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A Degree Theory for Multivalued Perturbations of Densely Defined (S_+)-Operators and Application to a Feedback Control Problem

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Abstract

We study the topological degree for densely defined operators of class (S_+) with compact multivalued perturbations. It is assumed that a perturbation can be represented as a sum of two terms: a multivalued mapping which is itself a composition of a continuous single-valued mapping and an upper semicontinuous multivalued mapping with aspherical images, and a single-valued compact operator. This makes it possible to extend the standard degree theory for convex-valued operators to multivalued mappings whose values may have a complicated geometry, as well as to operators defined not on the whole space but only on a dense subset. Theoretical aspects concerning the definition and main properties of the topological degree for such multivalued mappings are discussed. In particular, it is shown that the introduced degree is homotopy invariant and can serve as a convenient tool for establishing the solvability of the corresponding operator inclusions. As an application, a feedback control problem for a class of nonlinear elliptic equations is considered. Using the proposed topological degree, the solvability of this problem in the weak setting is proved.

Keywords: topological degree; monotone operators; multivalued mappings; densely defined operators; aspherical sets; feedback control problem; nonlinear elliptic equations

MSC: 47H11; 47H14; 35J60; 49J53

1. Introduction

Operator equations with multivalued mappings, or operator inclusions, arise naturally in a wide range of problems in mathematics and its applications, such as optimal control, differential equations and inclusions, variational inequalities, game theory, and mathematical economics [1–4]. One of the most efficient tools for analyzing such equations is the apparatus of topological characteristics [5–7]; first of all the degree theory, which makes it possible not only to establish the solvability of the corresponding problems but also to describe various qualitative properties of their solutions. One of the central objects of study in this direction is multivalued perturbations of monotone-type operators [8–12]—a class of mappings which, on the one hand, is wide enough for applications, and, on the other hand, possesses properties ensuring that the theory is closed with respect to passage to the limit.

To date, the degree theory for multivalued perturbations of monotone-type operators has been developed in sufficient detail, especially for (S_+)-mappings, which naturally arise in the study of nonlinear boundary and initial-boundary value problems [13–15]

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and variational inequalities [10,16,17]. Browder [8] has defined the degree for bounded, demicontinuous (S_+)-mappings with maximal monotone perturbations. His results were further developed in [9,10]. Later, in [11,12], a degree theory was constructed for compact multivalued perturbations of (S_+)-mappings with aspherical images. On the basis of this theory, various feedback control problems in systems described by nonlinear equations of mathematical physics were studied [18–20]. However, the assumptions used there include the condition that the unperturbed operator is defined on the whole Banach space, whereas in many important applications it is only densely defined.

The degree theory for densely defined mappings of type (S_+) was first proposed by Kartsatos and Skrypnik [21]; its applications and important generalizations are considered in [22–26]. In particular, in [22] the index of a critical point for nonlinear elliptic operators with strong coefficient growth was studied. In [23,24], respectively, maximal monotone and compact perturbations of densely defined (S_+)-mappings are considered, with applications to elliptic problems. Later, in [25], analogous constructions were applied to nonlinear parabolic problems. Nevertheless, these works do not cover multivalued perturbations of general form—they consider perturbations of a special structure (maximal monotone, convex-valued, quasibounded, etc.). Thus, until now no degree theory has simultaneously met all of the following requirements:

- it admits densely defined operators of class (S_+);
- it allows one to consider multivalued perturbations with nonconvex, geometrically complicated values;
- it includes, as a separate term, a single-valued compact part of the perturbation.

The present paper fills this gap. Its main novelty consists in constructing a topological degree for multivalued mappings of the form $\alpha - \Phi$, where

- α is a single-valued densely defined operator satisfying the condition $(S_+)_{\mathbb{F}}$;
- $\Phi := \beta \circ \Psi + \gamma$ is a compact multivalued mapping such that Ψ is a multivalued mapping with aspherical images, β is a single-valued mapping, and γ is a single-valued compact operator.

Compared with known constructions, this means a simultaneous extension of the theory in three directions at once:

- with respect to the domain of the unperturbed operator—from operators defined on the whole space to densely defined operators;
- with respect to the geometry of the perturbation values—from convex-valued perturbations to multivalued mappings with aspherical images, which may have a complicated geometry;
- with respect to the structure of the perturbation—to perturbations decomposing into a "complicated" multivalued part $\beta \circ \Psi$ and a "regular" single-valued compact part γ.

The necessity of such an extension is dictated by applications. A typical situation in which the unperturbed operator is defined only on a dense set arises in the study of boundary value problems for quasilinear elliptic and parabolic equations with strongly growing coefficients [21,22]: the domain of the operator naturally turns out to be a dense but not coinciding with the whole space subset. In turn, abandoning the convex-valuedness of the perturbation is essential, for example, in control problems where the set of admissible control values may have a nonconvex geometry, and the perturbation naturally decomposes into a multivalued part describing the feedback and a single-valued compact part such as an integral operator.

We note that it is precisely the combination of these three factors—the dense definiteness of the unperturbed operator, the nonconvex-valuedness of the multivalued perturbation, and the presence of a single-valued compact part—that creates the main difficulties in

constructing the degree theory. Classical constructions of the degree for (S_+)-operators essentially rely on the fact that the operator is defined on the whole space and the perturbation is either single-valued or convex-valued. When passing to densely defined operators, one has to use finite-dimensional approximations and prove stabilization of the corresponding degrees. When abandoning the convex-valuedness of the perturbation, one has to apply the apparatus of ε-approximations of multivalued mappings with aspherical images. Finally, the presence of a single-valued compact part requires separate account when passing to the limit and when verifying homotopy invariance. It is precisely the combination of these three aspects that constitutes the principal novelty of the proposed construction.

The construction of the degree follows the scheme below. First, the mapping $\alpha - \Phi$ is approximated by its finite-dimensional projections $\{\alpha_k - \Phi_k\}_{k=1}^{\infty}$, and for them the degree of multivalued mappings is defined. Then the stabilization of the obtained degrees as $k \to \infty$ is established, and the limiting value is declared to be the degree of the original mapping. The central technical results ensuring the correctness of the construction are:

- the preservation of the absence of solutions under passage to finite-dimensional projections (Lemma 2);
- the independence of the degree of the choice of the ε_k-approximation of the multivalued part of the perturbation (Equality (8));
- the stabilization of finite-dimensional degrees as $k \to \infty$ (Lemma 3).

The introduced characteristic possesses all standard properties of the topological degree: correctness of the definition, independence of the choice of approximations, and the homotopy invariance property. In the paper, the homotopy invariance of the degree is established (Theorem 1), and an analogue of the "main theorem" of the degree theory is proved (Theorem 2), asserting that the difference of the degree from zero guarantees the solvability of the corresponding operator inclusion.

In the concluding part of the paper, the constructed degree is applied to the study of a feedback control problem for a class of nonlinear elliptic equations. A boundary value problem is considered in which the state function and the control are related by a multivalued feedback mapping, and the nonlinearity in the equation admits strong growth of coefficients. Using the abstract degree theory, a theorem on the existence of a weak solution of this problem is proved under natural conditions on the data of the problem. This demonstrates that the developed abstract apparatus turns out to be a working tool in the study of concrete nonlinear problems of elliptic type, and its novelty is confirmed by an application going beyond the previously known degree theories. It is essential that in the application the multivaluedness of the feedback is not reduced to the convex-valued case, and the nonlinearity of the equation admits growth of coefficients typical for problems with strongly growing coefficients, which additionally illustrates the relevance of the proposed theory.

The rest of the paper is organized as follows. In Section 2, we give preliminary information: the definition of $(S_+)_{\mathbb{F}}$-operators, the notions of aspherical sets and $\mathcal{J}$-multimaps, as well as results on continuous approximations of multivalued mappings. In Section 3, the topological degree of compact multivalued perturbations of $(S_+)_{\mathbb{F}}$-mappings is constructed: the main assumptions are formulated, auxiliary assertions are proved, the definition of the degree is given, and some of its properties are established—homotopy invariance and the theorem on the solvability of inclusions under a nonzero degree. In Section 4, the constructed theory is applied to a feedback control problem for a nonlinear elliptic equation: the statement of the problem, assumptions on the data, the weak formulation, and the solvability theorem are presented. Finally, Section 5 provides our concluding remarks.

2. Preliminaries

This section provides the notions and statements that will be needed to obtain our main results.

Let X, Y, Z be metric spaces, E a real separable reflexive Banach space, E^* its dual.

We denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively. For elements $x \in E$ and $h \in E^*$, by $\langle h, x\rangle$ we denote the action of the functional h on the element x.

The symbol $\multimap$ is used to denote a multivalued map: the notation $A\colon X \multimap Y$ means that to each $x \in X$ a subset $A(x) \subset Y$ is assigned.

2.1. Inclusions with $(S_+)_{\mathbb{F}}$-Operators

Let $\{v_m\}_{m=1}^{\infty}$ be a fixed complete system in the space E, that is, the closed linear span of $\{v_m\}_{m=1}^{\infty}$ coincides with E. Assume that for each k the elements $v_1, \dots, v_k$ are linearly independent.

Let

$$\begin{aligned} \mathbb{F}_k &:= \text{the linear span of the elements } v_1, \dots, v_k, \\ \mathbb{F} &:= \bigcup_{k=1}^{\infty} \mathbb{F}_k. \end{aligned} \tag{1}$$

Consider a single-valued operator $\alpha\colon \operatorname{Dom}(\alpha) \to E^*$ such that

$$\mathbb{F} \subset \operatorname{Dom}(\alpha) \subset E.$$

Definition 1 (Condition $(S_+)_{\mathbb{F}}$)**.** *We say that the operator α satisfies the condition $(S_+)_{\mathbb{F}}$ if for any $h \in E^*$ and any sequence $\{u_j\} \subset \mathbb{F}$ such that $u_j \rightharpoonup u_0$ and*

$$\begin{aligned} &\limsup_{j\to\infty} \langle \alpha(u_j), u_j\rangle \le \langle h, u_0\rangle, \\ &\lim_{j\to\infty} \langle \alpha(u_j), v\rangle = \langle h, v\rangle, \quad \forall v \in \mathbb{F}, \end{aligned}$$

it follows that

$$u_j \to u_0, \quad u_0 \in \operatorname{Dom}(\alpha), \quad \alpha(u_0) = h.$$

The condition $(S_+)_{\mathbb{F}}$ is a generalization of the condition (S_+). In the papers [21,22], a degree theory for $(S_+)_{\mathbb{F}}$-mappings was constructed.

Let $\Phi\colon \operatorname{Dom}(\Phi) \multimap E^*$ be a multivalued mapping such that

$$\operatorname{Dom}(\alpha) \subset \operatorname{Dom}(\Phi) \subset E.$$

We will study solvability conditions for an inclusion of the form

$$\alpha(u) \in \Phi(u), \tag{2}$$

thereby generalizing and developing results from the fixed point theory [27–31].

2.2. Aspherical Sets, $\mathcal{J}$- and $\mathcal{J}_c$-Multimaps

Following [1,32], we introduce the following definitions.

For $\mathfrak{T} \subset X$ and $\varepsilon > 0$, denote

$$O_\varepsilon(\mathfrak{T}) := \{x \in X\colon \rho(x, \mathfrak{T}) < \varepsilon\},$$

where $\rho(x, \mathfrak{T})$ is the distance from x to the set $\mathfrak{T}$.

Definition 2 (Aspherical set)**.** *A nonempty compact subset $\mathfrak{S}$ of a metric space is called aspherical if for every $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that for every $n \in \{0, 1, \dots\}$ any continuous mapping $g\colon \mathbb{S}^n \to O_\delta(\mathfrak{S})$ can be extended to a continuous mapping $\widetilde{g}\colon \mathbb{B}^{n+1} \to O_\varepsilon(\mathfrak{S})$, where $\mathbb{S}^n$ and $\mathbb{B}^{n+1}$ denote the unit sphere and ball in $\mathbb{R}^{n+1}$, respectively.*

Definition 3 ($\mathcal{J}$-multimap)**.** *A multivalued mapping $\Psi\colon X \multimap Y$ is called a $\mathcal{J}$-multimap if it is upper semicontinuous and the set $\Psi(x)$ is aspherical for any $x \in X$. For brevity, we write $\Psi \in \mathcal{J}(X, Y)$.*

To note how wide the class of $\mathcal{J}$-multimaps is, we remark that examples of aspherical sets in a linear normed space are compact convex or contractible sets.

Definition 4 ($\mathcal{J}_c$-multimap)**.** *By $\mathcal{J}_c(X, Y)$ we denote the set of all multimaps $\Phi\colon X \multimap Y$ of the form $\Phi := \beta \circ \Psi + \gamma$, where*

- *$\Psi\colon X \multimap Z$ is a multivalued mapping belonging to the class $\mathcal{J}(X, Z)$;*
- *$\beta\colon Z \to Y$ is a continuous single-valued mapping;*
- *$\gamma\colon X \to Y$ is a single-valued compact operator.*

2.3. Continuous Approximations of Multivalued Mappings

Let $\Psi\colon X \multimap Y$ be a multivalued mapping and $\varepsilon > 0$.

Definition 5 (ε-approximation)**.** *A continuous mapping $\psi_\varepsilon\colon X \to Y$ is called an ε-approximation of Ψ if for every $x \in X$ there exists $x' \in O_\varepsilon(x)$ such that*

$$\psi_\varepsilon(x) \in O_\varepsilon(\Psi(x')).$$

The set of all ε-approximations of Ψ will be denoted by $\operatorname{appr}(\Psi, \varepsilon)$.

In the following lemma, we present an important property of ε-approximations; see [1] for details.

Lemma 1 (Preservation of the absence of solutions under continuous approximation)**.** *Let $f\colon X \to Y$ and $\beta\colon Z \to Y$ be continuous mappings. Let $\Psi\colon X \multimap Z$ be an upper semicontinuous multivalued mapping such that the set $\Psi(x)$ is compact for every $x \in X$. Suppose $\mathfrak{S}$ is a compact subset of X such that*

$$f(x) \notin \beta \circ \Psi(x), \quad \forall x \in \mathfrak{S}.$$

Then, if $\varepsilon > 0$ is sufficiently small and $\psi_\varepsilon \in \operatorname{appr}(\Psi, \varepsilon)$, we have

$$f(x) \neq \beta \circ \psi_\varepsilon(x), \quad \forall x \in \mathfrak{S}.$$

The following proposition was proved in [33].

Proposition 1 (Approximation properties of $\mathcal{J}$-multimaps)**.** *Let $\mathcal{X}$ be a locally contractible finite-dimensional compact and $\Psi \in \mathcal{J}(\mathcal{X}, Y)$. Then:*

(i) *the multimap Ψ is approximable, that is, for every $\varepsilon > 0$, there exists $\psi_\varepsilon \in \operatorname{appr}(\Psi, \varepsilon)$;*

(ii) *for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and for any two δ-approximations ψ_δ, $\psi'_\delta \in \operatorname{appr}(\Psi, \delta)$ there exists a continuous mapping $\widetilde{\psi}\colon \mathcal{X} \times [0, 1] \to Y$ such that*

$$\widetilde{\psi}(\cdot, 0) = \psi_\delta, \quad \widetilde{\psi}(\cdot, 1) = \psi'_\delta$$

and $\widetilde{\psi}(\cdot, t) \in \operatorname{appr}(\Psi, \varepsilon)$ for every $t \in [0, 1]$.

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3. Degree of Compact Multivalued Perturbations of $(S_+)_{\mathbb{F}}$-Operators

3.1. Setting and Assumptions

Our goal is to introduce the notion of the topological degree for $\mathcal{J}_c$-perturbations of $(S_+)_{\mathbb{F}}$-operators.

Assume that

(A.1) the operator α satisfies the condition $(S_+)_{\mathbb{F}}$;

(A.2) for every $v \in \mathbb{F}$ and $k \in \mathbb{N}$, the function $a_{v,k}\colon \mathbb{F}_k \to \mathbb{R}$ defined by

$$a_{v,k}(u) := \langle \alpha(u), v \rangle,$$

is continuous.

For a multivalued mapping $\Phi\colon \mathrm{Dom}(\Phi) \multimap E^*$ such that

$$\mathrm{Dom}(\alpha) \subset \mathrm{Dom}(\Phi) \subset E,$$

we suppose the following conditions to hold:

(A.3) $\Phi := \beta \circ \Psi + \gamma$ belongs to the class $\mathcal{J}_c(\mathrm{Dom}(\Phi), E^*)$;

(A.4) for every bounded set $\mathfrak{S} \subset E$, the set $\Psi(\mathrm{Dom}(\Phi) \cap \mathfrak{S})$ is relatively compact in Z.

Let $\mathcal{V}$ be an open bounded subset of E. By the symbols $\overline{\mathcal{V}}$ and $\partial\mathcal{V}$ we denote the closure and the boundary of the set $\mathcal{V}$, respectively.

Let us assume that

(A.5) for every $k \in \mathbb{N}$, the set $\overline{\mathcal{V} \cap \mathbb{F}_k}$ is locally contractible;

(A.6) the inclusion

$$\alpha(u) \in \Phi(u), \quad u \in \mathrm{Dom}(\alpha),$$

has no solutions belonging to $\partial\mathcal{V}$.

Under conditions (A.1)–(A.6) we will define $\deg(\alpha - \Phi, \mathcal{V}, 0)$—the degree of the multivalued mapping $\alpha - \Phi$ on the set $\mathcal{V}$ with respect to $0 \in E^*$. In order to present the construction of this degree, we will need some auxiliary assertions.

3.2. Auxiliary Statements

Introduce the finite-dimensional (linear) operator $\ell_{\mathbb{F}_k}\colon E^* \to \mathbb{F}_k$ by

$$\ell_{\mathbb{F}_k}(h) := \sum_{i=1}^{k} \langle h, v_i \rangle v_i.$$

Denote

$$\alpha_k := \ell_{\mathbb{F}_k} \circ \alpha, \quad \Phi_k := \ell_{\mathbb{F}_k} \circ \Phi, \quad \beta_k := \ell_{\mathbb{F}_k} \circ \beta, \quad \gamma_k := \ell_{\mathbb{F}_k} \circ \gamma.$$

Lemma 2 (Preservation of the absence of solutions under finite-dimensional approximation)**.** *Let $\mathfrak{S}$ be a closed bounded subset of E. Suppose that*

$$\alpha(u) \notin \Phi(u), \quad \forall u \in \mathrm{Dom}(\alpha) \cap \mathfrak{S}. \tag{3}$$

Then there exists k_0 such that, for any $k \geq k_0$, we have

$$\alpha_k(u) \notin \Phi_k(u), \quad \forall u \in \mathbb{F}_k \cap \mathfrak{S}.$$

Proof. We prove this assertion by contradiction. Assume that there exists a sequence $\{u_j\}$ belonging to $\mathbb{F}_{k_j} \cap \mathfrak{S}$ such that

$$\alpha_{k_j}(u_j) \in \Phi_{k_j}(u_j)$$

and $k_j \to \infty$ as $j \to \infty$. In this case, there exist sequences $\{g_j\}$ and $\{c_j\}$ such that

$$g_j \in \beta(\Psi(u_j)), \quad c_j = \gamma(u_j)$$

and

$$\langle \alpha(u_j) - g_j - c_j, v_i \rangle = 0, \ i = 1, \ldots, k_j. \tag{4}$$

Since the sequence $\{u_j\}$ is bounded, without loss of generality it can be assumed that

$$u_j \rightharpoonup u_0 \in E.$$

as $j \to \infty$. Moreover, by condition (A.4), we may assume that

$$g_j \to g_0 \in E^* \tag{5}$$

and from the compactness of γ it follows that

$$c_j = \gamma(u_j) \to c_0 \in E^*. \tag{6}$$

From (4) it follows that

$$\langle \alpha(u_j), u_j \rangle = \langle g_j + c_j, u_j \rangle,$$

whence, by (5) and (6), we obtain

$$\lim_{j \to \infty} \langle \alpha(u_j), u_j \rangle = \langle g_0 + c_0, u_0 \rangle.$$

Similarly,

$$\lim_{j \to \infty} \langle \alpha(u_j), v \rangle = \langle g_0 + c_0, v \rangle$$

for every $v \in \mathbb{F}$. From this, taking into account the closedness of the set $\mathfrak{S}$ and condition (A.1), we obtain:

$$u_j \to u_0, \quad u_0 \in \mathrm{Dom}(\alpha) \cap \mathfrak{S}, \quad \alpha(u_0) = g_0 + c_0.$$

In view of $\mathrm{Dom}(\alpha) \subset \mathrm{Dom}(\Phi)$, we have $u_0 \in \mathrm{Dom}(\Phi)$. Since

$$u_j \to u_0, \quad g_j \to g_0, \quad g_j \in \beta(\Psi(u_j))$$

and the multimap $\beta \circ \Psi$ is upper semicontinuous, we have

$$g_0 \in \beta(\Psi(u_0)).$$

From the continuity of γ it follows that $c_0 = \gamma(u_0)$. Consequently,

$$\alpha(u_0) \in \beta(\Psi(u_0)) + \gamma(u_0) = \Phi(u_0)$$

and

$$u_0 \in \mathrm{Dom}(\alpha) \cap \mathfrak{S},$$

which contradicts (3). Thus, Lemma 2 is proved. □

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Let $k \geq k_0$. Define the degree of the multivalued mapping $\alpha_k - \Phi_k \colon \mathcal{V} \cap \mathbb{F}_k \multimap \mathbb{F}_k$ by the formula:

$$\deg(\alpha_k - \Phi_k, \mathcal{V} \cap \mathbb{F}_k, 0) := \mathrm{d}(\alpha_k - \beta_k \circ \psi_{\varepsilon_k} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0), \tag{7}$$

where

- the symbol d on the right-hand side of (7) denotes the Brouwer degree (the degree of finite-dimensional mappings);
- ε_k is a sufficiently small positive number;
- ψ_{ε_k} is an ε_k-approximation of the multimap $\Psi|_{\overline{\mathcal{V} \cap \mathbb{F}_k}}$.

Note that the existence of an ε_k-approximation of $\Psi|_{\overline{\mathcal{V} \cap \mathbb{F}_k}}$ follows from Proposition 1. Let us show that the definition of deg given by (7) is correct.

First, note that, for a sufficiently small ε_k, the Brouwer degree on the right-hand side of (7) is well-defined, since

$$\alpha_k(u) - \beta_k \circ \psi_{\varepsilon_k}(u) - \gamma_k(u) \neq 0, \quad \forall u \in \partial\mathcal{V} \cap \mathbb{F}_k.$$

This relation follows from condition (A.6), Lemmas 1 and 2.

Next, we show that the degree on the right-hand side of (7) does not depend on the choice of the ε_k-approximation, that is,

$$\mathrm{d}(\alpha_k - \beta_k \circ \psi_{\varepsilon_k} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0) = \mathrm{d}(\alpha_k - \beta_k \circ \psi'_{\varepsilon_k} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0) \tag{8}$$

for any $\psi_{\varepsilon_k}, \psi'_{\varepsilon_k} \in \mathrm{appr}(\Psi|_{\overline{\mathcal{V} \cap \mathbb{F}_k}}, \varepsilon_k)$ for a sufficiently small ε_k.

From Lemmas 1 and 2, Proposition 1 and condition (A.6) it follows that the approximations ψ_{ε_k} and ψ'_{ε_k} can be joined by a continuous deformation $\widetilde{\psi} \colon \overline{\mathcal{V} \cap \mathbb{F}_k} \times [0,1] \to Z$ such that

$$\widetilde{\psi}(\cdot, 0) = \psi_{\varepsilon_k}, \quad \widetilde{\psi}(\cdot, 1) = \psi'_{\varepsilon_k}$$

and

$$\alpha_k(u) - \beta_k \circ \widetilde{\psi}(u, t) - \gamma_k(u) \neq 0, \quad \forall u \in \partial\mathcal{V} \cap \mathbb{F}_k,\ t \in [0,1],$$

whence, by the homotopy invariance property of the Brouwer degree, we obtain Equality (8).

The following assertion is valid.

Lemma 3 (Stabilization of degrees as $k \to \infty$)**.** *There exists k_1 such that for $k \geq k_1$ the value of* $\deg(\alpha_k - \Phi_k, \mathcal{V} \cap \mathbb{F}_k, 0)$ *is independent of k.*

Proof. For our proof, we will adopt the approach of [11,24]. Clearly, to prove Lemma 3 it suffices to establish that

$$\deg(\alpha_{k+1} - \Phi_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0) = \deg(\alpha_k - \Phi_k, \mathcal{V} \cap \mathbb{F}_k, 0) \tag{9}$$

for sufficiently large k.

We assume that in (7) the quantities ε_k are chosen so that the sequence $\{\varepsilon_k\}$ is monotonically decreasing and $\varepsilon_k \to 0$ as $k \to \infty$.

Let h_j be a functional from the space E^* satisfying the following condition

$$\langle h_j, v_i \rangle = \delta_{ij}, \quad i \in \{1, \ldots, j\},$$

where δ_{ij} is the Kronecker symbol.

Consider the mapping $\tau_{k+1}\colon \mathcal{V} \cap \mathbb{F}_{k+1} \to \mathbb{F}_{k+1}$ defined by

$$\tau_{k+1}(u) := \alpha_k(u) - \beta_k \circ \psi_{\varepsilon_{k+1}}(u) - \gamma_k(u) + \langle h_{k+1}, u\rangle v_{k+1}.$$

Let us show that for sufficiently large k

$$\begin{aligned} &s(\alpha_{k+1}(u) - \beta_{k+1} \circ \psi_{\varepsilon_{k+1}}(u) - \gamma_{k+1}(u)) \\ &\quad + (1-s)\tau_{k+1}(u) \neq 0, \quad \forall (u,s) \in (\partial\mathcal{V} \cap \mathbb{F}_{k+1}) \times [0,1]. \end{aligned} \tag{10}$$

Assume the contrary. Then, without loss of generality, we may assert that there exist sequences $\{s_m\}$ and $\{u_m\}$ such that

$$s_m \in [0,1], \quad u_m \in \partial\mathcal{V} \cap \mathbb{F}_m, \quad u_m \rightharpoonup u_0$$

such that

$$s_m(\alpha_m(u_m) - \beta_m \circ \psi_{\varepsilon_m}(u_m) - \gamma_m(u_m)) + (1-s_m)\tau_m(u_m) = 0.$$

The last equality is equivalent to the following relations:

$$\langle \alpha(u_m) - \beta \circ \psi_{\varepsilon_m}(u_m) - \gamma(u_m), v_i\rangle = 0, \quad i = 1, \dots, m-1, \tag{11}$$

$$s_m\langle \alpha(u_m) - \beta \circ \psi_{\varepsilon_m}(u_m) - \gamma(u_m), v_m\rangle + (1-s_m)\langle h_m, u_m\rangle = 0. \tag{12}$$

We can assume without loss of generality that $s_m \neq 0$ for any $m \in \mathbb{N}$.

In view of condition (A.4) and Definition 5, it can be assumed that

$$\psi_{\varepsilon_m}(u_m) \to q_0$$

for some $q_0 \in Z$ as $m \to \infty$. Moreover, by the compactness of γ, we may assume that

$$\gamma(u_m) \to c_0$$

for some $c_0 \in E^*$ as $m \to \infty$. Therefore, for every $v \in \mathbb{F}$, we have

$$\begin{aligned} \lim_{m\to\infty} \langle \alpha(u_m), v\rangle &= \lim_{m\to\infty} \langle \beta \circ \psi_{\varepsilon_m}(u_m) + \gamma(u_m), v\rangle \\ &= \langle \beta(q_0) + c_0, v\rangle. \end{aligned} \tag{13}$$

Let us now estimate the value of $\limsup_{m\to\infty}\langle \alpha(u_m), u_m\rangle$.

Since $u_m \in \mathbb{F}_m$, the element u_m can be represented in the form

$$u_m = \sum_{i=1}^{m} \eta_{mi} v_i,$$

where $\eta_{m1}, \dots, \eta_{mm}$ are real numbers.

Taking into account Equalities (11) and (12), we obtain

$$\begin{aligned} \langle \alpha(u_m), u_m\rangle &= \sum_{i=1}^{m} \eta_{mi}\langle \alpha(u_m), v_i\rangle \\ &= \sum_{i=1}^{m} \eta_{mi}\langle \beta \circ \psi_{\varepsilon_m}(u_m) + \gamma(u_m), v_i\rangle - \eta_{mm}\frac{1-s_m}{s_m}\langle h_m, u_m\rangle \\ &= \langle \beta \circ \psi_{\varepsilon_m}(u_m) + \gamma(u_m), u_m\rangle - \eta_{mm}\frac{1-s_m}{s_m}\langle h_m, u_m\rangle. \end{aligned} \tag{14}$$

Since

$$\begin{aligned}\langle h_m, u_m\rangle &= \langle h_m, \sum_{i=1}^{m} \eta_{mi} v_i\rangle \\ &= \sum_{i=1}^{m} \eta_{mi}\langle h_m, v_i\rangle \\ &= \sum_{i=1}^{m} \eta_{mi}\delta_{im} \\ &= \eta_{mm},\end{aligned}$$

from (14) it follows that

$$\langle \alpha(u_m), u_m\rangle \leq \langle \beta \circ \psi_{\varepsilon_m}(u_m) + \gamma(u_m), u_m\rangle.$$

Therefore,

$$\limsup_{m\to\infty} \langle \alpha(u_m), u_m\rangle \leq \langle \beta(q_0) + c_0, u_0\rangle. \tag{15}$$

From condition (A.1) and Relations (13) and (15) it follows that

$$u_m \to u_0, \quad u_0 \in \mathrm{Dom}(\alpha) \cap \partial\mathcal{V} \quad \alpha(u_0) = \beta(q_0) + c_0.$$

Taking into account the following convergences

$$u_m \to u_0, \quad \psi_{\varepsilon_m}(u_m) \to q_0, \quad \varepsilon_m \to 0$$

and the upper semicontinuity of the multimap Ψ, we obtain $q_0 \in \Psi(u_0)$. Moreover, from the continuity of γ it follows that $c_0 = \gamma(u_0)$. Therefore,

$$\alpha(u_0) = \beta(q_0) + c_0 \in \beta(\Psi(u_0)) + \gamma(u_0) = \Phi(u_0),$$

which contradicts condition (A.6). Thus, the validity of Relation (10) is proved.

In view of the homotopy invariance property of the Brouwer degree, from (10) it follows that

$$\mathrm{d}(\alpha_{k+1} - \beta_{k+1} \circ \psi_{\varepsilon_{k+1}} - \gamma_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0) = \mathrm{d}(\tau_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0). \tag{16}$$

Moreover, using the Leray–Schauder lemma (see, for example, [14], Chapter 2), we deduce

$$\mathrm{d}(\alpha_k - \beta_k \circ \psi_{\varepsilon_{k+1}} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0) = \mathrm{d}(\tau_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0). \tag{17}$$

Next, combining (16) and (17), we obtain

$$\begin{aligned}&\mathrm{d}(\alpha_{k+1} - \beta_{k+1} \circ \psi_{\varepsilon_{k+1}} - \gamma_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0) \\ &= \mathrm{d}(\alpha_k - \beta_k \circ \psi_{\varepsilon_{k+1}} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0).\end{aligned} \tag{18}$$

By the independence of the degree defined in formula (7) of the choice of an ε_k-approximation, the quantity on the right-hand side of (18) can be used for computing $\deg(\alpha_k - \Phi_k, \mathcal{V} \cap \mathbb{F}_k, 0)$.

On the other hand, it is obvious that the degree on the left-hand side of (18) determines $\deg(\alpha_{k+1} - \Phi_{k+1}, \mathcal{V} \cap \mathbb{F}_{k+1}, 0)$. Thus, we arrive at the required Equality (9), which completes the proof of Lemma 3. □

3.3. Definition and Some Properties of Degree

Assume that conditions (A.1)–(A.6) hold.

Definition 6 (Degree of a multivalued perturbation of an $(S_+)_{\mathbb{F}}$-operator)**.** *The degree of the multimap $\alpha - \Phi$ on the set $\mathcal{V}$ with respect to $0 \in E^*$ is the integer* $\deg(\alpha - \Phi, \mathcal{V}, 0)$ *given by*

$$\deg(\alpha - \Phi, \mathcal{V}, 0) := \lim_{k\to\infty} \deg(\alpha_k - \Phi_k, \mathcal{V} \cap \mathbb{F}_k, 0).$$

Remark 1. *In the cases when $\Phi \equiv \{0\}$ or $\gamma \equiv 0$, Definition 6 coincides with the definitions of the degree from the works [21] and [24], respectively.*

Consider a mapping $\widetilde{\alpha}\colon \operatorname{Dom}(\widetilde{\alpha}) \to E^*$ such that

$$\operatorname{Dom}(\widetilde{\alpha}) \subset E \times [0,1]$$

and denote

$$\operatorname{Dom}(\widetilde{\alpha}(\cdot,t)) := \{u \in E : (u,t) \in \operatorname{Dom}(\widetilde{\alpha})\}.$$

Let us assume that

$$\mathbb{F} \subset \operatorname{Dom}(\widetilde{\alpha}(\cdot,t)), \quad \forall t \in [0,1],$$

and, for every $v \in \mathbb{F}$ and $k \in \mathbb{N}$, the function $\widetilde{a}_{v,k}\colon \mathbb{F}_k \times [0,1] \to \mathbb{R}$ defined by

$$\widetilde{a}_{v,k}(u,t) := \langle \widetilde{\alpha}(u,t), v\rangle$$

is continuous.

Definition 7 (Condition $(\widetilde{S}_+)_{\mathbb{F}}$)**.** *We say that the operator $\widetilde{\alpha}$ satisfies the condition $(\widetilde{S}_+)_{\mathbb{F}}$ if for any $h \in E^*$ and any sequences $\{u_j\} \subset \mathbb{F}$, $\{t_j\} \subset [0,1]$ such that $u_j \rightharpoonup u_0$, $t_j \to t_0$ and*

$$\limsup_{j\to\infty} \langle \widetilde{\alpha}(u_j,t_j), u_j\rangle \le \langle h, u_0\rangle,$$
$$\lim_{j\to\infty} \langle \widetilde{\alpha}(u_j,t_j), v\rangle = \langle h, v\rangle, \quad \forall v \in \mathbb{F},$$

it follows that

$$u_j \to u_0, \quad u_0 \in \operatorname{Dom}(\widetilde{\alpha}(\cdot,t_0)), \quad \widetilde{\alpha}(u_0,t_0) = h.$$

Consider mappings $\alpha_i\colon \operatorname{Dom}(\alpha_i) \cap \overline{\mathcal{V}} \to E^*$ and $\Phi_i\colon \operatorname{Dom}(\Phi_i) \cap \overline{\mathcal{V}} \multimap E^*$ given by

$$\Phi_i := \beta_i \circ \Psi_i + \gamma_i, \quad i = 0,1,$$

such that

$$\mathbb{F} \subset \operatorname{Dom}(\alpha_i) \subset \operatorname{Dom}(\Phi_i), \quad i = 0,1.$$

Definition 8 (Homotopy)**.** *We say that the multivalued mappings $\alpha_0 - \Phi_0$ and $\alpha_1 - \Phi_1$ are homotopic if the five conditions are satisfied:*

(i) *there exists a mapping $\widetilde{\alpha}$ satisfying the condition $(\widetilde{S}_+)_{\mathbb{F}}$ such that*

$$\widetilde{\alpha}(\cdot,0) = \alpha_0, \quad \widetilde{\alpha}(\cdot,1) = \alpha_1;$$

(ii) *there exists a multimap $\widetilde{\Psi}\colon \operatorname{Dom}(\widetilde{\Psi}) \multimap Z$ such that*

$$\operatorname{Dom}(\widetilde{\alpha}) \subset \operatorname{Dom}(\widetilde{\Psi}) \subset E \times [0,1], \quad \widetilde{\Psi} \in \mathcal{J}(\operatorname{Dom}(\widetilde{\Psi}), Z),$$

$$\widetilde{\Psi}(\cdot,0) = \Psi_0, \quad \widetilde{\Psi}(\cdot,1) = \Psi_1$$

and, for every bounded set $\mathfrak{S} \subset E \times [0,1]$, *the set* $\widetilde{\Psi}\left(\mathrm{Dom}(\widetilde{\Psi}) \cap \mathfrak{S}\right)$ *is relatively compact in the space* Z;

(iii) *there exists a continuous mapping* $\widetilde{\beta}\colon Z \times [0,1] \to E^*$ *such that*

$$\widetilde{\beta}(\cdot,0) = \beta_0, \quad \widetilde{\beta}(\cdot,1) = \beta_1;$$

(iv) *there exists a single-valued compact operator* $\widetilde{\gamma}\colon \mathrm{Dom}(\widetilde{\gamma}) \to E^*$ *such that*

$$\mathrm{Dom}(\widetilde{\alpha}) \subset \mathrm{Dom}(\widetilde{\gamma}) \subset E \times [0,1],$$

$$\widetilde{\gamma}(\cdot,0) = \gamma_0, \quad \widetilde{\gamma}(\cdot,1) = \gamma_1;$$

(v) *the inclusion*

$$\widetilde{\alpha}(u,t) \in \widetilde{\Phi}(u,t), \quad (u,t) \in \mathrm{Dom}(\widetilde{\alpha}),$$

where

$$\widetilde{\Phi}(u,t) := \widetilde{\beta}(\widetilde{\Psi}(u,t),t) + \widetilde{\gamma}(u,t),$$

has no solutions belonging to $\partial\mathcal{V} \times [0,1]$.

Using the homotopy invariance property of the Brouwer degree and Proposition 1, it is not difficult to establish the validity of the following assertion.

Theorem 1 (Homotopy invariance property)**.** *If the multivalued mappings* $\alpha_0 - \Phi_0$ *and* $\alpha_1 - \Phi_1$ *are homotopic, then*

$$\deg(\alpha_0 - \Phi_0, \mathcal{V}, 0) = \deg(\alpha_1 - \Phi_1, \mathcal{V}, 0).$$

One of the properties of the introduced degree, key for applications, is stated in the following theorem.

Theorem 2 (Solvability of inclusions under a nonzero degree)**.** *Suppose that*

$$\deg(\alpha - \Phi, \mathcal{V}, 0) \neq 0. \tag{19}$$

Then inclusion (2) *has at least one solution belonging to* $\mathcal{V} \cap \mathrm{Dom}(\alpha)$.

Proof. Assume the contrary. Then, by Lemma 2, for sufficiently large k the inclusion

$$\alpha_k(u) \in \Phi_k(u), \quad u \in \mathbb{F}_k,$$

has no solutions belonging to the set $\overline{\mathcal{V}}$. In this case, by Lemma 1,

$$\alpha_k(u) - \beta_k \circ \psi_{\varepsilon_k}(u) - \gamma_k(u) \neq 0, \quad u \in \overline{\mathcal{V} \cap \mathbb{F}_k}. \tag{20}$$

Here, ψ_{ε_k} is an ε_k-approximation of $\Psi|_{\overline{\mathcal{V} \cap \mathbb{F}_k}}$ with a sufficiently small ε_k.

Using Definition 6 and Relation (20), we derive

$$\begin{aligned} \deg(\alpha - \Phi, \mathcal{V}, 0) &= \mathrm{d}(\alpha_k - \beta_k \circ \psi_{\varepsilon_k} - \gamma_k, \mathcal{V} \cap \mathbb{F}_k, 0) \\ &= 0, \end{aligned}$$

which contradicts (19). This contradiction proves Theorem 2. □

4. Application of Degree in Feedback Control Theory

The purpose of this section is to show that the degree theory constructed in Section 3 is a working tool for a concrete nonlinear problem and that this problem exhibits features that are not covered simultaneously by the previously known degree theories. Specifically, we consider a feedback control problem for a quasilinear elliptic equation in which the state and the control are related by a multivalued feedback mapping. This example is chosen because it exhibits all three features that motivated our construction: the leading elliptic operator is naturally defined only on a dense subset of $W_0^{1,p}(\Omega)$ (because of the strongly growing coefficient ζ), the set of admissible controls is not assumed to be convex, and the lower-order term contains a single-valued compact contribution. The proof of solvability (Theorem 3) follows the standard degree-theoretic scheme: we rewrite the problem as an operator inclusion, verify that the corresponding mappings satisfy assumptions (A.1)–(A.6), establish a homotopy with a model problem whose degree equals one, and then apply Theorem 2.

4.1. Statement of Feedback Control Problem

Let $\Omega \subset \mathbb{R}^3$ be a domain with a sufficiently smooth boundary Γ, and let ζ, a_i, f_i be given functions. Consider the following problem:

$$\begin{cases} \displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left\{\zeta^2(v)\frac{\partial v}{\partial x_i} + a_i\left(x, v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}\right)\right\} \\ \quad = f_1(x, v, u) + f_2(x, v) \ \text{ in } \Omega, \\ v|_\Gamma = 0, \\ u \in \mathcal{U}(v), \end{cases} \tag{21}$$

where $\mathcal{U}$ is a multivalued mapping governing the feedback control in the system.

The unknowns are the state function v and the control u. We will consider problem (21) in the weak setting.

4.2. Assumptions on Data

By p we denote the summability exponent of the gradient, with

$$2 \le p < 3.$$

Moreover, set

$$p^* := \frac{3p}{3-p} \ \text{(the Sobolev critical exponent)}.$$

Assume that the following conditions are satisfied.

(B.1) The function $\zeta : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the estimate

$$0 \le \zeta(t) \le \mu\left\{\left|\int_0^t \zeta(s)\,ds\right| + 1\right\}^r, \quad t \in \mathbb{R},$$

with constants μ and r such that

$$\mu > 0, \quad 0 \le r < 3.$$

(B.2) For fixed $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$, the function $a_i(\cdot, s, \xi)$ is measurable on Ω, $i = 1, 2, 3$.

(B.3) For almost all $x \in \overline{\Omega}$, the function $a_i(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^3$, $i = 1, 2, 3$.

(B.4) There exist positive constants p_1, ν_1, ν_2 and a function g_0 such that

$$\sum_{i=1}^{3}(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi_i')\geq \nu_1|\xi-\xi'|^p,$$

$$|a_i(x,s,\xi)|\leq \nu_2(|s|^{p_1}+|\xi|)^{p-1}+g_0(x)$$

for all $x\in\overline{\Omega}$, $s\in\mathbb{R}$, $\xi,\xi'\in\mathbb{R}^3$, where

$$g_0\in L^{\frac{p}{p-1}}(\Omega),\quad p_1<\frac{3}{3-p}.$$

(B.5) The function $f_1\colon \overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions: for almost all $x\in\overline{\Omega}$ the function $f_1(x,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}$, and for all $(y_1,y_2)\in\mathbb{R}\times\mathbb{R}$ the function $f_1(\cdot,y_1,y_2)$ is measurable on Ω. Moreover, f_1 satisfies the estimate

$$|f_1(x,y_1,y_2)|\leq C_1(|y_1|^{\rho_1}+|y_2|^{\rho_2})+g_1(x),$$

where

$$g_1\in L^{\frac{p}{p-1}}(\Omega),\quad C_1>0,$$

and the exponents ρ_1 and ρ_2 are defined by the following equalities:

$$\rho_1:=\frac{p_2(p-1)}{p},\quad \rho_2:=\frac{q(p-1)}{p},$$

with

$$1<p_2<p^*,\quad q>1.$$

(B.6) The function $f_2\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory conditions: for almost all $x\in\overline{\Omega}$ the function $f_2(x,\cdot)$ is continuous on $\mathbb{R}$, and for all $y\in\mathbb{R}$ the function $f_2(\cdot,y)$ is measurable on Ω. Moreover, f_2 satisfies the estimate

$$|f_2(x,y)|\leq C_2|y|^{\theta}+g_2(x),$$

where

$$g_2\in L^{\frac{p}{p-1}}(\Omega),\quad C_2>0,\quad \theta:=\frac{p_2(p-1)}{p}.$$

(B.7) The multivalued mapping $\mathcal{U}\colon L^{p_2}(\Omega)\multimap L^{q}(\Omega)$ is upper semicontinuous and the set $\mathcal{U}(v)$ is aspherical for every $v\in L^{p_2}(\Omega)$.

Remark 2. *We note that the growth condition* (B.1) *is not restrictive. This condition is satisfied even by the exponential function* $\zeta(t)\equiv\exp(t)$ *with* $\mu=1$ *and* $r\geq 1$.

4.3. Weak Formulation of Problem and Main Result

By $W_0^{1,p}(\Omega)$ we denote the subspace of the Sobolev space $W^{1,p}(\Omega)$ obtained as the closure of the set of all C^∞ functions with compact support in Ω with respect to the $W^{1,p}$-norm. The dual of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$, where $p':=p/(p-1)$.

Definition 9 (Weak solution)**.** *A weak solution of problem* (21) *is a pair of functions* (v,u) *such that*

- $(v,u)\in W_0^{1,p}(\Omega)\times L^q(\Omega)$;
- $\zeta^2(v)\frac{\partial v}{\partial x_i}\in L^{\frac{p}{p-1}}(\Omega)$, $i=1,2,3$;
- $u\in\mathcal{U}(v)$;

- *the following equality holds:*

$$\sum_{i=1}^{3}\int_{\Omega}\left\{\zeta^2(v)\frac{\partial v}{\partial x_i}+a_i\left(x,v,\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_3}\right)\right\}\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x = \int_{\Omega}(f_1(x,v,u)+f_2(x,v))\phi\,\mathrm{d}x$$

for every test function $\phi \in W_0^{1,p}(\Omega)$.

Remark 3. *Since* p^* *is the Sobolev critical exponent, the condition* $p_2 < p^*$ *ensure the compactness of the following embedding (see* [34,35]):

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p_2}(\Omega),$$

which is essentially used in the proof of the existence of solutions.

To formulate the existence theorem for problem (21), we introduce an auxiliary one-parameter family of problems:

$$\begin{cases} \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left\{\zeta^{2t}(v)\frac{\partial v}{\partial x_i}+ta_i\left(x,v,\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_3}\right)+(1-t)\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x_i}\right\} \\ = t\left(f_1(x,v,u)+f_2(x,v)\right) \ \text{ in } \Omega, \\ v|_{\Gamma}=0, \\ u\in\mathcal{U}(v), \end{cases} \tag{22}$$

with the parameter $t \in [0,1]$.

Denote

$$\mathfrak{A} := \{v \in W_0^{1,p}(\Omega) : \text{ there exist } u_0 \in L^q(\Omega) \text{ and } t_0 \in [0,1] \text{ such that } (v,u_0) \text{ is a weak solution of problem (22) for } t=t_0\}.$$

Theorem 3 (Existence of weak solutions)**.** *Suppose that conditions* (B.1)–(B.6) *are satisfied and, moreover, that the set* $\mathfrak{A}$ *is bounded in the space* $W_0^{1,p}(\Omega)$*. Then problem* (21) *has at least one weak solution.*

Proof. To prove the theorem we use the degree theory constructed in Section 3. First, we give an operator interpretation of the problem under consideration.

By $C_0^{\infty}(\Omega)$ denote the set of all C^{∞} functions with compact support in the domain Ω. Fix $\{v_m\}_{m=1}^{\infty} \subset C_0^{\infty}(\Omega)$—a complete system of elements in the space $W_0^{1,p}(\Omega)$. Assume that for each k the elements $v_1,\dots,v_k$ are linearly independent and (1) holds.

Introduce the operator $\widetilde{\alpha}\colon \mathrm{Dom}(\widetilde{\alpha}) \subset W_0^{1,p}(\Omega)\times[0,1] \to W^{-1,p'}(\Omega)$ by the formula

$$\langle \widetilde{\alpha}(v,t),\phi\rangle := \sum_{i=1}^{3}\int_{\Omega}\left\{\zeta^{2t}(v)\frac{\partial v}{\partial x_i}+ta_i\left(x,v,\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2},\frac{\partial v}{\partial x_3}\right)+(1-t)\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x_i}\right\}\frac{\partial \phi}{\partial x_i}\,\mathrm{d}x,$$

where

$$\phi \in W_0^{1,p}(\Omega),$$
$$\mathrm{Dom}(\widetilde{\alpha}) := \left\{(v,t)\in W_0^{1,p}(\Omega)\times[0,1] : \ \zeta^{2t}(v)\frac{\partial v}{\partial x_i} \in L^{\frac{p}{p-1}}(\Omega),\ i=1,2,3\right\}.$$

According to [21], the operator $\widetilde{\alpha}$ satisfies the condition $(\widetilde{S}_+)_{\mathbb{F}}$.

Consider the multivalued mapping $\widetilde{\Psi}\colon W_0^{1,p}(\Omega)\times[0,1] \multimap L^{p_2}(\Omega)\times L^q(\Omega)$ defined by

$$\widetilde{\Psi}(v,t) := \{\mathrm{id}(v)\}\times \mathcal{U}(\mathrm{id}(v)),$$

where $\mathrm{id}\colon W_0^{1,p}(\Omega)\to L^{p_2}(\Omega)$ is the embedding operator. In view of Remark 3, the operator id is well defined and compact. Therefore, using condition (B.7), we deduce that, for every bounded set $\mathcal{G}\subset W_0^{1,p}(\Omega)\times[0,1]$, the set $\widetilde{\Psi}(\mathcal{G})$ is relatively compact in $L^{p_2}(\Omega)\times L^q(\Omega)$. Moreover, we have

$$\widetilde{\Psi}\in \mathcal{J}(W_0^{1,p}(\Omega)\times[0,1], L^{p_2}(\Omega)\times L^q(\Omega)).$$

Define the mapping $\widetilde{\beta}\colon L^{p_2}(\Omega)\times L^q(\Omega)\times[0,1]\to W^{-1,p'}(\Omega)$ by the formula

$$\langle \widetilde{\beta}(v,u,t),\phi\rangle := t\int_\Omega f_1(x,v,u)\phi(x)\,\mathrm{d}x,$$

for every $\phi\in W_0^{1,p}(\Omega)$. From the Krasnoselskii theorem on the continuity of the superposition operator (see, for example, [14], Proposition 1.1) and condition (B.5) it follows that the operator $\widetilde{\beta}$ is continuous.

Define the single-valued operator $\widetilde{\gamma}\colon W_0^{1,p}(\Omega)\times[0,1]\to W^{-1,p'}(\Omega)$ by the formula

$$\langle \widetilde{\gamma}(v,t),\phi\rangle := t\int_\Omega f_2(x,v)\phi(x)\,\mathrm{d}x,\quad \phi\in W_0^{1,p}(\Omega).$$

From condition (B.6) and Remark 3 it follows that the operator $\widetilde{\gamma}$ is compact.

Let

$$\widetilde{\Phi}(v,t) := \widetilde{\beta}\big(\widetilde{\Psi}(v,t),t\big)+\widetilde{\gamma}(v,t),\quad v\in W_0^{1,p}(\Omega),\ t\in[0,1].$$

To prove Theorem 3 it suffices to show that the inclusion

$$\widetilde{\alpha}(v,1)\in\widetilde{\Phi}(v,1) \tag{23}$$

has at least one solution $v\in W_0^{1,p}(\Omega)$.

In view of the condition of Theorem 3, there exists a positive number R such that

$$\mathfrak{A}\subset\mathcal{B}_R := \Big\{v\in W_0^{1,p}(\Omega):\ \|v\|_{W_0^{1,p}(\Omega)}<R\Big\}.$$

By the above-mentioned properties of the operators $\widetilde{\alpha}$, $\widetilde{\Psi}$, $\widetilde{\beta}$, and $\widetilde{\gamma}$, the mappings $\widetilde{\alpha}(\cdot,0)-\widetilde{\Phi}(\cdot,0)$ and $\widetilde{\alpha}(\cdot,1)-\widetilde{\Phi}(\cdot,1)$ are homotopic in the sense of Definition 8 on $\mathcal{V}:=\mathcal{B}_R$. Therefore, applying Theorem 1, we obtain

$$\deg(\widetilde{\alpha}(\cdot,1)-\widetilde{\Phi}(\cdot,1),\mathcal{B}_R,0)=\deg(\widetilde{\alpha}(\cdot,0)-\widetilde{\Phi}(\cdot,0),\mathcal{B}_R,0).$$

Since $\widetilde{\Phi}(\cdot,0)=0$, we have

$$\widetilde{\alpha}(\cdot,0)-\widetilde{\Phi}(\cdot,0)=\widetilde{\alpha}(\cdot,0);$$

hence the computation of $\deg(\widetilde{\alpha}(\cdot,1)-\widetilde{\Phi}(\cdot,1),\mathcal{B}_R,0)$ reduces to finding $\deg(\widetilde{\alpha}(\cdot,0),\mathcal{B}_R,0)$.

Note that

$$\mathrm{Dom}(\widetilde{\alpha}(\cdot,0))=W_0^{1,p}(\Omega)$$

and

$$\langle\widetilde{\alpha}(v,0),v\rangle>0,\quad v\in\partial\mathcal{B}_R,$$

whence, by the standard properties of the degree of (S_+)-mappings [14], we obtain

$$\deg(\widetilde{\alpha}(\cdot,0),\mathcal{B}_R,0)=1.$$

Consequently, we have

$$\deg(\widetilde{\alpha}(\cdot,1)-\widetilde{\Phi}(\cdot,1),\mathcal{B}_R,0)=1.$$

From the last equality and Theorem 2 it follows that inclusion (23) has at least one solution, and hence problem is solvable in the weak setting (21). Thus, Theorem 3 is proved. □

5. Conclusions

The paper constructs a topological degree theory for multivalued perturbations of densely defined operators of class (S_+). The proposed construction covers mappings of the form $\alpha-\Phi$, where the unperturbed operator α is defined on a dense subset of a reflexive Banach space, and the perturbation Φ admits the representation $\Phi=\beta\circ\Psi+\gamma$. Here the multivalued part $\beta\circ\Psi$ may have nonconvex, geometrically complicated values, while the single-valued part γ is a compact operator.

The main results of the paper are as follows.

1. Conditions for the correctness of finite-dimensional approximations of the mapping $\alpha-\Phi$ are formulated, and a lemma on the preservation of the absence of solutions under passage to projections onto finite-dimensional subspaces is proved.
2. The independence of the finite-dimensional degree from the choice of the ε-approximation of the multivalued part of the perturbation is established, which ensures the correctness of the definition of the degree.
3. The stabilization of finite-dimensional degrees as the dimension of the approximating subspace tends to infinity is proved, which made it possible to define the degree of the original mapping as the limit of the corresponding finite-dimensional degrees.
4. The main properties of the introduced degree are established: homotopy invariance and an analogue of the main theorem of the degree theory, according to which the difference of the degree from zero guarantees the solvability of the operator inclusion $\alpha(u)\in\Phi(u)$.
5. The constructed theory is applied to a feedback control problem for a nonlinear elliptic equation with strongly growing coefficients. A theorem on the existence of a weak solution is proved. It is essential that the multivaluedness of the feedback is not reduced to the convex-valued case, and the nonlinearity of the equation admits growth typical for problems with strong growth of coefficients.

Thus, the proposed construction extends the known degree theories in three directions at once: with respect to the domain of the unperturbed operator, with respect to the geometry of the values of the multivalued perturbation, and with respect to the structure of the perturbation. The obtained results show that the developed abstract apparatus is a working tool in the study of concrete nonlinear problems of elliptic type.

As possible directions for further research, one can mention the extension of the proposed approach to parabolic problems, to the case of unbounded domains, as well as to operators acting in nonreflexive Banach spaces. It also seems interesting to study applications of the constructed theory to optimal control problems and variational inequalities with nonconvex constraints.

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