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Adaptive Adams-Type Methods Based on Algebraic, Trigonometric, and Exponential Interpolation

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Abstract

We consider an adaptive Adams-type multistep method for the numerical solution of ordinary differential equations based on algebraic, trigonometric, and exponential interpolation (the ATE method). For each interpolation class, the corresponding schemes are derived following the same construction as classical Adams methods. The stability properties of the trigonometric and exponential variants are analyzed, highlighting their differences relative to the algebraic method. The adaptive ATE algorithm selects one of the three schemes at each step according to a local interpolation indicator. This enables the method to adjust to different types of solution behavior. Numerical experiments are presented across a variety of differential equations, and the accuracy and execution times of the methods are compared.

Keywords: ordinary differential equation; Adams–Bashforth method; linear multistep method; trigonometric Adams method; exponential Adams method; stability analysis; convergence analysis

MSC: 65L06; 65L05; 65L20; 65D05

1. Introduction

Among the classical approaches for initial value problems, linear multistep methods are widely used because they reuse information from previous steps. In particular, Adams–Bashforth methods approximate the next value of the solution by applying polynomial interpolation to previously computed values of the right-hand side of the differential equation. This approach is effective for many smooth problems, but its efficiency may decrease when the solution has a particular structure, such as oscillatory behavior or rapid exponential growth or decay. One way to improve accuracy is to use a variable step size with a prescribed error tolerance [1–5]. This leads to variable coefficients of the multistep method that need to be recalculated at each change of the step size. A similar idea explored in [6] leads to constant coefficients but introduces two additional initial value problems that need to be solved in conjunction with the original one. If information about the qualitative structure of the solution is known in advance, then a natural modification of numerical schemes would be to adapt the method to this structure. However, in many practical problems, the qualitative behavior of the solution is not known in advance or may change over the integration interval. In such cases, selecting a single interpolation basis a priori may be inadequate. This motivates the adaptive approach proposed in this paper, which does

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not require prior knowledge of the solution's qualitative behavior. Instead, the interpolation basis is selected locally by comparing derivative-based criteria, allowing the method to respond to changes in the solution during the integration process. The natural mathematical framework for such adaptations is provided by generalized interpolation using different classes of interpolating functions. Classical polynomial interpolation is one possible choice; trigonometric and exponential bases provide alternatives that may be better suited to oscillatory or exponentially varying solutions. The use of exponential sums in step-by-step integration was considered by Brock and Murray [7], while Gautschi introduced methods based on trigonometric polynomials for oscillatory problems [8]. Further developments include trigonometrically fitted Adams–Bashforth–Moulton schemes [9,10], exponentially fitted Adams-type methods [11], and general frameworks for exponential fitting [12]. A review of trigonometrically fitted methods and their applications to oscillatory problems is given in [13].

Following this general interpolation framework, we consider Adams-type multistep methods constructed from three different interpolation bases: algebraic, trigonometric, and exponential. These three variants should not be viewed as unrelated numerical schemes but rather as members of the same family obtained by replacing the interpolation basis while preserving the underlying Adams construction [14–16]. Earlier studies laid the foundation for an interpolation framework based on algebraic, trigonometric, and exponential Adams-type formulas and introduced adaptive strategies for their combined application [14–16]. These contributions demonstrated the potential of selecting the approximation basis according to the local behavior of the problem. However, the stability properties of the resulting fixed-basis schemes have not been investigated in detail, and a systematic comparison of these schemes remains to be developed.

Building on the interpolation-based Adams framework and the adaptive strategies developed in [14–16], the present study provides the following contributions:

1. Third-order convergence is established for the trigonometric and exponential Adams-type methods in Theorems 3 and 4, respectively. The analysis uses bounds on the interpolation derivatives that are uniform in the step size and explicitly stated conditions on the accuracy of the starting values.
2. Third-order convergence is established for the adaptive ATE algorithm itself in Theorem 6, providing a convergence result for the procedure that selects among the three interpolation-based formulas.
3. The stability of the fixed trigonometric and exponential methods with respect to the test equation $y' = qy$ is analyzed. Their stability regions are compared with that of the classical Adams–Bashforth method, and an explicit analytical stability condition is derived using the Schur criterion in Proposition 2.
4. An expanded numerical study compares the fixed-basis methods and the adaptive ATE procedure in terms of accuracy and execution time. The tests include a nonlinear problem, a smooth mixed-component problem with oscillatory frequencies different from the fitting frequency, and stability tests in which the adaptive ATE procedure produces stable numerical results while individual fixed-basis methods become unstable.

The paper is organized as follows. Section 2 introduces the algebraic, trigonometric, and exponential Adams-type methods. Section 3 examines the convergence and stability properties of the individual schemes. Section 4 presents the local selection criterion, the adaptive ATE algorithm, and its convergence. Section 5 contains numerical experiments comparing the fixed-basis methods with the adaptive approach. Finally, Section 6 summarizes the main conclusions.

2. Interpolation-Based Adams-Type Methods

The construction of Adams-type methods is closely connected with the choice of an approximation basis for the right-hand side of the differential equation. In the classical case, this basis is algebraic and leads to the standard Adams–Bashforth formulas. However, the same idea can be extended by using other classes of interpolating functions. This section introduces the general framework of linear multistep methods and explains how algebraic, trigonometric, and exponential Adams-type schemes can be viewed as particular cases of the same interpolation-based construction. This unified viewpoint allows the resulting schemes to be compared systematically [17]. Let us consider the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{1}$$

where f is assumed to be sufficiently smooth on the considered interval. Let $H > 0$ be the actual constant mesh step and let $x_n = x_0 + nH$, $y_n \approx y(x_n)$, $f_n = f(x_n, y_n)$ for $n = 1, 2, \ldots, N$ and $N \in \mathbb{N}$. Linear multistep methods approximate the solution at a new mesh point by using information from several previously computed points. In their general form, k-step linear multistep methods can be written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = H \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{2}$$

where α_j and β_j are real coefficients defining the particular multistep method. Explicit methods correspond to $\beta_j = 0$, whereas $\beta_j \neq 0$ yields implicit schemes. In the present paper, we focus on Adams–Bashforth-type formulas constructed from three derivative values. Therefore, the general k-step form is specialized to the three-step case. For the fitted Adams-type methods considered, it is convenient to follow the notation in which the distance between two neighboring nodes is written as $2h$. Therefore, throughout the fitted Adams derivations, we set

$$H = 2h. \tag{3}$$

Thus, H denotes the actual step size of the numerical method, while h is the half-step parameter used in the fitted coefficient formulas.

2.1. Adams–Bashforth

As is well known, Adams-type methods can be derived from the integral form of the initial value problem. To motivate the construction of trigonometric and exponential fitting, we briefly recall the procedure [18]. Integrating the differential equation over the interval $[x_n, x_{n+1}]$, we obtain

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx. \tag{4}$$

Assume that the values of the derivative (or their approximations) are available at the points x_{n-2}, x_{n-1}, and x_n. This corresponds, after a simple shift of indices, to the three-step case ($k = 3$) of the general linear multistep formula (2). Since $y' = f(x, y)$, the integrand $f(x, y(x))$ in (4) is approximated by the quadratic Lagrange interpolating polynomial

$$P_2(x) = y'_{n-2} L_{n-2}(x) + y'_{n-1} L_{n-1}(x) + y'_n L_n(x),$$

where

$$L_k(x) = \prod_{\substack{j=n-2 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}, \qquad k = n-2, n-1, n.$$

Substituting this approximation,

$$f(x, y(x)) \approx P_2(x),$$

into (4) and evaluating the integral give the classical three-step Adams–Bashforth method

$$y_{n+1} = y_n + \frac{h}{6}(23y'_n - 16y'_{n-1} + 5y'_{n-2}), \tag{5}$$

where $h = H/2$ according to (3).

The trigonometric and exponential Adams-type methods are obtained by replacing the algebraic interpolation polynomial with the corresponding fitted interpolation functions while preserving the same integral construction.

2.2. Trigonometric Fitting

The algebraic Adams–Bashforth formula is obtained by approximating the right-hand side with an algebraic interpolation polynomial. For problems with oscillatory behavior, however, it is advantageous to replace the polynomial interpolation basis by a trigonometric one [14,15,19]. A trigonometric polynomial of order m has the form

$$T_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kx + b_k \sin kx),$$

and is determined by $2m+1$ interpolation conditions. In the present three-step construction, $m = 1$, and therefore the interpolant T_1 is determined by the three derivative values at x_{n-2}, x_{n-1}, and x_n. For $m = 1$, the general expression reduces to $T_1(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$. The following formula is its equivalent representation, obtained by imposing the three interpolation conditions; its basis functions belong to $\mathrm{span}\{1, \cos x, \sin x\}$. We approximate with the trigonometric interpolating function

$$T_1(x; y') = \sum_{k=n-2}^{n} y'_k \prod_{\substack{j=n-2 \\ j\neq k}}^{n} \frac{\sin\left(\frac{x - x_j}{2}\right)}{\sin\left(\frac{x_k - x_j}{2}\right)},$$

which satisfies $T_1(x_k; y') = y'_k$ for $k = n-2,\ n-1,\ n$ [14,15]. Replacing $f(x, y(x))$ in (4) by $T_1(x; y')$ gives

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} T_1(x; y')\, dx. \tag{6}$$

Evaluating the integral, we get the three-step trigonometrically fitted Adams method

$$y_{n+1} = y_n + h\big[t_1(h)y'_n - t_2(h)y'_{n-1} + \big(2 - t_1(h) + t_2(h)\big)y'_{n-2}\big], \tag{7}$$

where

$$t_1(h) = \frac{2h\cos h + \sin(3h) - \sin(5h)}{2h \sin h \sin(2h)}, \quad t_2(h) = \frac{2h\cos(2h) + \sin(2h) - \sin(4h)}{2h \sin^2 h}.$$

2.3. Exponential Fitting

An exponential interpolant of order m may be written equivalently in the hyperbolic basis

$$E_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cosh kx + b_k \sinh kx).$$

It contains $2m+1$ coefficients and is therefore determined by $2m+1$ interpolation conditions. In the present construction, the algebraic interpolant is replaced by the corresponding exponential interpolation formula. This formula provides the exponential analogue of the trigonometric interpolation formula and allows the same Adams-type integral construction to be carried out using a hyperbolic interpolation basis. For $m=1$, the general exponential interpolant reduces to $E_1(x) = \frac{a_0}{2} + a_1 \cosh x + b_1 \sinh x$. The following representation is obtained by imposing the three interpolation conditions. The products of the half-angle hyperbolic sine functions belong to the space spanned by 1, $\cosh x$, and $\sinh x$, establishing the equivalence of the two representations.

Similarly to the trigonometric case, we construct the interpolant of y'

$$E_1(x;y') = \sum_{k=n-2}^{n} y'_k \prod_{\substack{j=n-2 \\ j\neq k}}^{n} \frac{\sinh\left(\frac{x-x_j}{2}\right)}{\sinh\left(\frac{x_k-x_j}{2}\right)},$$

which satisfies $E_1(x_k;y') = y'_k$ for $k = n-2, n-1, n$. Replacing $f(x,y(x))$ in (4) by $E_1(x;y')$ gives

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} E_1(x;y')\,dx. \tag{8}$$

Evaluating this integral yields the three-step exponentially fitted Adams method

$$y_{n+1} = y_n + h\left[e_1(h)y'_n - e_2(h)y'_{n-1} + \left(2 - e_1(h) + e_2(h)\right)y'_{n-2}\right], \tag{9}$$

where

$$e_1(h) = \frac{\sinh(5h) - \sinh(3h) - 2h\cosh h}{2h\sinh h \sinh(2h)}, \quad e_2(h) = \frac{\sinh(4h) - \sinh(2h) - 2h\cosh(2h)}{2h\sinh^2 h}.$$

3. Convergence Analysis

In this section, we establish the convergence of the trigonometric and exponential Adams–Bashforth methods. For this purpose, we recall some classic definitions. In their original context, they refer to linear multistep methods with constant coefficients, i.e., methods of the form (2). However, we can use them in deriving a sufficient condition for the convergence of the method

$$y_{n+1} = y_n + H\sum_{j=0}^{k-1} \beta_j(H,n) f_{n-j}, \tag{10}$$

under the conditions stated in Theorem 2. Then, we will prove two auxiliary results necessary for establishing the convergence of the methods. Note that the β_j coefficients depend on H and n and that a constant step size H is used throughout. In practical use, the coefficients $\beta_j(H,n)$ are constant throughout the trigonometric and exponential Adams–Bashforth numerical procedure. Let us further state explicitly that when considering matters of convergence, we will assume that $H < 1$.

Definition 1 ([20]). *A multistep method is said to be of order p if the local error $y(x_n) - y_n$ is $\mathcal{O}(H^{p+1})$ for all sufficiently smooth problems of the form (1).*

Definition 2 ([20]). *A multistep method is called zero-stable if the polynomial*

$$\rho(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \ldots + \alpha_0$$

satisfies the following conditions:

1. *The roots of $\rho(z)$ lie on or within the unit circle;*
2. *The roots on the unit circle are simple.*

Definition 3 ([20]). *A linear multistep method is called convergent if for all initial value problems (1) such that f is continuous on*

$$D = \{(x, y) : x \in [x_0, \hat{x}], \|y(x) - y\| \leq \chi\},$$

where $y(x)$ denotes the solution to (1) and $\hat{x} \in \mathbb{R}$, $\hat{x} > x_0$, $\chi > 0$ and f satisfies the Lipschitz condition with respect to its second argument, it holds that

$$y(x_n) - y_n \to 0, \qquad \text{for } H \to 0,$$

whenever the starting values satisfy

$$y(x_j) - y_j \to 0, \qquad \text{for } H \to 0, j = 0, 1, \ldots, k-1.$$

A linear multistep method is called convergent of order p if f additionally is sufficiently differentiable and there exists a positive H_0 such that

$$\|y(x_n) - y_n\| \leq CH^p, \qquad \text{for } H \leq H_0$$

whenever the starting values satisfy

$$\|y(x_j) - y_j\| \leq C_0 H^p, \qquad \text{for } H \leq H_0, j = 0, 1, \ldots, k-1$$

for $C, C_0 \geq 0$.

Theorem 1 ([20]). *If a multistep method is stable and of order 1, then it is convergent. If the method is stable and of order p, then it is convergent of order p.*

Theorem 2. *For the initial-value problem (1), let f be continuous on*

$$D = \{(x, y) : x \in [x_0, x_N], \|y(x) - y\| \leq \chi\},$$

where $y(x)$ denotes the solution to (1) where $\chi > 0$ and f satisfies the Lipschitz condition with respect to its second argument. Let the method (10) be of order p and the starting values satisfy

$$\|y(x_j) - y_j\| \leq C_0 H^p, \qquad \text{for } H \leq H_0, j = 0, 1, \ldots, k-1$$

for $C_0 \geq 0$ and $H_0 > 0$ and $|\beta_j(H, n)| \leq M$ for $M > 0$, $j = 0, 1, \ldots, k-1$, $n \geq k-1$. Then the method (10) is convergent of order p.

Proof. Let $e_n = y(x_n) - y_n$, $E_n = \max_{j=0,1,2,\ldots n} \|e_j\|$ and τ_n be the truncation error at x_n. Then by examining the expression $y(x_{n+1}) - y_{n+1}$ for $n \geq k-1$, we reach the difference equation

$$e_{n+1} = e_n + H \sum_{j=0}^{k-1} \beta_j(H, n) \left[f(x_{n-j}, y(x_{n-j})) - f(x_{n-j}, y_{n-j}) \right] + \tau_{n+1}.$$

Let the Lipschitz constant of $f(x,y)$ with respect to its second argument be L and for the truncation error to hold $\|\tau_n\| \leq CH^{p+1}$ for $C > 0$ and all $n \geq k-1$. Then

$$\begin{aligned}\|e_{n+1}\| &\leq \|e_n\| + H\sum_{j=0}^{k-1}|\beta_j(H,n)|\left\|f(x_{n-j},y(x_{n-j})) - f(x_{n-j},y_{n-j})\right\| + CH^{p+1}\\ &\leq \|e_n\| + HML\sum_{j=0}^{k-1}\|e_{n-j}\| + CH^{p+1} \leq (1+HMLk)E_n + CH^{p+1}.\end{aligned}$$

Thus, setting $A = HMLk$ and observing that from $x_N = x_0 + NH$, it follows that $H = \frac{x_N - x_0}{N}$, from the chain of inequalities

$$\begin{aligned}E_{n+1} &\leq (1+A)E_n + CH^{p+1} \leq (1+A)^2E_{n-1} + [1+(1+A)]CH^{p+1}\\ &\leq \ldots \leq (1+A)^{n-k+2}E_{k-1} + \left[1+(1+A)+\ldots+(1+A)^{n-k+1}\right]CH^{p+1}\\ &\leq (1+A)^{N+2}E_{k-1} + \frac{(1+A)^{n-k+2}-1}{A}CH^{p+1}\\ &\leq (1+MLk)^2e^{(x_N-x_0)MLk}C_0H^p + \frac{(1+MLk)^2e^{(x_N-x_0)MLk}-1}{MLk}CH^p = C_2H^p,\end{aligned}$$

where $C_2 = (1+MLk)^2e^{(x_N-x_0)MLk}C_0 + \frac{(1+MLk)^2e^{(x_N-x_0)MLk}-1}{MLk}C$, we conclude that $\|y(x_{n+1}) - y_{n+1}\| \leq C_2H^p$, i.e., the method is convergent of order p. □

Remark 1. *A method of the form* (10) *is zero-stable. Indeed, the characteristic polynomial is* $\rho(z) = z^{k-1}(z-1)$*, from which the method is clearly zero-stable. Thus, there is no conceptual contradiction with Theorem 1.*

We will also need the following proposition:

Proposition 1 ([15])**.** *Let* $f \in C^{2m+1}[a,b]$*, and let the nodes* $x_0, x_1, \ldots, x_{2m} \in [a,b]$*. Then for every* $x \in [a,b]$*, there exists a point* $\xi_x \in (\min\{x, x_0, x_1, \ldots, x_{2m}\}, \max\{x, x_0, x_1, \ldots, x_{2m}\})$ *such that*

$$\begin{aligned}f(x) - T(x;f) &= \frac{\frac{d^{2m+1}}{dz^{2m+1}}[f(z) - T_m(z;f)]|_{z=\xi_x}}{(2m+1)!}\prod_{j=0}^{2m}(x-x_j),\\ f(x) - E(x;f) &= \frac{\frac{d^{2m+1}}{dz^{2m+1}}[f(z) - E_m(z;f)]|_{z=\xi_x}}{(2m+1)!}\prod_{j=0}^{2m}(x-x_j).\end{aligned}$$

As will be seen in the proof of Theorem 3, Proposition 1 shows that using the interpolant T_1 (or E_1) to interpolate sufficiently smooth functions leads to a third-order interpolation error. First, we will establish a uniform bound of $\frac{d^3}{dz^3}T_1(z;f)$ and $\frac{d^3}{dz^3}E_1(z;f)$ on an interval of the form $[x_{n-2}, x_{n+1}]$.

Lemma 1. *Let* $f \in C^2[a,b]$*,* $x_0, x_1, x_2, x_3 \in [a,b]$*,* $x_j = x_0 + 2jh$*,* $j = 0,1,2,3$*,* $h = \frac{H}{2}$*,* $H \in (0,1)$*,* $\|f'\| \leq C_1$ $\|f''\| \leq C_2$*. Then it holds that*

$$\begin{aligned}\left\|\frac{d^3}{dz^3}T_1(z;f)|_{z=x}\right\|_1 &\leq \frac{4C_1+C_2}{2\sin(1)\sin\left(\frac{1}{2}\right)},\\ \left\|\frac{d^3}{dz^3}E_1(z;f)|_{z=x}\right\|_1 &\leq \frac{1}{2}\left(\sinh\left(\frac{5}{2}\right)+\sinh\left(\frac{3}{2}\right)\right)(2C_1+C_2),\end{aligned}$$

where $\|\cdot\|_1$ *is the norm* $\|\cdot\|$ *restricted to the interval* $[x_0, x_3]$ *and* $T_1(z;f)$ *and* $E_1(z;f)$ *are defined via the nodes* x_0, x_1, *and* x_2.

Proof. We will first consider $T_1(x;f)$. Routine calculations lead to

$$\begin{aligned}\frac{d^3}{dz^3}T_1(z;f)|_{z=x} =& \frac{1}{4\cos(h)\sin^2(h)}[(f(x_2)-f(x_1))\sin(x_0+h-x)\\ &+(f(x_0)-f(x_1))\sin(x_0+3h-x)].\end{aligned}$$

Let $A = \frac{1}{4\cos(h)\sin^2(h)}$. Using the Taylor expansion of $f(x)$, we can establish the equalities $f(x_2)-f(x_1) = 2hf'(x_1)+2h^2f''(\xi)$ and $f(x_0)-f(x_1) = -2hf'(x_1)+2h^2f''(\eta)$ for $\xi \in (x_1, x_2)$, $\eta \in (x_0, x_1)$. Thus, we obtain

$$\begin{aligned}\left\|\frac{d^3}{dz^3}T(z;f)|_{z=x}\right\|_1 &\leq 2Ah|f'(x_1)|(|\sin(x_0+h-x)|+|\sin(x_0+3h-x)|)\\ &\quad+2Ah^2(|f''(\xi)||\sin(x_0+h-x)|+|f''(\eta)||\sin(x_0+3h-x)|)\\ &\leq 2AhC_1(|\sin(x_0+h-x)|+|\sin(x_0+3h-x)|)+4Ah^2C_2.\end{aligned}$$

Moreover, since $x \in [x_0, x_3]$ and $h = \frac{H}{2}$, it follows that $x_0+h-x \in [-5h, h]$, $h \in \left(0, \frac{1}{2}\right)$, and via the inequality $|\sin(z)| \leq |z|$ for $z \in \mathbb{R}$, we find $|\sin(x_0+h-x)| \leq |x_0+h-x| \leq 5h$. Analogously, due to $x_0+3h-x \in [-3h, 3h]$, we obtain $|\sin(x_0+3h-x)| \leq 3h$. Finally, due to the supremum of $\frac{x^2}{\sin^2(x)}$ over $\left(0, \frac{1}{2}\right)$ being at $x = \frac{1}{2}$, it is true that

$$Ah^2 = \frac{h^2}{4\cos(h)\sin^2(h)} \leq \frac{1}{16\cos\left(\frac{1}{2}\right)\sin^2\left(\frac{1}{2}\right)} = \frac{1}{8\sin(1)\sin\left(\frac{1}{2}\right)}.$$

Finally, we conclude that

$$\left\|\frac{d^3}{dz^3}T_1(z;f)|_{z=x}\right\|_1 \leq 16Ah^2C_1+4Ah^2C_2 \leq \frac{4C_1+C_2}{2\sin(1)\sin\left(\frac{1}{2}\right)}.$$

Let us now consider $\frac{d^3}{dz^3}E_1(z;f)|_{z=x}$. From

$$\begin{aligned}\frac{d^3}{dz^3}E_1(z;f)|_{z=x} =& \frac{1}{4\cosh(h)\sinh^2(h)}[(f(x_1)-f(x_2))\sinh(x_0+h-x)\\ &+(f(x_1)-f(x_0))\sinh(x_0+3h-x)]\end{aligned}$$

and the Taylor expansion of $f(x)$, we obtain

$$\left\|\frac{d^3}{dz^3}E(z;f)|_{z=x}\right\|_1 \leq (2BhC_1+2Bh^2C_2)(|\sinh(x_0+h-x)|+|\sinh(x_0+3h-x)|), \quad (11)$$

where $B = \frac{1}{4\cosh(h)\sinh^2(h)}$. Since $x_0+h-x \in [-5h, h]$, via the Taylor expansion of $\sinh(x)$, it follows that

$$\begin{aligned}|\sinh(x_0+h-x)| &\leq \sum_{k=0}^{\infty}\frac{(5h)^{2k+1}}{(2k+1)!} = h\sum_{k=0}^{\infty}\frac{5^{2k+1}h^{2k}}{(2k+1)!} \leq h\sum_{k=0}^{\infty}\frac{5^{2k+1}}{2^{2k}(2k+1)!}\\ &= 2h\sum_{k=0}^{\infty}\frac{5^{2k+1}}{2^{2k+1}(2k+1)!} = 2h\sinh\left(\frac{5}{2}\right).\end{aligned}$$

Analogously, $|\sinh(x_0+3h-x)| \le 2h\sinh\left(\frac{3}{2}\right)$. For $x \in \left(0, \frac{1}{2}\right)$, the supremum of $\frac{x^2}{\sinh^2(x)}$ is $\lim_{x\to 0+} \frac{x^2}{\sinh^2(x)} = 1$. It follows that $Bh^2 = \frac{h^2}{4\cosh(h)\sinh^2(h)} \le \frac{1}{4}$. From these estimations and (11), we conclude that

$$\begin{aligned}\frac{d^3}{dz^3}E_1(z;f)|_{z=x} &\le (2BhC_1 + 2Bh^2C_2)(|\sinh(x_0+h-x)| + |\sinh(x_0+3h-x)|)\\ &\le \frac{(4C_1+4hC_2)}{4}\left(\sinh\left(\frac{5}{2}\right)+\sinh\left(\frac{3}{2}\right)\right)\\ &\le \frac{1}{2}\left(\sinh\left(\frac{5}{2}\right)+\sinh\left(\frac{3}{2}\right)\right)(2C_1+C_2). \quad \square\end{aligned}$$

Lemma 2. *For $h \in \left(0, \frac{1}{2}\right)$, the functions $t_1(h)$, $t_2(h)$ are decreasing and $e_1(h), e_2(h)$ are increasing.*

Proof. Since establishing these claims is a matter of routine calculations, we will present the proof only for $e_2(h)$. From common identities of hyperbolic functions of a multiple of h, we obtain

$$e_2(h) = 4\frac{\cosh^2(h)\coth(h)}{h} - 3\frac{\coth(h)}{h} - \coth^2(h) - 1.$$

Further,

$$\begin{aligned}e_2'(h) =& 12\frac{\cosh(h)^2}{h} - 4\left[\frac{\cosh^2(h)\coth(h)}{h^2} + \frac{\cosh^2(h)\coth^2(h)}{h}\right]\\ &+ 3\left[\frac{\coth(h)}{h^2} + \frac{\coth^2(h)}{h}\right] - \frac{3}{h} + 2\frac{\coth(h)}{\sinh^2(h)}.\end{aligned}$$

From $\cosh^2(h) \ge 1$, it follows that

$$\begin{aligned}e_2'(h) &\ge 9\frac{\cosh(h)^2}{h} - 4\left[\frac{\cosh^2(h)\coth(h)}{h^2} + \frac{\cosh^2(h)\coth^2(h)}{h}\right]\\ &\quad + 3\left[\frac{\coth(h)}{h^2} + \frac{\coth^2(h)}{h}\right] + 2\frac{\coth(h)}{\sinh^2(h)}\\ &= \frac{\coth(h)}{8h^2\sinh^2(h)}[16h^2 - 8 + 12\cosh(2h) - 4\cosh(4h)\\ &\quad - h(14\sinh(2h) - 5\sinh(4h))].\end{aligned}$$

Let $g(h) = 16h^2 - 8 + 12\cosh(2h) - 4\cosh(4h) - h(14\sinh(2h) - 5\sinh(4h))$. Then $g(0) = g'(0) = g''(0) = 0$ and

$$g'''(h) = 320h\cosh(4h) - 112h\cosh(2h) - 16\sinh(4h) - 72\sinh(2h).$$

Since $z\cosh(z) \ge \sinh(z) \ge \sinh\left(\frac{z}{2}\right)$ and $\cosh(2z) \ge \cosh(z)$ for $z \ge 0$, it follows that $g'''(h) \ge 0$ for $h \in \left(0, \frac{1}{2}\right)$, from which $g(h) \ge 0$ and $e_2'(h) \ge 0$, that is, $e_2(h)$ is increasing on $\left(0, \frac{1}{2}\right)$. $\square$

Theorem 3. *For the initial-value problem (1), let f be continuous on*

$$D = \{(x,y) : x \in [x_0, x_N], \|y(x) - y\| \le \chi\},$$

where $y(x)$ denotes the solution to (1) and $\chi > 0$, and f satisfies the Lipschitz condition with respect to its second argument. Let $y(x) \in C^4[x_0, x_N]$ and the starting values satisfy

$$\|y(x_j) - y_j\| \leq C_0 H^3, \qquad \text{for } H \leq H_0, j = 0, 1, 2$$

for $C_0 \geq 0$ and $H_0 > 0$. Then the trigonometric Adams–Bashforth method (7) is convergent of order 3.

Proof. To determine the method's order, we estimate the local truncation error. Assuming that $y_{n-1}, y_{n-2}, y_{n-3}$ are exact, we subtract (6) from (4) and apply Proposition 1 with $m = 1$ to obtain

$$\begin{aligned}\|y(x_n) - y_n\| &= \left\| \int_{x_{n-1}}^{x_n} \left(f(x,y) - T_1(x;y')\right) dx \right\| \leq \int_{x_{n-1}}^{x_n} \|f(x,y) - T_1(x;y')\| \, dx \\ &\leq \int_{x_{n-1}}^{x_n} \left\| \frac{\frac{d^3}{dz^3}[f(z) - T_1(z;f)]|_{z=\xi_x}}{6} \prod_{j=0}^{2} (x - x_{n-1-j}) \right\| dx,\end{aligned}$$

where $\xi_x \in [x_{n-3}, x_n]$ and $\|\cdot\|$ is a suitable norm taken over $[x_{n-3}, x_n]$. Because $y(x) \in C^4[x_0, x_N]$, it follows that the composition $g(x) = f(x,y) = y'(x) \in C^3[x_0, x_N]$ and $\|g^{(3)}(x)\| \leq M_1$ for $M_1 > 0$. Further, from Lemma 1, it follows that $\left\| \frac{d^3}{dz^3} T(z;y') \right\| \leq M_2$ for $M_2 = \frac{4C_1 + C_2}{2\sin(1)\sin\left(\frac{1}{2}\right)}$, where $\|g'(x)\| \leq C_1$ and $\|g''(x,y)\| \leq C_2$. Thus, we conclude that $\|[f(z) - T(z;f)]^{(3)}\| \leq M_1 + M_2$ and

$$\|y(x_n) - y_n\| \leq \frac{M_1 + M_2}{6} \int_{x_{n-1}}^{x_n} \left| \prod_{j=0}^{2} (x - x_{n-1-j}) \right| dx = \frac{3(M_1 + M_2)}{8} H^4 = \mathcal{O}(H^4).$$

The method has a local truncation error of $O(H^4)$, hence it is of order 3. From Lemma 2, it follows that $|\beta_0(H)| \leq \lim_{h \to 0+} t_1(h) = \frac{23}{6}$, $|\beta_1(H)| \leq \lim_{h \to 0+} t_2(h) = \frac{8}{3}$ and $|\beta_2(H)| \leq 2 - t1\left(\frac{1}{2}\right) + \lim_{h \to 0+} t_2(h) = \frac{14}{3} - t1\left(\frac{1}{2}\right)$. Consequently, taking $M = \frac{23}{6}$, by Theorem 2, the method is convergent of order 3. □

Theorem 4. *For the initial-value problem (1), let f be continuous on*

$$D = \{(x,y) : x \in [x_0, x_N], \|y(x) - y\| \leq \chi\},$$

where $y(x)$ denotes the solution to (1) and $\chi > 0$ and f satisfies the Lipschitz condition with respect to its second argument. Let $y(x) \in C^4[x_0, x_N]$ and the starting values satisfy

$$\|y(x_j) - y_j\| \leq C_0 H^3, \qquad \text{for } H \leq H_0, j = 0, 1, 2$$

for $C, C_0 \geq 0$ and $H_0 > 0$. Then the exponential Adams–Bashforth method (9) is convergent of order 3.

Proof. The proof is analogous to that of Theorem 3, where an upper bound of the local error is found by subtracting (8) from (4) and using Proposition 1, Lemmas 1 and 2, and Theorem 2 for $E_1(x;y')$. □

After establishing convergence, it is customary to continue the analysis by considering the test equation $y' = qy$ for $q \in \mathbb{C}$, $\mathrm{Re}(q) < 0$ [21]. Substituting $y' = qy$ in (7) yields the difference equation

$$y_{n+1} - (1 + hqt_1(h))y_n + hqt_2(h)y_{n-1} - hq(2 - t_1(h) + t_2(h))y_{n-2} = 0. \tag{12}$$

Substituting the *ansatz* $y_n = w^n$, $n = 0, 1, \ldots$ into (12) leads to the characteristic equation

$$w^3 - (1 + hqt_1(h))w^2 + hqt_2(h)w - hq(2 - t_1(h) + t_2(h)) = 0. \tag{13}$$

In order for the method to be stable with respect to the test problem, we require that (13) has bounded solutions as $n \to \infty$. Usually one determines the region of z values for which (13) has bounded solutions. However, since the coefficients t_1 and t_2 depend on h, we cannot construct the regular stability regions in the complex plane. Thus, we will restrict our analysis to stability regions for fixed step sizes.

We determine the stability regions via the boundary locus method. Solving (13) for q, we get the expression

$$q = \frac{w^3 - w^2}{h[t_1(h)w^2 - t_2(h)w + 2 - t_1(h) + t_2(h)]}.$$

By tracing out the q values for which $|w| = 1$ in the complex plane, we obtain the boundary of the stability region. Some boundary points do not correspond to stable behavior of the trigonometric method, as they may be generated by a value of w that is a root of (13) with multiplicity greater than 1.

Analogously, the stability regions for the exponential Adams–Bashforth method are generated using the characteristic equation

$$w^3 - (1 + hqe_1(h))w^2 + hqe_2(h)w - hq(2 - e_1(h) + e_2(h)) = 0 \tag{14}$$

and the expression for q

$$q = \frac{w^3 - w^2}{h[e_1(h)w^2 - e_2(h)w + 2 - e_1(h) + e_2(h)]}.$$

Figures 1–3 show the stability region boundary (or part of it) for the three methods at various values of H. The stability region is enclosed by the curve. In Figure 1, we can clearly see that the trigonometric method has a larger stability region than the classical method with a small exception in a neighborhood on the imaginary axis close to the origin. The exponential method, on the other hand, has a smaller stability region. As H decreases, the differences become smaller, leading to visually overlapping stability regions. Therefore, only portions of the boundaries are shown in Figures 2 and 3. We observe that the trigonometric method retains a larger stability region, while the exponential method retains a smaller one. However, as the step size decreases, these differences become less significant. This follows from the fact that $\lim_{H\to 0} t_1(h) = \lim_{H\to 0} e_1(h) = \frac{23}{6}$ and $\lim_{H\to 0} t_2(h) = \lim_{H\to 0} e_2(h) = \frac{16}{6}$, i.e., the coefficients of the trigonometric and exponential methods tend to the coefficients of the classic method as the step size tends to 0.

The boundary locus method provides a convenient way to construct the stability region and verify whether z lies in it. However, an analytical expression would make checking stability significantly easier. To derive such an expression, we will restrict ourselves to the open stability region, that is, the values of z for which all roots of (13) or (14) satisfy $|w| < 1$. If a polynomial's roots satisfy this property, we will refer to it as strongly stable [21]. To find an analytical condition, we will use the following theorem (namely, the Schur criterion for a polynomial to be strongly stable):

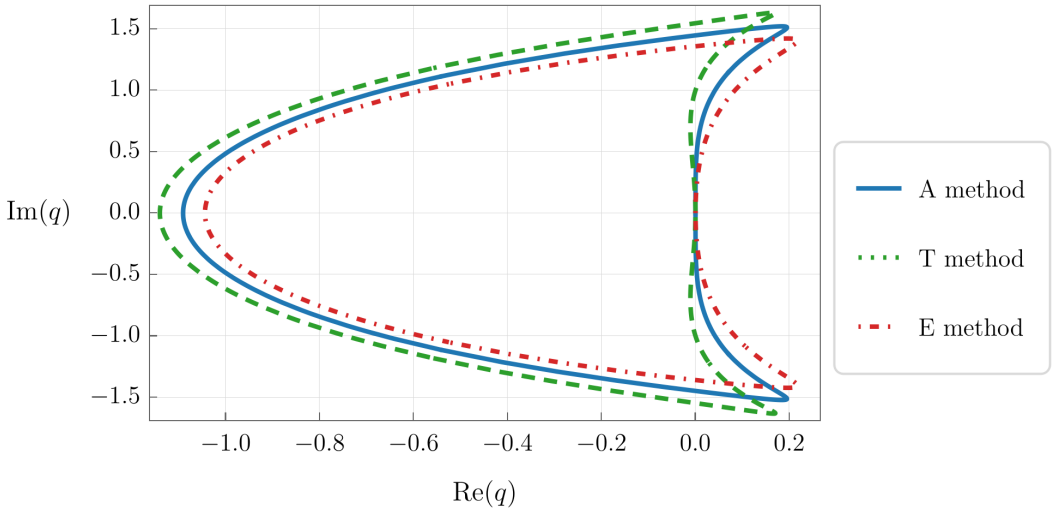


Figure 1. Boundary of the stability region of the A, T, and E methods with $H = 0.5$.

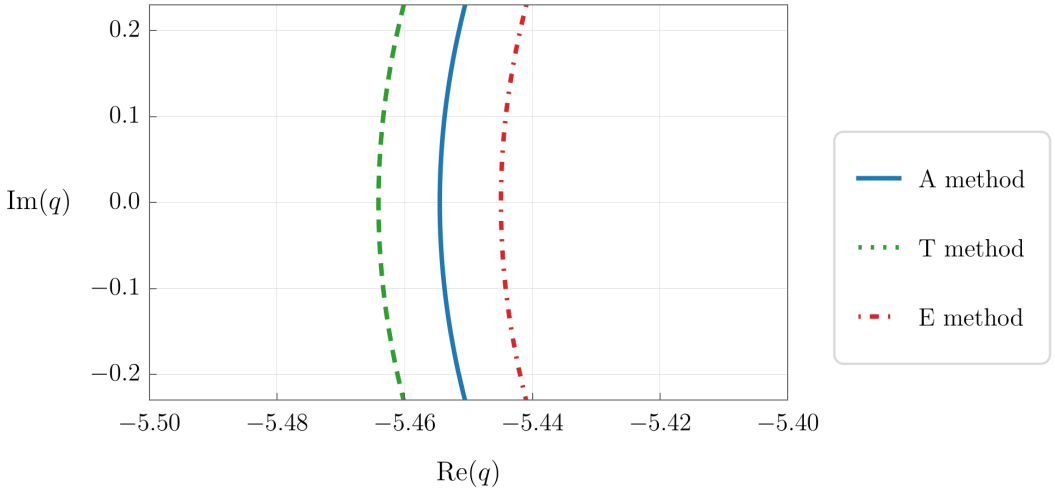


Figure 2. Part of the boundary of the stability region of the A, T, and E methods with $H = 0.1$.

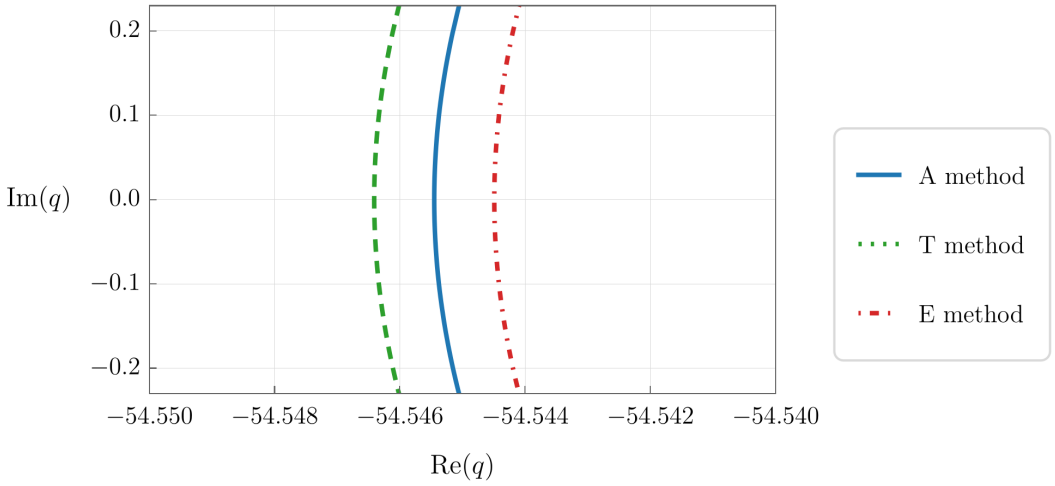


Figure 3. Part of the boundary of the stability region of the A, T, and E methods with $H = 0.01$.

Theorem 5 ([21]). *A polynomial P_n, given by*

$$P_n(w) = a_0 w^n + a_1 w^{n-1} + \cdots + a_{n-1} w + a_n,$$

where $a_0 \neq 0$ and $n \geq 2$ is strongly stable if and only if $|a_0|^2 > |a_n|^2$ and P_{n-1} is strongly stable, where

$$P_{n-1}(w) = (\overline{a_0}a_0 - a_n\overline{a_n})w^{n-1} + (\overline{a_0}a_1 - a_n\overline{a_{n-1}})w^{n-2} + \cdots + (\overline{a_0}a_{n-1} - a_n\overline{a_1}).$$

Using Theorem 5, we can find the open stability region of both the T and E Adams–Bashforth methods.

Proposition 2. *Let $z = hq$, $q_1 = t_1(h)z$, and $q_2 = t_2(h)z$ (respectively, $q_1 = e_1(h)z$ and $q_2 = e_2(h)z$) in (13) (respectively, in (14)). Then the polynomial in (13) (respectively, in (14)) is strongly stable if and only if $A > 0$ and $A^2 > |B|^2 + |B\overline{C} - AC|$, where $A = 1 - |q_1 - q_2 - 2z|^2$, $B = q_1 - 2z + (q_1 - q_2 - 2z)\overline{q_1}$, and $C = 1 + q_1 + (q_1 - q_2 - 2z)\overline{q_2}$.*

Proof. A straightforward application of Theorem 5 leads to the conditions

$$\begin{aligned} A &> 0, \\ |A| &> |B|, \\ \left|A^2 - |B|^2\right| &> |B\overline{C} - AC|. \end{aligned}$$

The second and third conditions are equivalent to $A^2 > |B|^2 + |B\overline{C} - AC|$. □

4. Adaptive ATE Algorithm

The Adams-type methods derived in Section 2, together with the convergence and stability analysis presented in Section 3, motivate their use as candidate schemes in the adaptive ATE procedure. The role of the adaptive criterion is to determine, at each integration step, which interpolation basis gives the most accurate local prediction of the derivative.

The selection is made by an auxiliary extrapolation test. At step n, the derivative value

$$y'_n = f(x_n, y_n)$$

is already known from the current numerical approximation. To decide which Adams-type formula should be applied next, each interpolation basis is used to predict this known value from previously computed derivative values. Thus, the comparison is not made with the exact solution but with the available quantity y'_n.

This extrapolation test requires one additional derivative value beyond the three used in the Adams-type formulas. The starting values y_1, y_2, and y_3, which are required before the multistep procedure can be applied, are computed by a suitable one-step method. The extrapolation test uses y'_{n-3}, y'_{n-2}, and y'_{n-1} to predict the derivative at x_n, and the result is compared with y'_n. Thus, y'_{n-3} is used only in the adaptive selection criterion.

The three extrapolated derivative values are given by

$$\begin{aligned} P_2(x_n; y') &= y'_{n-3} + 3(y'_{n-1} - y'_{n-2}), \\ T_1(x_n; y') &= y'_{n-3} + \frac{\sin(3h)}{\sin(h)}(y'_{n-1} - y'_{n-2}), \\ E_1(x_n; y') &= y'_{n-3} + \frac{\sinh(3h)}{\sinh(h)}(y'_{n-1} - y'_{n-2}). \end{aligned}$$

The corresponding deviations from the known derivative value $y'_n = f_n$ are

$$\delta_n = \left| P_2(x_n; y') - y'_n \right|, \tag{15}$$
$$\tau_n = \left| T_1(x_n; y') - y'_n \right|, \tag{16}$$
$$\varepsilon_n = \left| E_1(x_n; y') - y'_n \right|. \tag{17}$$

Let $\mu_n = \min\{\delta_n, \tau_n, \varepsilon_n\}$. The interpolation model attaining this minimum is selected, and the corresponding Adams-type formula is used to compute y_{n+1}.

The ATE method can be expressed as

$$y_{n+1} = y_n + H \sum_{j=0}^{2} \beta_j(H, n) f_{n-j}, \tag{18}$$

where $\beta_j(H, n)$ are the coefficients of the classical, trigonometric, or exponential Adams-Bashforth method, depending on μ_n.

The complete ATE solver is presented in Algorithm 1.

Algorithm 1: Solving Equation (1) with the ATE method.

Input: x_0, h, N, $f(x, y)$, y_0
Output: The numerical solution $\{y_n\}_{n=0}^{N}$.
Compute y_1, y_2, and y_3 with a suitable one-step method (in all further experiments: fourth-order Runge–Kutta method).
for $n = 3, 4, \ldots, N-1$ **do**
  $\delta_n \leftarrow$using (15)
  $\tau_n \leftarrow$using (16)
  $\varepsilon_n \leftarrow$using (17)
  $\mu_n \leftarrow \min\{\delta_n, \tau_n, \varepsilon_n\}$
  if $\mu_n = \delta_n$ **then**
    $y_{n+1} \leftarrow$ using (5)
  end
  else if $\mu_n = \tau_n$ **then**
    $y_{n+1} \leftarrow$ using(7)
  end
  else
    $y_{n+1} \leftarrow$ using (9)
  end
end
return $\{y_n\}_{n=0}^{N}$

Theorem 6. *For the initial-value problem* (1)*, let* f *be continuous on*

$$D = \{(x, y) : x \in [x_0, x_N], \|y(x) - y\| \leq \chi\},$$

where $y(x)$ *denotes the solution to* (1) *and* $\chi > 0$*, and* f *satisfies the Lipschitz condition with respect to its second argument. Let* $y(x) \in C^4[x_0, x_N]$ *and the starting values satisfy*

$$\|y(x_j) - y_j\| \leq C_0 H^3, \qquad \text{for } H \leq H_0, j = 0, 1, 2, 3$$

for $C, C_0 \geq 0$ *and* $H_0 > 0$*. Then the ATE method is convergent of order* 3*.*

Proof. Let $g(x) = f(x, y)$, $\|g^{(3)}(x)\| \leq M_1$ for $M_1 > 0$, $M_2 = \frac{4C_1 + C_2}{2\sin(1)\sin\left(\frac{1}{2}\right)}$ and $M_3 = \frac{1}{2}\left(\sinh\left(\frac{5}{2}\right) + \sinh\left(\frac{3}{2}\right)\right)(2C_1 + C_2)$, where $\|g'(x)\| \leq C_1$ and $\|g''(x)\| \leq C_2$. Then

using Theorems 3 and 4 and a well-known upper bound for the local truncation error of the A method [21] (assuming that $y_{n-1}, y_{n-2}, y_{n-3}$ are exact), we obtain

$$\|y(x_n) - y_n\| \leq \begin{cases} \frac{3M_1}{8}H^4, & \mu_{n-1} = \delta_{n-1} \\ \frac{3(M_1+M_2)}{8}H^4, & \mu_{n-1} = \tau_{n-1} \\ \frac{3(M_1+M_3)}{8}H^4, & \mu_{n-1} = \varepsilon_{n-1} \end{cases} \leq \frac{3(M_1+M_3)}{8}H^4 = O(H^4).$$

Thus, the method is of order 3. Further, for $\beta_j(H,n)$ in (18), using Lemma 2, we calculate that $|\beta_0(H,n)| \leq e_1\left(\frac{1}{2}\right)$, $|\beta_1(H,n)| \leq e_2\left(\frac{1}{2}\right)$ and $|\beta_2(H,n)| \leq e_2\left(\frac{1}{2}\right) - \frac{11}{6}$, from which $|\beta_j(H,n)| \leq e_1\left(\frac{1}{2}\right)$ for $n \geq 3$. Consequently, by Theorem 2, the ATE method is convergent of order 3. □

Due to the nature of the ATE algorithm, the procedure for establishing stability with respect to the equation $y' = qy$ would be more involved and therefore deferred to a future study.

The adaptive algorithm is employed in the numerical examples presented in the next section.

5. Numerical Examples

The numerical experiments illustrate the accuracy of the fixed-basis Adams-type methods and the adaptive ATE method. They also show the behavior of the ATE selection criterion and demonstrate situations in which an unsuitable fitted basis may lead to instability or loss of accuracy. For each method, the numerical approximation is compared with the exact solution at the mesh points. Since the adaptive ATE method requires one additional starting value to evaluate the local selection criterion, the errors are computed only on the common comparison interval. We therefore exclude the starting nodes and use $i = 4, 5, \ldots, N$. The maximum error and the root mean square error are defined by

$$L_\infty = \max_{4 \leq i \leq N} |y_i - y(x_i)| \tag{19}$$

and

$$\mathrm{RMSE} = \left(\frac{1}{N-3} \sum_{i=4}^{N} |y_i - y(x_i)|^2 \right)^{1/2}. \tag{20}$$

The fixed algebraic, trigonometric, and exponential Adams-type methods require the values y_0, y_1, y_2 before the recurrence can be started. These consist of the prescribed initial value and two additional values computed by the fourth-order Runge–Kutta method. The adaptive ATE method additionally requires y_3 because its selection criterion compares interpolation models using four known derivative values. Therefore, the reported errors are evaluated from $i = 4$ onward. In the graphical comparison, we plot the decimal logarithm of the absolute error. This representation is used because the errors of the different methods may differ by several orders of magnitude.

Unless stated otherwise, all experiments are performed on the interval $[0, 10]$. The computations are carried out with $N = 100$ and $N = 1000$, which correspond to the mesh sizes $H = 0.1$ and $H = 0.01$, respectively. Since the fitted coefficients are written in terms of the half-step parameter, we use $h = H/2$. The required starting values are computed using the fourth-order Runge–Kutta method. The errors are evaluated, as defined in (19) and (20).

At each integration step, the adaptive algorithm evaluates the selection criterion for the three available Adams-type formulas and chooses the formula corresponding to the smallest interpolation deviation. The selection is recorded by incrementing the correspond-

ing counter N_A, N_T, or N_E, initially set to zero. Thus, the triple (N_A, N_T, N_E) gives the total numbers of steps performed using the algebraic, trigonometric, and exponential formulas, respectively. For example, $(N_A, N_T, N_E) = (74, 6, 17)$ indicates that the algebraic formula was selected at 74 integration steps, the trigonometric formula at 6 steps, and the exponential formula at 17 steps. The same counting procedure produces the other reported triples.

Execution times were measured in Wolfram Mathematica using the built-in function `RepeatedTiming`, which evaluates an expression repeatedly and reports a trimmed mean of the elapsed wall-clock times in seconds, discarding the lower and upper quartiles of the measurements. All methods were timed under the same computational environment. The timed computations include the generation of the starting values and, for the ATE method, the evaluation of the selection criterion at each adaptive step.

The first three examples are deliberately chosen to illustrate the adaptive selection criterion in cases with a clearly identifiable preferred interpolation basis. The subsequent examples complement these tests with a nonlinear problem and a smooth mixed-component right-hand side.

5.1. Example 1

We first consider the polynomial initial value problem

$$y' = x^3 - x^2, \quad y(0) = 0.$$

The exact solution is

$$y(x) = \frac{x^4}{4} - \frac{x^3}{3}.$$

The numerical results are presented in Table 1, and the corresponding error behavior is shown in Figure 4. For $(H = 0.1)$, the adaptive method selects the formulas according to $(N_A, N_T, N_E) = (74, 6, 17)$, while for $(H = 0.01)$, the corresponding selection is $(N_A, N_T, N_E) = (762, 66, 169)$. The predominance of the algebraic selections reflects the polynomial structure of the test problem. The occasional trigonometric and exponential selections occur mainly near the beginning of the integration interval, where the derivative values are still small and the local deviation indicators associated with the three formulas may be close in magnitude. Since the adaptive criterion selects, at each step, the formula corresponding to the smallest local deviation indicator, a non-algebraic formula may occasionally be selected locally even though the underlying problem is polynomial. As the integration proceeds and the polynomial behavior becomes more pronounced in the available derivative history, the algebraic indicator becomes dominant.

Table 1. Numerical results for Example 1.

H	Method	L_∞	RMSE	Time (s)
0.1	A	2.21×10^{-2}	1.29×10^{-2}	6.66×10^{-3}
	T	3.48×10^{-1}	1.36×10^{-1}	6.26×10^{-3}
	E	3.04×10^{-1}	1.13×10^{-1}	6.17×10^{-3}
	ATE	1.83×10^{-2}	9.77×10^{-3}	1.52×10^{-2}
0.01	A	2.25×10^{-5}	1.30×10^{-5}	6.71×10^{-2}
	T	3.59×10^{-4}	1.37×10^{-4}	6.28×10^{-2}
	E	3.14×10^{-4}	1.14×10^{-4}	6.22×10^{-2}
	ATE	1.94×10^{-5}	1.05×10^{-5}	1.53×10^{-1}

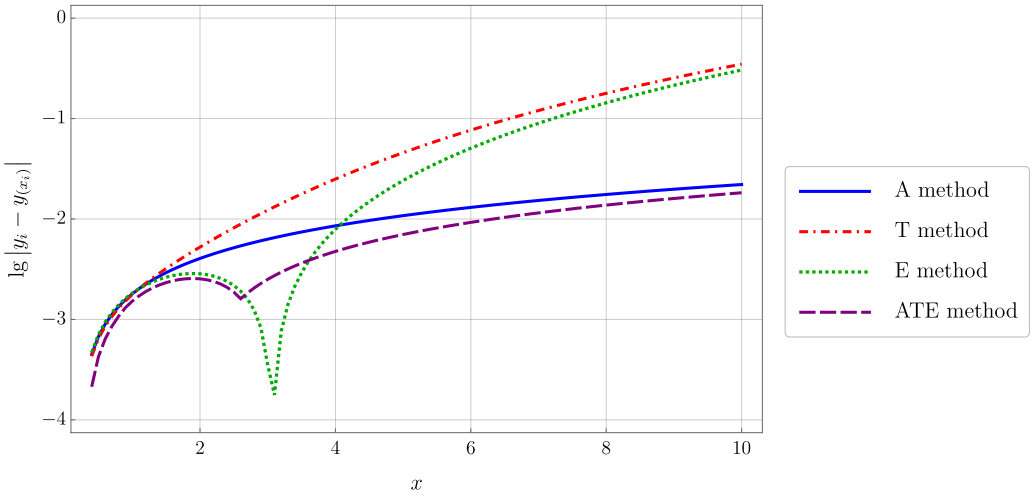


Figure 4. Absolute errors along the x-axis for Example 1 ($H = 0.1$).

5.2. Example 2

We next consider the trigonometric initial value problem

$$y' = \sin x,\ y(0) = 0.$$

The exact solution is

$$y(x) = 1 - \cos x.$$

The numerical results are presented in Table 2, and the corresponding error behavior is shown in Figure 5. The T method produces the smallest errors among the fixed-basis schemes, while the adaptive ATE method gives errors of the same order. For $H = 0.1$, the adaptive method selects the formulas according to $(N_A, N_T, N_E) = (0, 97, 0)$, while for $H = 0.01$, the corresponding selection is $(N_A, N_T, N_E) = (0, 997, 0)$. These choices show that the local selection criterion consistently identifies the trigonometric basis as the appropriate one for this oscillatory test problem. The small difference between the pure T method and the ATE method is due to the starting procedure: the fixed three-step methods require only y_0, y_1, y_2, whereas the adaptive selection criterion also requires y_3, so the two computations do not start from exactly the same set of initial numerical values. This difference is practically eliminated when the step size is reduced, as can be seen from the results for $H = 0.01$.

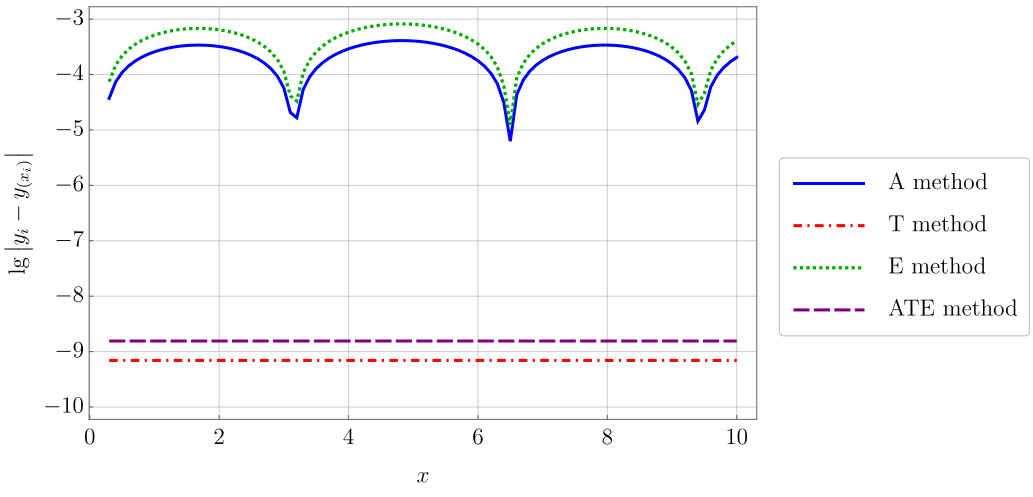


Figure 5. Absolute errors along the x-axis for Example 2 ($H = 0.1$).

Table 2. Numerical results for Example 2.

H	Method	L_∞	RMSE	Time (s)
0.1	A	4.09×10^{-4}	2.55×10^{-4}	5.42×10^{-3}
	T	6.92×10^{-10}	6.92×10^{-10}	5.13×10^{-3}
	E	8.19×10^{-4}	5.10×10^{-4}	4.99×10^{-3}
	ATE	1.55×10^{-9}	1.55×10^{-9}	1.98×10^{-2}
0.01	A	3.78×10^{-7}	2.58×10^{-7}	5.64×10^{-2}
	T	2.73×10^{-14}	1.79×10^{-14}	5.15×10^{-2}
	E	7.57×10^{-7}	5.17×10^{-7}	5.18×10^{-2}
	ATE	2.86×10^{-14}	1.77×10^{-14}	1.41×10^{-1}

5.3. Example 3

We now examine the hyperbolic initial value problem

$$y' = \sinh x, \qquad y(0) = 0.$$

The exact solution is

$$y(x) = \cosh x - 1.$$

The numerical results are presented in Table 3, and the corresponding error behavior is shown in Figure 6. The same remark as in Example 2 applies here: the small difference between the pure E method and the adaptive ATE method is caused by the additional starting value required by the adaptive selection criterion. The E method produces the smallest errors among the fixed-basis schemes, while the adaptive ATE method gives errors of the same order. For $H = 0.1$, the adaptive method selects the formulas according to $(N_A, N_T, N_E) = (0, 0, 97)$, while for $H = 0.01$, the corresponding selection is $(N_A, N_T, N_E) = (0, 0, 997)$. These choices show that the local selection criterion consistently identifies the exponential basis as the appropriate one for this hyperbolic test problem.

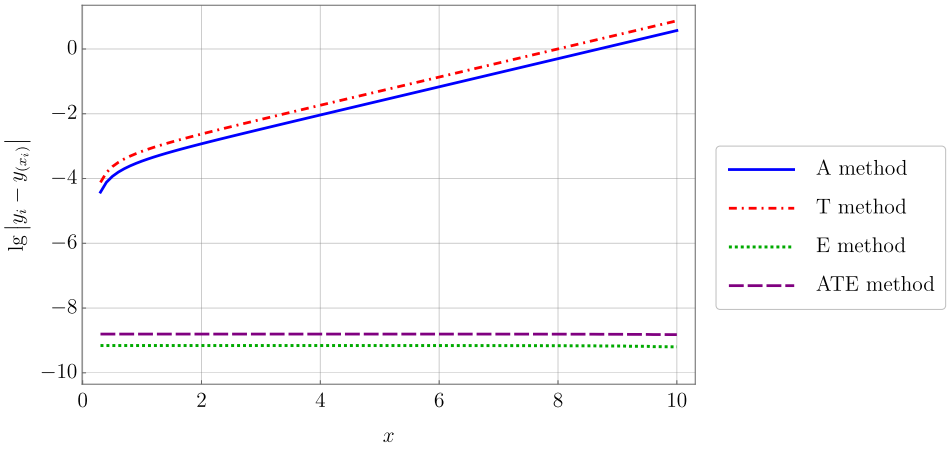


Figure 6. Absolute errors along the x-axis for Example 3 ($H = 0.1$).

Table 3. Numerical results for Example 3.

H	Method	L_∞	RMSE	Time (s)
0.1	A	3.71	8.86×10^{-1}	6.14×10^{-3}
	T	7.42	1.77	5.10×10^{-3}
	E	6.97×10^{-10}	6.89×10^{-10}	5.24×10^{-3}
	ATE	1.57×10^{-9}	1.57×10^{-9}	1.46×10^{-2}

Table 3. *Cont.*

H	Method	L_∞	RMSE	Time (s)
0.01	A	4.09×10^{-3}	9.20×10^{-4}	6.14×10^{-2}
	T	8.17×10^{-3}	1.84×10^{-3}	5.76×10^{-2}
	E	8.97×10^{-10}	2.02×10^{-10}	6.06×10^{-2}
	ATE	8.97×10^{-10}	2.02×10^{-10}	1.42×10^{-1}

5.4. Example 4

We next consider the nonlinear initial value problem

$$y' = 3\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{3}\right), \qquad y(0) = \frac{3\pi}{2}.$$

The exact solution used for the error computation is

$$y(x) = 6\arctan\left(e^{3\left(1-\cos\left(\frac{x}{3}\right)\right)}\right).$$

The numerical results are reported in Table 4, while Figure 7 illustrates the corresponding error distribution. In this nonlinear example, the adaptive ATE method produces smaller errors than each of the fixed methods. For $H = 0.1$, the selected methods are distributed as $(N_A, N_T, N_E) = (16, 29, 52)$, whereas for $H = 0.01$, the corresponding distribution is $(N_A, N_T, N_E) = (174, 289, 534)$. These results show that the adaptive criterion responds to changes in the local behavior of the problem by alternating between the available Adams-type formulas, rather than selecting one interpolation model throughout the entire interval.

Table 4. Numerical results for Example 4.

H	Method	L_∞	RMSE	Time (s)
0.1	A	3.37×10^{-4}	1.22×10^{-4}	8.26×10^{-3}
	T	2.00×10^{-4}	1.09×10^{-4}	7.75×10^{-3}
	E	7.90×10^{-4}	2.99×10^{-4}	7.71×10^{-3}
	ATE	9.95×10^{-5}	4.19×10^{-5}	1.73×10^{-2}
0.01	A	3.47×10^{-7}	1.23×10^{-7}	8.64×10^{-2}
	T	2.25×10^{-7}	1.20×10^{-7}	7.78×10^{-2}
	E	8.39×10^{-7}	3.14×10^{-7}	7.56×10^{-2}
	ATE	1.05×10^{-7}	4.01×10^{-8}	1.79×10^{-1}

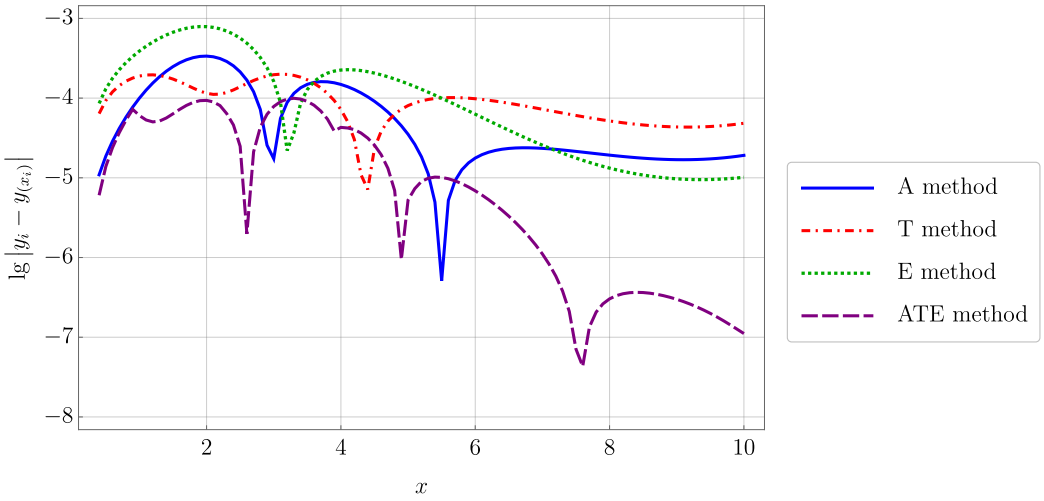


Figure 7. Absolute errors along the x-axis for Example 4 ($H = 0.1$).

5.5. Example 5

We finally examine the initial value problem

$$y' = \frac{(x-4)^2}{10} + \frac{3}{10}\sin\left(\frac{6}{5}(x-4)\right) + \frac{1}{10}e^{x-6} + \frac{1}{100}\sin\left(\frac{17x}{10}\right), \quad y(0) = 0.$$

The exact solution used for the error computation is

$$y(x) = \frac{(x-4)^3 + 64}{30} + \frac{\cos(24/5) - \cos\left(\frac{6}{5}(x-4)\right)}{4} + \frac{e^{x-6} - e^{-6}}{10} + \frac{1 - \cos\left(\frac{17x}{10}\right)}{170}.$$

This example combines algebraic, trigonometric, and exponential components within a single smooth function. Moreover, the oscillatory terms have frequencies different from the prescribed trigonometric fitting frequency. Consequently, the right-hand side does not belong exactly to any of the three interpolation spaces used by the considered Adams-type methods. The numerical results are reported in Table 5, and the corresponding error behavior is shown in Figure 8. For both step sizes, the adaptive ATE method produces smaller maximum and root mean square errors than each of the fixed methods. In particular, the RMSE of ATE is approximately four times smaller than that of the most accurate fixed method.

Table 5. Numerical results for Example 5.

H	Method	L_∞	RMSE	Time (s)
0.1	A	1.90×10^{-3}	5.50×10^{-4}	1.32×10^{-2}
	T	4.43×10^{-3}	1.13×10^{-3}	1.33×10^{-2}
	E	1.08×10^{-3}	6.57×10^{-4}	1.30×10^{-2}
	ATE	4.39×10^{-4}	1.24×10^{-4}	2.46×10^{-2}
0.01	A	2.07×10^{-6}	5.61×10^{-7}	1.34×10^{-1}
	T	4.82×10^{-6}	1.16×10^{-6}	1.33×10^{-1}
	E	1.11×10^{-6}	6.68×10^{-7}	1.32×10^{-1}
	ATE	4.29×10^{-7}	1.31×10^{-7}	2.62×10^{-1}

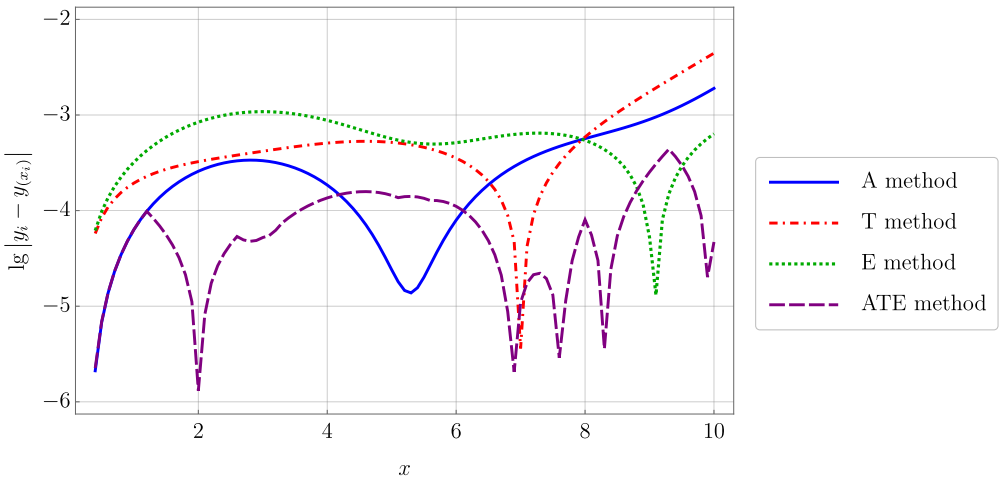


Figure 8. Absolute errors along the x-axis for Example 5 ($H = 0.1$).

For $H = 0.1$, the selected methods are distributed as $(N_A, N_T, N_E) = (31, 32, 34)$, whereas for $H = 0.01$, the corresponding distribution is $(N_A, N_T, N_E) = (329, 319, 349)$. These results show that the adaptive criterion frequently alternates between all three Adams-type formulas rather than relying predominantly on a single interpolation model.

The distribution of the selected methods, together with the reduction in numerical errors, illustrates the potential benefit of adaptive selection when the right-hand side contains different functional components and does not exactly match any single interpolation basis.

The preceding examples illustrate the accuracy of the methods and the behavior of the adaptive selection criterion. We now turn to numerical examples whose main purpose is to demonstrate the stability properties of the schemes.

5.6. Instability Cases

We consider the linear test problems

$$y' = -5.46y \qquad \text{and} \qquad y' = -5.45y.$$

For both problems, we take $y_0 = 0$. Both problems have exponentially decaying exact solutions. However, although the two parameters are very close, the numerical behavior of the methods differs significantly for $H = 0.1$. This is related to the stability regions shown in Figure 2, where the stability of the algebraic, trigonometric, and exponential methods changes near these values of the test parameter.

The computations are carried out with $N = 20{,}000$ and $H = 0.1$. The first stability test is $y' = -5.46y$. The corresponding numerical results are presented in Table 6, and the error behavior is shown in Figure 9. In agreement with the stability information in Figure 2, the A and E methods exhibit unstable behavior, which leads to a rapid error growth. By contrast, the T method remains stable and produces an accurate approximation. The ATE method avoids the unstable choices and produces a stable result.

Table 6. Numerical results for $y' = -5.46y$.

H	Method	L_∞	RMSE	Time (s)
0.1	A	4.19×10^{5}	6.92×10^{4}	2.11
	T	1.68×10^{-2}	9.14×10^{-4}	2.07
	E	4.04×10^{19}	4.02×10^{18}	2.04
	ATE	9.12×10^{-3}	5.13×10^{-4}	4.08

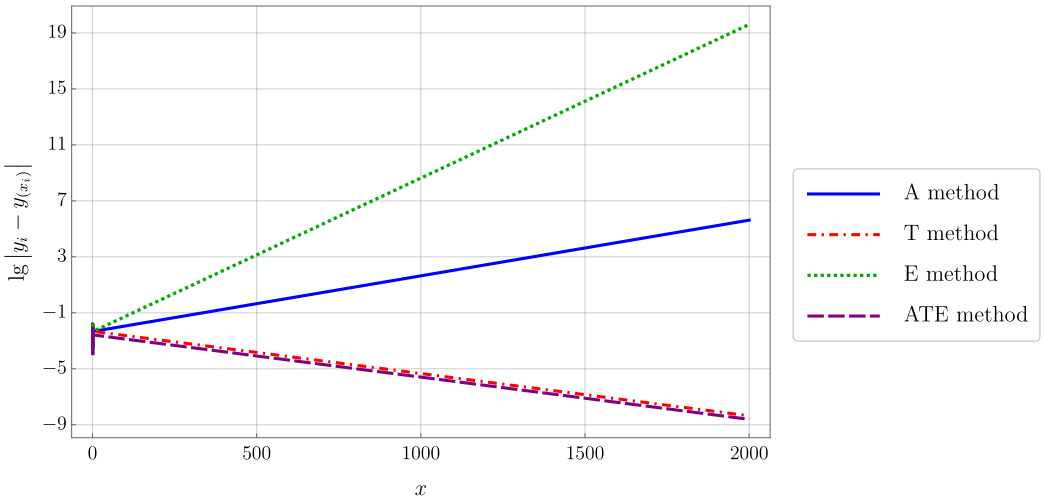


Figure 9. Absolute errors along the x-axis for $y' = -5.46y$ ($H = 0.1$).

An analytical approach via Proposition 2 is also possible. Let A_T,B_T, and C_T denote the quantities defined in Proposition 2 associated with the T method, and let A_E,B_E, and

C_E denote the corresponding quantities for the E method. Rounded to six decimal places, their values are

$$\begin{aligned} A_T &= 0.948178, \quad B_T = -0.736394, \quad C_T = -0.209922, \\ A_E &= 0.948309, \quad B_E = -0.740765, \quad C_E = -0.214320, \end{aligned}$$

from which it follows that

$$\begin{aligned} A_T^2 &= 0.899042, \quad |B_T|^2 + |B_T\overline{C_T} - A_T C_T| = 0.895905, \\ A_E^2 &= 0.899290, \quad |B_E|^2 + |B_E\overline{C_E} - A_E C_E| = 0.910735. \end{aligned}$$

Clearly the conditions for the T method are fulfilled, whereas the conditions for the E method are not. As for the A method, using Theorem 5, one can find the classic bound $-\frac{6}{11} < Hq < 0$ for $q \in \mathbb{R}$ or $-\frac{3}{11} < hq < 0$. When $q = -5.46$ and $H = 0.1$, then $hq < -\frac{3}{11}$. We can also check its characteristic equation for the problem $y' = qy$ with $z = hq$:

$$w^3 - \left(1 + \frac{23}{6}z\right)w^2 + \frac{8}{3}zw - \frac{5}{6}z = 0. \tag{21}$$

When $q = -5.46$ and $H = 0.1$, the equation has a root $w_1 = -1.000920$, which is greater than 1 in absolute value and is consistent with the demonstrated instability of the A method.

The second stability test is $y' = -5.45y$. The results are shown in Table 7. In this case, the E method is unstable, while the A and T methods remain stable. The adaptive ATE procedure again selects formulas that avoid unstable behavior and preserve the stability of the numerical solution.

Table 7. Numerical results for $y' = -5.45y$.

H	Method	L_∞	RMSE	Time (s)
0.1	A	1.62×10^{-2}	8.40×10^{-4}	2.17
	T	1.67×10^{-2}	5.06×10^{-4}	2.10
	E	9.99×10^{4}	1.72×10^{4}	2.06
	ATE	9.07×10^{-3}	2.88×10^{-4}	3.94

An application of Proposition 2 leads to

$$\begin{aligned} A_T &= 0.948368, \quad B_T = -0.734611, \quad C_T = -0.207404, \\ A_E &= 0.948498, \quad B_E = -0.738973, \quad C_E = -0.211792, \end{aligned}$$

and

$$\begin{aligned} A_T^2 &= 0.899401, \quad |B_T|^2 + |B_T\overline{C_T} - A_T C_T| = 0.888709, \\ A_E^2 &= 0.899649, \quad |B_E|^2 + |B_E\overline{C_E} - A_E C_E| = 0.903474. \end{aligned}$$

Again the conditions for the T method are fulfilled, whereas the conditions for the E method are not. For $q = -5.45$ and $H = 0.1$, the condition $hq > -\frac{3}{11}$ is fulfilled and the root with the largest modulus of (21) is $w_1 = -0.999236$, which is in alignment with the stability of the A method.

The adaptive ATE method generally requires more execution time than the fixed-basis methods, reflecting the additional work involved in evaluating the interpolation indicators and selecting a scheme. In several examples, this additional cost is accompanied by improved or competitive accuracy. The reported timings supplement the accuracy results and do not constitute a comprehensive assessment of computational efficiency.

6. Conclusions

In this paper, Adams-type methods based on algebraic, trigonometric, and exponential interpolation are studied. The methods share the same integral construction but differ in the interpolation basis used for the derivative approximation. Such an approach for constructing numerical procedures for ODEs has not gained much attention in the literature, and further study of it reduces the likelihood of it falling into obscurity. Convergence of order 3 of the trigonometric and exponential Adams–Bashforth methods is established, and the stability characteristics of these methods are studied via the locus method and the Schur criterion, deriving a sufficient criterion for stability with respect to $y' = qy$. Convergence of order 3 of the ATE method is also proven.

The numerical results show that the choice of basis affects both accuracy and stability. In problems with algebraic, trigonometric, or hyperbolic structure, the adaptive ATE method generally selects the corresponding fitted scheme. In the nonlinear example, the method switches between the available formulas and gives lower errors in the reported tests.

The stability examples illustrate that an unsuitable fitted basis may lead to instability or loss of accuracy. The adaptive procedure reduces this effect in the considered cases by selecting among the available schemes locally.

Future work may include higher-order adaptive Adams-type formulas, implicit variants, adaptive step size control, and a study of the stability of the ATE method with respect to $y' = qy$.

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