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On Some Karhunen–Loève Expansions Related to SU(2) with Application to Tests of Symmetry

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Abstract

Karhunen–Loève expansions play a key role in mathematical statistics. Two kernels associated with the well-known Anderson–Darling statistic admit such developments involving Legendre polynomials and their derivatives. We give an interpretation of these developments in terms of special functions occurring as the matrix elements of irreducible unitary representations of the compact Lie group $SU(2)$. This interpretation leads us to introduce several families of bilinear expansions defined over $SU(2) \times SU(2)$. In this framework, the Anderson–Darling test for uniformity has an interpretation in terms of rotational invariance of some probability distributions. We provide a similar geometric interpretation of our new developments. We outline statistical applications.

Keywords: special functions; group representations; directional statistics; $V-$statistics

MSC: 33C45; 62H11

1. Introduction

What is the shape of space? If our universe, at a given time, is a finite, homogeneous, 3-dimensional space with constant positive curvature, models are provided by the Euclidean sphere S^4 and its quotient submanifolds as the projective and lens spaces. These manifolds have been studied from the point of view of mathematical physics in the seminal article [1], and from the point of view of Riemannian geometry and Lie groups, see [2] (2.4.10 Corollary, p. 69 and §7.5, p. 225), [3] (Example 12.13, p. 269) and [4] (p. 350).

The present paper is motivated by the statistical challenge associated with this issue from cosmology. This challenge might be added to those surveyed, e.g., in [5–7].

The branch of Statistics dealing with samples from general manifolds is directional statistics. In this field a fundamental reference is [8], in particular Chapter 13, which deals with general sample spaces. See also [9] for a recent survey. Two widely used general frameworks for statistics on manifolds were discussed by Beran and Giné, see §6.3.7 and §10.8 in [8] for references to original papers. Group actions are a fundamental tool in Beran's framework, whereas eigenfunctions of the Laplacian operator play a fundamental role in Giné's. Interestingly, the connection between special functions and group invariant operators, such as the Laplacian, is at the heart of the theory of special functions expounded by Vilenkin in the seminal book [10]. The properties of special functions from Vilenkin's point of view have not been, to our knowledge, exploited systematically in directional statistics. The need for developing such a systematic framework seems to be justified by the recent advances in cosmology mentioned above. In other words, the object of the present

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paper is at the crossroads of directional statistics, the representation theory of Lie groups, and cosmology.

Our main tools are orthogonal expansions of bilinear kernels, also called Karhunen–Loève (K–L) expansions, which play a key role in mathematical statistics. If (X, μ) is a measure space endowed with the usual inner product

$$(f, g) \mapsto \int_X f(x)\overline{g(x)}d\mu(x)$$

and the associated norm, we call K–L expansion of the symmetric kernel $K : X \times X \to \mathbb{R}$, an equality in quadratic mean

$$K(x, y) = \sum_{k=1}^{\infty} \mu_k \cdot f_k(x)\overline{f_k(y)},$$

with $\mu_k \in [0, \infty)$, $f_k \in L^2(X)$, and

$$\int_X K(x, y)f_k(y)d\mu(y) = \mu_k f_k(x), \quad \int_X f_k(x)\overline{f_\ell(x)}d\mu(x) = \mathbb{1}_{\{k=\ell\}},$$

for $k, \ell = 1, 2, \ldots$. The real numbers μ_k are called the eigenvalues and the functions f_k the eigenfunctions of this expansion.

Some classical examples from mathematical statistics involve trigonometric functions, and Jacobi polynomials defined for $\alpha, \beta > -1$ by Rodrigues' formula

$$P_k^{(\alpha,\beta)}(x) = \frac{(-1)^k}{2^k k!(1-x)^\alpha(1+x)^\beta}\frac{d^k}{dx^k}[(1-x)^{k+\alpha}(1+x)^{k+\beta}], \quad k = 0, 1, \cdots$$

(see [11] (§5.2.2, p. 213), or [12] (§6.3.8, (4), p. 290)). For instance, if $X_1, X_2, \cdots$ is a sequence of independent random variables with uniform distribution over $(0, 1)$, we observe the convergences in distribution

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K_C(X_i, X_j) = \int_0^1 B(t)^2 dt, \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K_A(X_i, X_j) = \int_0^1 \frac{B(t)^2}{t(1-t)} dt, \tag{2}$$

with

$$\begin{aligned} K_C(s, t) &= \frac{s^2 + t^2 - 2\max(s, t)}{2} + \frac{1}{3} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \cdot 2\cos(k\pi s)\cos(k\pi t), \end{aligned} \tag{3}$$

$$\begin{aligned} K_A(s, t) &= \int_0^1 \frac{[\mathbb{1}_{\{s\le u\}} - u][\mathbb{1}_{\{t\le u\}} - u]}{u(1-u)}\, du \\ &= -\log\{\, e[1 - \min(s, t)]\max(s, t)\,\} \\ &= \sum_{\ell=1}^{\infty} \frac{1}{\ell(\ell+1)} \cdot (2\ell+1)P_\ell^{(0,0)}(2s-1)P_\ell^{(0,0)}(2t-1) \end{aligned} \tag{4}$$

(see [13] (Example 12.13, pp. 170–171) for K_C, [14] (Example 6.3.1, p. 240) and [11] (p. 239) for K_A). In (1) and (2), the sums on the left-hand sides are the classical Cramér-von Mises and Anderson–Darling statistics, defined as $V-$statistics with kernels K_C and K_A, respectively; the process $\{B(t) : 0 < t < 1\}$ denotes a Brownian bridge, i.e., a

centred Gaussian random process with covariance function $(s,t) \mapsto \min(s,t)[1-\max(s,t)]$. The covariance functions of $B(t)$ and $B(t)/\sqrt{t(1-t)}$ admit the K–L expansions

$$\min(s,t)[1-\max(s,t)] = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \cdot 2\sin(k\pi s)\sin(k\pi t), \tag{5}$$

$$\frac{\min(s,t)[1-\max(s,t)]}{\sqrt{s(1-s)t(1-t)}} = \sum_{\ell=1}^{\infty} \frac{1}{\ell(\ell+1)} \cdot \frac{4(2\ell+1)\sqrt{s(1-s)}P'_\ell(2s-1)\sqrt{t(1-t)}P'_\ell(2t-1)}{\ell(\ell+1)}, \tag{6}$$

where $P_\ell = P_\ell^{(0,0)}$ is the Legendre polynomial with degree ℓ (see Proposition 5.3.1. p. 213 and Theorem 5.4.1. p. 225 in [15]).

The first aim of the present paper is to provide a geometric interpretation of the presence of Jacobi polynomials in (6), in connection with the Lie group $SU(2)$. This interpretation will enable us to derive new K–L expansions, associated with potential generalizations of the Anderson–Darling statistic.

Before stating our results, let us briefly outline our approach. Some natural analogues of sine and cosine functions for Euclidean spheres of dimensions 2 and 3 are the matrix elements of irreducible representations of $SU(2)$. Surprisingly, these analogues, discussed in [10] (Chapter III) and in the more recent and enlarged edition [12] (Chapter 6), have not received systematic attention concerning their use in directional statistics. Recall that they are defined, in terms Jacobi polynomials and Euler angles (ϕ,θ,ψ) (see Section 2), by

$$t^\ell_{m,n}(u) := i^{m-n}e^{-i(m\phi+n\psi)}P^\ell_{m,n}(\cos\theta), \tag{7}$$

$$\text{with } P^\ell_{m,n}(z) = 2^{-m}\Big[\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\Big]^{\frac{1}{2}}(1-z)^{\frac{m-n}{2}}(1+z)^{\frac{m+n}{2}}P^{(m-n,m+n)}_{\ell-m}(z)$$

$$\text{if } m-n \in \mathbb{N}, \tag{8}$$

$$\text{and } P^\ell_{m,n}(z) = (-1)^{m+n}P^\ell_{n,m}(z) = (-1)^{m-n}P^\ell_{-m,-n}(z) = (-1)^{\ell+n}P^\ell_{-m,n}(-z), \tag{9}$$

where (ℓ,m,n) equals $(0,0,0)$ or belongs to the index set

$$I_{SU(2)} := \Big\{(\ell,m,n) : 2\ell \in \mathbb{N}^* \text{ and } m,n \in \{-\ell,-\ell+1,\ldots,\ell\}\Big\} \tag{10}$$

(see [12] ((5), p. 285 and (1), p. 288) and [10] ((6), p. 120 and (2), p. 125)).

To our knowledge, the connection between the Anderson–Darling statistic and irreducible representations of $SU(2)$ has not been exploited. Indeed, if we set $s = \cos\theta_2 \le t = \cos\theta_1$ with $0 < \theta_1 \le \theta_2 < \pi$, and in virtue of relations

$$P'_\ell(z) = [(\ell+1)/2]P^{(1,1)}_{\ell-1}(z), \quad P^\ell_{1,0}(z) = \frac{1}{2}\Big[\frac{\ell+1}{\ell}\Big]^{\frac{1}{2}}\sqrt{1-z^2}P^{(1,1)}_{\ell-1} \tag{11}$$

(use [12] ((2), p. 290) with $\alpha = \beta = 0$, and (8)), it is easily seen for (4) and it will be shown in Lemma 3 for (6) that these two K–L expansions can be rewritten, for $-1 < z_1 \le z_2 < 1$, as

$$-\frac{1}{2}\log\Big[\frac{e(1-z_1)(1+z_2)}{4}\Big] = \sum_{\ell=1}^{\infty}\frac{1}{\ell(\ell+1)}\cdot(\ell+1/2)P^\ell_{0,0}(z_1)P^\ell_{0,0}(z_2), \tag{12}$$

$$\frac{1}{2}\left[\frac{(1+z_1)(1-z_2)}{(1-z_1)(1+z_2}\right]^{1/2} = \sum_{\ell=1}^{\infty}\frac{1}{\ell(\ell+1)}\cdot(\ell+1/2)P^\ell_{1,0}(z_1)P^\ell_{1,0}(z_2). \tag{13}$$

In view of orthogonality relations

$$(\ell + 1/2) \int_{-1}^{1} P^{\ell}_{m,n}(z) P^{\ell'}_{m,n}(z) dz = \mathbb{1}_{\{\ell=\ell'\}}, \quad 2\ell \in \mathbb{N} \tag{14}$$

(see [12] ((2), p. 365) and [10] ((7), p. 161)), developments (4) and (6) rewritten in the form (12) and (13), now clearly appear as the first two members of a wider family of K–L expansions. Eigenfunctions of these K–L expansions would be the orthonormalized versions of functions $P^{\ell}_{m,n}$, the eigenvalues being for all of them the inverse of those of the Laplacian on $SU(2)$, i.e., $[\ell(\ell+1)]^{-1}$ for $\ell = 1/2, 1, 3/2 \cdots$ (see [12] (§6.10.3, (8), p. 372)). In this setting, one of our main results is the following generalization of (12) and (13). It involves the hypergeometric function

$$z \mapsto {}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{15}$$

where $(x)_0 = 1$ and $(x)_k = x(x+1)\cdots(x+k-1)$ if k is positive (see [16] (6.1.21, p. 256, 15.1.1, p. 557])).

Theorem 1. *Assume*

$$m \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\},\ n \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\},\ m - n \in \mathbb{N}. \tag{16}$$

The K–L expansion

$$\frac{\Omega^0_{m,n}(z_1)\Omega^0_{m,-n}(-z_2)}{W_{m,n}} = \sum_{\ell=m}^{\infty} \frac{1}{\ell(\ell+1)} \cdot \frac{(2\ell+1) P^{\ell}_{m,n}(z_1)\, P^{\ell}_{m,n}(z_2)}{2}$$
$$(-1 < z_1 \le z_2 < 1) \tag{17}$$

holds, with

$$\Omega^0_{m,n}(z) = \frac{\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}}{\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}}} {}_2F_1\left(n, n+1; m+n+1; \frac{1+z}{2}\right),$$

and where $W_{m,n}$ is the constant function satisfying

$$W_{m,n} = (1-z^2)\left\{ \frac{d}{dz}[\Omega^0_{m,n}(z)]\Omega^0_{m,-n}(-z) - \Omega^0_{m,n}(z)\frac{d}{dz}[\Omega^0_{m,-n}(-z)] \right\} \quad (-1 < z < 1).$$

In Section 2, we recall basic facts and notations about $SU(2)$ and its irreducible unitary representations. Section 3 is devoted to the proof of Theorem 1. We also state, as a corollary of (17), a K–L expansion over $SU(2) \times SU(2)$. In Section 4 we state K–L expansions derived from group properties of matrix elements $(t^{\ell}_{m,n})$. In Section 5 we give some K–L expansions associated with some of the remarkable submanifolds of $SU(2)$. In Section 6 we outline statistical tests for various types of symmetries for distribution over $SU(2)$. A by-product is a goodness-of-fit test for uniformity over a lens space. And as announced in this Introduction, the Anderson–Darling test finds a natural interpretation in this framework. In Section 7, we outline the way our tests should be implemented and give general results related to their Bahadur efficiency.

Technical lemmas that would disrupt the flow of the main ideas, theorems and propositions structuring the article are postponed to Section 8.

2. The Lie Group $SU(2)$

In this Section we use notations and results from [10] (Chapter III) and [12] (Chapter 6), keeping in mind that misprints are numerous in the latter, see [17] about this topic.

2.1. Euler Angles and Metrics over $SU(2)$

Following [10] (§III-1.1., (1), p. 98), or [12] (§6.1.1. (1), p. 269), consider the special unitary group

$$SU(2) = \{u(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha,\beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1\}.$$

The adjoint of $u \in SU(2)$ is u^* given by

$$u^* = u^{-1} = \overline{u'} = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix},$$

where u' denotes the transpose of u. We know from [10] (§III-1.1., (4), p. 99), or [, §6.1.1., (4), p. 269] that every $u \in SU(2)$ can be written in the form

$$u(\phi,\theta,\psi) = \begin{pmatrix} \cos\frac{\theta}{2} e^{\frac{i(\phi+\psi)}{2}} & i\sin\frac{\theta}{2} e^{\frac{i(\phi-\psi)}{2}} \\ i\sin\frac{\theta}{2} e^{\frac{i(\psi-\phi)}{2}} & \cos\frac{\theta}{2} e^{-\frac{i(\phi+\psi)}{2}} \end{pmatrix}, \quad (\phi,\theta,\psi) \in [0,2\pi) \times [0,\pi] \times [-2\pi,2\pi).$$

Angle θ is unique, determined by the relation

$$\cos\theta = 2|\alpha|^2 - 1. \tag{18}$$

If $\alpha = 0$, we have $\theta = \pi$ and we can choose $\phi = 0$, with ψ being then determined by the relation $\beta/i = e^{-i\psi/2}$. If $\beta = 0$, we have $\theta = 0$ and again we can set $\phi = 0$, with ψ being determined by relation $\alpha = e^{i\psi/2}$. If $\alpha\beta \neq 0$, then $(\phi,\psi) \in [0,2\pi) \times [-2\pi,2\pi)$ is the unique solution of the system

$$e^{i\phi} = -\frac{i\alpha\beta}{|\alpha\beta|}, \quad e^{\frac{i\psi}{2}} = \frac{\alpha e^{-\frac{i\phi}{2}}}{|\alpha|}, \tag{19}$$

[10] (§III-1.1., (5)–(5″), p. 99), or [12] (§6.1.1., (5), p. 269).

The coordinates (ϕ,θ,ψ) are called Euler's angles of the element u, and we will write $u = u(\phi,\theta,\psi)$. A particular case is the 2×2 identity matrix $I_2 = u(0,0,0)$. The trace of $u \in SU(2)$ is given by

$$\mathrm{tr}[u(\phi,\theta,\psi)] = 2\Re(\alpha) = 2\cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2} \in [-2,2],$$

with $\mathrm{tr}(u) = \pm 2$ if and only if $u = \pm I_2$. Given the two elements

$$u_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\overline{\beta}_j & \overline{\alpha}_j \end{pmatrix} \in SU(2), \quad j \in \{1,2\},$$

an important role will be played by the product

$$u_1 u_2^* = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\overline{\beta}_1 & \overline{\alpha}_1 \end{pmatrix} \begin{pmatrix} \overline{\alpha_2} & -\beta_2 \\ \overline{\beta}_2 & \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\overline{\alpha_2} + \beta_1\overline{\beta_2} & \beta_1\alpha_2 - \beta_2\alpha_1 \\ \overline{\beta_2}\alpha_1 - \overline{\beta_1}\alpha_2 & \overline{\alpha_1}\alpha_2 + \overline{\beta_1}\beta_2 \end{pmatrix}. \tag{20}$$

The Frobenius distance $d(u_1, u_2)$ between u_1 and u_2 is the Euclidean distance in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4$, given by

$$d^2(u_1, u_2) = |\alpha_1 - \alpha_2|^2 + |\beta_1 - \beta_2|^2 = \frac{1}{2}\mathrm{tr}[(u_1 - u_2)(u_1 - u_2)^*] = 2 - \mathrm{tr}(u_1 u_2^*).$$

If for $j = 1, 2$, we set $\alpha_j = a_j + ib_j$ and $\beta_j = c_j + id_j$ with $(a_j, b_j, c_j, d_j) \in \mathbb{R}^4$, we have $a_j^2 + b_j^2 + c_j^2 + d_j^2 = 1$. Consequently we can identify u_j with the unit vector $(a_j, b_j, c_j, d_j) \in S^3$. The dot product in $\mathbb{R}^4$ takes the form

$$u_1 \cdot u_2 = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2 = \Re(\alpha_1 \overline{\alpha_2} + \beta_1 \overline{\beta_2}) = \frac{1}{2}\mathrm{tr}(u_1 u_2^*).$$

Consequently, the associated spherical (or intrinsic) distance $\delta(u_1, u_2)$ measured on the sphere S^3 is given by the equivalent relations

$$\delta(u_1, u_2) = \arccos(u_1 \cdot u_2) = \arccos \frac{\mathrm{tr}(u_1 u_2^*)}{2}, \quad \cos \delta(u_1, u_2) = 1 - \frac{d^2(u_1, u_2)}{2}. \tag{21}$$

Note that the latter is the particular case $n = 3$ of formula [18] ((2.1.7), p. 37).

2.2. Matrix Elements of Irreducible Representations

We know from [10] (§III-6.1., (7), and §III-6.2., (1), p. 60) or [12] (§1.4.1, (1), p. 275 and §6.10.1, (1), p. 365) that functions defined above by (7) satisfy the orthogonality relations

$$(2\ell + 1) \int_{SU(2)} t^\ell_{m,n}(u) \overline{t^s_{p,q}(u)} du = \mathbb{1}_{\{(\ell,m,n)=(s,p,q)\}}, \quad du = \frac{1}{16\pi^2} \sin\theta d\phi \, d\theta \, d\psi, \tag{22}$$

du denoting the Haar measure on $SU(2)$, which satisfies

$$\int_{SU(2)} du = \int_{SU(2)} t^0_{0,0}(u) \overline{t^0_{0,0}(u)} du = 1.$$

The function $P^\ell_{m,n}$ is a solution of the differential equation

$$[(1 - z^2)y'(z)]' - \frac{m^2 + n^2 - 2mnz}{1 - z^2} y(z) = -\ell(\ell + 1)y(z) \quad (-1 < z < 1) \tag{23}$$

(see §III-4.5., (6), p. 138, or [12] (§6.7.5., (2), p. 341)). In order to use classical results from Sturm–Liouville differential equations, we need the following supplementary result.

Proposition 1. *For every $(\ell, m, n) \in I_{SU(2)}$, the function $y = P^\ell_{m,n}$ satisfies the boundary conditions*

$$\lim_{z \to 1^-} (1 - z^2)y'(z) = \lim_{z \to -1^+} (1 - z^2)y'(z) = 0. \tag{24}$$

Proof. In view of identities (9), it suffices to prove the result for $m \geq n \geq 0$. In this case, we see from (8) that there exist $\alpha, \beta \in \{0, 1/2\}$ and a polynomial Q such that $y = (1 - z)^\alpha (1 + z)^\beta Q$. In all cases the function

$$\phi(z) := (1 - z^2)[(1 - z)^\alpha (1 + z)^\beta]' = (1 - z)^\alpha (1 + z)^\beta [-\alpha(1 + z) + \beta(1 - z)]$$

satisfies $\phi(\pm 1) = 0$. Elementary calculus gives

$$(1 - z^2)y'(z) = \phi(z)Q(z) + (1 - z)^{\alpha+1}(1 + z)^{\beta+1}Q'(z),$$

and the result follows. □

2.3. Characters

Given $\ell \in \mathbb{N}^*/2$, the mapping $u \mapsto T_\ell(u) = (t^\ell_{m,n}(u))_{(\ell,m,n)\in I_{SU(2)}}$ is a representation of $SU(2)$. The value $\ell = 0$ induces the trivial representation $u \mapsto T_0(u) = t^0_{0,0}(u) = 1$. We know from [10] (Theorem 1 p. 114) that every irreducible unitary representation of $SU(2)$ is equivalent to one of the representations T_ℓ, $\ell = 0, 1/2, 1, \ldots$. The dimension of T_ℓ is $2\ell + 1$. It follows from Peter–Weyl Theorem (see [10] (Theorem 1, p. 45 and §III.6.2) and [12] (§2.3.4) for the general, as well as our particular, cases) that the functions $\sqrt{2\ell+1}t^\ell_{m,n}(u)$ form a complete orthonormal system of functions relative to the invariant measure du. The character χ_ℓ of representation T_ℓ is defined to be

$$\chi_\ell(u) = \mathrm{tr}[T_\ell(u)] = \sum_{m=-\ell}^{\ell} t^\ell_{m,m}(u) \quad (\ell = 0, \frac{1}{2}, 1, \ldots).$$

A first elementary finite K–L expansion associated with representation T_ℓ is

$$\sum_{m=-\ell}^{\ell}\sum_{n=-\ell}^{\ell} t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)} = \sum_{m=-\ell}^{\ell}\sum_{n=-\ell}^{\ell} t^\ell_{m,n}(u_1)t^\ell_{n,m}(u_2^{-1}) = \sum_{m=-\ell}^{\ell} t^\ell_{m,m}(u_1u_2^{-1})$$
$$= \chi_\ell(u_1u_2^{-1}) = \chi_\ell(u_1u_2^*).$$

The first equality follows from the fact that T_ℓ is unitary, so that $T_\ell(u^{-1}) = \overline{T_\ell(u)'}$; the second is the product formula of two matrices. This elementary K–L expansion has the eigenvalue 1, with multiplicity $(2\ell+1)^2 = \dim T^2_\ell$. It can be seen as canonically associated with representation T_ℓ.

We also know from [12] ((3′), p. 359) that in terms of Euler angles, characters satisfy

$$\chi_\ell(u) = \frac{\sin(\ell + \frac{1}{2})t}{\sin\frac{t}{2}}, \quad 0 \le t = t(u) \le 2\pi, \tag{25}$$

$$\text{with} \quad \cos\frac{t(u)}{2} = \frac{\mathrm{tr}(u)}{2} = \Re(\alpha) = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}. \tag{26}$$

In view of (21), in the K–L expansion

$$\sum_{m=-\ell}^{\ell}\sum_{n=-\ell}^{\ell} t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)} = \chi_\ell(u_1u_2^*) \tag{27}$$

just derived above, the bivariate kernel $(u_1, u_2) \mapsto \chi_\ell(u_1u_2^*)$ is a function of $\delta(u_1, u_2)$. In other words, this kernel enjoys the invariance property, making it suitable for being used as the kernel of a $U-$statistic; regarding this concept, see [19] (Chapter 6).

2.4. From Complex to Real K–L Expansions

Before proceeding further, let us show in which way the K–L expansions of a real symmetric kernel, in terms of complex or real functions, are related. Assume $(m, n) \neq (0, 0)$, so that $t^\ell_{m,n} \neq t^\ell_{-m,-n}$. In view of (22), given $\lambda > 0$, the elementary (complex) K–L expansion

$$K(u_1, u_2) = \lambda \cdot (2\ell+1)[t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)} + t^\ell_{-m,-n}(u_1)\overline{t^\ell_{-m,-n}(u_2)}]$$

is that of a real-valued kernel with eigenvalue λ of multiplicity 2. Elementary computations give

$$t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)} + t^\ell_{-m,-n}(u_1)\overline{t^\ell_{-m,-n}(u_2)} = 2\Re\{t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)}\}$$
$$= 2\Re\{i^{m-n}e^{-i(m\phi_1+n\psi_1)}P^\ell_{m,n}(\cos\theta_1)\overline{[i^{m-n}e^{-i(m\phi_2+n\psi_2)}P^\ell_{m,n}(\cos\theta_2)]}\}$$

Thus, keeping in mind that $P^{\ell}_{m,n}$ is real-valued, we can write

$$K(u_1,u_2) = \lambda \cdot (2\ell+1)[2\cos(m\phi_1+n\psi_1)\cos(m\phi_2+n\psi_2)]P^{\ell}_{m,n}(\cos\theta_1)P^{\ell}_{m,n}(\cos\theta_2) + 2\sin(m\phi_1+n\psi_1)\sin(m\phi_2+n\psi_2)]P^{\ell}_{m,n}(\cos\theta_1)P^{\ell}_{m,n}(\cos\theta_2),$$

and we have

$$\begin{aligned}
&\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi}(2\ell+1)2[\cos(m\phi+n\psi)P^{\ell}_{m,n}(\cos\theta)]^2\,\frac{\sin\theta d\theta\, d\phi\, d\psi}{16\pi^2}\\
&=\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}\int_{0}^{\pi}(2\ell+1)2[\sin(m\phi+n\psi)P^{\ell}_{m,n}(\cos\theta)]^2\,\frac{\sin\theta d\theta\, d\phi\, d\psi}{16\pi^2}\\
&=2\int_{0}^{\pi}(\ell+1/2)[P^{\ell}_{m,n}(\cos\theta)]^2\sin\theta d\theta\cdot\int_{-2\pi}^{2\pi}\int_{0}^{2\pi}[1\pm\cos(2m\phi+2n\psi)]\frac{d\phi\, d\psi}{16\pi^2}\\
&=\frac{2\times 1\times 8\pi^2}{16\pi^2}=1
\end{aligned}$$

In view of orthogonality relations (14), we have proved that K admits a K–L expansion with real eigenfunctions, and eigenvalue λ of multiplicity 2. In other words, the two K–L expansions are equivalent in terms of eigenvalues and their multiplicities.

Let us now derive K–L expansions associated with certain subsets of $I_{SU(2)}$.

3. K–L Expansions Involving $t^{\ell}_{m,n}$ with (m,n) Fixed

This section is devoted to the proof of Theorem 1. The kernel involved is symmetric, defined by

$$K_{m,n}(z_1,z_2) = \mathfrak{Q}^0_{m,n}(z_1)\mathfrak{P}^0_{m,n}(z_2) \quad (-1 < z_1 \le z_2 < 1),$$

where the two components are given by

$$\mathfrak{Q}^0_{m,n}(z) = \frac{\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}}{\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}}}\,{}_2F_1\left(n,n+1;m+n+1;\frac{1+z}{2}\right), \tag{28}$$

$$\mathfrak{P}^0_{m,n}(z) = \mathfrak{Q}^0_{m,-n}(-z) = \frac{\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}}}{\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}}\,{}_2F_1\left(-n,-n+1;m-n+1;\frac{1-z}{2}\right). \tag{29}$$

Let us apply classical results from Sturm–Liouville theory (see, e.g., [20] (Chapter 2)) by rewriting (23) with the following notations. Consider the differential operator

$$Ly(z) = [p(z)y'(z)]' - q(z)y(z) \quad (-1 < z < 1)$$

and the boundary conditions

$$\lim_{z\to -1^+} p(z)y'(z) = 0, \quad \lim_{z\to 1^-} p(z)y'(z) = 0 \tag{30}$$

with $p(z) = (1-z^2)$ and $q(z) = (m^2+n^2-2mnz)/(1-z^2)$ for $-1 < z < 1$. We recognize the Sturm–Liouville operator (17.1) p. 64 and the initial conditions (17.5) and (17.6) with $\alpha = \beta = \pi/2$ in [20]. We will apply results from this reference in the case $(a,b) = (-1,1)$, with $y_1 = \mathfrak{Q}^0_{m,n}$ and $y_2 = \mathfrak{P}^0_{m,n}$.

Proposition 2. *Assume $2m, 2n \in \mathbb{N}$, with m positive and $m-n \in \mathbb{N}$. The two functions $\mathfrak{P}^0_{m,n}$ and $\mathfrak{Q}^0_{m,n}$ are solutions, for $\ell = 0$, of the differential Equation (23). They are linearly independent. Furthermore, they satisfy the limit conditions*

$$\lim_{z\to -1^+}(1-z^2)\frac{d}{dz}\mathfrak{Q}^0_{m,n}(z)=\lim_{z\to 1^-}(1-z^2)\frac{d}{dz}\mathfrak{P}^0_{m,n}(z)=0. \tag{31}$$

and there exists a real number $W_{m,n}\neq 0$ *such that*

$$W_{m,n}:=(1-z^2)\left\{\frac{d}{dz}[\mathfrak{Q}^0_{m,n}(z)]\mathfrak{P}^0_{m,n}(z)-\mathfrak{Q}^0_{m,n}(z)\frac{d}{dz}[\mathfrak{P}^0_{m,n}(z)]\right\}\quad(-1<z<1). \tag{32}$$

Proof. We know from Lemma 7 that for the two values $e=\pm 1$, the function

$$(\frac{1-ez}{2})^{(m-en)/2}(1-\frac{1-ez}{2})^{-(m+en)/2}{}_2F_1(-en,-en+1;m-en+1;\frac{1-ez}{2})\quad(-1<z<1)$$

is a solution of (23). We obtain $\mathfrak{P}^0_{m,n}(z)$ for $e=1$, and $\mathfrak{Q}^0_{m,n}(z)$ for $e=-1$. Next, we see from definition (28) that in the neighborhood of $z=-1$, there exists $\alpha\in\{0,1/2\}$ and an entire function E such that $y(z)=\mathfrak{Q}^0_{m,n}(z)=(1+z)^\alpha E(z)$. Elementary calculus gives

$$(1-z^2)y'(z)=(1-z)(1+z)^\alpha[\alpha E(z)+(1+z)E'(z)].$$

For both values of α we find $\lim_{z\to -1^+}(1-z^2)y'(z)=0$, and the first equality in (31) is proved. The second boundary condition is proved in a similar way.

Let us prove the linear independence of the two functions. Directly from (28) for $\mathfrak{Q}^0_{m,n}$, from (29) and [16] (15.1.1(b)) with $(a,b,c)=(-n,-n+1,m-n+1)$ for $\mathfrak{P}^0_{m,n}$, we infer that there exist $C_1,C_2>0$, such that, as $z\to -1^+$,

$$\mathfrak{Q}^0_{m,n}(z)\sim C_1(1+z)^{(m+n)/2},\quad \mathfrak{P}^0_{m,n}(z)\sim\frac{C_2}{(1+z)^{(m+n)/2}}.$$

Since $m+n\geq m>0$, the independence is established. The existence and definition of $W_{m,n}$ follows from [20] ((17.5)–(17.7), p. 65), with $C=-W_{m,n}$. □

Let us determine the eigenfunctions and eigenvalues of our kernel $K_{m,n}$.

Proposition 3. *Assume* $2m,2n\in\mathbb{N}$, *with* $m-n\in\mathbb{N}$. *For every* $z_1\in(-1,1)$, *the bivariate kernel* $K_{m,n}$ *satisfies*

$$\int_{-1}^1 K_{m,n}(z_1,z_2)P^\ell_{m,n}(z_2)dz_2=\frac{W_{m,n}}{\ell(\ell+1)}P^\ell_{m,n}(z_1)\quad(\ell=m,m+1,\cdots). \tag{33}$$

Proof. In view of Proposition 2 and relations ([20] (17.5)–(17.7)), we have proved that $G=K_{m,n}/W_{m,n}$ is the Green's function of the differential equation $Ly=0$ with boundary conditions (30). Therefore the equivalence

$$Ly(z)+\lambda y(z)=0\text{ with (30)}\iff\lambda\int_{-1}^1\frac{K_{m,n}(z,z')}{W_{m,n}}y(z')dz'=y(z)$$

holds (use (16.12), p. 63 and (17.14), p. 68 with $r=1$ in [20]). If $\lambda=\ell(\ell+1)$, differential equation (8) and limit conditions (24) satisfied by $P^\ell_{m,n}$ are therefore equivalent to the claimed result. □

Let us show that $K_{m,n}$ is a Hilbert–Schmidt kernel.

Proposition 4. *Assume* $2m,2n\in\mathbb{N}$, *with* m *positive and* $m-n\in\mathbb{N}$. *The bivariate kernel*

$$K_{m,n}(z_2,z_1)=K_{m,n}(z_1,z_2)=\mathfrak{Q}^0_{m,n}(z_1)\mathfrak{P}^0_{m,n}(z_2)\quad(-1<z_1\leq z_2<z_2) \tag{34}$$

satisfies

$$\int_{-1}^{1}\int_{-1}^{1} K_{m,n}^2(z_1,z_2)dz_1dz_2 < \infty. \tag{35}$$

Proof. By symmetry, and by using Fubini's theorem, we find

$$\begin{aligned} A &:= \int_{-1}^{1}\int_{-1}^{1} K_{m,n}^2(z_1,z_2)dz_1dz_2 \\ &= 2\int_{\{-1<z_1<z_2<1\}} K_{m,n}^2(z_1,z_2)dz_1dz_2 \\ &= 2\int_{\{-1<z_1<z_2<1\}} [\mathfrak{Q}_{m,n}^0(z_1)\mathfrak{P}_{m,n}^0(z_2)]^2 dz_1dz_2. \end{aligned}$$

Let us use definitions (28) and (29). On the first hand, if $-1 < z_1 < z_2 < 1$ we get

$$\frac{\left(\frac{1+z_1}{2}\right)^{\frac{m+n}{2}}}{\left(\frac{1-z_1}{2}\right)^{\frac{m-n}{2}}}\frac{\left(\frac{1-z_2}{2}\right)^{\frac{m-n}{2}}}{\left(\frac{1+z_2}{2}\right)^{\frac{m+n}{2}}} = \left(\frac{1+z_1}{1+z_2}\right)^{\frac{m+n}{2}}\cdot\left(\frac{1-z_2}{1-z_1}\right)^{\frac{m-n}{2}} < 1.$$

On the other hand we know from Lemma 4 that there exist positive constants M_1, M_2, and M_3 such that

$$\begin{aligned} |{}_2F_1(n,n+1;m+n+1;\frac{1+z_1}{2})\,{}_2F_1(-n,-n+1;m-n+1;\frac{1-z_2}{2})| \leq \\ M_1|M_2\mathbb{1}_{\{m>n\}} + M_3\mathbb{1}_{\{m=n\}}\log\frac{2}{1-z_2}]|. \end{aligned}$$

Thus

$$\begin{aligned} A &\leq 2\int_{\{-1<z_1<z_2<1\}} [M_1|M_2 + M_3\log\frac{2}{1-z_2}|]^2 dz_1dz_2 \\ &= 2\int_{-1}^{1}\left\{\int_{-1}^{z_2} dz_1\right\}[M_1|M_2 + M_3\log\frac{2}{1-z_2}|]^2 dz_2 \\ &= 2\int_{-1}^{1}(z_2+1)[M_1|M_2 + M_3\log\frac{2}{1-z_2}|]^2 dz_2 < \infty, \end{aligned}$$

and the result is established. □

Proposition 4 means that $K_{m,n}$ is of the Hilbert–Schmidt type. In view of orthogonality relations (14), functions $(P_{m,n}^{\ell})_{\ell\geq m}$ form an orthogonal set, and we know from [10] ((8), p. 161) that this set is complete. Expansion (17) follows from the classical theory of the Hilbert–Schmidt operator, see, e.g., [21] (Theorem 14, p. 63), the last equality p. 50 in [22]. Consequently, Theorem 1 is established.

A closely related K–L expansion with variables in $SU(2)$ is the following.

Corollary 1. *Assume* $2m, 2n \in \mathbb{N}$, *with* m *positive and* $m-n \in \mathbb{N}$. *The K–L expansion*

$$\begin{aligned} &4\cos[m(\phi_1-\phi_2)+n(\psi_1-\psi_2)]K_{m,n}(\cos\theta_1,\cos\theta_2) = \\ &\sum_{(\ell,p,q)\in I(m,n)} \frac{W_{m,n}}{\ell(\ell+1)}(2\ell+1)t_{p,q}^{\ell}(u_1)\overline{t_{p,q}^{\ell}(u_2)} \end{aligned} \tag{36}$$

holds, with

$$I(m,n) := \{(\ell,p,q) \in I_{SU(2)} : (p,q) = \pm(m,n)\}.$$

Proof. Set $a_\ell = W_{m,n}(\ell+1/2)/\ell(\ell+1)$. The sum on the right-hand side can be successively rewritten as

$$\sum_{(\ell,p,q)\in I(m,n)} (2a_\ell)t^\ell_{p,q}(u_1)\overline{t^\ell_{p,q}(u_2)} = 2\sum_{\ell=m}^{\infty} a_\ell[t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_2)} + t^\ell_{-m,-n}(u_1)\overline{t^\ell_{-m,-n}(u_2)}]$$
$$= 4\sum_{\ell=m}^{\infty} a_\ell \cos[m(\phi_1-\phi_2)+n(\psi_1-\psi_2)]P^\ell_{m,n}(\cos\theta_1)P^\ell_{m,n}(\cos\theta_2)$$
$$= 4\cos[m(\phi_1-\phi_2)+n(\psi_1-\psi_2)]\sum_{\ell=m}^{\infty} a_\ell P^\ell_{m,n}(\cos\theta_1)P^\ell_{m,n}(\cos\theta_2)$$
$$= 4\cos[m(\phi_1-\phi_2)+n(\psi_1-\psi_2)]K_{m,n}(\cos\theta_1,\cos\theta_2).$$

The result is proved. □

Let us examine more closely the case $n = 0$.

4. K–L Expansions Involving $t^\ell_{m,0}$ with m Fixed

In this section we derive step-by-step some K–L expansions involving various subsets of matrix elements. In the particular case $n = 0$, Theorem 1 becomes

Proposition 5. *If m is a positive integer, the K–L expansions*

$$\left[\frac{(1+z_1)(1-z_2)}{(1-z_1)(1+z_2)}\right]^{\frac{m}{2}} = \sum_{\ell=m}^{\infty}\frac{2m}{\ell(\ell+1)}\cdot(\ell+1/2)P^\ell_{m,0}(z_1)\,P^\ell_{m,0}(z_2)$$
$$(-1<z_1\le z_2<1),$$
$$\left[\frac{(1-\cos\theta_1)(1+\cos\theta_2)}{(1+\cos\theta_1)(1-\cos\theta_2)}\right]^{\frac{m}{2}} = \sum_{\ell=m}^{\infty}\frac{2m}{\ell(\ell+1)}\cdot(\ell+1/2)P^\ell_{m,0}(\cos\theta_1)\,P^\ell_{m,0}(\cos\theta_2)$$
$$(0<\theta_1\le\theta_2<\pi)$$

hold.

Proof. Note that the two K–L expansions are equivalent, since $0<\theta_1\le\theta_2<\pi$ if and only if $-1<\cos\theta_2\le\cos\theta_2<1$. Let us prove the first one. We only have to prove that $W_{m,0}=2m$. Set $\alpha=m/2$, $y_1(z)=[(1+z)/(1-z)]^\alpha$ and $y_2(z)=[(1-z)/(1+z)]^\alpha$. Equalities $y_1'=2\alpha y_1/(1-z^2)$ and $y_2'=-2\alpha y_2/(1-z^2)$ yield $y_1'y_2-y_1y_2'=4\alpha y_1y_2/(1-z^2)=2m/(1-z^2)$. Consequently $W_{m,0}=(1-z^2)[y_1'y_2-y_1y_2']=2m$, so the result is established. □

Given $u_j(\phi_j,\theta_j,\psi_j)\in SU(2)$, $j=1,2$, let us denote the kernel appearing in the preceding proposition by

$$\rho=\rho(\theta_1,\theta_2)=\rho(\theta_2,\theta_1)=\sqrt{\frac{(1-\cos\theta_1)(1+\cos\theta_2)}{(1+\cos\theta_1)(1-\cos\theta_2)}}=\frac{\tan(\theta_1/2)}{\tan(\theta_2/2)}\quad(0\le\theta_1\le\theta_2\le\pi).$$

We know from [23] (2. §1.448, p. 41) that

$$2\sum_{p=1}^{\infty}\frac{r^p}{p}\cos(p\phi)=-\log[1-2r\cos\phi+r^2],\ (|r|<1,\phi\in\mathbb{R}) \tag{37}$$

This identity, when combined with the preceding proposition, leads naturally to the following result.

Proposition 6. *Assume m is a positive integer and let*

$$I(m) := \{(\ell, pm, 0) \in I_{SU(2)} : p \in \mathbb{Z}^*\}.$$

For every $\rho \in (0,1)$, the K–L expansion

$$\sum_{(\ell,m',0)\in I(m)} \frac{1}{\ell(\ell+1)} \cdot (2\ell+1) t^{\ell}_{m',0}(u_1)\overline{t_{m',0}(u_2)} = -\frac{1}{m}\log[1 - 2\rho^m \cos(m[\phi_1 - \phi_2]) + \rho^{2m}]$$

holds.

Proof. Set $a_\ell = (2\ell+1)/[\ell(\ell+1)]$, so that Proposition 5 allows us to write, for $p > 0$,

$$\rho^{pm}(\theta_1, \theta_2) = pm \sum_{\ell=pm}^{\infty} a_\ell \cdot P^{\ell}_{pm,0}(\cos\theta_1)\, P^{\ell}_{pm,0}(\cos\theta_2) \quad (-1 < z_1 \le z_2 < 1).$$

We will write $t^{\ell}_{k,0}(\phi,\theta,\psi) = t^{\ell}_{k,0}(\phi,\theta)$, since the value does not depend upon ψ. Computations give

$$\begin{aligned}
&\sum_{(\ell,mp,0)\in I(m)} a_\ell t^{\ell}_{mp,0}(\phi_1,\theta_1)\, \overline{t^{\ell}_{mp,0}(\phi_2,\theta_2)} \\
&= \sum_{(\ell,mp,0)\in I(m)} a_\ell\, i^{mp} e^{-imp\phi_1} P^{\ell}_{mp,0}(\cos\theta_1) \overline{\left(i^{pm} e^{-ipm\phi_2} P^{\ell}_{pm,0}(\cos\theta_2)\right)} \\
&= \sum_{(\ell,mp,0)\in I(m)} a_\ell\, e^{ipm(\phi_2-\phi_1)} P^{\ell}_{pm,0}(\cos\theta_1) P^{\ell}_{pm,0}(\cos\theta_2) \\
&= \sum_{p=1}^{\infty}\sum_{\ell=pm}^{\infty} a_\ell \{e^{ipm(\phi_2-\phi_1)} P^{\ell}_{pm,0}(\cos\theta_1) P^{\ell}_{pm,0}(\cos\theta_2) \\
&\quad + e^{-ipm(\phi_2-\phi_1)} P^{\ell}_{-pm,0}(\cos\theta_1) P^{\ell}_{-pm,0}(\cos\theta_2)\} \\
&= 2\sum_{p=1}^{\infty}\sum_{\ell=pm}^{\infty} a_\ell \cos[pm(\phi_2-\phi_1)] P^{\ell}_{pm,0}(\cos\theta_1) P^{\ell}_{pm,0}(\cos\theta_2) \\
&= 2\sum_{p=1}^{\infty} \cos[pm(\phi_2-\phi_1)] \left\{ \sum_{\ell=pm}^{\infty} a_\ell P^{\ell}_{pm,0}(\cos\theta_1) P^{\ell}_{pm,0}(\cos\theta_2) \right\} \\
&= 2\sum_{p=1}^{\infty} \cos[pm(\phi_2-\phi_1)] \frac{\rho^{pm}(\theta_1,\theta_2)}{pm} = \frac{2}{m}\sum_{p=1}^{\infty} \cos[pm(\phi_2-\phi_1)] \frac{[\rho^m(\theta_1,\theta_2)]^p}{p}
\end{aligned}$$

and the result follows from (37). □

5. K–L Expansions on Submanifolds

From the point of view of Riemannian geometry, several remarkable manifolds can be seen as submanifolds of $SU(2)$.

5.1. 3-Manifolds

First, we can consider the 3-dimensional Euclidean sphere $S^3 = SU(2)$ itself. Then the quotient of $SU(2)$ with respect to the equivalence relation

$$u(\alpha,\beta) \sim -u(\alpha,\beta) = u(-\alpha,-\beta) \tag{38}$$

is the real projective space $\mathbb{P}^3 = SO(3)$. In terms of Euler angles this equivalence relation becomes $(\phi,\theta,\psi) \sim (\phi,\theta,\psi - 2\pi)$, for $0 \le \psi < 2\pi$. Endowed with the metric δ, $\mathbb{P}^3$ is called

the elliptic 3-space (see [18] (§2.2)). The fact that $\mathbb{P}^3$ can be identified with $SO(3)$ results from the Lie group morphism defined by

$$R_3 : SU(2) \to SO(3)$$
$$u(\phi,\theta,\psi) \mapsto$$
$$\begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix},$$

which is a 2-covering with kernel $\pm u(0,0,0)$. For $0 \le \psi < 2\pi$ we have $R_3u(\phi,\theta,\psi-2\pi) = R_3u(\phi,\theta,\psi)$. More generally, given two coprime positive integers $a \ge b$, consider the equivalence relation $u(\alpha,\beta) \sim u(\zeta_a\alpha, \zeta_a^b\beta)$, with $\zeta_a = e^{2i\pi/a}$. The quotient space of $SU(2)$ associated with this equivalence relation is the lense space $L(a,b)$. In particular $L(1,1)$ is $SU(2)$ itself, whereas $L(2,1)$ can be seen as $SO(3)$, in view of the equivalence relation (38).

5.2. 2-Manifolds

Another equivalence relation over $SU(2)$ can be defined in terms of Euler angles by $(\theta,\phi,\psi) \sim (\theta,\phi,0)$. The quotient space is the 2-dimensional sphere S^2, since the equivalence class of $u \in SU(2)$ can be identified with $R_3u(\phi,\theta,\psi)(0,0,1)'$, i.e., the point

$$\xi(\theta,\phi) = (\sin\phi\sin\theta, -\cos\phi\sin\theta, \cos\theta) \in S^2,$$

with geographical coordinates $(\theta, \phi - \frac{\pi}{2})$, not depending upon ψ.

Clearly, in view of definition (7) and if $(\ell,m,n) \neq (0,0,0)$, we have

$$\forall\psi \in [0,2\pi),\ t^\ell_{m,n}(\theta,\phi,\psi-2\pi) = t^\ell_{m,n}(\theta,\phi,\psi) \iff (\ell,m,n) \in I_{SO(3)},$$
$$\forall\psi \in [-2\pi,2\pi),\ t^\ell_{m,n}(\theta,\phi,\psi) = t^\ell_{m,n}(\theta,\phi,0) \iff (\ell,m,n) \in I_{S^2},$$

with

$$I_{SO(3)} := I_{SU(2)} \cap \mathbb{N}\times\mathbb{Z}\times\mathbb{Z}, \quad I_{S^2} := I_{SU(2)} \cap \mathbb{N}\times\mathbb{Z}\times\{0\}. \tag{39}$$

The K–L expansions associated with these submanifolds are the following.

Theorem 2. *Assume*

$$u_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\overline{\beta}_j & \overline{\alpha}_j \end{pmatrix}, \quad j = 1,2.$$

and let $t = t(u_1,u_2) \in [0,2\pi]$ *be defined by* $\cos\frac{t}{2} = \Re(\alpha_1\overline{\alpha_2} + \beta_1\overline{\beta_2})$. *The K–L expansions*

$$\sum_{(\ell,p,q)\in I_{SU(2)}} \frac{1}{\ell(\ell+1)} \cdot (2\ell+1) t^\ell_{p,q}(u_1)\overline{t^\ell_{p,q}(u_2)} = -1 + (2\pi - t)\cot\frac{t}{2}, \tag{40}$$

$$\sum_{(\ell,p,q)\in I_{SO(3)}} \frac{1}{\ell(\ell+1)} \cdot (2\ell+1) t^\ell_{p,q}(u_1)\overline{t^\ell_{p,q}(u_2)} = -1 + (\pi - t)\cot\frac{t}{2}, \tag{41}$$

$$\sum_{(\ell,m,0)\in I_{S^2}} \frac{1}{\ell(\ell+1)} \cdot (2\ell+1) t^\ell_{m,0}(u_1)\overline{t^\ell_{m,0}(u_2)} = -\log[1 - 2\rho\cos(\phi_1-\phi_2) + \rho^2] \tag{42}$$

are valid.

Proof. In virtue of (27) we can write

$$G(u_1,u_2) := \sum_{(\ell,p,q)\in I_{SU(2)}} \frac{1}{\ell(\ell+1)}\cdot(2\ell+1)t^{\ell}_{p,q}(u_1)\overline{t^{\ell}_{p,q}(u_2)} = \sum_{\ell\in\frac{\mathbb{N}^*}{2}} \frac{(2\ell+1)\chi_\ell(u_1u_2^*)}{\ell(\ell+1)}.$$

The first equality then follows from Lemma 8, and in view of (20).

Now assume $0 \le \psi < 2\pi$. We have $u(\phi,\theta,\psi-2\pi) = -u(\phi,\theta,\psi)$. Furthermore it is easily seen from definitions (7) and (39) that $t^{\ell}_{p,q}(\phi,\theta,\psi-2\pi) = \pm t^{\ell}_{p,q}(\phi,\theta,\psi-2\pi)$, according to $(\ell,p,q)\in I_{S0(3)}$ or $(\ell,p,q)\in I_{SU(2)}\setminus I_{S0(3)}$. Therefore

$$2\sum_{(\ell,p,q)\in I_{SO(3)}} t^{\ell}_{p,q}(u_1)\overline{t^{\ell}_{p,q}(u_2)} = G(u_1,u_2)+G(u_1,-u_2)$$

On the other hand, if

$$\cos\frac{t(u_1,u_2)}{2} = \Re(\alpha_1\overline{\alpha_2}+\beta_1\overline{\beta_2}),$$

we have

$$\cos\frac{t(u_1,-u_2)}{2} = \Re(\alpha_1\overline{(-\alpha_2)}+\beta_1\overline{(-\beta_2)}) = -\cos\frac{t(u_1,u_2)}{2} = \cos\frac{2\pi-t(u_1,u_2)}{2}.$$

Thus $t(\phi,\theta,\psi-2\pi) = 2\pi - t(\phi,\theta,\psi)$. Therefore we obtain

$$\begin{aligned} G(u_1,u_2)+G(u_1,-u_2) &\\ &= -2+(2\pi-t)\cot\frac{t}{2}+(2\pi-[2\pi-t])\cot\frac{2\pi-t}{2}\\ &= -2+(2\pi-t)\cot\frac{t}{2}-t\cot\frac{t}{2} = -2+2(\pi-t)\cot\frac{t}{2}, \end{aligned}$$

and the second equality follows. The third equality is the particular case $m=1$ of Proposition 6. □

Our proof of the preceding proposition used the fact that the kernel on the right-hand side of (41) can be obtained from the kernel in (40), say G, via the average $[G(u_1,u_2)+G(u_1,-u_2)]/2$. The second variable takes all possible values in the equivalence class of u_2 associated with relation (38).

Let us apply this averaging process to the case of the lens space $L(a,b)$. To this end set

$$G_{L(a,b)}(u_1(\alpha_1,\beta_1),u_2(\alpha_2,\beta_2)) = \frac{1}{a}\sum_{j=0}^{a-1} G\Big(u_1(\alpha_1,\beta_1),u_2(\zeta_a^j\alpha_2,\zeta_a^{bj}\beta_2)\Big), \tag{43}$$

$$I_{L(a,b)} = \{(\ell,m,n)\in I_{SU(2)} : m+n+b(m-n)\in a\mathbb{Z}\}. \tag{44}$$

It will be shown in Lemma 2 that

$$\frac{1}{a}\sum_{j=0}^{a-1} t^{\ell}_{m,n}(\alpha\zeta_a^j,\beta\zeta_a^{bj}) = \begin{cases} t^{\ell}_{m,n}(\alpha,\beta) & \text{if } (\ell,m,n)\in I_{L(a,b)},\\ 0 & \text{otherwise.}\end{cases} \tag{45}$$

This equality illustrates the fact that the functions $t^{\ell}_{m,n}$ behave particularly nicely with respect to operators associated with subgroup structures. Relations (40), (43) and (45) evidently yield the following K–L expansion.

Proposition 7. *With notations as above, we have*

$$G_{L(a,b)}(u_1,u_2)=\sum_{(\ell,p,q)\in I_{L(a,b)}}\frac{1}{\ell(\ell+1)}\cdot(2\ell+1)t^{\ell}_{p,q}(u_1)\overline{t^{\ell}_{p,q}(u_2)}, \tag{46}$$

with convergence in the quadratic mean.

Let us now see how these kernels can be used as kernels of specific $U-$ or $V-$statistics.

6. Some Goodness-of-Fit Tests

By a probability density function (pdf) we mean a summable real valued function f defined over $SU(2)$ and satisfying

$$f(u)\geq 0,\quad \int_{SU(2)}f(u)du=1,\quad \int_{SU(2)}f^2(u)du<\infty.$$

For instance the constant function $f(u)=t^0_{0,0}(u)=1$ is the pdf of the uniform distribution. Keeping in mind that

$$\int_{SU(2)}t^{\ell}_{m,n}(u)du=\int_{SU(2)}t^{\ell}_{m,n}(u)\overline{t^0_{0,0}(u)}du=\mathbb{1}_{\{(\ell,m,n)=(0,0,0)\}},$$

every pdf can be expanded into the series

$$f(u)=1+\sum_{(\ell,m,n)\in I_{SU(2)}}a^{\ell}_{m,n}\sqrt{2\ell+1}t^{\ell}_{m,n}(u), \tag{47}$$

$$\text{with}\quad a^{\ell}_{m,n}=\sqrt{2\ell+1}\int_{SU(2)}f(u)\overline{t^{\ell}_{m,n}(u)}du, \tag{48}$$

the convergence being in quadratic mean (use [10] ((1)–(2) p.163) with $\alpha^{\ell}_{m,n}=\sqrt{2\ell+1}a^{\ell}_{m,n}$). In this setting the Parseval equality [10] (III.6.2. (6)) takes the simple form

$$\int_{SU(2)}|f(u)|^2\,du=1+\sum_{(\ell,m,n)\in I_{SU(2)}}|a^{\ell}_{m,n}|^2. \tag{49}$$

Given a pdf f, we will write $f\in D(I)$ for $I\subset I_{SU(2)}$, if the Fourier expansion of f is of the form

$$f(\phi,\theta,\psi)=1+\sum_{(\ell,p,q)\in I}a^{\ell}_{p,q}\,\sqrt{2\ell+1}\,t^{\ell}_{p,q}(\phi,\theta,\psi).$$

For instance $D(I_{SU(2)})$ is the set of all pdf, whereas $D(\emptyset)$ reduces to the uniform distribution. Since a pdf is a real function, and keeping in mind that $\overline{t^{\ell}_{m,n}}=t^{\ell}_{-m,-n}$, we will consider index sets I satisfying $(\ell,m,n)\in I\iff(\ell,-m,-n)\in I$. Given such an index set I, consider the real valued symmetric kernel defined by the K–L expansion

$$G(I;u_1,u_2):=\sum_{(\ell,p,q)\in I}\frac{1}{\ell(\ell+1)}\cdot(2\ell+1)t^{\ell}_{p,q}(u_1)\,\overline{t^{\ell}_{p,q}(u_2)}. \tag{50}$$

For instance if $I=I_{SU(2)}$, the first K–L expansion of Theorem 2 can be rewritten as

$$G(I_{SU(2)};u_1,u_2)=-1+[2\pi-t(u_1u_2^*)]\cot\frac{t(u_1u_2^*)}{2}.$$

Now suppose that two distinct index sets J_0 and J_1 satisfy $J_0 \subset J_1$. Consider the statistical hypotheses

$$H_0 : f \in D(J_0),$$
$$H_1 : f \in D(J_1) \setminus D(J_0).$$

Throughout this Section, $\{C_{p,q}^{\ell} : (\ell, m, n) \in I_{SU(2)}\}$ will denote a set of independent chi-square random variables with one degree of freedom. Recall that the characteristic function of $C_{p,q}^{\ell}$ is

$$E(e^{itC_{p,q}^{\ell}}) = (1 - 2it)^{-1/2} \quad (t \in \mathbb{R}).$$

We will use repeatedly the closely connected relation

$$E\big(\exp\{it\,\frac{C_{p,q}^{\ell} - 1}{\lambda}\}\big) = (1 - \frac{2it}{\lambda})^{-1/2} e^{-it/\lambda} \quad (t \in \mathbb{R}). \tag{51}$$

for $\lambda > 0$.

Theorem 3. *Assume $u_1, u_2, \cdots$ is a sequence of independent random matrices from $SU(2)$, following the same distribution with pdf f given by (47). Consider the $U-$statistic*

$$U_n = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} [G(J_1; u_i, u_j) - G(J_0; u_i, u_j)] \tag{52}$$

Under H_0, we have the convergence in distribution

$$\lim_{n \to \infty} (nU_n) = \sum_{(\ell,p,q) \in J_1 \setminus J_0} \frac{C_{p,q}^{\ell} - 1}{\ell(\ell+1)}, \tag{53}$$

whereas under H_1,

$$\lim_{n \to \infty} U_n = \sum_{(p,q,\ell) \in J_1 \setminus J_0} \frac{|a_{p,q}^{\ell}|^2}{\ell(\ell+1)}.$$

Proof. Set $b_\ell = (2\ell+1)/[\ell(\ell+1)]$. In order to apply [14] (Theorem 4.3.1, p. 138), let us check that assumptions (4.3.2) and (4.3.5) from this reference are satisfied. Note that

$$\sum_{(\ell,p,q) \in I_{SU(2)}} \frac{1}{[\ell(\ell+1)]^2} = \sum_{\ell \in \mathbb{N}^*/2}^{\infty} \sum_{p=-\ell}^{\ell} \sum_{q=-\ell}^{\ell} \frac{1}{[\ell(\ell+1)]^2} = \sum_{\ell \in \mathbb{N}^*/2}^{\infty} \frac{(2\ell+1)^2}{[\ell(\ell+1)]^2} < \infty.$$

Therefore, $G(J_0; ., .)$ and $G(J_1; ., .)$ are square-summable. It implies that for almost every $u_1 \in SU(2)$, the functions $G(J_0; u_1, .)$ and $G(J_1; u_1, .)$ are, as $f(.)$, square-summable. Consequently, for almost every $u_1 \in SU(2)$, we can interchange the summation and integration change to obtain

$$\int_{SU(2)} [G(J_1; u_1, u_2) - G(J_0; u_1, u_2)] f(u_2) du_2 =$$
$$\int_{SU(2)} \{ \sum_{(\ell,p,q) \in J_1 \setminus J_0} b_\ell t_{p,q}^{\ell}(u_1) \overline{t_{p,q}^{\ell}(u_2)} \} \{1 + \sum_{(\ell',p',q') \in J_0} a_{p',q'}^{\ell'} \sqrt{2\ell'+1}\, t_{p',q'}^{\ell'}(u_2) \}\, du_2 = 0. \tag{54}$$

In other words, under H_0, the statistic U_n is degenerate with respect to f. Orthogonality relations (22) imply the claimed result under H_0. Under H_1, keeping in mind that

$$b_\ell \int_{SU(2)} t_{p,q}^{\ell}(u) \overline{t_{p',q'}^{\ell'}(u)} du = \frac{\mathbb{1}_{\{\ell=\ell', p=p', q=q'\}}}{\ell(\ell+1)},$$

the law of large numbers for $U-$statistics yields

$$\lim U_n = \iint\limits_{SU(2)\times SU(2)} [G(J_1;u_1,u_2) - G(J_0;u_1,u_2)]f(u_1)f(u_2)du_1du_2 =$$
$$\iint\limits_{SU(2)\times SU(2)} \{\sum_{(k,m,n)\in J_1\setminus J_0} b_k t^k_{m,n}(u_1)\,\overline{t^k_{m,n}(u_2)}\}\} \times$$
$$\overline{\{\sum_{(\ell,p,q)\in J_1} \sqrt{2\ell+1}\,a^\ell_{p,q}t^\ell_{p,q}(u_1)\}}\{\sum_{(\ell',p',q')\in J_1} \sqrt{2\ell'+1}\,a^{\ell'}_{p',q'}t^{\ell'}_{p',q'}(u_2)\}\,du_1\,du_2$$
$$= \sum_{(k,m,n)\in J_1\setminus J_0} (a^k_{m,n})^2 b_k \int_{SU(2)} t^k_{m,n}(u_1)\overline{t^k_{m,n}(u_1)}du_1 \times (2k+1)\int_{SU(2)} t^k_{m,n}(u_2)\overline{t^k_{m,n}(u_2)}du_2$$
$$= \sum_{(k,m,n)\in J_1\setminus J_0} \frac{|a^\ell_{m,n}|^2}{\ell(\ell+1)}.$$

The result is established. □

The preceding theorem means that the statistical test rejecting H_0 for large values of U_n is consistent.

Example 1. *Assume we wish to test uniformity over $SU(2)$ against any alternative, i.e.,*

$$H_0 : f = 1,$$
$$H_1 : f \neq 1.$$

We have $J_0 = \emptyset$ and $J_1 = I_{SU(2)}$. In view of (40), the kernel of the $U-$statistic is

$$G(u_1,u_2) = -1 + [2\pi - t(u_1u_2^*)]\cot\frac{t(u_1u_2^*)}{2}.$$

One has $(\ell,m,n) \in J_1 \setminus J_0$ if and only if $\ell = (k-1)/2$ for some integer $k > 1$. In this case the eigenvalue $1/[\ell(\ell+1)] = 4/(k^2-1)$ has multiplicity $2\ell+1 = k$. Therefore, in view of (51), the characteristic functions of the limiting random variable is given by

$$t \mapsto \prod_{k=2}^{\infty} \left[(1 - \frac{8it}{k^2-1})^{-1/2} e^{-4it/(k^2-1)}\right]^k.$$

Example 2. *Assume we wish to test whether f is a pdf defined over $S0(3)$, in other words $f(-u) = f(u)$. In terms of Euler angles our hypotheses are*

$$H_0 : f(\phi,\theta,\cdot) = (\phi,\theta,\cdot - 2\pi);$$
$$H_1 : f(\phi,\theta,\cdot) \neq (\phi,\theta,\cdot - 2\pi)$$

We have $J_0 = I_{SO(3)}$ and $J_1 = I_{SU(2)}$. In view of (40) and (41), the kernel of the $U-$statistic is given by

$$G(u_1,u_2) = \{-1 + [2\pi - t(u_1u_2^*)]\cot\frac{t(u_1u_2^*)}{2}\} - \{-1 + [\pi - t(u_1u_2^*)]\cot\frac{t(u_1u_2^*)}{2}\}$$
$$= \pi\cot\frac{t(u_1u_2^*)}{2}$$

One has $(\ell, m, n) \in J_1 \setminus J_0$ *if and only if* $\ell = (2k-1)/2$ *for some* $k \in \mathbb{N}^*$. *In this case the eigenvalue* $1/[\ell(\ell+1)] = 4/(4k^2-1)$ *has multiplicity* $2\ell+1 = 2k$. *Therefore, the characteristic functions of the limiting random variable are*

$$t \mapsto \prod_{k=2}^{\infty} \left[\left(1 - \frac{8it}{4k^2-1}\right)^{-1/2} e^{-4it/(4k^2-1)} \right]^{2k}.$$

Example 3. *Assume we wish to test whether* f *is a pdf defined over* $L(a,b)$. *We have* $J_0 = I_{L(a,b)}$ *and* $J_1 = I_{SU(2)}$. *The kernel of the* $U-$*statistic is given by*

$$G(u_1, u_2) = G(I_{SU(2)}; u_1, u_2) - G(I_{L(a,b)}; u_1, u_2).$$

7. Implementation and Efficiency of the New Tests

So far our results remain at a formal level. How should a practitioner implement our new tests? What about their statistical efficiency? A full discussion would be beyond the scope of the present paper. Let us just outline some properties of the statistics introduced above.

The first issue we have to mention is the difficulty of deriving percentiles of the random variable on the right-hand side (53). To our knowledge, no significant progress has been made since the article [24]. Four methods are described in [15] (§5.2, pp. 212–213). In the general case, only simulations allow one to obtain tables. Explicit formulas are available for the limit of the Anderson–Darling statistic, see [15] (Exercise 2, p. 227) and [25] (§4.5.2, p. 105 and Table 4.3, p. 106) for its use with finite samples.

The second issue is the statistical efficiency of our tests. Instead of simulations for particular cases, we propose general results related to the concept of Bahadur efficiency. A reference about this subject is [26], in which Bahadur, Hodges–Lehmann, and Chernoff efficiencies of non-parametric tests are discussed. Bahadur efficiency is a particularly well suited tool for our statistics, since orthogonal expansions associated with K–L expansions prove useful in this field. In directional statistics, a prototypical example illustrating this fact is provided by Watson's statistic U_n^2, defined by

$$nU_n^2 := \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, x_j) = \int_0^2 [n^{-1/2} \sum_{i=1}^{n} Y(x_i, t)]^2 dt$$

with

$$\begin{aligned} Y(x,t) &= \mathbb{1}_{\{x<t\}} - t + x - \frac{1}{2} \quad (0 < x, t < 1), \\ K(x,y) &= \int_0^1 Y(x,t) Y(y,t) dt \\ &= \min(x,y) - xy + \frac{1}{2}(x^2 + y^2) - \frac{1}{2}(x+y) + \frac{1}{12} \\ &= \sum_{k=1}^{\infty} \frac{2\cos(2\pi x)\cos(2\pi y) + 2\sin(2\pi x)\sin(2\pi y)}{4k^2\pi^2}. \end{aligned}$$

In this K–L expansion, the maximal eigenvalue is

$$\lambda_0 := \max\{(4k^2\pi^2)^{-1} : k = 1, 2, \ldots\} = (4\pi^2)^{-1}.$$

In view of this value it is possible to prove the large-deviation relation

$$\lim_{n\to\infty} n^{-1}\log[P(U_n^2 \geq a)] = -\frac{a}{2\lambda_0} + o(a) \quad (a \to a) \tag{55}$$

(see [26] (Theorem 2.5.3, p. 70)). The proof is based upon the Sethuraman-Rao theorem restated in [26] (Theorem 1.6.3, p. 23). We can use this theorem thanks to the inequality

$$\int_0^1 Y^2(x,t)dt = \frac{1}{12} < \infty,$$

ensuring that assumption (1.6.9) of [26] (Theorem 1.6.3, p. 23) is satisfied. This large-deviation type result is the most difficult to check in Bahadur's fundamental theorem. About this topic see the comments following its restatement in [26] (Theorem 1.2.2, pp 6–7).

Once large deviations under the null hypothesis are available, it is easy to compute the Bahadur exact slope against a given alternative.

Let us determine for what sequences of eigenvalues inequality (1.6.9) of [26] (Theorem 1.6.3, p. 23) can be expected to be satisfied.

Proposition 8. *Assume* $\{\lambda_\ell : \ell = \frac{1}{2}, 1, \ldots\}$ *is a non-negative sequence. The series*

$$H(u,v) := \sum_{\ell\in\mathbb{N}^*/2} \sqrt{\lambda_\ell} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} (2\ell+1) t_{m,n}^{\ell}(u)\, \overline{t_{m,n}^{\ell}(v)} \quad u,v \in SU(2) \tag{56}$$

converges in quadratic mean over $SU(2)\times SU(2)$ *if and only if*

$$\sum_{\ell} (2\ell+1)^2 \lambda_\ell < \infty. \tag{57}$$

In the case when (56) *and* (57) *hold, suppose furthermore that*

$$\int_{SU(2)} |H(u,v)|^2 dv \leq M^2 \quad u \in SU(2) \tag{58}$$

for some $M > 0$. *Then the series*

$$K(u,v) := \sum_{\ell\in\mathbb{N}^*/2} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \lambda_\ell (2\ell+1) t_{m,n}^{\ell}(u)\, \overline{t_{m,n}^{\ell}(v)} \quad u,v \in SU(2), \tag{59}$$

converges in quadratic mean over $SU(2)\times SU(2)$ *and satisfies*

$$K(u,v) = \int_{SU(2)} H(u,w)H(w,v)dw \quad u,v \in SU(2). \tag{60}$$

Furthermore, given the sequence $u_1, u_2, \ldots \in SU(2)$, *we have*

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} K(u_i,u_j) = \int_{SU(2)} |\, n^{-1} \sum_{i=1}^{n} H(u_i,u)\,|^2\, du \leq M^2, \tag{61}$$

for every positive integer n.

Proof. Note that H is real-valued. In (56), the eigenvalue $\lambda_\ell^{1/2}$ occurs $(2\ell+1)^2$ times. Therefore, the equivalence between (56) and (57) is a direct consequence of orthogonality relations (22), which imply that given $\ell_0 \leq \ell_1$, we have

$$\iint_{SU(2)\times SU(2)} \left| \sum_{\ell_0\le\ell\le\ell_1} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \lambda_\ell^{1/2}(2\ell+1) t_{m,n}^{\ell}(u)\, \overline{t_{m,n}^{\ell}(v)} \right|^2 du dv = \sum_{\ell_0\le\ell\le\ell_1} (2\ell+1)^2 \lambda_\ell.$$

The same orthogonality relations imply that

$$\begin{aligned} &\sum_{\ell_0\le\ell\le\ell_1} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \lambda_\ell (2\ell+1) t_{m,n}^{\ell}(u)\, \overline{t_{m,n}^{\ell}(v)} \\ &\quad = \int_{SU(2)} \Big[\sum_{\ell_0\le\ell\le\ell_1} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \lambda_\ell^{1/2}(2\ell+1) t_{m,n}^{\ell}(u)\, \overline{t_{m,n}^{\ell}(w)} \Big] \\ &\qquad \times \Big[\sum_{\ell_0\le\ell'\le\ell_1} \sum_{m'=-\ell'}^{\ell'} \sum_{n'=-\ell}^{\ell'} \lambda_{\ell'}^{1/2}(2\ell'+1) t_{m',n'}^{\ell'}(w)\, \overline{t_{m',n'}^{\ell'}(v)} \Big] dw. \end{aligned}$$

Relations (59) and (60) then readily follow from (56). Then (60) allows us to write

$$\sum_{i=1}^{n}\sum_{j=1}^{n} K(u_i,u_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{SU(2)} H(u_i,w)H(u_j,w)dw = \int_{SU(2)} |\sum_{i=1}^{n} H(u_i,w)|^2 dw,$$

from which follows the equality in (61). Assumption (58) and the triangle inequality for the L^2 norms yield

$$\left[\int_{SU(2)} |\sum_{i=1}^{n} H(u_i,u)|^2\, du\right]^{1/2} \le \sum_{i=1}^{n} \left[\int_{SU(2)} |H(u_i,u)|^2 du\right]^{1/2} \le nM,$$

from which follows the inequality in (61). □

We are now equipped to deal with the problem of testing the hypothesis

$$H_0 : f(u) = 1,$$

against the alternative

$$H_1(a) : f(u) = 1 + a \sum_{(\ell,m,n)\in I_{SU(2)}} a_{m,n}^{\ell} \sqrt{2\ell+1}\, t_{m,n}^{\ell}(u),$$

where a is some positive real number.

Theorem 4. *Assume assumptions of Proposition 8 hold. Consider the statistic*

$$T_n = \left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K(u_i,u_j)\right]^{1/2}, \tag{62}$$

If $H_1(a)$ obtains, we have the almost sure convergence

$$\lim_{n\to\infty} n^{-1/2} T_n = a \left(\sum_{(\ell,p,q)\in I_{SU(2)}} \lambda_\ell |a_{p,q}^{\ell}|^2 \right)^{1/2}. \tag{63}$$

Under the null hypothesis of uniformity, we have the large deviation identity

$$\lim_{n\to\infty} n^{-1} \log[P(T_n \ge \sqrt{n}t)] = -\frac{t^2}{2\lambda_0} + o(t^2) \quad (t\to 0), \tag{64}$$

with

$$\lambda_0 := \max\{\lambda_\ell : \ell \in \mathbb{N}^*/2\}.$$

Consequently, the Bahadur exact slope satisfies

$$c(a) = \frac{a^2}{\lambda_0} \sum_{(\ell,m,n)\in I_{SU(2)}} \lambda_\ell |a^\ell_{m,n}|^2 + o(a^2).$$

Proof. The limit under $H_1(a)$ is a restatement of the last equality in Theorem 3. The proof of the large deviation relation under H_0 is the same as that of (55), where $n^{-1/2}T_n$ plays the role of U_n, a being replaced by t^2. We refer to the proof of [26] (Theorem 2.5.3, p. 70). The expression for the slope is given by Bahadur's formula

$$c(a) = 2f(b(a)),$$

with $f(t) = t^2/(2\lambda_0) + o(t^2)$ and $b(a)^2 = a^2 \sum_{(\ell,p,q)\in I_{SU(2)}} \lambda_\ell |a^\ell_{p,q}|^2$, see [27] (Theorem 7.2) or its restatement [26] (Theorem 1.1.2, p. 6). □

Define for two pdfs, say f_1 and f_2, the Kullback–Leibler information number

$$K(f_1, f_2) := \int_{SU(2)} \log \frac{f_1(x)}{f_2(x)} f_1(x)dx$$

(see [26] ((1.2.9), p. 8)). A fundamental result in Bahadur's theory is the existence of an upper bound for the exact slope, in terms of the Kullback–Leibler information number. See [26] ((1.2.15)–(1.2.16), p. 9)) for definitions of *asymptotic optimality*, or *local asymptotic optimality* in the Bahadur sense. A corollary of the preceding theorem is as follows.

Corollary 2. *Assume assumptions of Proposition 8 hold. With notations as above, the V−statistic V_n is locally asymptotically optimal in the Bahadur sense if and only if*

$$\sum_{(\ell,m,n)\neq 0} \frac{\lambda_\ell}{\lambda_0} |a^\ell_{m,n}|^2 = \sum_{(\ell,m,n)\neq 0} |a^\ell_{m,n}|^2.$$

8. Useful Technical Results

8.1. Kullback–Leibler Information Number

Lemma 1. *Suppose that for every a belonging to some neighborhood of zero, $u \mapsto f_a(u) := 1 + a\phi(u)$ is a pdf over $SU(2)$. As $a \to 0$, the Kullback–Leibler information number satisfies*

$$K(f_a, f_0) = \int_{SU(2)} f_a(u) \log[f_a(u)]\, du = \frac{a^2}{2} \int_{SU(2)} \phi(u)^2 du + o(a^2).$$

Proof. Taking into account

$$\int \phi(u)du = 0,$$

elementary calculus gives

$$\begin{aligned}\int [1 + a\phi(u)] \log[1 + a\phi(u)]\, du &= \int [1 + a\phi(u)][a\phi(u) - \frac{[a\phi(u)]^2}{2}]\, du + o(a^2) \\ &= \int [a\phi(u) + \frac{[a\phi(u)]^2}{2}]\, du + o(a^2) = \frac{a^2}{2} \int \phi(u)^2 du + o(a^2).\end{aligned}$$

□

8.2. Lens Spaces

Recall that given the two positive coprime integers a and b, we use notations $\zeta_a = e^{2i\pi/a}$ and $I_{L(a,b)} = \{(\ell, m, n) \in I_{SU(2)} : m + n + b(m - n) \in a\mathbb{Z}\}$.

Lemma 2. *Assume $u = (\alpha, \beta) \in SU(2)$. We have*

$$t^{\ell}_{m,n}(\alpha\zeta_a, \beta\zeta_a^b) = \zeta_a^{-m-n-b(m-n)} t^{\ell}_{m,n}(\alpha, \beta).$$

Consequently,

$$\frac{1}{a}\sum_{k=0}^{a-1} t^{\ell}_{m,n}(\alpha\zeta_a^k, \beta\zeta_a^{bk}) = \begin{cases} t^{\ell}_{m,n}(\alpha, \beta) & \text{if } (\ell, m, n) \in I(a, b), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will use definitions (18) and (19). Put $g = u(\alpha, \beta)$, with Euler angles (ψ, θ, ϕ). Write $g' = u(\alpha', \beta')$ with $(\alpha', \beta') = (\alpha\zeta_a, \beta\zeta_a^b)$. The Euler angles of g', say (ϕ', θ', ψ'), satisfy the following relations. Firstly,

$$\cos\theta' = 2|\alpha'|^2 - 1 = 2|\alpha|^2 - 1 = \cos\theta,$$

so that $\theta = \theta'$. Secondly,

$$e^{i\phi'} = -\frac{i\alpha'\beta'}{|\alpha'\beta'|} = -\frac{i\alpha\beta}{|\alpha\beta|} e^{2i\pi/a} e^{2ib\pi/a} = e^{i\phi} e^{2i\pi(1+b)/a}.$$

Thirdly,

$$\begin{aligned} e^{i\psi'/2} &= \frac{\alpha' e^{-i\phi'/2}}{|\alpha'|} = \frac{\alpha e^{2i\pi/a} e^{-i\phi/2} e^{-i\pi(1+b)/a}}{|\alpha|} \\ &= e^{i\pi(1-b)/a} \frac{\alpha e^{-i\phi/2}}{|\alpha|} = e^{i\pi(1-b)/a} e^{i\psi/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} t^{\ell}_{m,n}(g') &= i^{m-n} e^{-im\phi'} e^{-in\psi'} P^{\ell}_{m,n}(\cos\theta') \\ &= i^{m-n} [e^{i\phi} e^{2i\pi(1+b)/a}]^{-m} [e^{2i\pi(1-b)/a} e^{i\psi}]^{-n} P^{\ell}_{m,n}(\cos\theta) \\ &= i^{m-n} e^{-im\phi} e^{-in\psi} P^{\ell}_{m,n}(\cos\theta) \exp\{-2i\pi \frac{m(1+b) + n(1-b)}{a}\} \\ &= t^{\ell}_{m,n}(g) \exp\{2i\pi \frac{-m-n-b(m-n)}{a}\}. \end{aligned}$$

The first equality is proved. From it, and after a simple recursion, we infer that

$$\frac{1}{a}\sum_{k=0}^{a-1} t^{\ell}_{m,n}(\alpha\zeta_a^k, \beta\zeta_a^{bk}) = \frac{t^{\ell}_{m,n}(\alpha, \beta)}{a} \sum_{k=0}^{a-1} [\zeta_a^{-m-n-b(m-n)}]^k.$$

Next, we know that $m + n + b(m - n)$ is an integer, and that given the integer k, $\zeta_a^k = 1 \iff k \in a\mathbb{Z}$. Therefore, if $m + n + b(m - n) \in a\mathbb{Z}$, we have

$$\sum_{k=0}^{a-1} [\zeta_a^{-m-n-b(m-n)}]^k = \sum_{k=0}^{a-1} 1^k = a.$$

If $m+n+b(m-n) \notin a\mathbb{Z}$, we obtain

$$\sum_{k=0}^{a-1} [\zeta_a^{-m-n-b(m-n)}]^k = \frac{\zeta_a^{[-m-n-b(m-n)]a} - 1}{\zeta_a^{-m-n-b(m-n)} - 1} = \frac{1-1}{\zeta_a^{-m-n-b(m-n)} - 1} = 0.$$

The second equality readily follows. □

8.3. K–L Expansion of Anderson–Darling Covariance

Lemma 3. *Assume* $z_1 = 2s-1$ *and* $z_2 = 2t-1$ *with* $0 < s \leq t < 1$. *Expressions appearing in* (6) *can be rewritten as*

$$\left(\frac{s-st}{\sqrt{s(1-s)t(1-t)}} \right)^{1/2} = \left(\frac{(1+z_1)(1-z_2)}{(1-z_1)(1+z_2} \right)^{1/2}$$

and

$$\frac{4(2\ell+1)}{\ell(\ell+1)} \cdot \sqrt{s(1-s)} P'_\ell(2s-1) \cdot \sqrt{t(1-t)} P'_\ell(2t-1) = (2\ell+1) P^\ell_{1,0}(z_1) P^\ell_{1,0}(z_2).$$

Consequently, developments (6) *and* (13) *are equivalent.*

Proof. On the first hand,

$$\left(\frac{s-st}{\sqrt{s(1-s)t(1-t)}} \right)^{1/2} = \left(\frac{s(1-t)}{t(1-s)} \right)^{1/2}$$

and the first equality readily follows from relations $2s = 1+z_1, 2(1-s) = 1-z_1, 2(1-t) = 1-z_2$, and $2t = 1-z_1$. On the other hand in virtue of relations (11), by substitution we obtain

$$\begin{aligned} \frac{4(2\ell+1)}{\ell(\ell+1)} \cdot \sqrt{s(1-s)} P'_\ell(2s-1) \cdot \sqrt{t(1-t)} P'_\ell(2t-1) = \\ \frac{4(2\ell+1)}{\ell(\ell+1)} \cdot \frac{\sqrt{1-z_1^2}}{2} \frac{\ell+1}{2} P^{(1,1)}_{\ell-1}(z_1) \cdot \frac{\sqrt{1-z_2^2}}{2} \frac{\ell+1}{2} P^{(1,1)}_{\ell-1}(z_2) \\ = \frac{(2\ell+1)(\ell+1)}{4\ell} \cdot \sqrt{1-z_1^2} P^{(1,1)}_{\ell-1}(z_1) \cdot \sqrt{1-z_2^2} P^{(1,1)}_{\ell-1}(z_2) \\ = (2\ell+1) P^\ell_{1,0}(z_1) P^\ell_{1,0}(z_2). \end{aligned}$$

The result easily follows. □

8.4. Some Properties of Hypergeometric Functions

Recall that given the three complex numbers a, b and c, the hypergeometric function

$$z \mapsto_2 F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

is an entire function over the open disc $|z| < 1$, the solution of the differential equation

$$x(1-x)y''(x) + [c-(a+b+1)x]y'(x) - aby(x) = 0 \quad (0 < x < 1).$$

Lemma 4. *Assume* $2m \in \mathbb{N}^*$, $2n \in \mathbb{Z}$ *and* $m-n \in \mathbb{N}$. *There exist positive constants* M_1, M_2, *and* M_3 *such that*

$$|{}_2F_1(-n,-n+1;m-n+1;\frac{1-z}{2})| \le M_1 \quad (-1 \le z \le 1), \tag{65}$$

$$0 \le {}_2F_1(n,n+1;m+n+1;\frac{1+z}{2}) \le M_2 \mathbb{1}_{\{m>n\}} + M_3 \mathbb{1}_{\{m=n\}} \log \frac{2}{1-z} \quad (-1 \le z < 1). \tag{66}$$

Proof. We know from [16] (15.1.1 (b)), that if $a+b-c<0$, function $z \mapsto_2 F_1(a,b;c;z)$ is continuous over the disk $[z| \le 1$. For the function on the left-hand side of (65), we find $a+b-c = -n-n+1-(m-n+1) = -m-n \le -m < 0$. Thus, $z \mapsto_2 F_1(-n,-n+1;m-n+1;z)$ is continuous over $[-1,1]$, and $z \mapsto_2 F_1(-n,-n+1;m-n+1;[1-z]/2)$ is also continuous over $[-1,1]$. The first inequality readily follows.

The hypergeometric function in (66) is clearly nonnegative. Furthermore we find $a+b-c = n+n+1-(m+n+1) = n-m$. If $n<m$, we conclude as in the preceding case that there exists $M_2>0$ such that ${}_2F_1(n,n+1;m+n+1;\frac{1+z}{2}) \le M_2$. If $m=n$, in view of the equality $(a)_k = \Gamma(k+a)/\Gamma(a)$, this function satisfies, for $0 \le z < 1$,

$$\begin{aligned} 0 \le {}_2F_1(m,m+1;2m+1;z) &= \sum_{k=0}^{\infty} \frac{(m)_k (m+1)_k}{(2m+1)_k} \cdot \frac{z^k}{k!} \\ &= \frac{\Gamma(2m+1)}{\Gamma(m)\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+m)\Gamma(k+m+1)}{\Gamma(k+2m+1)} \cdot \frac{z^k}{\Gamma(k+1)}. \end{aligned}$$

As $k \to \infty$, we infer from [16] (6.1.46, p. 247) the equivalence

$$\frac{\Gamma(k+m)\Gamma(k+m+1)}{\Gamma(k+2m+1)\Gamma(k+1)} \sim k^{(m+m+1)-(2m+1+1)} = k^{-1}.$$

Consequently there exists $M_3>0$ such that

$${}_2F_1(m,m+1;m+n+1;z) \le M_3 \sum_{k=1}^{\infty} \frac{z^k}{k} = M_3 \log \frac{1}{1-z} \quad (0 \le z < 1).$$

Now, $-1 \le z < 1$ is equivalent to $0 \le (1+z)/2 < 1$, and (66) is established. □

8.5. Some Differential Equations

Lemma 5. *If w is a solution of the hypergeometric differential equation*

$$x(1-x)w'' + [c-(a+b+1)x]w' - abw = 0 \quad (0<x<1),$$

then the function $y = wx^{-\alpha}(1-x)^{-\beta}$ satisfies the differential equation

$$x(1-x)y'' + (2\alpha + c - [a+b+1+2\alpha+2\beta]x)y' + \frac{Ax^2 - Bx + C}{x(1-x)} = 0,$$

with

$$\begin{aligned} A &= \alpha(\alpha+1) + \beta(\beta+1) + 2\alpha\beta + (\alpha+\beta)(a+b+1) + +ab, \\ &= (\alpha+\beta)(\alpha+\beta+a+b) + ab \\ B &= 2\alpha(\alpha-1) + 2\alpha\beta + c(\alpha+\beta) + \alpha(a+b+1) + ab, \\ C &= \alpha(\alpha-1) + \alpha c. \end{aligned}$$

Proof. Set $\phi = x^{\alpha}(1-x)^{\beta}$. We have

$$w = \phi y, \quad w' = \phi' y + \phi y', \quad w'' = \phi'' y + 2\phi' y' + \phi y''$$

so that the differential equation satisfied by w becomes

$$x(1-x)[\phi''y + 2\phi'y' + \phi y''] + [c-(a+b+1)x][\phi'y + \phi y'] - ab\phi y = 0,$$

which can be rewritten as

$$x(1-x)\phi y'' + \{2x(1-x)\phi' + [c-(a+b+1)x]\phi\}y' + \{x(1-x)\phi'' + [c-(a+b+1)x]\phi' - ab\phi\}y = 0. \quad (67)$$

Using relations

$$x(1-x)\frac{\phi'}{\phi} = \alpha(1-x) - \beta x$$

and

$$\frac{\phi''}{\phi} = \frac{\alpha(\alpha-1)}{x^2} + \frac{\beta(\beta-1)}{(1-x)^2} - \frac{2\alpha\beta}{x(1-x)},$$

Equation (67) becomes, after dividing by ϕ,

$$x(1-x)y'' + \{2\alpha - 2(\alpha+\beta)x + [c-(a+b+1)x]\}y' + \{\frac{\alpha(\alpha-1)(1-x)}{x} + \frac{\beta(\beta-1)x}{1-x} - 2\alpha\beta + [c-(a+b+1)x](\frac{\alpha}{x} - \frac{\beta}{1-x}) - ab\}y = 0, \quad (68)$$

which can be rewritten as

$$x(1-x)y'' + (2\alpha + c - [a+b+1+2\alpha+2\beta]x)y' + \frac{Ax^2 - Bx + C}{x(1-x)}y = 0$$

with

$$Ax^2 - Bx + C = \alpha(\alpha-1)(1-x)^2 + \beta(\beta-1)x^2 - 2\alpha\beta x(1-x) + [c-(a+b+1)x][\alpha(1-x) - \beta x] - abx(1-x).$$

By identification we obtain

$$\begin{aligned} A &= \alpha(\alpha-1) + \beta(\beta-1) + 2\alpha\beta + (a+b+1)(\alpha+\beta) + ab \\ &= (\alpha+\beta)(\alpha+\beta+a+b) + ab \\ B &= 2\alpha(\alpha-1) + 2\alpha\beta + c(\alpha+\beta) + \alpha(a+b+1) + ab \\ C &= \alpha(\alpha-1) + \alpha c \end{aligned}$$

and the result is established. □

Lemma 6. *Assume $2m \in \mathbb{N}$, $2n \in \mathbb{Z}$ with m positive and $m-n \in \mathbb{N}$. Set*

$$w(n,x) = {}_2F_1(-n, -n+1; m-n+1; x) \quad (0 < x < 1). \quad (69)$$

The function $y(n;x) = w(n;x)x^{(m-n)/2}(1-x)^{-(m+n)/2}$ satisfies the differential equation

$$x(1-x)y'' + (1-2x)y' - \frac{4mnx + (m-n)^2}{4x(1-x)} = 0 \quad (0 < x < 1).$$

Proof. The function w is an hypergeometric function satisfying the differential equation of the preceding lemma, with the values

$$a = -n,\ b = -n+1,\ c = m-n+1,\ \alpha = \frac{n-m}{2},\ \beta = \frac{m+n}{2}.$$

With the notation of this lemma, we obtain

$$\alpha+\beta=n,\ a+b=-2n+1,\ 2\alpha+c=1,\ a+b+1+2\alpha+2\beta=2,$$

so that

$$\begin{aligned}
A &= (\alpha+\beta)(\alpha+\beta+a+b)+ab = n(-n+1)+(-n)(-n+1)=0\\
B &= 2\alpha(\alpha-1)+2\alpha\beta+c(\alpha+\beta)+\alpha(a+b+1)+ab\\
&= (n-m)(\frac{n-m}{2}-1)+(n-m)\frac{m+n}{2}+(m-n+1)n+\frac{n-m}{2}(-2n+2)+n(n-1)\\
&= mn,\\
C &= \alpha(\alpha-1)+\alpha c = \frac{n-m}{2}(\frac{n-m}{2}-1)+\frac{n-m}{2}(m-n+1) = -\frac{(m-n)^2}{4}.
\end{aligned}$$

We obtain

$$\frac{Ax^2-Bx+C}{x(1-x)} = \frac{-Bx+C}{x(1-x)} = \frac{-4Bx+4C}{4x(1-x)} = \frac{-4mnx-(m-n)^2}{4x(1-x)}$$

and the result is established as a consequence of the preceding lemma. □

Before stating the following lemma, note that if $e=\pm1$, the pair (m,n) satisfies the assumptions of the preceding lemma if and only if (m,en) does.

Lemma 7. *With the notations of the preceding lemma, assume further more that $e=\pm1$ and set*

$$f(z)=y(en,x),\ x=\frac{1-ez}{2}\quad(-1<z<1,\ 0<x<1).$$

Then f satisfies

$$(1-z^2)f''(z)-2zf'(z)-\frac{m^2+n^2-2mnz}{1-z^2}f(z)=0\quad(-1<z<1).$$

Proof. We know from the preceding lemma that $y(x)=w(en,x)$ satisfies

$$x(1-x)y''+(1-2x)y'-\frac{(m-en)^2+4mnex}{4x(1-x)}y=0.$$

By writing

$$y(x)=y(en,\frac{1-ez}{2})=f(z),$$

we find, by differentiation with respect to the variable z

$$\begin{aligned}
y(x)=f(z),\quad -\frac{e}{2}y'(x)=f'(z),\quad \frac{1}{4}y''(x)=f''(z)\\
x(1-x)y''(x)=2x\cdot2(1-x)y''(x)/4=(1+ez)(1-ez)f''(z)=(1-z^2)f''(z)\\
(1-2x)y'(x)=(ez)[-2ef'(z)]=-2zf'(z),\\
\frac{(m-en)^2+4mnex}{4x(1-x)}=\frac{m^2+n^2-2emn+2mne(1-ez)}{1-z^2}=\frac{m^2+n^2-2mnz}{1-z^2},
\end{aligned}$$

and the result follows. □

8.6. Sums of Matrix Elements

Lemma 8. *Consider $u \in SU(2)$ and $t \in [0, 2\pi]$ given by*

$$u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{\frac{i(\phi+\psi)}{2}} & i\sin\frac{\theta}{2} e^{\frac{i(\phi-\psi)}{2}} \\ i\sin\frac{\theta}{2} e^{\frac{i(\psi-\phi)}{2}} & \cos\frac{\theta}{2} e^{-\frac{i(\phi+\psi)}{2}} \end{pmatrix},$$

$$\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2} = \Re(\alpha).$$

We have

$$\sum_{\ell \in \frac{\mathbb{N}^*}{2}} \frac{(2\ell+1)\chi_\ell(u)}{\ell(\ell+1)} = \sum_{\ell \in \frac{\mathbb{N}^*}{2}} \frac{(2\ell+1)\sin(\ell+\frac{1}{2})t}{\ell(\ell+1)\sin\frac{t}{2}} = -1 + (2\pi - t)\cot\frac{t}{2}.$$

Proof. The first equality follows from (26). On the other hand, with the integral index $k = 2\ell$ and the angle $\theta = t/2 \in [0, \pi]$, the second member of the equality can be rewritten as

$$\sum_{\ell \in \frac{\mathbb{N}^*}{2}} \frac{(2\ell+1)\sin(\ell+\frac{1}{2})t}{\ell(\ell+1)\sin\frac{t}{2}} = 4\sum_{k\in\mathbb{N}^*} \frac{(k+1)\sin[(k+1)\theta]}{k(k+2)\sin\theta}.$$

Elementary computations give

$$\begin{aligned} 2\sum_{k\in\mathbb{N}^*} \frac{(k+1)\sin[(k+1)\theta]}{k(k+2)} &= \sum_{k\geq 1} \frac{\sin[(k+1)\theta]}{k} + \sum_{k\geq 1} \frac{\sin[(k+1)\theta]}{k+2} \\ &= \sum_{k\geq 1} \frac{\sin[(k+1)\theta]}{k} + \sum_{k\geq 3} \frac{\sin[(k-1)\theta]}{k} \\ &= \sum_{k\geq 1} \frac{\cos\theta\sin[k\theta] + \sin\theta\cos[k\theta]}{k} + \sum_{k\geq 3} \frac{\cos\theta\sin[k\theta] - \sin\theta\cos[k\theta]}{k} \\ &= 2\sum_{k\geq 1} \frac{\cos\theta\sin(k\theta)}{k} - \cos\theta(\sin\theta + \frac{\sin(2\theta)}{2}) + \sin\theta(\cos\theta + \frac{\cos(2\theta)}{2}) \\ &= (\pi - \theta)\cos\theta - \frac{\sin\theta}{2}, \end{aligned}$$

where for the last equality we used the identities

$$\sum_{k\geq 1} \frac{\sin(k\theta)}{k} = \frac{\pi-\theta}{2} \quad (0 < \theta < \pi), \; -\cos\theta\frac{\sin(2\theta)}{2} + \sin\theta\frac{\cos(2\theta)}{2} = -\frac{\sin\theta}{2}$$

(see [28] ((2), p. 148)). Multiplying both sides by $2/\sin\theta$ gives

$$4\sum_{k\in\mathbb{N}^*} \frac{\sin[(k+1)\theta]}{k(k+2)\sin\theta} = 2(\pi-\theta)\cot\theta - 1.$$

Since $\theta = t/2$, we obtain the claimed result □

9. Discussion

We have provided an interpretation of the K-K expansion of the Anderson–Darling kernel in terms of matrix elements of unitary irreducible representations of the Lie group $SU(2)$. This interpretation enabled us to generalize this K–L expansion. We have derived several families of K–L expansions of kernel defined over $SU(2) \times SU(2)$.

The new K–L expansion that initially motivated our research is that of Theorem 1. It is the result of tedious analytical computations. On the other hand, some of our new K–L

expansions directly follow from algebraic relations associated with group representations. This double aspect illustrates the nature of special functions, when considered from the point of view of group representations as in [10,12].

A point deserving discussion is the choice of eigenvalues of our K–L expansions. Systematically, we have privileged the eigenvalues of the Laplacian operator. Whereas in our framework, the eigenfunctions of K–L expansions have to be the matrix elements of irreducible representations, eigenvalues might have been chosen in view of statistical applications. If (λ_ℓ) denotes the sequence of eigenvalues, the convergence in quadratic mean of K–L expansion

$$K(u_1, u_2) = \sum_{(\ell,m,n)\in I_{SU(2)}} \lambda_\ell t^\ell_{m,n}(u_1)\overline{t^\ell_{m,n}(u_1)}$$

requires

$$\sum_{(\ell,m,n)\in I_{SU(2)}} (2\ell+1)\lambda_\ell^2 < \infty,$$

making of the integral operator associated with K a Hilbert–Schmidt operator. But in order to get results about Bahadur efficiency, we have met the stronger condition

$$\sum_{(\ell,m,n)\in I_{SU(2)}} (2\ell+1)\lambda_\ell < \infty,$$

ensuring that K induces a trace operator. The latter condition is not satisfied if $\lambda_\ell = [\ell(\ell+1)]^{-1}$. A natural alternative choice might be $\lambda_\ell = \rho^\ell$ for some $\rho \in (0,1)$. The associated kernel would be closely related to the so-called Poisson kernel of Jacobi polynomials and has been widely studied in the theory of special functions (see, e.g., Theorem 4.7.8 and Remark 4.7.3, p. 112–113 in [29]).

The only way to check the statistical interest of the statistical tests associated with our kernels and the associated $U-$ or $V-$ statistics would be to use real data, or carry out simulations. These tasks will be the object of forthcoming papers.

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