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Oscillation of Noncanonical Neutral-Type Delay Differential Equations

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Abstract

The present paper is devoted to the study of oscillation phenomena in a class of second-order nonlinear neutral delay differential equations. We focus on the less studied noncanonical case. By applying a suitable transformation, the problem is transformed into a canonical form, yielding new oscillation criteria. The new results improve the existing oscillation criteria for noncanonical neutral-type delay differential equations. Examples are included to demonstrate the effectiveness of the obtained criteria and to highlight the importance of the results.

Keywords: second-order; neutral equation; canonical; noncanonical; oscillation

MSC: 34C10; 34K11; 34K40

1. Introduction

The study of qualitative properties of differential equations has been an active area of research for many years. In particular, neutral delay differential equations (NDEs) have attracted considerable attention due to their wide range of applications and mathematical complexity. These equations differ from classical delay equations in that they involve derivatives of the unknown function evaluated at both present and earlier instances. Classical results on NDEs can be found in monographs [1–4]. Over time, many authors have extended these results and developed new techniques; see, for example, [5–9]. These works mainly focus on the oscillatory and asymptotic behavior of solutions.

In this work, we study the oscillatory behavior of the second-order nonlinear neutral delay differential equation

$$\left(a(t)[x(t)+p(t)x^{\alpha}(\tau(t))]'\right)'+q(t)x^{\beta}(\sigma(t))=0, \qquad t\geq t_0>0, \tag{1}$$

where it is assumed that the functions a, p, q, τ, and σ are sufficiently smooth to ensure that solutions exist on $[t_0,\infty)$. The exponents α and β are ratios of odd positive integers.

To motivate our study, we mention some related works on such equations. For instance, Baculíková and Džurina [5] investigated the second-order neutral differential equation

$$(r(t)[x(t)+p(t)x(\tau(t))]')'+q(t)x(\sigma(t))=0,$$

and derived oscillation criteria under the conditions $0\leq p(t)\leq p_0<\infty$ and $\tau(\sigma(t))=\sigma(\tau(t))$. Tamilvanan et al. [9] considered the nonlinear equation

$$(a(t)[x(t)+p(t)x^{\alpha}(\tau(t))]')'+q(t)x^{\beta}(\sigma(t))=0,$$

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where α and β are the ratio of odd positive integers with $\alpha \in (0,1]$, and the oscillation theorems have been established for different ranges of β. Grace and Graef [8] further studied this problem by reducing it to a linear equation of the same form. In another study, Džurina et al. [7] examined neutral equations with several deviating arguments of the form

$$(a(t)[x(t) + \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t))]')' + q(t)x^{\beta}(\sigma(t)) = 0,$$

using comparison techniques together with suitable auxiliary functions. It is worth mentioning that most existing results in the literature make the assumption that $p(t) > 0$. Therefore, we will consider (1) when $p(t) < 0$. One of the few works addressing this case is due to Baculíková et al. [6], where the authors focused on the canonical case

$$\int_{t_0}^{\infty} \frac{ds}{a(s)} = \infty.$$

Indeed, this condition is used in most works in the literature to obtain certain monotonicity properties for the solution or an associated function, which are not easily established when this condition fails. In this paper, we consider the less-studied noncanonical case and provide examples that highlight this limitation, meaning that we will allow the above improper integral to be convergent:

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty. \tag{2}$$

We will assume that the following hypotheses (H) hold:

$$0 < \alpha \le 1,$$
$$a(t) > 0,\ q(t) > 0 \text{ and } p(t) < 0 \text{ for all } t \ge t_0,$$
$$\tau(t),\ \sigma(t) \le t, \tau'(t),\ \sigma'(t) > 0, \text{ and } \lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

A function $x \in C([T_x, \infty), \mathbb{R})$, $T_x \ge t_0$, is said to be a solution of (1) if $a(t)\big(x(t) + p(t)x^{\alpha}(\tau(t))\big)' \in C^1([T_x, \infty), \mathbb{R})$, and Equation (1) holds for all $t \ge T_x$. Naturally, we restrict ourselves to the solutions that are nontrivial in the neighborhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros, and nonoscillatory, otherwise.

Following our method in [10], we will employ an oscillation preserving transformation. Such transformations have been used very successfully in earlier works; see [11–13]. For further related results, we refer the reader to [14–22] and the references therein.

2. Oscillation Results

Assume that (2) holds and define

$$v(t) = \int_{t}^{\infty} \frac{ds}{a(s)}. \tag{3}$$

The following transformation lemma is well known; see [10,11].

Lemma 1. *Let v be defined by (3). Then, given a solution $x(t)$ of (1), the function $y(t) = x(t)/v(t)$ solves the associated equation:*

$$(a_1(t)[y(t) + p_1(t)\, y^{\alpha}(\tau(t))]')' + q_1(t)\, y^{\beta}(\sigma(t)) = 0, \qquad t \ge t_0, \tag{4}$$

where

$$a_1(t) = v^2(t)\, a(t); \tag{5}$$
$$p_1(t) = \frac{v^{\alpha}(\tau(t))}{v(t)}\, p(t); \tag{6}$$
$$q_1(t) = v(t)\, v^{\beta}(\sigma(t))\, q(t). \tag{7}$$

Conversely, given a solution $y(t)$ of (4), the function $x(t) = v(t)y(t)$ becomes a solution of (1).

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Proof. Let $x(t)$ be a solution of (1). Setting $y(t) = x(t)/v(t)$, and

$$u(t) = v(t)y(t) + p(t)v^{\alpha}(\tau(t))y^{\alpha}(\tau(t)),$$

we can write that $u(t) = v(t)z(t)$, where

$$z(t) = y(t) + p_1(t)y^{\alpha}(\tau(t)). \tag{8}$$

Next, suppressing the variable t, we find

$$\begin{aligned}(a_1z')' &= \left(v^2a\left(u/v\right)'\right)' \\ &= (au')'v - u(av')' \\ &= (au')'v \\ &= -q_1\, y^{\beta}(\sigma(t)),\end{aligned}$$

and hence

$$(a_1(t)z')' + q_1(t)\, y^{\beta}(\sigma(t)) = 0, \quad t \geq t_0.$$

This establishes that $y(t) = x(t)/v(t)$ is a solution of (4) whenever $x(t)$ is a solution of (1). The converse is done in a similar manner. □

We will also need the following auxiliary lemmas in the proofs of the main results.

Lemma 2. *Let v, a_1, p_1, q_1, and z be given by (3), (5)–(8), respectively. Assume (2) holds and $-1 < p_0 \leq p_1(t) < 0$ for all $t \geq t_0$. Let $x(t)$ be a positive solution of (1). Then the function z satisfies one of the following:*

(i) $z(t) > 0, z'(t) > 0$, *and* $(a_1(t)z'(t))' < 0$;
(ii) $z(t) < 0, z'(t) > 0$, *and* $(a_1(t)z'(t))' < 0$,

for all $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Proof. Assume $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$, some $t_1 \geq t_0$. From the definition of z and (4) we have

$$(a_1(t)z'(t))' = -q_1(t)y^{\beta}(\sigma(t)) < 0.$$

Hence $a_1(t)z'(t)$ is decreasing and has a fixed sign for large t, i.e., there exists $t_2 \geq t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$. Suppose $z'(t) < 0$ for $t \geq t_2$. Then $a_1(t)z'(t) \leq c < 0$ for $t \geq t_2$, where $c = a_1(t_2)z'(t_2) < 0$. Hence

$$z(t) \leq z(t_2) + c\int_{t_2}^{t} \frac{ds}{a_1(s)}.$$

Since

$$v'(t) = -\frac{1}{a(t)}, \quad v(t) > 0, \quad \lim_{t\to\infty} v(t) = 0,$$

it follows that

$$\int_{t_0}^{\infty} \frac{dt}{a_1(t)} = \int_{t_0}^{\infty} -\frac{v'(t)}{v^2(t)}\, dt = \int_{t_0}^{\infty} \left(\frac{1}{v(t)}\right)' dt = \infty.$$

Therefore, $\lim_{t\to\infty} z(t) = -\infty$. We consider two cases:

Case 1: y is unbounded.

Then there exists a sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k = \infty, \qquad \lim_{k\to\infty} y(t_k) = \infty,$$

where

$$y(t_k) = \max\{y(s) : t_0 \le s \le t_k\}.$$

Since $\lim_{t\to\infty} \tau(t) = \infty$, we have $\tau(t_k) > t_0$ for all sufficiently large k. Moreover, because $\tau(t) \le t$, it follows that

$$y(\tau(t_k)) \le y(t_k).$$

Using $0 < \alpha \le 1$ and the boundedness of $p_1(t)$, we obtain

$$\begin{aligned} z(t_k) &= y(t_k) + p_1(t_k)\, y^{\alpha}(\tau(t_k)) \\ &\ge y(t_k) + p_1(t_k)\, y^{\alpha}(t_k) \\ &= \left(1 + p_1(t_k)\, y^{\alpha-1}(t_k)\right) y(t_k) \longrightarrow \infty \quad \text{as} \quad k \to \infty, \end{aligned}$$

which contradicts the assumption that

$$\lim_{t\to\infty} z(t) = -\infty.$$

Case 2: y is bounded. If y is bounded and p_1 is bounded, then $z(t)$ is also bounded. This again contradicts the assumption that

$$\lim_{t\to\infty} z(t) = -\infty.$$

Therefore, both cases lead to contradictions. Hence, $z'(t) > 0$ eventually, and consequently one of the alternatives (i) or (ii) must hold. This completes the proof. □

Define

$$R(t) = \int_{t_1}^{t} \frac{ds}{a_1(s)}, \quad t \ge t_1 \ge t_0.$$

Lemma 3. *Let v, a_1, p_1, q_1, and z be given by (3), (5)–(8), respectively. Assume that (2) holds and that $-1 < p_0 \le p_1(t) < 0$ for all $t \ge t_0$. Let x be a positive solution of (1) such that case (i) of Lemma 2 holds. Then,*

$$y(t) \ge z(t) \ge R(t)\, a_1(t) z'(t), \tag{9}$$

for all $t \ge t_1$ and $z(t)/R(t)$ is eventually decreasing.

Proof. It follows directly from the definition of z that $y(t) \ge z(t)$ for all $t \ge t_1$. Since $z'(t) > 0$ and $(a_1(t)z'(t))' < 0$, it follows that $a_1(t)z'(t)$ is decreasing on $[t_1, \infty)$. Hence,

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{a_1(s)z'(s)}{a_1(s)}\, ds > a_1(t)z'(t) \int_{t_1}^{t} \frac{1}{a_1(s)}\, ds = R(t)\, a_1(t) z'(t), \quad t \ge t_1. \tag{10}$$

Using (10), we obtain

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{a_1(t)R(t)z'(t) - z(t)}{a_1(t)R^2(t)} < 0,$$

which shows that $z(t)/R(t)$ is strictly decreasing for $t \ge t_1$.

□

Theorem 1. *Assume $\beta < \alpha$ and $\sigma(t) < \tau(t)$. In addition, suppose that (2) holds and that $-1 < p_0 \le p_1(t) < 0$ for all $t \ge t_0$. If*

$$\int_{t_1}^{\infty} q_1(t)\, R^{\beta}(\sigma(t))\, dt = \infty, \tag{11}$$

and

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{a_1(s)} \left(\int_{s}^{t} q_1(u)\, p_1^{-\beta/\alpha}\left(\tau^{-1}(\sigma(u))\right) du \right) ds < 0, \tag{12}$$

then every solution of (1) is oscillatory.

Proof. Suppose there exists a nonoscillatory solution x of (1) with $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. By Lemma 2, the function z satisfies either (i) or (ii).

Case (i): since $y \geq z$ and $y^\beta(\sigma(t)) \geq z^\beta(\sigma(t))$, it follows from (4) that

$$(a_1(t)z'(t))' + q_1(t)z^\beta(\sigma(t)) \leq 0.$$

Let $w(t) = a_1(t)z'(t) > 0$ for $t \geq t_1$. Then, using (9), we derive the delay inequality

$$w'(t) + q_1(t)\, R^\beta(\sigma(t))\, w^\beta(\sigma(t)) \leq 0, \qquad t \geq t_1. \tag{13}$$

Since $z'(t) > 0$, we have $w(t) > 0$ for $t \geq t_1$. Applying ([3], Theorem 3.9.3) with q replaced by $q_1(t)\, R^\beta(\sigma(t))$, and using (11), we see that

$$w'(t) + q_1(t)\, R^\beta(\sigma(t))\, w^\beta(\sigma(t)) = 0$$

is oscillatory. However, this contradicts $w(t) > 0$ for $t \geq t_1$, showing that case (i) cannot occur.

Case (ii): here $z(t) < 0$ and $z'(t) > 0$. From $z(t) = y(t) + p_1(t)y^\alpha(\tau(t)) < 0$, we obtain

$$y(\tau(t)) \geq \left(\frac{z(t)}{p_1(t)}\right)^{1/\alpha}. \tag{14}$$

Substituting (14) into (4) gives

$$(a_1(t)z'(t))' + q_1(t)\, p_1^{-\beta/\alpha}\big(\tau^{-1}(\sigma(t))\big)\, z^{\beta/\alpha}\big(\tau^{-1}(\sigma(t))\big) \leq 0. \tag{15}$$

Since z is negative and increasing, $\lim_{t\to\infty} z(t) = c \leq 0$. If $c < 0$, then for large t we have $z(t) \leq c$ and $z(\tau^{-1}(\sigma(t))) \leq c$, so

$$z^{\beta/\alpha}\big(\tau^{-1}(\sigma(t))\big) \leq c^{\beta/\alpha}. \tag{16}$$

After Substituting (16) into (15), integrating the resulting inequality from t to ∞, and then integrating once more from t_1 to ∞, we obtain

$$z(t_1) \leq -c^{\beta/\alpha} \int_{t_1}^{\infty} \frac{1}{a_1(u)} \Big(\int_u^{\infty} q_1(s)\, p_1^{-\beta/\alpha}\big(\tau^{-1}(\sigma(s))\big)\, ds \Big) du,$$

which is a contradiction, since (12) implies that the right-hand side diverges to $-\infty$. Therefore, $\lim_{t\to\infty} z(t) = 0$ with $z(t) < 0$ and $z'(t) > 0$. Integrating (15) from s to t $(t > s)$ gives

$$-a_1(s)z'(s) \leq -\int_s^t q_1(u)\, p_1^{-\beta/\alpha}\big(\tau^{-1}(\sigma(u))\big)\, z^{\beta/\alpha}\big(\tau^{-1}(\sigma(u))\big)\, du.$$

Then integrating the last inequality in s from $\tau^{-1}(\sigma(t))$ to t and using the fact that $z(t)$ is increasing, we obtain

$$z(\tau^{-1}(\sigma(t))) - z(t) \leq -z^{\beta/\alpha}\big(\tau^{-1}(\sigma(t))\big) \int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{a_1(s)} \Big(\int_s^t q_1(u)\, p_1^{-\beta/\alpha}\big(\tau^{-1}(\sigma(u))\big)\, du \Big) ds$$

or

$$\frac{z(\tau^{-1}(\sigma(t)))}{z^{\beta/\alpha}\big(\tau^{-1}(\sigma(t))\big)} \geq -\int_{\tau^{-1}(\sigma(t))}^{t} \frac{1}{a_1(s)} \Big(\int_s^t q_1(u)\, p_1^{-\beta/\alpha}\big(\tau^{-1}(\sigma(u))\big)\, du \Big) ds.$$

Since $\frac{z(\tau^{-1}(\sigma(t)))}{z^{\beta/\alpha}\left(\tau^{-1}(\sigma(t))\right)} = |z(\tau^{-1}(\sigma(t)))|^{1-\beta/\alpha}$ and $1 - \beta/\alpha > 0$, and $\lim_{t\to\infty} z(t) = 0$, taking lim sup as $t \to \infty$ contradicts (12). Thus, case (ii) is also impossible, and every solution must be oscillatory. □

Example 1. *Consider the second-order neutral differential equation*

$$\left(t\ln^2 t\left(x(t)-\frac{1}{2}\frac{(\ln(t/2))^{1/3}}{\ln t}x^{1/3}(t/2)\right)'\right)'+\ln t(\ln(t/3))^{1/7}x^{1/7}(t/3)=0,\qquad t\geq 2. \tag{17}$$

Here $a(t)=t\ln^2 t$, $p(t)=-\frac{1}{2}\frac{(\ln(t/2))^{1/3}}{\ln t}$, $q(t)=\ln t(\ln(t/3))^{1/7}$, $\tau(t)=t/2$, $\sigma(t)=t/3$, $\alpha=1/3$, *and* $\beta=1/7$. *We directly compute*

$$v(t)=\int_t^\infty\frac{ds}{a(s)}=\int_t^\infty\frac{ds}{s\ln^2 s}=\frac{1}{\ln t},\quad a_1(t)=v^2(t)a(t)=t,$$

$$R(t)=\int_2^t\frac{ds}{a_1(s)}=\int_2^t\frac{ds}{s}=\ln\left(\frac{t}{2}\right),\quad p_1(t)=\frac{v^\alpha(\tau(t))}{v(t)}p(t)=-\frac{1}{2}>-1,$$

$$q_1(t)=v(t)\,v^\beta(\sigma(t))\,q(t)=1,$$

$$\tau^{-1}(t)=2t,\quad \sigma(t)=\frac{t}{3},\quad \tau^{-1}(\sigma(t))=\frac{2t}{3}.$$

Next, we verify conditions (11) *and* (12). *We have*

$$\int_{t_1}^\infty q_1(t)\,R^\beta(\sigma(t))\,dt=\int_2^\infty\left(\ln\left(\tfrac{t}{6}\right)\right)^{1/7}dt=\infty$$

and hence condition (11) *is satisfied. Moreover,*

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^t\frac{1}{a_1(s)}\left(\int_s^t q_1(u)\,p_1^{-\beta/\alpha}\,du\right)ds\\
&=\limsup_{t\to\infty}\int_{2t/3}^t\frac{1}{s}\left(\int_s^t -2^{3/7}\,du\right)ds\\
&=\limsup_{t\to\infty}2^{3/7}\int_{2t/3}^t\frac{s-t}{s}\,ds\\
&=\limsup_{t\to\infty}2^{3/7}t\left(\frac{1}{3}-\ln\left(\frac{3}{2}\right)\right)=-\infty.
\end{aligned}$$

Therefore, condition (12) *is also satisfied. All the conditions of Theorem 1 are satisfied; therefore, every solution of* (17) *is oscillatory.*

Theorem 2. *Let* $\beta=1$. *Suppose that* (2) *holds and that* $-1<p_0\leq p_1(t)<0$ *for all* $t\geq t_0$. *If*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^t q_1(s)\,R(\sigma(s))\,ds>\frac{1}{e}, \tag{18}$$

then every solution of (1) *is either oscillatory or tends to zero as* $t\to\infty$.

Proof. Let x be a nonoscillatory solution with $x>0$, $x(\tau(t))>0$, $x(\sigma(t))>0$ for all $t\geq t_1$. By Lemma 2, one of the cases (i) or (ii) holds.

Case (i): in view of the proof of Theorem 1 with $\beta=1$, $w(t)=a_1(t)z'(t)$ is a positive solution of

$$w'(t)+q_1(t)R(\sigma(t))\,w(\sigma(t))\leq 0. \tag{19}$$

By a known nonexistence criterion for eventually positive solutions of such inequalities (see [3] Theorem 2.1.1), inequality (19) admits no positive solution in view of (18). Hence, a contradiction is obtained.

Case (ii): here $z(t)<0$, $z'(t)>0$ and hence $\lim_{t\to\infty}z(t)=c\leq 0$, so z is bounded. As in Lemma 2, boundedness of z implies y is bounded; hence $\lim_{t\to\infty}x(t)=\lim_{t\to\infty}v(t)y(t)=0$. □

Example 2. *Consider the second-order neutral differential equation*

$$\left(t^2\left(x(t)-\frac{1}{2t^{2/3}}x^{1/3}(t/2)\right)'\right)'+t\,x(t/3)=0,\qquad t\geq 1. \tag{20}$$

Here $a(t) = t^2$, $p(t) = -\frac{1}{2t^{2/3}}$, $q(t) = t$, $\tau(t) = t/2$, $\sigma(t) = t/3$, $\alpha = 1/3$, *and* $\beta = 1$. *We directly compute*

$$v(t) = \int_t^\infty \frac{ds}{a(s)} = \frac{1}{t}, \quad a_1(t) = v^2(t)a(t) = 1,$$

$$R(t) = \int_1^t \frac{ds}{a_1(s)} = t - 1, \quad p_1(t) = \frac{v^\alpha(\tau(t))}{v(t)}\, p(t) = -2^{-2/3} > -1,$$

$$q_1(t) = v(t)\, v^\beta(\sigma(t))\, q(t) = \frac{3}{t}.$$

Next, we verify condition (18). *We have*

$$\begin{aligned}\int_{\sigma(t)}^t q_1(s)\, R(\sigma(s))\, ds &= \int_{t/3}^t \frac{3}{s}\left(\frac{s}{3} - 1\right) ds \\ &= \frac{2t}{3} - 3\ln 3.\end{aligned}$$

Hence,

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t q_1(s)\, R(\sigma(s))\, ds = \liminf_{t\to\infty} (\frac{2t}{3} - 3\ln 3) = \infty.$$

Therefore, condition (18) *is satisfied. All the conditions of Theorem* 2 *are satisfied; therefore, every solution of* (20) *is either oscillatory or tends to zero as* $t \to \infty$.

Theorem 3. *Let* $\beta > 1$. *Suppose that* (2) *holds and that* $-1 < p_0 \le p_1(t) < 0$ *for all* $t \ge t_0$. *If there exists a positive nondecreasing function* $\eta(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_2}^t \left(\eta(s)\, q_1(s) \; - \; \frac{a_1(\sigma(s))\, (\eta'(s))^2}{4\beta\, K^{\beta-1}\, \eta(s)\, \sigma'(s)} \right) ds = \infty, \tag{21}$$

for any constant $K > 0$, *every solution of* (1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Proceeding as in Theorem 1, we conclude that one of the cases in Lemma 2 must occur for $t \ge t_1$.

Case (i): by the proof of Theorem 1 in case (i), it follows that

$$(a_1(t)z'(t))' + q_1(t)\, z^\beta(\sigma(t)) \le 0, \qquad t \ge t_1. \tag{22}$$

Define

$$w(t) = \eta(t)\, \frac{a_1(t)z'(t)}{z^\beta(\sigma(t))}, \qquad t \ge t_1.$$

It is clear that $w(t) > 0$. Differentiating w and using (22), we obtain

$$w'(t) \le -\eta(t)q_1(t) + \frac{\eta'(t)}{\eta(t)}\, w(t) \; - \; \frac{\beta\, a_1(t)z'(t)z'(\sigma(t))z^{\beta-1}(\sigma(t))\sigma'(t)\eta(t)}{z^{2\beta}(\sigma(t))}. \tag{23}$$

Since $a_1(t)z'(t)$ is nonincreasing, we have $a_1(\sigma(t))z'(\sigma(t)) \ge a_1(t)z'(t)$; substituting this into (23), we obtain

$$w'(t) \le -\eta(t)q_1(t) + \frac{\eta'(t)}{\eta(t)}\, w(t) \; - \; \frac{\beta\, \left(a_1(t)z'(t)\right)^2 z'(\sigma(t))z^{\beta-1}(\sigma(t))\sigma'(t)\eta(t)}{a_1(\sigma(t))z^{2\beta}(\sigma(t))}. \tag{24}$$

Moreover, since $z(t)$ is increasing and positive and $\beta > 1$, there exists a $K > 0$ such that $z^{\beta-1}(\sigma(t)) \ge K^{\beta-1} > 0$ for all $t \ge t_2 \ge t_1$. Using this in (24), we obtain

$$w'(t) \le -\eta(t)q_1(t) + \frac{\eta'(t)}{\eta(t)}\, w(t) \; - \; \frac{\beta\, K^{\beta-1}\, \sigma'(t)}{\eta(t)\, a_1(\sigma(t))}\, w^2(t).$$

Completing the square leads to

$$w'(t) \le -\eta(t)q_1(t) + \frac{a_1(\sigma(t))\,(\eta'(t))^2}{4\beta\,K^{\beta-1}\,\eta(t)\,\sigma'(t)}, \qquad t \ge t_2.$$

Integrating from t_2 to t gives

$$\int_{t_2}^{t}\left(\eta(s)\,q_1(s) - \frac{a_1(\sigma(s))\,(\eta'(s))^2}{4\beta\,K^{\beta-1}\,\eta(s)\,\sigma'(s)}\right)ds \;<\; w(t_2) \;<\; \infty,$$

which contradicts (21).

Case (ii): here $z(t) < 0$ and $z'(t) > 0$ for $t \ge t_1$. Proceeding as in Theorem 2, we conclude $\lim_{t\to\infty} x(t) = 0$. □

Example 3. *Consider the second-order neutral differential equation*

$$\left(t^3\left(x(t) - \frac{1}{10^{6/7}\,t^{8/7}}\,x^{3/7}(t/5)\right)'\right)' + t^5\,x^{5/3}(t/2) = 0, \qquad t \ge 1. \tag{25}$$

Here $a(t) = t^3$, $p(t) = -\frac{1}{10^{6/7}\,t^{8/7}}$, $q(t) = t^5$, $\tau(t) = t/5$, $\sigma(t) = t/2$, $\alpha = 3/7$, *and* $\beta = 5/3$. *We directly compute*

$$v(t) = \int_t^\infty \frac{ds}{s^3} = \frac{1}{2t^2}, \quad a_1(t) = \frac{1}{4t},$$

$$R(t) = \int_1^t \frac{ds}{a_1(s)} = \int_1^t 4s\,ds = 2(t^2-1),$$

$$p_1(t) = \frac{v^\alpha(\tau(t))}{v(t)}\,p(t) = -2^{-2/7} > -1,$$

$$q_1(t) = v(t)\,v^\beta(\sigma(t))\,q(t) = 2^{2/3}t^{-1/3}.$$

Next, we verify condition (21) *by taking* $\eta(t) = 1$. *Thus, we have*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\eta(s)\,q_1(s) \;-\; \frac{a_1(\sigma(s))\,(\eta'(s))^2}{4\beta\,K^{\beta-1}\,\eta(s)\,\sigma'(s)}\right)ds$$
$$= \limsup_{t\to\infty}\int_{t_2}^{t} 2^{2/3}s^{-1/3}\,ds = \infty.$$

Therefore, condition (21) *is satisfied. All the conditions of Theorem 3 are fulfilled; hence, every solution of* (25) *is either oscillatory or tends to zero as* $t \to \infty$.

Remark 1. *It is important to note that the results in [6] are not applicable to the examples, since the corresponding equations are of noncanonical type, i.e.,*

$$\int_{t_0}^{\infty}\frac{dt}{a(t)} < \infty.$$

Hence, our results extend the existing theory and contribute new findings to the literature.

3. Conclusions

We studied the oscillatory behavior of a class of second-order nonlinear neutral delay differential equations in the noncanonical case. By applying an oscillation-preserving transformation, we obtained several new oscillation criteria for different ranges of β. Namely, we investigated the cases $0 < \beta < 1$, $\beta = 1$, and $\beta > 1$, since each required a different approach. The examples presented for each case demonstrated the applicability of the results and

showed that they can be applied for noncanonical equations for which several existing criteria were not applicable. Thus, the obtained results extend and complement the current literature on oscillation theory for neutral differential equations.

Future studies may include generalizing the present approach to problems with multiple time delays, mixed nonlinear neutral terms, and possibly distributed delay arguments. In addition, extending the equivalence transformation technique to higher-order neutral delay equations represents a worthwhile research direction. Such developments would contribute to the development of the oscillation theory of functional differential equations.

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