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Diagonal Stability of Two-Block Systems with Reciprocal Single-Channel Coupling

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Abstract

This paper considers coupled linear subsystems $\dot{x} = -Ax - By$ and $\dot{y} = -Cx - Dy$, whose local blocks A and D may have different dimensions. For reciprocal single-channel coupling $B = \beta e_i e_j^T$ and $C = \gamma e_j e_i^T$, let $\delta_i(A)$ and $\delta_j(D)$ denote the directional diagonal-Lyapunov margins of the respective blocks: each is the supremum of a normalized coordinatewise Lyapunov reserve over admissible positive diagonal Lyapunov matrices. I prove that the full matrix is diagonally stable if and only if both local blocks are diagonally stable and $\beta\gamma < \delta_i(A)\delta_j(D)$. The strict proof does not presume attainment of the local suprema. I also give a general Schur-complement test, constructive relative diagonal-Lyapunov scalings, and a dimension-free criterion for arbitrary one-way coupling. Closed planar formulas recover the classical acyclic threshold, while reproducible examples with cyclic internal graphs illustrate the broader theorem and show that Hurwitz stability need not imply diagonal stability.

Keywords: diagonal stability; diagonal Lyapunov function; coupled systems; rank-one channel; block matrix; Schur complement; P-matrix

MSC: 15A45; 15B48; 34D20; 93D05; 93D30

1. Introduction

Diagonal stability asks for a positive diagonal Lyapunov matrix. Under the positive-definite convention used in this paper, a real square matrix M is diagonally stable if there exists a positive diagonal matrix P such that

$$PM + M^T P \succ 0. \tag{1}$$

For a linear system written as $\dot{z} = -Mz$, condition (1) provides the separable quadratic Lyapunov function $V(z) = z^T Pz$. Diagonal stability has long-standing connections with P-matrices, positive systems, large-scale systems, network dynamics, and structured Lyapunov analysis; see, among others, [1–6].

This is a very appealing approach due to the fact that there is no free Lyapunov matrix in the final test. For 2×2 matrices, diagonal stability is equivalent to the P-matrix property. In three dimensions, Refs. [2,7] give necessary and sufficient criteria in different equivalent forms; Ref. [4] also provides an equivalent Hadamard-product test. The equivalence of common diagonal Lyapunov functions from [8] and the alternative characterization in [9] help to make things clearer in this domain. The condition of positive minors is necessary in general but not sufficient without additional structure, such as the graph acyclicity

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assumption in [3]. In more than three dimensions, there is no similar entrywise description available for arbitrary matrices.

A parallel characterization for common Lyapunov diagonal stability in terms of Khatri–Rao products has been presented in [10]. While that study deals with common certificates for matrices, the present study deals with optimizing two local certificates for the prescribed reciprocal channel.

One structural result in classical form follows from the work of Berman and Hershkowitz [3], where positivity of all principal minors is equivalent to diagonal stability when the graph of the matrix is acyclic. Thus, the theory is well developed for tree and forest structures. Other exact conditions include those for cyclic interconnection [11] and rank-one perturbation of a positive diagonal matrix [12]. Hence, it is crucial to recognize when an apparently novel block interconnection theorem is actually just a hidden specialization of known graph families.

Newer studies employ structured Lyapunov inequalities for the analysis of delays and uncertainties of interconnections in networks [13,14]. This delay-system literature provides relevant context for structured Lyapunov certificates; the present result isolates a finite-dimensional, delay-free diagonal quadratic certificate for reciprocal coupling. Other techniques for stability assessment include indefinite Lyapunov–Krasovskii functionals for fractional-order delayed neural networks [15] and stabilization with fixed-time feedback for nonlinear systems [16]. These examples consider delays, fractional systems, or nonlinear/fixed-time objectives, while the current study considers a precise diagonal quadratic Lyapunov certificate for finite-dimensional systems. The two perspectives are different but not interchangeable.

Related robustness questions are addressed through matrix-nearness and structured singular-value methods for economic models [17] and through several notions of $\mathfrak{D}$-stability for interval matrix families [18]. These are distinct from the diagonal Lyapunov condition characterized below; in particular, ordinary D-stability and diagonal stability should not be conflated. This distinction matters for the reciprocal-channel result because the threshold $\beta\gamma < \delta_i(A)\delta_j(D)$ is a condition for a diagonal Lyapunov certificate, not a D-stability criterion.

Consider the two-block system

$$\begin{cases} \dot{x}(t) = -Ax(t) - By(t), \\ \dot{y}(t) = -Cx(t) - Dy(t), \end{cases} \qquad x(t) \in \mathbb{R}^n, \quad y(t) \in \mathbb{R}^m, \tag{2}$$

with associated matrix

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{3}$$

The internal matrices A and D are unrelated, and the directional couplings B and C may also be different.

A concrete realization refers to the linearization of two compartments that are internally stable, where the compartments communicate via one chosen sensor/actuator pair coordinate from each compartment. The coefficients β and γ encode possibly asymmetric feedback strengths and signs. Focusing on a single channel isolates the smallest genuine feedback loop and allows an exact scalar optimization, even when either internal subsystem has many states and a cyclic graph. The central case is a reciprocal scalar communication channel between one coordinate of each subsystem,

$$B = \beta e_i e_j^T, \qquad C = \gamma e_j e_i^T, \tag{4}$$

where $e_i \in \mathbb{R}^n$ and $e_j \in \mathbb{R}^m$ are standard basis vectors. Although each directional map has rank one, the off-diagonal perturbation $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ has rank two when $\beta\gamma \neq 0$. Unlike the diagonal-plus-rank-one form $\Lambda + uv^T$ in [12], the present matrix contains two full, possibly cyclic diagonal blocks; its Lyapunov Schur complement becomes a rank-one *downdate* only after two local diagonal certificates are chosen.

The contributions of the present study are as follows. First, I define a coordinate-dependent diagonal Lyapunov margin $\delta_i(A)$ for an arbitrary diagonally stable matrix A. The margin is scale invariant and finite and admits a rank-one semidefinite characterization. Second, for arbitrary dimensions n and m, I prove that the reciprocal single-channel interconnection (4) is diagonally stable if and only if

$$\beta\gamma < \delta_i(A)\delta_j(D), \tag{5}$$

where $\delta_i(A)$ and $\delta_j(D)$ are the directional reserves of the respective diagonally stable blocks, defined below. The proof is exact and constructive: it identifies the relative scaling of the two local Lyapunov functions that minimizes the coupling penalty. Third, I prove a dimension-free one-way result: for each of the triangular matrices $\mathcal{A}_+ = \left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right]$ and $\mathcal{A}_- = \left[\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right]$, with completely arbitrary B or C, diagonal stability holds if and only if both diagonal blocks are diagonally stable. This does *not* assert unconditional stability for arbitrary simultaneous two-way feedback. Fourth, for 2×2 blocks, I evaluate the margins in closed form, recovering an explicit fourth-order criterion. I do not present the latter equivalence as bypassing the classical acyclic-matrix theorem: for two planar blocks connected by one channel, the nondirected graph is a forest, and the same threshold follows from the P-matrix characterization in [3]. The planar formulas are nevertheless useful because they evaluate the general margins explicitly and provide a direct Lyapunov certificate. Finally, a 5×5 example with a triangular internal graph shows that the dimension-free theorem is not restricted to acyclic global sparsity patterns.

The paper is organized as follows. Section 2 discusses directional margins and their properties. Section 3 presents the two-block reduction and the one-sided case. Section 4 proves the reciprocal-channel theorem without dimensions. Section 5 provides the planar closed form and ties it to the classical acyclic theory. Section 6 contains the construction formulas and examples. Conclusions are presented in Section 7.

2. Directional Diagonal Lyapunov Margins

Let $\mathscr{D}^r_{++} := \{P = \operatorname{diag}(p_1, \ldots, p_r) : p_k > 0\}$ denote all positive diagonal $r \times r$ matrices. This calligraphic notation is different from the subsystem matrix D. Let $A \in \mathbb{R}^{n\times n}$ be a square matrix and let $P \in \mathscr{D}^n_{++}$. Then,

$$H_A(P) := PA + A^T P. \tag{6}$$

If $H_A(P) \succ 0$, write $p_i = e_i^T P e_i$ and define the local directional reserve

$$\eta_i(A; P) := \frac{1}{2p_i\, e_i^T H_A(P)^{-1} e_i}. \tag{7}$$

Definition 1. *Suppose $A \in \mathbb{R}^{n\times n}$ is diagonally stable. For $i \in \{1, \ldots, n\}$, define*

$$\delta_i(A) := \sup_{\substack{P\in\mathscr{D}^n_{++} \\ H_A(P)\succ 0}} \eta_i(A; P). \tag{8}$$

The following lemma states the basic properties that are needed subsequently.

Lemma 1. *Suppose that A is diagonally stable. Then, for all admissible P and all $c > 0$,*

$$\eta_i(A; cP) = \eta_i(A; P). \tag{9}$$

Moreover,

$$0 < \delta_i(A) \le a_{ii}. \tag{10}$$

In particular, every diagonal entry of a diagonally stable matrix is positive.

Proof. Since $H_A(cP) = cH_A(P)$, its inverse is $c^{-1}H_A(P)^{-1}$, and (9) follows immediately. Positive definiteness of $H_A(P)$ gives

$$2p_i a_{ii} = e_i^T H_A(P) e_i > 0,$$

and so $a_{ii} > 0$. For any positive definite matrix H, the Cauchy–Schwarz inequality $1 = (e_i^T e_i)^2 \le (e_i^T H e_i)(e_i^T H^{-1} e_i)$ gives $(H^{-1})_{ii} \ge (H_{ii})^{-1}$; equality holds exactly when e_i is an eigenvector of H. Therefore,

$$\eta_i(A; P) = \frac{1}{2p_i (H_A(P)^{-1})_{ii}} \le \frac{H_A(P)_{ii}}{2p_i} = a_{ii}.$$

Positivity follows immediately from (7). □

Proposition 1. *For every admissible P,*

$$\eta_i(A; P) = \max\Big\{\mu \ge 0 : H_A(P) - 2\mu p_i e_i e_i^T \succeq 0\Big\}. \tag{11}$$

Consequently, by the scale invariance shown in Lemma 1,

$$\delta_i(A) = \sup\left\{\mu : \begin{array}{l} P \in \mathscr{D}^n_{++},\ p_i = 1,\ H_A(P) \succ 0, \\ H_A(P) - 2\mu e_i e_i^T \succeq 0 \end{array}\right\}. \tag{12}$$

For each fixed $\varepsilon > 0$, the constraint $H_A(P) \succeq \varepsilon I$ gives an SDP-representable inner *approximation of this supremum. Given that each strictly feasible P has a positive minimum eigenvalue, the approximation recovers the supremum as $\varepsilon \downarrow 0$; no optimizer or strict feasibility at the limiting semidefinite boundary is assumed.*

Proof. For $H \succ 0$ and $u \neq 0$, congruence by $H^{-1/2}$ gives $H - tuu^T \succeq 0$ precisely when $1 - t\|H^{-1/2}u\|^2 \ge 0$, i.e.,

$$H - tuu^T \succeq 0 \iff t \le \frac{1}{u^T H^{-1} u}.$$

For the strictly positive definite downdate, the inequality is *strict*:

$$H - tuu^T \succ 0 \iff t < \frac{1}{u^T H^{-1} u}.$$

The equality case belongs to the semidefinite margin only and is never accepted as a strict diagonal Lyapunov certificate.

Apply this with $H = H_A(P)$, $u = e_i$, and $t = 2\mu p_i$ to obtain (11). Formula (12) follows by taking the supremum and normalizing $p_i = 1$ using (9). □

Remark 1. *The margin $\delta_i(A)$ is not a spectral abscissa or an eigenvalue margin. It measures the largest normalized amount that can be removed from the ith coordinate of the symmetric Lyapunov*

form while retaining positive semidefiniteness after optimizing over all diagonal Lyapunov scalings. It is exactly this directional quantity that becomes part of a reciprocal scalar coupling.

Remark 2. *A practical lower bound is obtained without optimization: any diagonal $P \succ 0$ satisfying $H_A(P) \succ 0$ gives $\underline{\delta}_i = \eta_i(A;P) \le \delta_i(A)$, while Lemma 1 gives $\delta_i(A) \le a_{ii}$. In cases of sparse, large-scale blocks, the calculation of the lower bound may be performed through the linear system $H_A(P)w = e_i$ instead of calculating the inverse; a diagonally dominant scaling, if it exists, is a cheap starting point for P. Bounds may be improved through approximate solutions of the margin SDP using sparsity-exploiting conic optimization or first-order methods, monitoring the distance from a_{ii}. Such approximate bounds provide a* sufficient *coupling test if their product is larger than $\beta\gamma$, but failure does not certify instability.*

3. General Two-Block Reductions

I first record an exact Schur-complement formulation valid for arbitrary block sizes and arbitrary coupling matrices. The Schur-complement step is classical; see, e.g., [19].

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$. The matrix $\mathcal{A}$ in (3) is diagonally stable if and only if there exist $P \in \mathscr{D}^n_{++}$ and $Q \in \mathscr{D}^m_{++}$ such that*

$$H_A(P) \succ 0 \tag{13}$$

and

$$H_D(Q) - (B^TP + QC)H_A(P)^{-1}(PB + C^TQ) \succ 0. \tag{14}$$

Whenever (13) and (14) hold, $\mathrm{diag}(P,Q)$ *is a diagonal Lyapunov matrix for $\mathcal{A}$.*

Proof. For $\mathcal{P} = \mathrm{diag}(P,Q)$,

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = \begin{bmatrix} H_A(P) & PB + C^TQ \\ B^TP + QC & H_D(Q) \end{bmatrix}. \tag{15}$$

Positive definiteness of (15) is equivalent to (13) together with the Schur-complement inequality (14). □

Proposition 2. *If $\mathcal{A}$ is diagonally stable, then A and D are diagonally stable.*

Proof. The two diagonal blocks of the positive definite matrix in (15) are positive definite principal submatrices. □

The next result is dimension-free and will also be used to clarify the zero-gain limits of the main theorem.

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$. For arbitrary $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, the matrices*

$$\mathcal{A}_+ = \begin{bmatrix} A & B \\ 0_{m\times n} & D \end{bmatrix}, \qquad \mathcal{A}_- = \begin{bmatrix} A & 0_{n\times m} \\ C & D \end{bmatrix}.$$

are diagonally stable if and only if both A and D are diagonally stable. Here $\mathcal{A}_+$ describes $C = 0$, while $\mathcal{A}_-$ describes $B = 0$; this is the meaning of arbitrary one-way *coupling.*

Proof. Necessity follows from Proposition 2. For sufficiency, choose $P \in \mathscr{D}^n_{++}$ and $Q \in \mathscr{D}^m_{++}$ such that $H_A(P) \succ 0$ and $H_D(Q) \succ 0$.

For $\mathcal{A}_+$, use the Lyapunov matrix $\mathrm{diag}(P, \tau Q)$. The Schur complement with respect to $H_A(P)$ is

$$\tau H_D(Q) - B^T P H_A(P)^{-1} P B,$$

which is positive definite for all sufficiently large $\tau > 0$.

For $\mathcal{A}_-$, the corresponding Schur complement is

$$\tau H_D(Q) - \tau^2 Q C H_A(P)^{-1} C^T Q = \tau\big[H_D(Q) - \tau Q C H_A(P)^{-1} C^T Q\big],$$

which is positive definite for all sufficiently small $\tau > 0$. □

Remark 3. *Theorem 2 is a relative-scaling statement. A feedforward cross term can always be absorbed by increasing or decreasing the Lyapunov weight of the downstream block. In a two-way loop, the same scalar simultaneously magnifies one directional contribution and attenuates the other; a genuine loop-gain restriction then appears.*

4. Reciprocal Single-Channel Coupling in Arbitrary Dimensions

Fix $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, and consider

$$\mathcal{A}_{ij}(\beta, \gamma) = \begin{bmatrix} A & \beta e_i e_j^T \\ \gamma e_j e_i^T & D \end{bmatrix}. \tag{16}$$

We first isolate the scalar optimization responsible for the exact threshold.

Lemma 2. *For $u, v \in \mathbb{R}$ and $\tau > 0$,*

$$f(\tau) = \frac{(u + \tau v)^2}{\tau} = \frac{u^2}{\tau} + 2uv + v^2\tau. \tag{17}$$

Then,

$$\inf_{\tau > 0} f(\tau) = 2|uv| + 2uv = \begin{cases} 4uv, & uv > 0, \\ 0, & uv \le 0. \end{cases} \tag{18}$$

If $uv > 0$, the minimum is attained at $\tau = |u/v|$. If $uv < 0$, f vanishes at $\tau = -u/v$. If exactly one of u, v is zero, the infimum 0 is approached only in the limit $\tau \to \infty$ (when $v = 0$) or $\tau \to 0^+$ (when $u = 0$); if both vanish, $f \equiv 0$.

Proof. The arithmetic–geometric mean inequality gives

$$\frac{u^2}{\tau} + v^2\tau \ge 2|uv|,$$

with equality at $\tau = |u/v|$ when $uv \neq 0$. Adding $2uv$ yields (18). □

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$. The matrix $\mathcal{A}_{ij}(\beta, \gamma)$ in (16) is diagonally stable if and only if the following three statements are all true:*

(i) *A is diagonally stable;*

(ii) *D is diagonally stable;*

(iii)

$$\beta\gamma < \delta_i(A)\delta_j(D). \tag{19}$$

Here, $\delta_i(A)$ and $\delta_j(D)$ are the directional margins defined by (8).

Proof. Conditions (i)–(ii) are necessary by Proposition 2. Assume henceforth that A and D are diagonally stable. Choose arbitrary local diagonal Lyapunov matrices $P \in \mathscr{D}^n_{++}$ and $Q \in \mathscr{D}^m_{++}$ and write

$$p_i = e_i^T P e_i, \qquad q_j = e_j^T Q e_j,$$
$$h_i = e_i^T H_A(P)^{-1} e_i, \qquad k_j = e_j^T H_D(Q)^{-1} e_j.$$

Because the relative weight of the two local Lyapunov functions is free, consider

$$\mathcal{P}(\tau) = \operatorname{diag}(P, \tau Q), \qquad \tau > 0.$$

The off-diagonal block of the corresponding Lyapunov matrix is

$$P(\beta e_i e_j^T) + (\gamma e_j e_i^T)^T(\tau Q) = (\beta p_i + \tau\gamma q_j) e_i e_j^T.$$

Applying the Schur complement with respect to $H_A(P)$ gives

$$\tau H_D(Q) - (\beta p_i + \tau\gamma q_j)^2 h_i e_j e_j^T \succ 0. \tag{20}$$

A rank-one downdate of the positive definite matrix $\tau H_D(Q)$ is positive definite exactly when

$$\frac{(\beta p_i + \tau\gamma q_j)^2}{\tau} < \frac{1}{h_i k_j}. \tag{21}$$

We distinguish the nonpositive and positive loop-gain cases. Throughout, the necessary diagonal stability of the two local blocks follows from Proposition 2.

Case 1: $\beta\gamma \le 0$. Since both directional margins are strictly positive, $\beta\gamma < \delta_i(A)\delta_j(D)$ holds automatically. For sufficiency, if $\beta\gamma < 0$, both gains are nonzero and the finite positive choice

$$\tau = \frac{|\beta| p_i}{|\gamma| q_j}$$

annihilates $\beta p_i + \tau\gamma q_j$, so that (20) reduces to $\tau H_D(Q) \succ 0$. If $\beta\gamma = 0$, at least one directional coupling vanishes; Theorem 2 supplies a strict diagonal certificate. Thus, no positive-feedback restriction is required in this case.

Case 2: $\beta\gamma > 0$. *Necessity.* Suppose the full matrix has a positive diagonal Lyapunov certificate $\mathcal{P}_0 = \operatorname{diag}(P_0, Q_0)$. Its two principal Lyapunov blocks are $H_A(P_0) \succ 0$ and $H_D(Q_0) \succ 0$. Set $P = P_0$, $Q = Q_0$ and $\tau = 1$ in (20) and (21); this uses the given certificate directly, as opposed to asserting a prescribed relative structure for an arbitrary certificate. By Lemma 2,

$$4\beta\gamma p_i q_j \le (\beta p_i + \gamma q_j)^2 < \frac{1}{h_i k_j}.$$

Consequently,

$$\beta\gamma < \eta_i(A; P_0)\eta_j(D; Q_0) \le \delta_i(A)\delta_j(D).$$

Sufficiency. Assume $\beta\gamma < \delta_i(A)\delta_j(D)$. Both margins are finite, positive suprema over nonempty sets. Since the inequality is strict, one can choose *actual admissible* diagonal matrices P, Q satisfying

$$\beta\gamma < \eta_i(A; P)\eta_j(D; Q).$$

No attainment of either supremum is needed. With the finite positive scaling $\tau_* = |\beta| p_i / (|\gamma| q_j)$, Lemma 2 yields

$$\frac{(\beta p_i + \tau_*\gamma q_j)^2}{\tau_*} = 4\beta\gamma p_i q_j < \frac{1}{h_i k_j},$$

which proves (20). Equality in the global threshold is excluded because every Lyapunov inequality used here is strict. □

Corollary 1. *If A and D are diagonally stable and $\beta\gamma \leq 0$, then $\mathcal{A}_{ij}(\beta, \gamma)$ is diagonally stable for arbitrary $|\beta|$ and $|\gamma|$.*

Corollary 2. *Let P and Q be local diagonal Lyapunov matrices such that*

$$\beta\gamma < \eta_i(A; P)\eta_j(D; Q). \tag{22}$$

If $\beta\gamma > 0$, then

$$\tau_* = \frac{|\beta| p_i}{|\gamma| q_j} \tag{23}$$

minimizes the left-hand side of (21), and

$$\operatorname{diag}(P, \tau_* Q)$$

is a diagonal Lyapunov matrix for $\mathcal{A}_{ij}(\beta, \gamma)$. If $\beta\gamma < 0$, the same τ_* cancels the cross term exactly. If $\gamma = 0$ and $\beta \neq 0$, any

$$\tau > \beta^2 p_i^2 h_i k_j \tag{24}$$

is sufficient; if $\beta = 0$ and $\gamma \neq 0$, any

$$0 < \tau < \frac{1}{\gamma^2 q_j^2 h_i k_j} \tag{25}$$

is sufficient.

Proof. The first two assertions are the minimizing and cancellation choices in Lemma 2. If $\gamma = 0$, condition (21) becomes $\beta^2 p_i^2/\tau < 1/(h_i k_j)$, which is (24); the case $\beta = 0$ is analogous. □

Remark 4. *The matrix class in Theorem 3 is different from the rank-one network form "diagonal matrix plus xy^T" analyzed in [12]. In this case, A and D are fully populated and independent, and the reciprocal off-diagonal perturbation has rank two provided that $\beta\gamma \neq 0$. Unlike the diagonal-plus-rank-one structure, which leads to a single global rank-one update, here the two directional maps occupy distinct off-diagonal blocks and produce a rank-two perturbation of $\operatorname{diag}(A, D)$. Their two local Lyapunov weights must first be balanced. Only thereafter does the Schur complement reduce the coupled problem to a rank-one downdate; this distinction is the mechanism of the new product-of-margins criterion.*

5. Planar Blocks: Closed Form and Relation to Acyclic Theory

I now specialize the directional margin to a planar matrix. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{26}$$

For $r > 0$, set $R(r) = \operatorname{diag}(r, 1)$. Then,

$$H_M(r) = R(r)M + M^T R(r) = \begin{bmatrix} 2ra & rb + c \\ rb + c & 2d \end{bmatrix}. \tag{27}$$

The matrix M is diagonally stable if and only if

$$a > 0, \qquad d > 0, \qquad \Delta := ad - bc > 0, \tag{28}$$

that is, if and only if it is a P-matrix [1,2].

Proposition 3. *Assume* (28). *Then,*

$$\delta_1(M) = \min\left\{a, \frac{\det M}{d}\right\} = \begin{cases} a, & bc \le 0, \\ \dfrac{\det M}{d}, & bc > 0, \end{cases} \tag{29}$$

$$\delta_2(M) = \min\left\{d, \frac{\det M}{a}\right\} = \begin{cases} d, & bc \le 0, \\ \dfrac{\det M}{a}, & bc > 0. \end{cases} \tag{30}$$

If $bc \neq 0$, both suprema are attained at

$$r_* = \left|\frac{c}{b}\right|. \tag{31}$$

Proof. The determinant of (27) is

$$\det H_M(r) = 4rad - (rb + c)^2.$$

Since

$$e_1^T H_M(r)^{-1} e_1 = \frac{2d}{\det H_M(r)}, \qquad e_2^T H_M(r)^{-1} e_2 = \frac{2ra}{\det H_M(r)},$$

one can obtain

$$\eta_1(M; R(r)) = a - \frac{(rb + c)^2}{4dr},$$
$$\eta_2(M; R(r)) = d - \frac{(rb + c)^2}{4ar}.$$

The common penalty satisfies

$$\frac{(rb + c)^2}{r} = b^2 r + 2bc + \frac{c^2}{r},$$

whose infimum over $r > 0$ is $2|bc| + 2bc$. When $bc \neq 0$, the minimizing ratio is (31); when $b = 0$ and $c \neq 0$, the infimum penalty 0 is approached as $r \to \infty$; when $c = 0$ and $b \neq 0$, it is approached as $r \to 0^+$. If $b = c = 0$, the penalty is identically zero, and every $r > 0$ is an optimizer. Thus, the directional margin remains well defined in each zero-entry case even though a finite optimal ratio may not exist. Substitution gives (29) and (30). □

Corollary 3. *Let*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

be diagonally stable, and let $i, j \in \{1, 2\}$. With $\bar{1} = 2$ and $\bar{2} = 1$,

$$\begin{bmatrix} A & \beta e_i e_j^T \\ \gamma e_j e_i^T & D \end{bmatrix} \text{ is diagonally stable} \tag{32}$$

if and only if

$$\beta\gamma < \min\left\{ a_{ii}d_{jj},\ \frac{a_{ii}\det D}{d_{\overline{jj}}},\ \frac{(\det A)d_{jj}}{a_{\overline{ii}}},\ \frac{(\det A)(\det D)}{a_{\overline{ii}}d_{\overline{jj}}} \right\}. \tag{33}$$

Proof. Combine Theorem 3 and Proposition 3. Indeed, all four quantities appearing in (33) are positive: diagonal stability of A, D implies $a_{ii}, a_{\overline{ii}}, d_{jj}, d_{\overline{jj}}, \det A, \det D > 0$. For arbitrary positive x_1, x_2, y_1, y_2,

$$\min\{x_1, x_2\}\min\{y_1, y_2\} = \min\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}.$$

Thus, (33) is *exactly* the product of the two planar directional margins; it is not a potentially invalid algebraic factorization. Equivalently, the threshold may be written as

$$\beta\gamma < \min\left\{ a_{ii}, \frac{\det A}{a_{\overline{ii}}} \right\} \min\left\{ d_{jj}, \frac{\det D}{d_{\overline{jj}}} \right\}.$$

□

Consistency with the Berman–Hershkowitz Acyclic Theorem

For two planar blocks joined by only one reciprocal channel, the nondirected graph of the 4×4 matrix in Corollary 3 is a forest: each planar block contributes at most one internal edge, and the coupling contributes one bridge. Therefore the classical result of Berman and Hershkowitz [3] applies: diagonal stability is equivalent to positivity of all principal minors.

This observation provides an independent check of (33). Assuming A and D are P-matrices, the only additional principal minors involving the bridge are

$$a_{ii}d_{jj} - \beta\gamma, \tag{34}$$
$$(\det A)d_{jj} - a_{\overline{ii}}\beta\gamma, \tag{35}$$
$$a_{ii}(\det D) - d_{\overline{jj}}\beta\gamma, \tag{36}$$
$$(\det A)(\det D) - a_{\overline{ii}}d_{\overline{jj}}\beta\gamma. \tag{37}$$

The four principal-minor inequalities are equivalent, respectively, to $\beta\gamma$ being smaller than

$$a_{ii}d_{jj}, \quad \frac{(\det A)d_{jj}}{a_{\overline{ii}}}, \quad \frac{a_{ii}\det D}{d_{\overline{jj}}}, \quad \frac{(\det A)(\det D)}{a_{\overline{ii}}d_{\overline{jj}}}.$$

All divisors are strictly positive. Requiring all four inequalities simultaneously is therefore equivalent to their four-term minimum, precisely (33). This independently verifies the factorized directional-margin expression above and the equivalence with the acyclic principal-minor test.

Remark 5. *Accordingly, the planar equivalence* (33) *should be viewed as a closed-form evaluation of the general directional-margin theorem and as a constructive Lyapunov formulation of a threshold already consistent with classical forest theory. The dimension-free Theorem 3 is not restricted to acyclic internal blocks: A and D may contain arbitrary graph cycles, as illustrated in Example 1 below.*

6. Certificates and Examples

For planar blocks with nonzero off-diagonal entries, Proposition 3 makes Corollary 2 completely explicit.

Proposition 4. *Assume the hypotheses of Corollary 3, suppose*

$$a_{12}a_{21} \neq 0, \qquad d_{12}d_{21} \neq 0,$$

and assume (33). Set

$$r_* = \left|\frac{a_{21}}{a_{12}}\right|, \qquad s_* = \left|\frac{d_{21}}{d_{12}}\right|, \tag{38}$$

and

$$\rho_i^* = e_i^T R(r_*) e_i, \qquad \sigma_j^* = e_j^T R(s_*) e_j.$$

If $\beta\gamma \neq 0$, define

$$\tau_* = \frac{|\beta|\rho_i^*}{|\gamma|\sigma_j^*}. \tag{39}$$

Then

$$\mathcal{P}_* = \operatorname{diag}\big(R(r_*), \tau_* R(s_*)\big) \tag{40}$$

is a diagonal Lyapunov matrix. For $\beta\gamma < 0$, the off-diagonal Lyapunov block vanishes identically. If one directional gain is zero, the explicit bounds (24) and (25) apply.

Proof. The choices (38) attain the planar directional margins by Proposition 3. The conclusion is then Corollary 2. More explicitly, the four-term bound in (33) is identical to $\delta_i(A)\delta_j(D)$, since all four factors in the underlying two minima are positive. Hence, for $\beta\gamma > 0$, the strict four-term inequality (33) ensures $4\beta\gamma\rho_i^*\sigma_j^* < 1/(h_i k_j)$ for the optimized local matrices $R(r_*), R(s_*)$, where $h_i = e_i^T H_A(R(r_*))^{-1} e_i$ and $k_j = e_j^T H_D(R(s_*))^{-1} e_j$. The scaling (39) then gives the strict Schur condition. For $\beta\gamma < 0$, the off-diagonal Lyapunov block cancels exactly. □

Example 1 (A cyclic 5×5 interconnection beyond forest theory)**.** *Consider*

$$A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Both matrices are symmetric positive definite and therefore diagonally stable with $P = I_3$ and $Q = I_2$. The nondirected graph of A is a triangle, and so the global graph obtained after adding a channel between the first coordinates is not acyclic. Thus, the forest theorem of [3] does not apply to the full matrix.

For $i = j = 1$, direct inversion gives

$$\eta_1(A; I_3) = \frac{10}{3}, \qquad \eta_1(D; I_2) = \frac{8}{3}.$$

Choose $\beta = 3$ and $\gamma = 1$. The full interconnected matrix and a diagonal Lyapunov certificate are

$$\mathcal{A}_{11}(3,1) = \begin{bmatrix} 4 & -1 & -1 & 3 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 0 \\ 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}, \qquad \mathcal{P} = \operatorname{diag}(1,1,1,3,3).$$

$$\beta\gamma = 3 < \frac{80}{9} = \eta_1(A; I_3)\eta_1(D; I_2) \leq \delta_1(A)\delta_1(D).$$

Corollary 2 gives $\tau_* = 3$ *and the explicit certificate*

$$\mathcal{P} = \mathrm{diag}(I_3, 3I_2).$$

For verification, the smallest eigenvalue of

$$\mathcal{P}\mathcal{A}_{11}(3,1) + \mathcal{A}_{11}(3,1)^T\mathcal{P}$$

is approximately $2.7477 > 0$. *In fact, the five eigenvalues of this symmetric Lyapunov matrix are approximately* 2.747736, 7.641878, 10.000000, 14.440478, *and* 25.169907. *This example illustrates the purpose of the general theorem: the internal block itself may carry cycles, while the reciprocal channel is handled exactly through directional local margins.*

For the initial condition $z(0) = (1, -1, 1/2, -1/2, 1)^T$, *the exact response is* $z(t) = \exp(-t\mathcal{A}_{11}(3,1))z(0)$; *its Euclidean norm and* $V(t) = z(t)^T\mathcal{P}z(t)$ *are reported in Figure 1.*

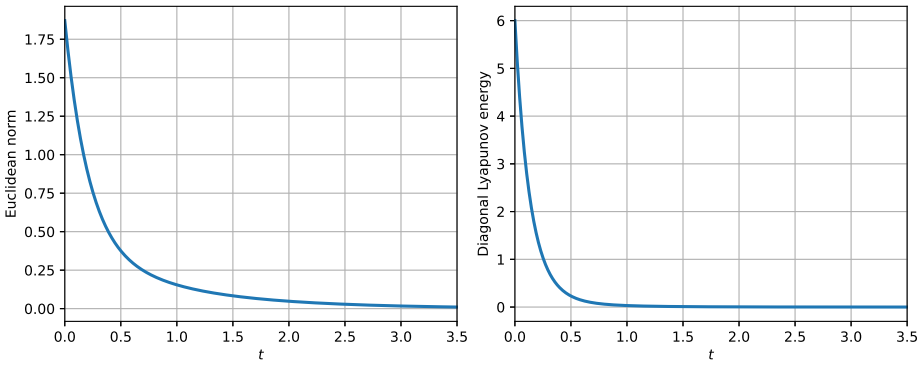


Figure 1. Exact matrix-exponential response for the cyclic 5×5 system: $\|z(t)\|_2$ and the separable Lyapunov energy $V(t)$.

Example 2 (Asymmetric fourth-order feedback). *Let*

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 1 \\ -\frac{1}{2} & 3 \end{bmatrix}. \tag{41}$$

Both are P-matrices and hence diagonally stable. Couple coordinate $i = 1$ *of the first subsystem to coordinate* $j = 2$ *of the second by*

$$B = 2e_1e_2^T, \qquad C = 3e_2e_1^T.$$

The resulting explicitly specified matrix is

$$\mathcal{A}_{12}(2,3) = \begin{bmatrix} 3 & -1 & 0 & 2 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 3 & 0 & -\frac{1}{2} & 3 \end{bmatrix}.$$

Since $a_{12}a_{21} = -2 < 0$ *and* $d_{12}d_{21} = -1/2 < 0$,

$$\delta_1(A) = 3, \qquad \delta_2(D) = 3.$$

Thus,

$$\beta\gamma = 6 < 9 = \delta_1(A)\delta_2(D).$$

To verify the four-term criterion independently, one has $\det A = 8$, $\det D = 13/2$, $a_{22} = d_{11} = 2$, *and hence*

$$\left\{ a_{11}d_{22}, \frac{a_{11}\det D}{d_{11}}, \frac{(\det A)d_{22}}{a_{22}}, \frac{(\det A)(\det D)}{a_{22}d_{11}} \right\} = \left\{ 9, \frac{39}{4}, 12, 13 \right\}.$$

The minimum is exactly 9. *For* $\beta\gamma = 6$, *the four cross-channel principal minors in* (34)–(37) *are, respectively,* 3, 12, 15/2, *and* 28, *all strictly positive.*

The closed certificate is

$$r_* = 2, \qquad s_* = \frac{1}{2}, \qquad \tau_* = \frac{4}{3},$$

and so

$$\mathcal{P} = \operatorname{diag}\left(2, 1, \frac{2}{3}, \frac{4}{3}\right). \tag{42}$$

A direct calculation gives

$$\mathcal{P}\mathcal{A} + \mathcal{A}^T\mathcal{P} = \begin{bmatrix} 12 & 0 & 0 & 8 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \succ 0,$$

whose eigenvalues are 4, 8/3, *and* $10 \pm 2\sqrt{17}$.

The smallest eigenvalue is $10 - 2\sqrt{17} \approx 1.7538 > 0$. *For* $z(0) = (1, -1, 1/2, -1/2)^T$, *Figure 2 plots* $z(t) = \exp(-t\mathcal{A}_{12}(2,3))z(0)$ *and the certified energy* $z(t)^T\mathcal{P}z(t)$.

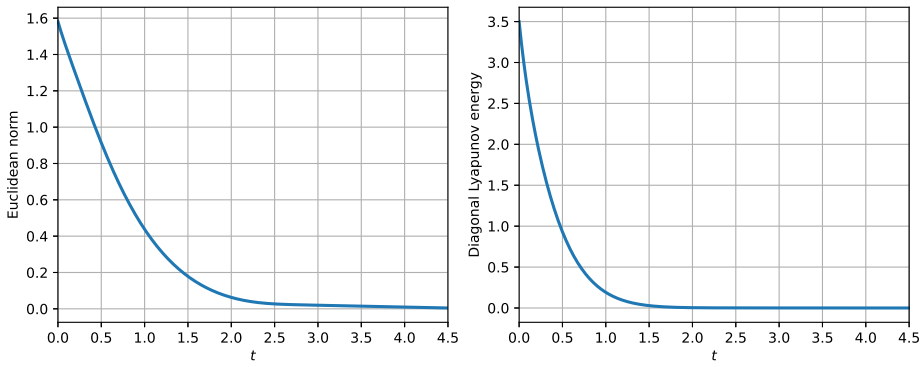


Figure 2. Exact dynamics and decreasing diagonal Lyapunov energy for Example 2.

Example 3 (Loss of diagonal stability while Hurwitz stability persists)**.** *Keep A and D from* (41), *but choose*

$$B = 3e_1e_2^T, \qquad C = 4e_2e_1^T.$$

The full matrix is

$$\mathcal{A}_{12}(3,4) = \begin{bmatrix} 3 & -1 & 0 & 3 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 4 & 0 & -\frac{1}{2} & 3 \end{bmatrix}.$$

The recalculated four candidate bounds are $9, 39/4, 12, 13$, *and so their minimum, equivalently* $\delta_1(A)\delta_2(D)$, *is* exactly 9. *Since* $\beta\gamma = 12 > 9$, *this example lies beyond the* strict diagonal-stability threshold *for the fixed local blocks; the inequality at* $\beta\gamma = 9$ *is strict, and equality does*

not admit a diagonal certificate. At $\beta\gamma = 12$*, the four relevant cross-channel principal minors are, respectively,*

$$9 - 12 = -3, \qquad 24 - 2(12) = 0, \qquad \tfrac{39}{2} - 2(12) = -\tfrac{9}{2}, \qquad 52 - 4(12) = 4.$$

In particular, the negative 2×2 *principal minor proves that the full matrix is not a P-matrix and hence cannot be diagonally stable. The number* 9 *is* not *asserted to be the Hurwitz spectral boundary.*

Nevertheless, its eigenvalues are approximately

$$6.1689, \qquad 2.2230, \qquad 1.3997, \qquad 0.2084,$$

and so the system matrix $-\mathcal{A}$ *is Hurwitz at this specified coupling value. The positive real parts are confirmed numerically (the smallest is about* 0.208394*). This demonstrates that the sharp diagonal-stability threshold* 9 *and the spectral-stability boundary need not coincide; no global spectral threshold is inferred from these particular numerical eigenvalues.*

For completeness, an ordinary (not necessarily diagonal) Lyapunov certificate is the unique symmetric solution X *of* $\mathcal{A}_{12}(3,4)^T X + X\mathcal{A}_{12}(3,4) = I_4$*. It is numerically given (to six decimal places) by*

$$X \approx \begin{bmatrix} 3.969280 & 1.693963 & -0.814022 & -3.698941 \\ 1.693963 & 1.096981 & -0.412709 & -1.673624 \\ -0.814022 & -0.412709 & 0.440379 & 0.761515 \\ -3.698941 & -1.673624 & 0.761515 & 3.611770 \end{bmatrix}.$$

Its computed eigenvalues are approximately 0.077892, 0.245875, 0.335973, *and* 8.458669*, all positive; this independently verifies Hurwitz stability. These numerical values are reproducible using the second-round verification script supplied with the revised source package. The same initial state as in Example 2 gives the norm response in Figure 3.*

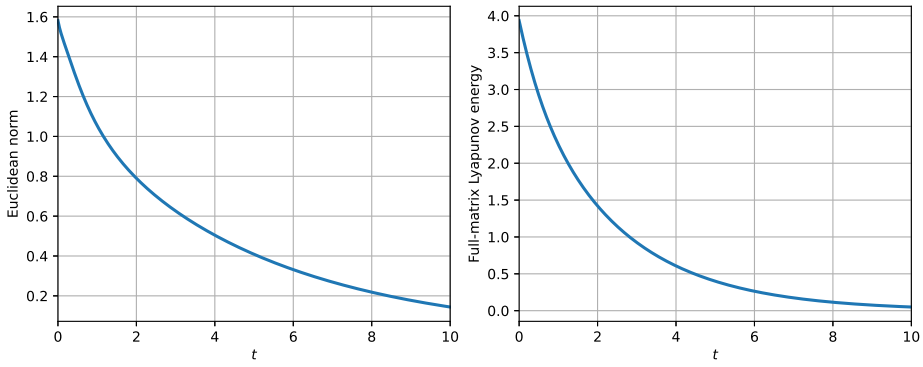


Figure 3. Matrix-exponential response for the Hurwitz, non-diagonally-stable feedback matrix in Example 3; a stable trajectory does not imply the existence of a diagonal quadratic Lyapunov certificate.

Example 4 (Large opposite-sign coupling)**.** *Again, use* (41) *and choose*

$$B = 5e_1e_2^T, \qquad C = -20e_2e_1^T.$$

Here, $\beta\gamma = -100 < 0$*. Corollary 1 guarantees diagonal stability despite the large coupling magnitudes. Taking* $r_* = 2$*,* $s_* = 1/2$*, and*

$$\tau_* = \frac{5 \cdot 2}{20 \cdot 1} = \frac{1}{2}$$

gives

$$\beta\rho_1^* + \tau_*\gamma\sigma_2^* = 10 - 10 = 0.$$

Hence, one may take

$$\mathcal{P} = \operatorname{diag}\left(2, 1, \frac{1}{4}, \frac{1}{2}\right),$$

and the cross block of the Lyapunov matrix is exactly zero.

Example 5 (Arbitrary feedforward block)**.** *Let A, D be as in (41), and set*

$$B = \begin{bmatrix} 10 & -7 \\ 4 & 12 \end{bmatrix}, \qquad C = 0.$$

The coupling is neither diagonal nor rank one. Theorem 2 nevertheless shows that

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

is diagonally stable because both diagonal blocks are diagonally stable.

7. Discussion and Conclusions

The main result can be interpreted as an exact diagonal Lyapunov small-gain theorem for a reciprocal scalar channel. The product $\beta\gamma$ is the loop gain, while $\delta_i(A)$ and $\delta_j(D)$ are optimized local reserves. The relative scale of the two local Lyapunov functions removes dependence on the individual magnitudes of β and γ and leaves only their product. The relative scaling exactly cancels out the coupling term whenever the signs are opposite, whereas the scalar minimization provides the exact threshold for positive feedback whenever the signs are the same.

The formulation also makes clear the connection with the known structured diagonal stability. The classical acyclic theorem [3] gives a complete explanation of the reason why the planar single-channel 4×4 case admits a principal-minor representation. Instead of treating that case as an independent new matrix class, I exploit it in proving the correctness of the directional-margin formulation and deriving the closed form of the certificate. The general theorem permits A and D to be arbitrary diagonally stable matrices of arbitrary sizes and thus considers global graphs with internal cycles, as illustrated by Example 1. It also differs from the previous diagonal-plus-rank-one cases [12], where the two internal blocks are full and the perturbation is rank two.

For practical computation in high dimensions, each local margin can be computed separately according to the semidefinite characterization (12). The appeal of this formulation lies in the fact that the two systems of computations are kept separate until the scalar coupling test is performed.

The numerical propagation of large coupled systems is a separate computational question. Krylov projection has been studied for large-scale differential systems [20], exponential-function collocation for linear matrix differential equations [21], and extended-block-Krylov methods for fractional differential systems [22]. These references motivate scalable computational directions but are not used to establish the exact diagonal-stability threshold in this paper.

For example, consider the fixed but uncertain blocks $A \in \mathcal{U}_A$ and $D \in \mathcal{U}_D$, which are all individually diagonally stable, with the lower bounds $\underline{\delta}_A := \inf_{A \in \mathcal{U}_A} \delta_i(A) > 0$ and $\underline{\delta}_D := \inf_{D \in \mathcal{U}_D} \delta_j(D) > 0$. If admissible scalar gains satisfy the condition $\sup\{\beta\gamma\} < \underline{\delta}_A \underline{\delta}_D$, the result of Theorem 3 guarantees diagonal stability in each case of freezing. This is a robust pointwise assertion; it does not, by itself, provide a common diagonal matrix for time-varying uncertainty and switching. It would require the common feasible local matrices P, Q and a common relative scaling τ satisfying the Schur inequality.

Several directions remain open. Two or more simultaneous reciprocal channels lead to a matrix-valued coupling penalty rather than the scalar expression in Lemma 2; exact elimination of the relative Lyapunov weights is then substantially more difficult. Other natural extensions include block-diagonal rather than fully diagonal Lyapunov functions, uncertain channel gains, and nonlinear interconnections whose local dissipation inequalities reduce to the present directional-margin framework.

In contrast to full quadratic Lyapunov matrices or Lyapunov–Krasovskii functionals, a diagonal quadratic form uses only $n + m$ weights, is separable across coordinates, and gives a transparent sharp loop-gain test for the present time-invariant single-channel class. Its limitation is conservatism for general Hurwitz matrices: a stable system can fail to admit any diagonal certificate, as in Example 3. Delay-dependent LKFs are needed to capture history effects, while the present method has no delay state or fractional derivative. Thus, it is an exact characterization of a structured certificate, not a universal test for dynamical stability.

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