeta_{ij} = a_i + a_j$, $W_{ij} = W(e_i, e_j)$ and $b_{ij} = QB(ie_i, ie_j) \in QX_0$. Equation (131) gives, for $t > 0$, an equality in Y and hence in X_0:

$$\frac{e^{tA_Q}W_{ij} - W_{ij}}{t} = \frac{e^{\zeta_{ij}t} - 1}{t}W_{ij} - \frac{2}{t}\int_0^t e^{(t-v)A_Q}e^{\zeta_{ij}v}b_{ij}\,\mathrm{d}v. \tag{132}$$

The first term tends to $\zeta_{ij}W_{ij}$ in Y, therefore in X_0. The integral average tends to b_{ij} in X_0. Thus, the right difference quotient in (132) is proven to converge in X_0; no stronger convergence topology is used. To see the latter directly, strong continuity on QX_0 gives $\sup_{0\leq u\leq t}\|(e^{uA_Q} - I)b_{ij}\|_{X_0} \to 0$; also, $\sup_{0\leq v\leq t}|e^{\zeta_{ij}v} - 1| \to 0$ and the semigroup is bounded on that finite interval. Subtracting b_{ij} inside the average and using these two suprema proves the limit. Thus,

$$\lim_{t\downarrow 0}\frac{e^{tA_Q}W_{ij} - W_{ij}}{t} = \zeta_{ij}W_{ij} - 2b_{ij} \quad \text{in } X_0. \tag{133}$$

For a strongly continuous semigroup on a Banach space, its generator domain is exactly the set of vectors for which this right difference quotient has a norm limit, and the limit is the generator applied to that vector. Lemma 4 identifies that generator here as A_Q with domain QX_1; the restriction is valid because Q commutes with the semigroup and preserves the full domain. Therefore, $W_{ij} \in QX_1$, and

$$(A_Q - a_i - a_j)W_{ij} = -2QB(ie_i, ie_j). \tag{134}$$

There are six symmetric pairs. Finite-dimensional norm equivalence assembles their domain membership into $W \in \mathcal{B}^2_{\mathrm{sym}}(E, QX_1)$. The bilinearity of (134) in a basis is precisely (130). Repeated slow eigenvalues are allowed because the basis is chosen only for this fixed self-adjoint operator; parameter differentiation is carried out at the Riesz-block level later.

The domain gain is a statement about the quadratic normal coefficient; the full graph and its higher-order remainder remain at the Y regularity specified above. □

Proposition 6 (Exact separation and uniqueness of the quotient jet)**.** *Under (H3), the slow eigenvalues, their quadratic sums and the fast spectrum satisfy*

$$\begin{gathered} a_i \in [-\alpha, 0], \qquad a_i + a_j \in [-2\alpha, 0], \qquad \operatorname{spec} A_Q \subset (-\infty, -\beta], \\ \operatorname{dist}(a_i + a_j, \operatorname{spec} A_Q) \geq \gamma := \beta - 2\alpha > 0. \end{gathered} \tag{135}$$

Every normalized locally invariant C^2 graph in the mild sense of Definition 1 has the same six normal coefficients. Consequently, (128) *is independent of the full graph and of its internal parameterization, once its linear tangent inputs are specified. No uniqueness of the full local invariant graph is asserted.*

Proof. The slow group bound evaluated on an eigenfield for $t \to -\infty$ implies $-a_i \leq \alpha$; the nonpositivity of A implies $a_i \leq 0$. For a fast eigenfield, the forward estimate as $t \to +\infty$ implies its positive rate is at least β. The compact self-adjoint spectral description in Lemma 3 contains no additional finite spectral points, so this accounts for its whole spectrum. Now, $\alpha < \eta$ and $3\eta < \beta$ imply $\beta - 2\alpha > 0$, including the case $\alpha = 0$.

If two graphs give coefficients $W_{ij}, \widetilde{W}_{ij}$, Proposition 5 puts both in QX_1 and their difference in the kernel of $A_Q - a_i - a_j$. This kernel is zero by (135). One can also see invertibility directly in a fast mass-orthonormal eigenbasis: the inverse multiplier is $(-\lambda_n - a_i - a_j)^{-1}$, of absolute value at most γ^{-1}, and the elliptic estimate raises its output by two derivatives. The full bounded inverse construction is recorded in Lemma 12. The equality of all six coefficients proves equality as a bilinear map. Equation (129) and the normalizing coordinate change prove the remaining quotient assertions. □

7.3. The Reduced Quadratic Dynamics

Proposition 7 (Tangential coefficient without differentiating the graph in its domain)**.** *For any graph in Proposition 5, its reduced equation and its quadratic term are*

$$\dot{z} = Lz + i^{-1} PB(K(z), K(z)), \tag{136}$$

$$\dot{z} = Lz + \tfrac{1}{2} r_2(z, z) + o(|z|^2), \qquad r_2(a, b) = 2i^{-1} PB(ia, ib). \tag{137}$$

Proof. Project (104) on the finite-dimensional space E. Its variation-of-constants formula has a continuous finite-dimensional integrand, so it is differentiable in time and gives (136). The expansion $K(z) = iz + \frac{1}{2} W(z, z) + o_Y(|z|^2)$ and the bound $B : Y \times Y \to X_0$ imply $B(K(z), K(z)) = B(iz, iz) + o_{X_0}(|z|^2)$. Applying the bounded projection and input identification gives (137). This derivation uses the bounded mild formulation for the graph expansion. Internal quadratic resonances are retained through the quadratic reduced term r_2 rather than removed by a normal-form transformation. □

The coefficient r_2 in (137) describes the quadratic tangential dynamics on the slow coordinates. The normal coefficient $\overline{W}_\varepsilon$ in (15) instead describes the quadratic displacement into the complementary space; these are distinct parts of the fixed-shape expansion.

8. Common-Domain Shape Differentiation

8.1. Transported Block Operators

Use T_ε to compare the spaces with $E_0 \oplus Q_0 X_0$ and define the operators in (16). They are block diagonal in the linear part and have the same common domains as the original realizations. The derivative formulas are

$$\overline{A}_0' = A_0' + [A_0, T_0'], \tag{138}$$

$$\overline{B}_0'(a,b) = B_0'(a,b) - T_0' B_0(a,b) + B_0(T_0' a, b) + B_0(a, T_0' b). \tag{139}$$

For each $r = s, s+1, s+2$, both terms of the commutator in (138) map H_σ^{r+2} to H_σ^r: T_0' is bounded on both these levels. Each term in (139) maps $H_\sigma^{r+1} \times H_\sigma^{r+1}$ to H_σ^r. Thus, these equations are norm identities in all the stated spaces. The output-transport term $-T_0' B_0$ includes the tangential component of B_0 and contributes to the derivative.

8.2. The Six-Output Resolvent Construction

For $r = s, s+1, s+2$, set

$$\mathcal{Z}_{1,r} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H_\sigma^{r+2}), \qquad \mathcal{Z}_{0,r} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H_\sigma^{r}).$$

We write $\mathcal{Z}_1 = \mathcal{Z}_{1,s}$ and $\mathcal{Z}_0 = \mathcal{Z}_{0,s}$. On these fixed spaces, $\mathscr{H}_\varepsilon$ is the operator in (17).

For the six base-point coefficient equations, put $\zeta = a_i + a_j$. The shifted fast resolvents use the separation

$$\zeta \in [-2\alpha, 0], \qquad \operatorname{spec}(A_{0,Q}) \subset (-\infty, -\beta], \qquad \operatorname{dist}(\zeta, \operatorname{spec} A_{0,Q}) \geq \gamma := \beta - 2\alpha > 0. \tag{140}$$

The operator mappings are recorded in Table 2. The two actions of L_0 on the finite-dimensional input space preserve $\mathcal{Z}_{1,r}$; the embedding into $\mathcal{Z}_{0,r}$ introduces no differential loss. The shifted inverse gains two derivatives by (144). Here, $r \in \{s, s+1, s+2\}$. Repeated or crossing eigenvalues inside the isolated slow block are handled through the Riesz block, without choosing a parameter-dependent eigenbasis.

Lemma 12 (Homological isomorphism and norm remainder)**.** *For each $r = s, s+1, s+2$, the map $\mathscr{H}_0 : \mathcal{Z}_{1,r} \to \mathcal{Z}_{0,r}$ is an isomorphism. In a sufficiently small shape interval, these inverses persist, are locally uniformly bounded, and are C^1 in $\mathcal{L}(\mathcal{Z}_{0,r}, \mathcal{Z}_{1,r})$. Moreover,*

$$(\mathscr{H}_0' W)(a,b) = \overline{A}_{0,Q}' W(a,b) - W(L_0' a, b) - W(a, L_0' b), \tag{141}$$

$$(\mathscr{H}^{-1})_0' = -\mathscr{H}_0^{-1} \mathscr{H}_0' \mathscr{H}_0^{-1}. \tag{142}$$

With $J_r = \mathscr{H}_0^{-1}$ and $E_H = \mathscr{H}_\varepsilon - \mathscr{H}_0 - \varepsilon \mathscr{H}_0'$, the explicit remainder is

$$\begin{aligned} &\| \mathscr{H}_\varepsilon^{-1} - J_r + \varepsilon J_r \mathscr{H}_0' J_r \|_{\mathcal{Z}_{0,r} \to \mathcal{Z}_{1,r}} \\ &\quad \leq \|J_r\|^2 \|E_H\|_{\mathcal{Z}_{1,r} \to \mathcal{Z}_{0,r}} + 2\|J_r\|^3 \|\mathscr{H}_\varepsilon - \mathscr{H}_0\|^2_{\mathcal{Z}_{1,r} \to \mathcal{Z}_{0,r}} = o(|\varepsilon|). \end{aligned} \tag{143}$$

Proof. Choose a mass-orthonormal eigenbasis only at the base parameter, writing $L_0 e_i = a_i e_i$. For $c \in \mathcal{Z}_{0,r}$, its six component equations are

$$(A_{0,Q} - a_i - a_j) W_{ij} = c_{ij}, \qquad 1 \leq i \leq j \leq 3.$$

By (135), the L^2 inverse multiplier on the fast eigenbasis has absolute value at most γ^{-1}, with $\gamma = \beta - 2\alpha$. It defines a unique weak resolvent solution. The scalar/velocity bootstrap

in Theorem 4 applies to that solution with the zeroth-order term ζW_{ij}, $\zeta = a_i + a_j \in [-2\alpha, 0]$. Interpolating this term in the elliptic estimate gives

$$\|(A_{0,Q} - \zeta)^{-1} c_{ij}\|_{H^{r+2}} \le C_{r,\alpha}(1 + \gamma^{-1})\|c_{ij}\|_{H^r}. \tag{144}$$

The solution remains in Q_0 because the shifted spectral inverse commutes with Q_0. This proves surjectivity into $Q_0 H^{r+2}_\sigma$ as well as injectivity. The norm of a symmetric bilinear map on the three-dimensional input space is equivalent to the sum of the norms of these six coefficients: evaluate on basis vectors in one direction and expand two unit vectors in that basis in the other. Thus, the six solutions assemble to a bounded inverse on the stated infinite-output spaces.

By Propositions 2 and 3, $\overline{A}_\varepsilon$ is C^1 from H^{r+2}_σ to H^r_σ and L_ε is a C^1 matrix on the fixed E_0. Hence, the three terms of $\mathscr{H}_\varepsilon W$ have Derivative (141) and a remainder in $\mathcal{L}(\mathcal{Z}_{1,r}, \mathcal{Z}_{0,r})$ bounded by the corresponding operator remainders times the fixed input constants. Parameter dependence is handled through the Riesz block, so repeated or crossing eigenvalues within the block cause no difficulty. Apply Lemma 5 at each of the three tensor spaces. The condition $\|J_r(\mathscr{H}_\varepsilon - \mathscr{H}_0)\|_{\mathcal{Z}_{1,r}\to\mathcal{Z}_{1,r}} < 1/2$ holds on a common small interval. Its derivative and remainder formulas are (142) and (143). Recentring this argument at any parameter proves continuity of the inverse derivative. This establishes the two homological mapping rows used below. □

Remark 1 (Repeated eigenvalues inside the Riesz block)**.** *A basis is fixed only for L_0 to display the six base-point resolvents. For $\varepsilon \neq 0$, the proof differentiates the Riesz block, the fixed-space operator L_ε and the homological operator itself. Repeated or crossing eigenvalues inside the rank-three block are therefore allowed as long as the block remains isolated from its complement.*

8.3. Raised-Scale Regularity of the Quadratic Coefficient

Proposition 8 (Raised-scale regularity of the quadratic jet)**.** *Under Assumption 1, the representative of the unique quotient jet satisfies*

$$\overline{W}_\varepsilon \in \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma), \qquad \varepsilon \longmapsto \overline{W}_\varepsilon \in C^1\big(\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma)\big). \tag{145}$$

In particular, at the base point,

$$\|\overline{W}_\varepsilon - \overline{W}_0 - \varepsilon \overline{W}_0'\|_{\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma)} = o(|\varepsilon|). \tag{146}$$

Its value and derivative have locally uniform bounds in this norm. This conclusion concerns the quadratic coefficient, not the regularity of the full graph.

Proof. At the raised level, set

$$\mathcal{Z}_{0,s+2} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma), \qquad \mathcal{Z}_{1,s+2} = \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma).$$

It suffices to establish

$$b_\varepsilon \in C^1\big(\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma)\big), \qquad \|b_\varepsilon - b_0 - \varepsilon b_0'\|_{\mathcal{Z}_{0,s+2}} = o(|\varepsilon|),$$

and

$$\mathscr{H}_\varepsilon \in C^1(\mathcal{L}(\mathcal{Z}_{1,s+2}, \mathcal{Z}_{0,s+2})), \qquad \|\mathscr{H}_\varepsilon - \mathscr{H}_0 - \varepsilon \mathscr{H}_0'\|_{\mathcal{L}(\mathcal{Z}_{1,s+2}, \mathcal{Z}_{0,s+2})} = o(|\varepsilon|).$$

The source loses one spatial derivative through B_ε, whereas $\mathscr{H}_\varepsilon$ loses two derivatives through $\overline{A}_{\varepsilon,Q}$; the finite-dimensional L_ε terms cause no spatial loss. The derivatives b_0' and

$\mathscr{H}_0$ act in the same displayed target topologies. We verify these mappings below. At the base point, choose a mass-orthonormal basis e_1, e_2, e_3 of E_0. The inverse acts on exactly the six symmetric pairs

$$(e_1,e_1),\ (e_1,e_2),\ (e_1,e_3),\ (e_2,e_2),\ (e_2,e_3),\ (e_3,e_3),$$

through the shifted fast resolvents in Lemma 12. Thus, the passage from an H^{s+2} source to an H^{s+4} output is a two-derivative elliptic gain for each of the six coefficients, obtained from the shifted resolvent estimates.

Set $U_\varepsilon = T_\varepsilon i = P_\varepsilon i$. The two-resolvent Formula (80) proves $U_\varepsilon \in C^1(\mathcal{L}(E_0, H^{s+4}_\sigma))$. In particular, both U_ε and U'_ε take values in H^{s+3}_σ. Use the specifically raised bilinear mapping in (75):

$$B_\varepsilon : H^{s+3}_\sigma \times H^{s+3}_\sigma \longrightarrow H^{s+2}_\sigma.$$

Its composition with the two U_ε's, with T_ε^{-1} on H^{s+2}_σ and with Q_0 on that level, gives

$$b_\varepsilon = -2Q_0 T_\varepsilon^{-1} B_\varepsilon(U_\varepsilon \cdot, U_\varepsilon \cdot) \in C^1\big(\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma)\big). \tag{147}$$

For example, its derivative at zero is

$$b_0'(a,b) = -2Q_0\big[B_0'(ia,ib) - T_0' B_0(ia,ib) + B_0(T_0' ia, ib) + B_0(ia, T_0' ib)\big]. \tag{148}$$

Every term belongs to H^{s+2}_σ by the displayed input and output spaces. The product remainder lemma proves that $\|b_\varepsilon - b_0 - \varepsilon b_0'\|_{\mathcal{Z}_{0,s+2}} = o(|\varepsilon|)$. Likewise, for every fixed $W \in \mathcal{Z}_{1,s+2}$, $\|\mathscr{H}_\varepsilon W - \mathscr{H}_0 W - \varepsilon \mathscr{H}_0' W\|_{\mathcal{Z}_{0,s+2}} = o(|\varepsilon|)\|W\|_{\mathcal{Z}_{1,s+2}}$, uniformly on the unit ball; equivalently, the remainder is $o(|\varepsilon|)$ in $\mathcal{L}(\mathcal{Z}_{1,s+2}, \mathcal{Z}_{0,s+2})$. This is an operator-norm parameter statement obtained from the product-remainder calculus.

Apply Lemma 12 at $r = s+2$, whose domain is $\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma)$ and whose range is the space in (147). It gives the solution $W^*_\varepsilon = \mathscr{H}_\varepsilon^{-1} b_\varepsilon$ with the asserted C^1 raised regularity. Included in the lower domain $\mathcal{Z}_1$, it solves the same equation as the coefficient obtained from Proposition 5 and the comparison T_ε. The lower-level inverse is unique, so $W^*_\varepsilon = \overline{W}_\varepsilon$.

The spatial generator g in the comparison class of Definition 2, explicitly given by (158) in Lemma 13, is bounded $H^{r+1}_\sigma \to H^r_\sigma$ on smooth first variations, while the parameter-dependent Piola family is differentiated in operator norm in the two-loss topology $H^{r+2}_\sigma \to H^r_\sigma$. The H^{s+4} output in (145) is obtained specifically from the $r = s+2$ row of $\mathscr{H}_\varepsilon^{-1}$ acting on the raised H^{s+2} source (147).

The bounds are explicit in their dependence: on the common parameter interval,

$$\begin{aligned}
&\|\overline{W}_\varepsilon\|_{\mathcal{Z}_{1,s+2}} \le C_H \|b_\varepsilon\|_{\mathcal{Z}_{0,s+2}},\\
&\|\overline{W}'_\varepsilon\|_{\mathcal{Z}_{1,s+2}} \le C_H\big(\|b'_\varepsilon\|_{\mathcal{Z}_{0,s+2}} + \|\mathscr{H}'_\varepsilon\|_{\mathcal{Z}_{1,s+2}\to\mathcal{Z}_{0,s+2}} \|\overline{W}_\varepsilon\|_{\mathcal{Z}_{1,s+2}}\big).
\end{aligned}$$

Here, C_H is the uniform raised inverse bound. The source bounds use the three raised input/output maps identified above, and C_H uses the raised elliptic constants and the separated six resolvents. These are locally uniform by the preceding propositions, which supplies the raised-level hypotheses directly at coefficient level. □

8.4. Derivative and All Three Remainder Norms

Proof of Theorem 1. Theorem 6 gives an invariant C^2 graph at each shape. Propositions 5 and 6 give its unique quadratic quotient coefficient and its domain gain. Transporting (130) by T_ε yields precisely (19). Lemma 12 and the source product calculus, including Proposition 8, give $\overline{W}_\varepsilon = \mathscr{H}_\varepsilon^{-1} b_\varepsilon$. Differentiating this identity between fixed Banach spaces proves (20), with the full derivative terms (69), (70), (79), (82), (141) and (148).

Here is the remainder at every level used. Write $W_0 = \overline{W}_0$, $W_1 = \overline{W}_0'$, $E_b = b_\varepsilon - b_0 - \varepsilon b_0'$, and $E_H = \mathscr{H}_\varepsilon - \mathscr{H}_0 - \varepsilon \mathscr{H}_0'$. Exact subtraction gives

$$\mathscr{H}_\varepsilon(\overline{W}_\varepsilon - W_0 - \varepsilon W_1) = E_b - E_H W_0 - \varepsilon(\mathscr{H}_\varepsilon - \mathscr{H}_0)W_1. \tag{149}$$

For each index r in the explicit set $\{s, s+1, s+2\}$, its inverse yields

$$\begin{aligned}&\|\overline{W}_\varepsilon - W_0 - \varepsilon W_1\|_{\mathcal{Z}_{1,r}} \\ &\quad \le C_{H,r}\big(\|E_b\|_{\mathcal{Z}_{0,r}} + \|E_H\|_{\mathcal{Z}_{1,r}\to\mathcal{Z}_{0,r}}\|W_0\|_{\mathcal{Z}_{1,r}} + |\varepsilon|\|\mathscr{H}_\varepsilon - \mathscr{H}_0\|_{\mathcal{Z}_{1,r}\to\mathcal{Z}_{0,r}}\|W_1\|_{\mathcal{Z}_{1,r}}\big) = o(|\varepsilon|).\end{aligned} \tag{150}$$

Thus, the three respective remainder spaces are

$$\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma), \qquad \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+3}_\sigma), \qquad \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+4}_\sigma).$$

The middle level follows from the inverse and source mappings at $r = s+1$, and the upper level is the proposition just proved. Recentring the product and inverse identities gives the continuity of the derivative in each norm. Finally, the quotient/complement diagram in (127) transports this norm derivative to the quotient statement. The shape derivative is therefore obtained directly from the coefficient equation on fixed Banach spaces. □

9. Physical Comparisons, Quotient Covariance and Derivative Loss

9.1. Four Separate Admissibility Conditions

The comparison direction is fixed by the physical diagram

$$\text{reference coordinate 1} \xrightarrow{\tau_\varepsilon} \text{physical field} \xleftarrow{\widehat{\tau}_\varepsilon} \text{reference coordinate 2},$$

so the map from reference coordinate 1 to reference coordinate 2 is

$$G_\varepsilon = \widehat{\tau}_\varepsilon^{-1}\tau_\varepsilon, \qquad \widehat{q} = G_\varepsilon q. \tag{151}$$

Consequently, the inverse placements are forced by the comparison directions:

$$\widehat{A}_\varepsilon = G_\varepsilon A_\varepsilon G_\varepsilon^{-1}, \qquad \widehat{B}_\varepsilon(a,b) = G_\varepsilon B_\varepsilon(G_\varepsilon^{-1}a, G_\varepsilon^{-1}b), \qquad D_\varepsilon = \widehat{T}_\varepsilon^{-1} G_\varepsilon T_\varepsilon. \tag{152}$$

With block restrictions of D_ε, the quadratic representative must therefore transform as

$$\widehat{\overline{W}}_\varepsilon = D_{\varepsilon,Q}\overline{W}_\varepsilon(D_{\varepsilon,E}^{-1}\cdot, D_{\varepsilon,E}^{-1}\cdot). \tag{153}$$

These formulas fix the comparison directions for the rest of the section. They agree with the scalar–vector convention in Proposition 1; the onset calculation uses the same comparison directions.

Definition 2 (Admissible state and spectral comparisons)**.** *The following four blocks of conditions are imposed independently.*

Physical state comparison. *The map G_ε is real and linear, $G_0 = I$, and is an isomorphism on each H^r_σ, $r = s, s+1, s+2, s+3, s+4$. It identifies the same physical equation by the exact relations*

$$\widehat{A}_\varepsilon = G_\varepsilon A_\varepsilon G_\varepsilon^{-1}, \qquad \widehat{B}_\varepsilon(a,b) = G_\varepsilon B_\varepsilon(G_\varepsilon^{-1}a, G_\varepsilon^{-1}b). \tag{154}$$

There is no translation in the state variable or change of the physical equilibrium.

Regularity of the comparison and its inverse. *Either $G_\varepsilon, G_\varepsilon^{-1}$ are C^1 in the endomorphism norm of each of those five spaces (bounded normalizations), or G_ε is a spatial Piola action $\mathsf{P}_{\phi_\varepsilon}$ or its*

inverse. In the spatial class, each ϕ_ε is a smooth orientation-preserving diffeomorphism, $\phi_0 = \mathrm{Id}$, and both ϕ_ε and ϕ_ε^{-1} depend C^1 on ε in C^{s+7} charts. The induced state maps and inverses are locally uniformly bounded and strongly continuous on the five spaces. Their norm expansions are required, and proven below, in

$$\|G_\varepsilon - I - \varepsilon g\|_{H^{r+2}_\sigma \to H^r_\sigma} = o(|\varepsilon|), \tag{155}$$

$$\|G_\varepsilon^{-1} - I + \varepsilon g\|_{H^{r+2}_\sigma \to H^r_\sigma} = o(|\varepsilon|), \qquad r = s, s+1, s+2, \tag{156}$$

where $g : H^{r+1}_\sigma \to H^r_\sigma$. At every other base parameter, the corresponding first-derivative family is continuous in these two-loss operator norms. Same-level operator-norm differentiability is not part of the spatial condition.

Regularity of the conjugated equation. *The operators in* (154) *have the common domains H^{r+2}_σ on H^r_σ and the C^1 operator rows for A, B in Table 2, for $r = s, s+1, s+2$. Their spectral constructions have the raised bounds used in Propositions 3 and 8. Their compatible realizations and smoothing bounds are those transferred by G_ε. These assertions follow from bounded composition in the first class and from the geometric coefficient argument in Proposition 9 in the second; they do not replace the preceding regularity condition on G_ε.*

Spectral comparison. *For each of the two operator families, take its canonical comparison, optionally composed on the right with a block-diagonal family S_ε on $E_0 \oplus Q_0 H^s_\sigma$, with $S_0 = I$. Both S_ε and its inverse must be C^1 bounded endomorphisms on H^r_σ, $r = s, s+1, s+2, s+3, s+4$. Thus, T_ε and $\widehat{T}_\varepsilon$ intertwine the corresponding projections, equal I at zero, and preserve all the required levels. All coefficient, comparison, inverse and remainder bounds in these four blocks are locally uniform.*

9.2. Spatial Piola Maps with Derivative Loss

For an orientation-preserving diffeomorphism, write

$$\mathsf{P}_\phi q = \left[D\phi\,(q/j_\phi)\right] \circ \phi^{-1}, \qquad \phi^* \mu_0 = j_\phi \mu_0. \tag{157}$$

This is the push-forward of a flux relative to the fixed reference area, not the unweighted push-forward of a velocity.

Lemma 13 (Two-loss norm differentiability of the spatial action)**.** *Under the spatial assumptions in Definition 2,* (157) *and its inverse are bounded and strongly continuous on H^r_σ for $r = s, \ldots, s+4$. If $V = \partial_\varepsilon \phi_\varepsilon|_0$, their derivatives at zero are g and $-g$, where*

$$gq = -[V, q] - (\mathrm{div}_0 V)q, \qquad g\psi = -V\psi \quad \textit{in scalar stream coordinates.} \tag{158}$$

For each $r = s, s+1, s+2$, there is a modulus $\omega_r(\varepsilon) \to 0$ such that

$$\|\mathsf{P}_{\phi_\varepsilon} - I - \varepsilon g\|_{H^{r+2}_\sigma \to H^r_\sigma} \le |\varepsilon| \omega_r(\varepsilon) + C_r \varepsilon^2. \tag{159}$$

The inverse obeys this estimate with g replaced by $-g$.

Proof. The change-of-variables identity for a test function a gives

$$\int_M da(\mathsf{P}_\phi q)\, \mathrm{d}\mu_0 = \int_M d(a \circ \phi)(q)\, \mathrm{d}\mu_0.$$

This proves preservation of the distributional divergence-free condition. The chain rule also proves $\mathsf{P}_\phi^{-1} = \mathsf{P}_{\phi^{-1}}$. In a pair of fixed local charts, write $\chi_\varepsilon = \phi_\varepsilon^{-1}$ and $M_\varepsilon = (D\phi_\varepsilon / j_{\phi_\varepsilon}) \circ \chi_\varepsilon$. Then, $\mathsf{P}_{\phi_\varepsilon} q = M_\varepsilon (q \circ \chi_\varepsilon)$. Uniform bounds on chart derivatives and inverse Jacobians

prove the H^r boundedness by ordinary chain rules and change of variables for each integer $r \le s+4$. Cutoff terms involve only the same finite derivatives.

At zero, $\chi_\varepsilon = \mathrm{Id} - \varepsilon V + e_\varepsilon$, with $\|e_\varepsilon\|_{C^{r+1}} = o(|\varepsilon|)$, and

$$M_\varepsilon = I + \varepsilon(DV - (\mathrm{div}_0 V)I) + F_\varepsilon, \qquad \|F_\varepsilon\|_{C^{r+1}} = o(|\varepsilon|).$$

These follow by differentiating the coordinate Jacobian relative to μ_0; its derivative is $\mathrm{div}_0 V$. Let $d_\varepsilon = \chi_\varepsilon - \mathrm{Id}$. The exact local Taylor identity for a smooth field component is

$$q(x+d_\varepsilon) - q(x) - Dq(x)d_\varepsilon = \int_0^1 (1-t)D^2q(x+td_\varepsilon)[d_\varepsilon, d_\varepsilon]\,\mathrm{d}t.$$

On slightly smaller fixed patches, the intermediate maps $x \mapsto x + td_\varepsilon(x)$ have uniformly bounded inverse Jacobians for small ε. The product and composition estimates therefore bound the remainder in H^r by $C_r\|d_\varepsilon\|^2_{C^{r+1}}\|q\|_{H^{r+2}} \le C_r\varepsilon^2\|q\|_{H^{r+2}}$. The linear part satisfies

$$\|Dq(d_\varepsilon + \varepsilon V)\|_{H^r} \le C_r\|d_\varepsilon + \varepsilon V\|_{C^{r+1}}\|q\|_{H^{r+1}}.$$

Multiplying by M_ε, the other discarded terms are $F_\varepsilon q$, bounded by $o(|\varepsilon|)\|q\|_{H^r}$, and the product of $M_\varepsilon - I$ with $q\circ\chi_\varepsilon - q$, bounded by $C_r\varepsilon^2\|q\|_{H^{r+1}}$. Adding these estimates and localizing proves (159). The first-order term is $DV\,q - (\mathrm{div}_0 V)q - Dq\,V$, namely, (158). In particular, it maps H^{r+1} to H^r. Density extends the estimate from smooth fields to the stated spaces. The substitution of ϕ_ε^{-1}, whose velocity at zero is $-V$, proves the inverse expansion.

Strong continuity on a fixed H^r level follows first on smooth fields and then by approximation and the uniform same-level bounds. At an arbitrary base parameter, let $V_\varepsilon = (\partial_\varepsilon\phi_\varepsilon)\circ\phi_\varepsilon^{-1}$. On smooth fields, $\partial_\varepsilon \mathsf{P}_{\phi_\varepsilon} = g(V_\varepsilon)\mathsf{P}_{\phi_\varepsilon}$. The coordinate first-difference estimate gives the continuity of the spatial action in $\mathcal{L}(H^{r+2}, H^{r+1})$, while $g(V_\varepsilon)$ is continuous in $\mathcal{L}(H^{r+1}, H^r)$. Splitting the difference of this product proves continuity in $\mathcal{L}(H^{r+2}, H^r)$ of its derivative. The inverse family is treated by its own curve ϕ_ε^{-1}. Finally, the identity $\iota_q\mu_0 = -d\psi$ transforms under a flux push-forward to the potential $\psi\circ\phi^{-1}$, giving $g\psi = -V\psi$ modulo constants. □

For comparison, translation on a periodic coordinate has derivative remainder multiplier $(e^{in\varepsilon} - 1)/\varepsilon - in$. Dividing by $(1+n^2)^{1/2}$ does not make its supremum tend to zero when $n\varepsilon$ remains a fixed nonzero constant. Thus, even one extra derivative does not in general give an operator-norm derivative. Lemma 13 uses two extra derivatives as a sufficient, explicit topology; it does not assume same-level differentiability of a spatial action.

Proposition 9 (The geometric conjugated operators retain the required scale)**.** *For a spatial comparison in Definition 2, the third block of that definition holds on* $H^{r+2}_\sigma \to H^r_\sigma$ *and* $H^{r+1}_\sigma \times H^{r+1}_\sigma \to H^r_\sigma$, $r = s, s+1, s+2$.

Proof. For $G_\varepsilon = \mathsf{P}_{\phi_\varepsilon}$, the embedding for the second coordinate system is $\widehat{R}_\varepsilon = R_\varepsilon \circ \phi_\varepsilon^{-1}$. The chain rule and the area change show exactly that its physical transport satisfies $\widehat{\tau}_\varepsilon G_\varepsilon = \tau_\varepsilon$. Its metric, divergence, pressure equation and convection are natural under this coordinate change, giving (154). The parameter differentiation of a spatial composition consumes one of the controlled spatial derivatives: $\widehat{R}_\varepsilon$ is C^1 in C^{s+6}, its metric and density in C^{s+5}, its connection in C^{s+4}, and its curvature and zeroth-order viscous coefficients in C^{s+3}.

These budgets suffice at the highest stated Sobolev levels. For $A : H^{s+4} \to H^{s+2}$, coefficient multiplication requires derivatives through $s+2$, whereas even its order-zero coefficients have $s+3$. For $B : H^{s+3} \times H^{s+3} \to H^{s+2}$, the one-derivative product uses the

same multiplier order. For Π on H^{s+4}, the scalar inverse is $H^{s+3} \to H^{s+5}$, supplied by Lemma 1 and Appendix A with a C^{s+5} metric. The weak pressure proof fixes constants as before. The two-resolvent Formula (80) consequently gives the raised projection and frame regularity. Applying the product and inverse remainder identities in these spaces proves the C^1 dependence of the conjugated operator families. The lower two indices are covered by the same finite index sets in the pressure lemma and (57). For $G_\varepsilon = \mathsf{P}_{\phi_\varepsilon}^{-1}$, replace ϕ_ε by its inverse, which has the identical prescribed regularity. Compatible domains and semigroup bounds also transfer directly by the bounded isomorphism G_ε and its inverse on each of the stated spaces. □

Remark 2 (An excluded nonsmooth symmetry)**.** *On the sphere, let G_ε be rotation through angle $|\varepsilon|$ about a fixed axis. The conjugated equation and its canonical spectral projection are constant, by rotational equivariance. But, on a degree-one function rotated into another one, the two one-sided derivatives of G_ε at zero have opposite signs. With canonical spectral comparisons, $D_\varepsilon = G_\varepsilon$, so D_0' does not exist. This is excluded by the second block of Definition 2, even though the third block alone would be satisfied.*

9.3. Exact Covariance and Differentiation of the Quotient Representative

Proof of the covariance assertions in Theorem 2. Resolvent conjugacy gives

$$\widehat{P}_\varepsilon = G_\varepsilon P_\varepsilon G_\varepsilon^{-1}.$$

Using the two intertwining identities, the relative map $D_\varepsilon = \widehat{T}_\varepsilon^{-1} G_\varepsilon T_\varepsilon$ satisfies $D_\varepsilon P_0 = P_0 D_\varepsilon$. Thus, its restrictions $D_{\varepsilon,E}$ and $D_{\varepsilon,Q}$ in (21) are genuine block maps, with bounded inverses on the specified levels.

For any normalized invariant graph of the first system,

$$\widehat{K}_\varepsilon(z) = G_\varepsilon K_\varepsilon(D_{\varepsilon,E}^{-1} z) \tag{160}$$

is invariant for the second system by the exact state conjugacy. It is C^2 at the fixed parameter because G_ε is bounded on Y. Moreover, $\widehat{P}_\varepsilon \widehat{K}_\varepsilon(z) = \widehat{T}_\varepsilon i z$ and $D\widehat{K}_\varepsilon(0) = \widehat{T}_\varepsilon i$, so it has the prescribed normalization. Taking its second derivative, using $\widehat{Q}_\varepsilon G_\varepsilon = G_\varepsilon Q_\varepsilon$, and then applying $\widehat{T}_\varepsilon^{-1}$ proves

$$\widehat{\overline{W}}_\varepsilon = D_{\varepsilon,Q} \overline{W}_\varepsilon (D_{\varepsilon,E}^{-1}\cdot, D_{\varepsilon,E}^{-1}\cdot).$$

This is the exact Law (22) in $\mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 Y)$ and on the raised levels at each fixed parameter. By Proposition 6, it is independent of the full graph selected. The quotient statement follows through the explicit comparison diagram in (127).

In the bounded class, all factors have ordinary same-level norm derivatives. In the spatial class, the inverse and product estimates for $T, \widehat{T}$ and Lemma 13 imply

$$D_\varepsilon = I + \varepsilon d + o_{\mathcal{L}(H_\sigma^{r+2}, H_\sigma^r)}(|\varepsilon|), \qquad d = -\widehat{T}_0' + g + T_0', \quad r = s, s+1, s+2. \tag{161}$$

The derivative is also a bounded map $H_\sigma^{r+1} \to H_\sigma^r$. Differentiating the exact block identity in this topology shows $P_0 d = d P_0$ on its domain. Restriction to the smooth, finite-dimensional input block makes $D_{\varepsilon,E}$ a C^1 matrix with derivative d_E and inverse derivative $-d_E$.

For the output, apply Proposition 8, which gives H^{s+4} output, together with the $r = s+2$ case of (161). Put $W_0 = \overline{W}_0$. Then,

$$\frac{(D_{\varepsilon,Q} - I) W_0}{\varepsilon} \longrightarrow d_Q W_0 \quad \text{in } \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H_\sigma^{s+2}),$$

because W_0 takes values in H^{s+4}_σ. The other part is

$$D_{\varepsilon,Q}\frac{\overline{W}_\varepsilon - W_0}{\varepsilon} \longrightarrow \overline{W}_0' \quad \text{in } \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma).$$

Indeed the difference quotient converges in that space, the $D_{\varepsilon,Q}$ are uniformly bounded there, and strong convergence $D_{\varepsilon,Q} \to I$ acts on each of the six fixed coefficients of $\overline{W}_0'$. Finite input dimension converts these six strong limits to the bilinear norm. Expanding the two finite-dimensional input inverses now gives (23), including both input correction terms. This proves the formula in X_1 using the two-loss topology established for spatial push-forwards.

If $W_0 = 0$, all three correction terms of (23) vanish, proving the comparison-independence assertion at a zero jet. All comparison maps in this statement agree at the base point; different base linear coordinates still transform the tensor itself in the usual way. □

Remark 3 (Analytic normalization counterexample)**.** *Consider bounded scalar normalization. The raw derivative is generally comparison dependent, and for $G_\varepsilon = e^{\sigma\varepsilon} I$ the comparison direction in* (151) *gives*

$$\widehat{A}_\varepsilon = A_\varepsilon, \qquad \widehat{B}_\varepsilon = e^{-\sigma\varepsilon} B_\varepsilon, \qquad \widehat{\overline{W}}_0' = \overline{W}_0' - \sigma \overline{W}_0. \tag{162}$$

Here, the canonical spectral comparisons coincide and the exact tensor law gives $\widehat{\overline{W}}_\varepsilon = e^{-\sigma\varepsilon}\overline{W}_\varepsilon$. Thus, whenever $\overline{W}_0 \neq 0$ and $\sigma \neq 0$, the raw first derivative changes even though the physical family is the same. At a zero jet, the correction disappears, which is precisely the comparison-independence used in the spherical onset theorem.

9.4. Tangential Coordinate Changes and Their Convention

For a pure tangential reparameterization, after the exact physical and tangent-input comparisons, the physically compared jet is constant and its compared first derivative vanishes.

Proposition 10 (Physical tangential covariance)**.** *For an exact reparameterization $R_\varepsilon = R_0 \circ \phi_\varepsilon$, the map from the moving flux coordinate to the fixed physical coordinate is $C_\varepsilon = \mathsf{P}_{\phi_\varepsilon}$. This satisfies*

$$F_\varepsilon = C_\varepsilon^{-1} F_0 C_\varepsilon, \qquad P_\varepsilon = C_\varepsilon^{-1} P_0 C_\varepsilon. \tag{163}$$

The physically compared quotient jet, with its tangent inputs compared by $C_\varepsilon|_{E_\varepsilon}$, is constant. Its first derivative is therefore zero. The same first-order conclusion holds for a shape curve having only tangential first velocity at the base.

Proof. For $R_\varepsilon = R_0 \circ \phi_\varepsilon$, its area density relative to the fixed reference metric is j_{ϕ_ε}. The chain rule in the physical Piola map gives $\tau_\varepsilon q = \tau_0 \mathsf{P}_{\phi_\varepsilon} q$ as a field at the same physical point. Therefore, C_ε maps the moving flux coordinate to the fixed physical coordinate. The naturality of the covariant derivatives and the unique normalized pressure equation proves the first identity in (163); resolvent conjugacy proves the second. In the first identity, the nonlinear meaning is $A_\varepsilon = C_\varepsilon^{-1} A_0 C_\varepsilon$ and $B_\varepsilon(a,b) = C_\varepsilon^{-1} B_0(C_\varepsilon a, C_\varepsilon b)$.

For $c = C_0'$, as specified in (158), differentiation on smooth fields gives

$$A' = Ac - cA, \qquad B'(a,b) = B(ca,b) + B(a,cb) - cB(a,b). \tag{164}$$

On smooth fields, these formulas are algebraic consequences of the conjugacy. Their extensions to $H^{r+2}_\sigma \to H^r_\sigma$ and $H^{r+1}_\sigma \times H^{r+1}_\sigma \to H^r_\sigma$, $r = s, s+1, s+2$, are the derivatives

already proven in Proposition 9. The combined commutator $Ac - cA$, with its principal cancellation, extends boundedly on $H^{r+2}_\sigma \to H^r_\sigma$ in this topology.

Mapping a graph by C_ε gives a graph for the unchanged equation F_0. Comparing its inputs by $C_\varepsilon|_{E_\varepsilon}$ and its quotient output by the induced map gives the unique quotient jet for F_0, by Proposition 6. It is thus constant for every ε, proving the zero physically compared derivative. Finally, a general embedding curve with first velocity dR_0V has the same first metric and operator variations as $R_0 \circ \phi_\varepsilon$ with initial velocity V. Equations (59)–(70) and the common-domain derivative formula show equality of their first jet variations after the same first-order comparison. This proves the last assertion and finishes Theorem 2. □

10. Anisotropy Onset at the Sphere

10.1. Mechanism of the Onset

At the round sphere, the slow block consists of the three rotational Killing fields. Their degree-one stream functions satisfy $\Delta_0\psi = -2\psi$, so the two symmetric convection brackets in (31) cancel and the quadratic normal jet vanishes. A nonscalar symmetric deformation breaks rotational isotropy: the first variation of a slow degree-one field contains a cubic harmonic component, while the degree-one correction is removed by the normal projection. The first surviving normal source therefore lies in spherical degree three. The Stokes generator on that degree is -10ν, so the inversion of the homological equation contributes one factor proportional to $1/\nu$. The calculation below makes these cancellations and constants explicit.

10.2. Conventions and Affine Geometry

We derive Theorem 3 in spherical stream coordinates, using the stream and sign conventions in Section 3.3 and the scalar–vector identification in Proposition 1. The sphere has outward orientation, $J_0 v = x \times v$, $\Delta_0 x_i = -2x_i$, and

$$\{a, b\}_0 = (x \times \nabla a) \cdot \nabla b, \qquad \{x_1, x_2\}_0 = x_3 \tag{165}$$

with cyclic counterparts. Euclidean extensions may be used in the bracket because the radial part drops out. We use the fixed, unnormalized tangent inputs $\psi_p = p \cdot x$ and $q_p = J_0 \operatorname{grad}_0 \psi_p = x \times p$; thus, $p \in \mathbb{R}^3$ identifies a specific element of E_0, rather than a unit-mass eigenvector. The mass of this field is $\|q_p\|^2_{L^2} = (8\pi/3)|p|^2$. Passing to another linear input frame transforms the two tensor inputs, as already specified.

Let D be any real symmetric matrix and $R_\delta(x) = (I + \delta D)x$. For $|\delta|\|D\| < 1$, this is a smooth embedding. Set

$$f = x^T D x, \qquad \tau = \operatorname{tr} D, \qquad d_p = (Dp) \cdot x.$$

The velocity Dx splits as

$$V_n = f, \qquad V_T = Dx - fx = \tfrac{1}{2} \operatorname{grad}_0 f. \tag{166}$$

Directly differentiating $g_\delta(v, w) = ((I + \delta D)v) \cdot ((I + \delta D)w)$ on tangent vectors gives

$$h(v, w) = 2v \cdot Dw, \qquad h = 2f g_0 + \mathcal{L}_{V_T} g_0, \qquad j_0' = \tau - f. \tag{167}$$

For the second identity, the spherical Hessian is $\operatorname{Hess}_0 f(v, w) = 2v \cdot Dw - 2f\, v \cdot w$, so $\mathcal{L}_{\frac{1}{2}\operatorname{grad} f} g_0 = \operatorname{Hess} f$. The density derivative is half the tangent trace of h. It is also obtained from $j_\delta = \det(I + \delta D)|(I + \delta D)^{-T} x|$.

The basic polynomial identities are

$$\operatorname{grad}_0 \psi_p = p - \psi_p x, \qquad \operatorname{grad}_0 f = 2(Dx - fx), \tag{168}$$

$$\Delta_0 \psi_p = -2\psi_p, \qquad \Delta_0 f = 2\tau - 6f, \tag{169}$$

$$\operatorname{grad}_0 f \cdot \operatorname{grad}_0 \psi_p = 2(d_p - f\psi_p), \qquad \Delta_0(f\psi_p) = 2\tau\psi_p + 4d_p - 12f\psi_p. \tag{170}$$

The last identity follows by applying the Laplacian product rule to the preceding three; every term has therefore been specified.

For a normal variation $h_N = 2fg_0$, Formula (62) in dimension two gives $\Delta_N' = -2f\Delta_0$. For the tangential part $h_T = \mathcal{L}_{V_T} g_0$, scalar naturality gives $\Delta_T' = V_T\Delta_0 - \Delta_0 V_T$. Since $V_T\psi_p = d_p - f\psi_p$, the affine variation is

$$\begin{aligned} \Delta_0'\psi_p &= 4f\psi_p + V_T(-2\psi_p) - \Delta_0(d_p - f\psi_p) \\ &= 6f\psi_p + \Delta_0(f\psi_p) = 4d_p + 2\tau\psi_p - 6f\psi_p. \end{aligned} \tag{171}$$

The normal curvature variation from (64) is $k_N' = -(\Delta_0 + 2)f = 4f - 2\tau$. The tangential part is $V_T k_0 = 0$, because $k_0 = 1$. Hence,

$$k_0' = 4f - 2\tau \tag{172}$$

for both the affine and the normal description. The affine calculation retains the tangential velocity and density contributions in addition to the normal metric variation.

10.3. The Cubic Harmonic Projection and the Frame Derivative

For a homogeneous cubic polynomial P, $\Delta_{\mathbb{R}^3} P$ is linear and $\Delta_{\mathbb{R}^3}(|x|^2\ell) = 10\ell$ for a linear polynomial ℓ. Thus,

$$\Pi_3 P = \left(P - \frac{|x|^2}{10} \Delta_{\mathbb{R}^3} P \right) \Big|_{S^2}. \tag{173}$$

The subtracted polynomial is $|x|^2$ times a linear harmonic, so its restriction has degree one; the remaining cubic is harmonic. The polar-coordinate identity for the Euclidean Laplacian gives eigenvalue $-3(3+1) = -12$ for its restriction. Since $\Delta_{\mathbb{R}^3}(f\psi_p) = 2\tau\psi_p + 4d_p$, we obtain

$$H_{3,p} = \Pi_3(f\psi_p) = f\psi_p - \tfrac{1}{5}(\tau\psi_p + 2d_p), \qquad \Delta_0 H_{3,p} = -12 H_{3,p}. \tag{174}$$

This gives the degree-three projection by an exact Euclidean polynomial calculation.

Let $\mathscr{L}_g = \Delta_g^2 + 2\operatorname{div}_g(k_g \operatorname{grad}_g)$ be the numerator in (29). On degree-one inputs, $\mathscr{L}_0\psi_p = 0$. Differentiate its four contributions, using $k_0 = 1$:

$$\begin{aligned} \mathscr{L}_0\psi_p &= \Delta_0'\Delta_0\psi_p + \Delta_0\Delta_0'\psi_p + 2\Delta_0'\psi_p + 2\operatorname{div}_0(k_0' \operatorname{grad}_0 \psi_p) \\ &= \Delta_0\Delta_0'\psi_p + 2\operatorname{div}_0(k_0' \operatorname{grad}_0 \psi_p). \end{aligned} \tag{175}$$

The cancellation here is $\Delta_0'(-2\psi_p) + 2\Delta_0'\psi_p = 0$. To evaluate the two remaining terms, use (170) and (172):

$$\Delta_0\Delta_0'\psi_p = \Delta_0(4d_p + 2\tau\psi_p - 6f\psi_p) = 72f\psi_p - 32d_p - 16\tau\psi_p, \tag{176}$$

$$\begin{aligned} 2\operatorname{div}_0(k_0' \operatorname{grad}_0 \psi_p) &= 2k_0'\Delta_0\psi_p + 2\operatorname{grad}_0 k_0' \cdot \operatorname{grad}_0 \psi_p \\ &= -32f\psi_p + 16d_p + 8\tau\psi_p. \end{aligned} \tag{177}$$

Their sum is

$$\mathscr{L}_0\psi_p = 40f\psi_p - 16d_p - 8\tau\psi_p = 40H_{3,p}. \tag{178}$$

Since $\mathscr{L}_0\psi_p = 0$, differentiating $A_g^\psi = \nu\Delta_g^{-1}\mathscr{L}_g$ produces no inverse-derivative contribution on this input. On degree three, $\Delta_0^{-1} = -1/12$, so

$$(A^\psi)_0'\psi_p = -\frac{10\nu}{3}H_{3,p}. \tag{179}$$

By Proposition 1, the symbols P_δ, T_δ and A_δ^ψ in this stream-coordinate calculation are the scalar Riesz projection, canonical comparison and scalar realization induced from the velocity objects; their quotient outputs are related by (49). The canonical slow frame is $\psi_{p,\delta} = T_\delta\psi_p = P_\delta\psi_p$. At zero, $L_0 = 0$. Differentiating the invariant-subspace equation $A_\delta^\psi T_\delta|_{E_0} = T_\delta|_{E_0}L_\delta$ gives

$$(A^\psi)_0'\psi_p + A_0^\psi\psi_{p,0}' = iL_0'p.$$

Its slow projection gives $L_0' = 0$, because (179) is entirely degree three. Its fast projection gives $A_{0,Q}^\psi\psi_{p,0}' = -Q_0(A^\psi)_0'\psi_p$. The identity $P_0P_0'P_0 = 0$ ensures that the canonical frame derivative has no slow part. As $A_0^\psi = \nu(\Delta_0 + 2) = -10\nu$ on degree three, we obtain

$$\psi_{p,0}' = -\tfrac{1}{3}H_{3,p}, \qquad L_0' = 0. \tag{180}$$

The derivation is carried out at the Riesz-block level and remains valid for repeated slow eigenvalues.

10.4. The Full Defect and the Source Coefficient

The quantity relevant to the convection is

$$\begin{aligned}\rho_p &= \frac{\mathrm{d}}{\mathrm{d}\delta}(\Delta_\delta\psi_{p,\delta} + 2\psi_{p,\delta})\Big|_0 = \Delta_0'\psi_p + (\Delta_0 + 2)\psi_{p,0}' \\ &= 4d_p + 2\tau\psi_p - 6f\psi_p + \frac{10}{3}H_{3,p} \\ &= \frac{8}{3}(d_p - f\psi_p) + \frac{4}{3}\tau\psi_p.\end{aligned} \tag{181}$$

This displays both the operator variation and the frame variation. Holding the degree-one frame fixed would omit its second term and change the coefficient.

At the sphere, (31) gives $B_0^\psi(\psi_p, \psi_q) = 0$, because the brackets with $\Delta_0\psi = -2\psi$ cancel. The homological inverse therefore gives $W_0 = 0$. For each parameter, write $\Delta_\delta\psi_{p,\delta} = -2\psi_{p,\delta} + r_{p,\delta}$, where $r_{p,0} = 0$ and $r_{p,0}' = \rho_p$. The bracket pair with -2ψ cancels identically, even at $\delta \neq 0$. When the remaining pair is differentiated at zero, derivatives of its first inputs or of the bracket coefficients multiply $r_{p,0} = 0$. Hence, its derivative is exactly

$$S_{p,q} = \{\psi_p, \rho_q\}_0 + \{\psi_q, \rho_p\}_0. \tag{182}$$

Substitute (181) and use the bracket product rule. The trace terms and the terms $f\{\psi_p, \psi_q\}$ cancel in symmetric pairs, leaving

$$S_{p,q} = \frac{8}{3}(\{\psi_p, d_q\}_0 + \{\psi_q, d_p\}_0 - C_{p,q}), \quad C_{p,q} = \psi_p\{\psi_q, f\}_0 + \psi_q\{\psi_p, f\}_0. \tag{183}$$

Here, $\{\psi_p, d_q\}_0 + \{\psi_q, d_p\}_0 = \psi_{p\times Dq+q\times Dp}$ is degree one. The polynomial $C_{p,q}$ is homogeneous cubic, so its spherical decomposition contains only degrees one and three. The normal projection at the sphere removes the degree-one part and retains exactly

$$Q_0 S_{p,q} = -\tfrac{8}{3}\Pi_3 C_{p,q}. \tag{184}$$

This explains both the disappearance of the Dp term from the normal source and the cancellation of the trace term.

The source in B^ψ has factor $-\frac{1}{2}\Delta^{-1}$. Its value before differentiation was zero, so the derivative of this inverse contributes nothing. Output comparison and output projection derivatives likewise multiply $B_0(E_0, E_0) = 0$. The input comparison derivatives are already included in ρ_p. On degree three,

$$Q_0 \overline{B}_0'(\psi_p, \psi_q) = \left(-\frac{1}{2}\right)\left(-\frac{1}{12}\right)\left(-\frac{8}{3}\right)\Pi_3 C_{p,q} = -\frac{1}{9}\Pi_3 C_{p,q}. \tag{185}$$

Finally, $W_0 = 0$ and $L_0 = 0$, so the differentiated homological equation is $A^\psi_{0,Q} W_0' = -2Q_0\overline{B}_0'$. The forcing is entirely degree three and its inverse factor is $-1/(10\nu)$. Therefore,

$$W_0'(p,q) = \left(-\frac{1}{10\nu}\right)(-2)\left(-\frac{1}{9}\right)\Pi_3 C_{p,q} = -\frac{1}{45\nu}\Pi_3 C_{p,q}. \tag{186}$$

The three displayed steps in (184)–(186) separate the cubic projection, the convection normalization and the homological inverse. The factor -2 comes from using $D^2K(0)$ as the jet; the Taylor factor $1/2$ belongs only to the graph reconstruction and is not included in W_0'. This proves the coefficient and sign in (24). Proposition 1, especially (50), identifies this scalar quotient representative with the velocity-space quadratic normal jet: in the fixed reference velocity coordinates, the result is $J_0 \operatorname{grad}_0 W_0'(p,q)$. Because the unperturbed jet is zero, no derivative of this scalar-to-physical representation changes the compared first tensor.

Remark 4 (Normal-deformation consistency calculation)**.** *For the purely normal family $\widetilde{R}_\delta(x) = (1+\delta f(x))x$, one has $\widetilde{h} = 2fg_0$, $\widetilde{\Delta}_0' = -2f\Delta_0$ and $\widetilde{k}_0' = 4f - 2\tau$. Repeating only the frame equation gives $\widetilde{\psi}_{p,0}' = \frac{2}{3}H_{3,p}$, rather than the affine value $-\frac{1}{3}H_{3,p}$, but the complete defect is unchanged:*

$$\widetilde{\rho}_p = 4f\psi_p + (\Delta_0 + 2)\frac{2}{3}H_{3,p} = \frac{8}{3}(d_p - f\psi_p) + \frac{4}{3}\tau\psi_p = \rho_p.$$

Hence, the degree-three source and the coefficient in (186) agree, providing an independent consistency check of the comparison law.

10.5. A Complete Component Calculation

Take $D = \operatorname{diag}(-1, 0, 1)$ and $p = q = e_1$, with $\psi_1 = x_1$. The deformation and its bracket are

$$f = x_3^2 - x_1^2, \tag{187}$$

$$\{\psi_1, f\}_0 = (0, x_3, -x_2)\cdot(-2x_1, 0, 2x_3) = -2x_2x_3. \tag{188}$$

Consequently, with $X = x_1x_2x_3$,

$$C_{11} = 2\psi_1\{\psi_1, f\}_0 = -4X. \tag{189}$$

Each second coordinate derivative of X is zero. Thus, $\Delta_{\mathbb{R}^3} C_{11} = 0$, and the exact projection Formula (173) gives

$$\Pi_3 C_{11} = -4X. \tag{190}$$

Here, $\tau = 0$ and $d_1 = -x_1$, so the complete defect and its symmetric bracket source can also be evaluated directly:

$$\rho_1 = -\frac{8}{3}(1+f)x_1, \qquad S_{11} = 2\{x_1, \rho_1\}_0 = \frac{32}{3}X.$$

This source is already degree three. The scalar convection factor, inverse Laplacian, normal-source coefficient, quadratic homological forcing and degree-three Stokes inverse contribute, respectively,

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{12}\right)\left(-\frac{8}{3}\right)(-2)\left(-\frac{1}{10\nu}\right) = -\frac{1}{45\nu}. \tag{191}$$

In particular,

$$Q_0 \overline{B}_0'(\psi_1, \psi_1) = \left(-\frac{1}{2}\right)\left(-\frac{1}{12}\right)\frac{32}{3}X = \frac{4}{9}X,$$

and the differentiated homological equation yields

$$W_0'(e_1, e_1) = \left(-\frac{1}{10\nu}\right)(-2)\frac{4}{9}X = \frac{4}{45\nu}X = -\frac{1}{45\nu}\Pi_3 C_{11}. \tag{192}$$

The positive sign of this component comes from $\Pi_3 C_{11} = -4X$; this is consistent with the negative coefficient of the general tensor in (186).

10.6. Components, Rotations and Nonvanishing

For $D = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$,

$$\{x_1, f\}_0 = 2(\alpha_2 - \alpha_3)x_2 x_3, \quad \{x_2, f\}_0 = 2(\alpha_3 - \alpha_1)x_3 x_1, \quad \{x_3, f\}_0 = 2(\alpha_1 - \alpha_2)x_1 x_2.$$

Let $X = x_1 x_2 x_3$, which is a harmonic cubic. The three diagonal components are

$$\begin{aligned} W_0'(e_1, e_1) &= -\frac{4}{45\nu}(\alpha_2 - \alpha_3)X, \\ W_0'(e_2, e_2) &= -\frac{4}{45\nu}(\alpha_3 - \alpha_1)X, \\ W_0'(e_3, e_3) &= -\frac{4}{45\nu}(\alpha_1 - \alpha_2)X. \end{aligned} \tag{193}$$

For example, $C_{12} = 2x_3[(\alpha_3 - \alpha_1)x_1^2 + (\alpha_2 - \alpha_3)x_2^2]$, with $\Delta_{\mathbb{R}^3} C_{12} = 4(\alpha_2 - \alpha_1)x_3$. Equation (173) then gives all three mixed components on S^2:

$$\begin{aligned} W_0'(e_1, e_2) &= -\frac{2}{45\nu}x_3\left[(\alpha_3 - \alpha_1)x_1^2 + (\alpha_2 - \alpha_3)x_2^2 - \frac{\alpha_2 - \alpha_1}{5}\right], \\ W_0'(e_2, e_3) &= -\frac{2}{45\nu}x_1\left[(\alpha_1 - \alpha_2)x_2^2 + (\alpha_3 - \alpha_1)x_3^2 - \frac{\alpha_3 - \alpha_2}{5}\right], \\ W_0'(e_3, e_1) &= -\frac{2}{45\nu}x_2\left[(\alpha_2 - \alpha_3)x_3^2 + (\alpha_1 - \alpha_2)x_1^2 - \frac{\alpha_1 - \alpha_3}{5}\right]. \end{aligned} \tag{194}$$

The final linear terms are the restrictions of the $|x|^2$ corrections inside the degree-three harmonic projection in (173). These six formulas give a component evaluation of the full bilinear expression.

Proposition 11 (Rotation covariance and the precise onset conclusion)**.** *The tensor in* (24) *is nonzero exactly when the real symmetric matrix D is nonscalar. For each fixed nonscalar D, the actual quadratic normal jet is nonzero for every sufficiently small $0 < |\delta| < \delta_0(D)$, with no explicit value asserted for $\delta_0(D)$.*

Proof. For $O \in SO(3)$, define $\mathsf{U}_O a(x) = a(O^T x)$. Then, $f_{ODO^T} = \mathsf{U}_O f_D$ and $\psi_{Op} = \mathsf{U}_O \psi_p$. Because $O(v \times w) = Ov \times Ow$ for an orientation-preserving orthogonal matrix, the gradient and bracket transformation give $\{\mathsf{U}_O a, \mathsf{U}_O b\}_0 = \mathsf{U}_O\{a, b\}_0$. The Euclidean Laplacian and $|x|^2$ are rotation invariant, so (173) implies $\Pi_3 \mathsf{U}_O = \mathsf{U}_O \Pi_3$. Consequently,

$$W'_{ODO^T}(Op, Oq) = \mathsf{U}_O W'_D(p, q). \tag{195}$$

The underlying surface equation, its pressure inverse and its canonical Riesz projection obey the same identity by naturality; thus, the transformation also respects the identified tensor, not just its polynomial expression.

Every real symmetric D has an orthogonal eigenbasis. Changing the sign of one eigenvector when necessary makes that basis orientation preserving without changing its diagonal eigenvalue matrix. Hence, (195) reduces vanishing to the diagonal case. If the three diagonal entries are not all equal, at least one coefficient in (193) is nonzero and X is not the zero function. If they are all equal, f is constant on S^2, so both brackets in (24) vanish. This proves the equivalence.

Near the sphere, the block and three-weight gap persist, as proven in Section 5. The C^1 result of Theorem 1 gives, in the bilinear Sobolev norm,

$$\overline{W}_\delta = \delta \overline{W}'_0 + o(|\delta|) \quad \text{in } \mathcal{B}^2_{\mathrm{sym}}(E_0, Q_0 H^{s+2}_\sigma). \tag{196}$$

If $\overline{W}'_0 \neq 0$, choose $\delta_0(D) > 0$ so that for $0 < |\delta| < \delta_0(D)$,

$$\|\overline{W}_\delta - \delta \overline{W}'_0\| \leq \tfrac{1}{2}|\delta| \|\overline{W}'_0\|.$$

Then, $\|\overline{W}_\delta\| \geq \tfrac{1}{2}|\delta| \|\overline{W}'_0\| > 0$. This C^1 argument establishes the existence of $\delta_0(D)$ without supplying an explicit numerical value. Finally, $W_0 = 0$ in the full velocity quotient and Theorem 2 remove all first-order admissible comparison corrections. This completes Theorem 3. □

For $D = \mathrm{diag}(-1, 0, 1)$ and $\nu = 1$, the diagonal coefficients of X are $4/45, -8/45, 4/45$. The ellipsoid with semiaxes $(24/25, 1, 26/25)$ occurs at $\delta = 1/25$ on this family, and Theorem 5 gives its independent spectral certificate. Proposition 11 is local near $\delta = 0$; because $\delta_0(D)$ is not quantified, it does not determine whether $1/25$ lies in the onset neighborhood or whether the continuum quadratic normal jet is nonzero on that ellipsoid.

11. Conclusions

The quadratic normal geometry of the selected surface-fluid dynamics is represented by a quotient-valued jet. Every admissible normalized C^2 mild invariant graph has the same quadratic tensor after tangent-input identification; the argument neither requires nor proves the uniqueness of the full graph. Finite-time invariance supplies the generator-domain gain for the six quadratic coefficients, and common-domain operator calculus yields their C^1 shape dependence with infinite-dimensional output. The exact comparison law separates the geometric tensor from its representative: when the base jet is nonzero, the raw derivative changes with an admissible normalization, whereas, at the spherical zero jet, the first variation is comparison independent.

For symmetric deformations of the sphere, the first surviving normal source is a cubic harmonic and gives the coefficient $-1/(45\nu)$. The tensor vanishes exactly for scalar deformations. Its nonzero onset for each fixed nonscalar D identifies the leading quadratic reconstruction into complementary modes, without asserting a stability change. There exists $\delta_0(D) > 0$ such that the jet is nonzero for $0 < |\delta| < \delta_0(D)$, but no explicit or numerical value is obtained. Independently, the ellipsoid $(24/25, 1, 26/25)$ satisfies the slow–fast separation by an exact rational min–max certificate. That certificate does not show that $\delta = 1/25$ lies in the onset neighborhood, nor that the ellipsoid has a nonzero quadratic normal jet.

A computable onset radius would require an explicit lower bound for $\|\overline{W}_0'\|$ in the chosen bilinear norm and an explicit bound for the remainder in (196). The latter would require quantitative control of the operator derivatives, elliptic and resolvent constants, spectral separation and their continuity along the shape family. A numerical continuation of the slow block and of the six homological coefficient equations could help to estimate these quantities, but certified error bounds would be needed to turn such approximations into a rigorous radius.

Extensions beyond the present setting require additional analysis. On higher-genus surfaces, harmonic vector fields must be included alongside stream functions, with a new spectral selection and gap check. At a nonzero equilibrium, the linearized convection and the equilibrium's shape dependence enter the operator calculus. Steady forcing would also require a compatible equilibrium family, while time-dependent forcing calls for a nonautonomous invariant-object formulation. The present results establish none of these extensions.

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Appendix A. Scalar Regularity and Local Uniformity on a Fixed Atlas

This appendix proves the scalar input to Lemma 1; it is independent of the surface–Stokes and invariant-graph arguments. Let g range over metrics on a fixed compact connected surface with a common ellipticity lower bound and common C^{s+5} coefficient bounds. Fix outer coordinate patches, smaller patches covering M, and cutoff functions equal to one on the smaller patches. Their derivatives do not depend on g.

Appendix A.1. Coercivity, Means and the Weak Inverse

The compact inclusions used here can be seen on the fixed atlas. Localize a bounded H^{k+1} sequence to a patch and extend it, with the fixed cutoff, to a larger coordinate torus. The H^k norm of its Fourier tail $|n| > N$ is at most $(1+N^2)^{-1/2}$ times its H^{k+1} bound. The finitely many modes $|n| \le N$ have a convergent subsequence. Diagonal extraction followed by the uniform tail bound gives convergence in H^k. A finite partition of unity proves the compactness of $H^{k+1}(M) \hookrightarrow H^k(M)$, also for vector fields and closed subspaces, at every nonnegative integer k used below.

The reference Poincaré inequality follows by contradiction: a sequence of reference-mean-zero functions with unit L^2 norm and gradients tending to zero has, by the compactness of $H^1 \hookrightarrow L^2$, a limit with zero gradient; connectedness makes it constant and its mean makes it zero, a contradiction. All metric L^2 and gradient norms are uniformly equivalent to the reference norms. If $m_g u = 0$ and $u_0 = u - m_0 u$, then

$$m_0 u = -m_g u_0, \qquad |m_g u_0| \le C\|u_0\|_{L^2_0}, \qquad \|u\|_{L^2} \le C\|\,\mathrm{d}u\|_{L^2}.$$

Thus, the Poincaré constant for the moving mean-zero condition is also uniform. The area is bounded away from zero, so m_g is bounded on H^{-1}: in reference distribution notation, its numerator pairs with the smooth density j.

For $F \in H^{-1}$ with $m_g F = 0$, the form $a_g(u,v) = \int g^{ij}\partial_i u \partial_j v \,\mathrm{d}\mu_g$ is bounded and coercive on $H^1/\mathbb{R}$. The weak equation $a_g(u,v) = -\langle F, v\rangle_g$ has a unique quotient solution with $\|u\|_{H^1/\mathbb{R}} \le C\|F\|_{H^{-1}}$. This is the Hilbert-space coercive-form theorem: a bounded form satisfying $a(u,u) \ge c\|u\|^2$ defines an invertible operator to the dual, with inverse norm at most c^{-1}. These hypotheses are verified by the uniform metric bound and Poincaré. Choosing $m_g u = 0$ gives the same uniform bound in H^1. Combining this solution with the constant $-m_g F$ gives the full-space inverse in (27), while the construction remains on the full scalar spaces.

Appendix A.2. The H^2 Gain from a Weak Solution

In an outer chart, write $b = \sqrt{|g|}$ and $a^{ij} = bg^{ij}$. The equation is $\partial_i(a^{ij}\partial_j u) = bF$. Let ζ be a fixed cutoff, and let $\delta_h u(x) = [u(x+he_k) - u(x)]/h$ in one coordinate direction. For small h, test the weak equation with $-\delta_{-h}(\zeta^2 \delta_h u)$. The identity $\int v\,\delta_{-h} w = -\int (\delta_h v) w$ gives a principal term

$$\int \zeta^2 a^{ij}(x+he_k)\partial_j \delta_h u\, \partial_i \delta_h u.$$

The coefficient-difference term contains $\delta_h a^{ij}\partial_j u$, and the cutoff terms contain $\partial\zeta\,\delta_h u$. Their absolute values, using $\|\delta_h a\|_\infty \le \|\partial_k a\|_\infty$ and $\|\delta_h u\|_{L^2} \le \|\partial_k u\|_{L^2}$, are bounded by

$$C\|u\|_{H^1}\|\zeta\nabla\delta_h u\|_{L^2} + C\|u\|^2_{H^1}.$$

The forcing term obeys

$$\left|\int bF\,\delta_{-h}(\zeta^2\delta_h u)\right| \le C\|F\|_{L^2}\big(\|\zeta\nabla\delta_h u\|_{L^2} + \|u\|_{H^1}\big).$$

Indeed, $\|\delta_{-h} v\|_{L^2} \le \|\partial_k v\|_{L^2}$, applied to $v = \zeta^2\delta_h u$, proves the last estimate without differentiating F. Ellipticity and Young's inequality now yield

$$\|\zeta\nabla\delta_h u\|^2_{L^2} \le C\big(\|F\|^2_{L^2} + \|u\|^2_{H^1}\big), \tag{A.1}$$

uniformly for small h and for all metrics in the neighborhood. A uniform difference-quotient bound for a first derivative implies its weak derivative is in L^2: weak compactness

identifies the distributional limit of the quotients, or equivalently Fatou's lemma applies to their Fourier multipliers. Therefore, $u \in H^2$ on the inner patch. Summing over the fixed finite covering gives

$$\|u\|_{H^2} \le C(\|\Delta_g u\|_{L^2} + \|u\|_{H^1}).$$

The local Fourier inequality $\|u\|_{H^1} \le a\|u\|_{H^2} + C_a\|u\|_{L^2}$, with a fixed sufficiently small a, absorbs the first term on the right. This proves (26) for $\ell = 0$, including the weak-to-strong conclusion. For a mean-zero solution, the weak coercivity estimate also controls its lower norms directly by the forcing.

Appendix A.3. Higher Integer Levels and Their Constants

Suppose the statement has been proved below an integer $\ell \ge 1$. In nondivergence notation on a chart, write $\Delta_g = g^{ij}\partial_i\partial_j + b^j\partial_j$, where b^j contains one derivative of g. For $|a| = \ell$, the differentiated equation is

$$\Delta_g(\partial^a u) = \partial^a F - [\partial^a, g^{ij}]\partial_i\partial_j u - [\partial^a, b^j]\partial_j u.$$

The commutators are explicitly

$$[\partial^a, c]w = \sum_{0<b\le a} \binom{a}{b} (\partial^b c)\partial^{a-b} w.$$

Every derivative of u on the right has order at most $\ell + 1$, because at least one derivative falls on a coefficient. Their L^2 norm is bounded by $C_\ell\|u\|_{H^{\ell+1}}$, with C_ℓ depending on coefficient derivatives through order $\ell + 1$. Localizing creates only first-order commutators with the fixed cutoff. Apply the preceding H^2 estimate to the localized derivatives and sum the finitely many derivatives and patches. One obtains

$$\|u\|_{H^{\ell+2}} \le C_\ell\big(\|F\|_{H^\ell} + \|u\|_{H^{\ell+1}}\big).$$

The induction hypothesis ensures that these operations are valid for a weak solution, first as distributional differentiated equations and then by the established H^2 gain. Equivalently, smooth difference quotients can be used until the last step and the uniform estimate passed to their weak limit. The interpolation estimate

$$\|u\|_{H^{\ell+1}} \le a\|u\|_{H^{\ell+2}} + C_{\ell,a}\|u\|_{L^2}$$

absorbs the lower term. Its Fourier proof splits $|\xi| \le N$ and $|\xi| > N$ and chooses N so that $(1 + |\xi|^2)^{-1/2} < a$ on the second region; localization contributes only fixed lower-order constants. This proves (26) for every $0 \le \ell \le s + 3$.

The proof uses one fixed atlas, a common positive lower bound for a^{ij}, fixed cutoff derivatives, and coefficient upper bounds through at most order $s + 4$. A bounded C^{s+5} metric family supplies all of them. All Young parameters can be chosen at the maximum of the finitely many resulting constants. Weak coercivity and the constant decomposition then give uniform inverse estimates for $\mathscr{D}_g$ at every level from $H^{-1} \to H^1$ to $H^{s+3} \to H^{s+5}$. This establishes the stated uniform scalar regularity.

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