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On Survival Time in Robust Optimization Over Time for Discrete Search Spaces

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Abstract

Robust Optimization Over Time (ROOT) addresses dynamic optimization problems where the goal is not to track the transient optimum but to identify solutions that remain effective across multiple future environments. Despite recent algorithmic advances, the theoretical foundations of ROOT remain limited, particularly regarding survival-time formulations in discrete search spaces. This paper bridges this gap by analyzing expected survival time in combinatorial ROOT under stochastic k-bit-flip dynamics. Using drift analysis and martingale techniques, we derive order-matching analytical bounds and phase transitions for dynamic OneMax, capturing gradual fitness degradation. For dynamic LeadingOnes, which models catastrophic structural failure, we prove that survival time follows an exact geometric distribution. Monte Carlo simulations empirically validate these analytical models. Ultimately, these results connect ROOT with classical runtime analysis, formalizing the mechanisms governing solution durability and providing a rigorous mathematical foundation for survival-aware heuristics in combinatorial environments.

Keywords: robust optimization over time; survival time; drift analysis; runtime analysis; k-bit-flip dynamics; OneMax; LeadingOnes; discrete optimization

MSC: 90C27; 90C31; 68W50; 60G42

1. Introduction

Many real-world optimization problems arise in environments that evolve over time. Changes in demand, prices, resource availability, weather conditions, or system states may alter the objective function and cause previously high-quality solutions to deteriorate [1]. To cope with such scenarios, the field of dynamic optimization has developed a wide range of evolutionary algorithms and metaheuristics capable of adapting to environmental changes and tracking moving optima [2–6]. This research direction has become known as Evolutionary Dynamic Optimization (EDO) and constitutes one of the most active areas of evolutionary computation [3].

Within EDO, Robust Optimization Over Time (ROOT) emerged as an alternative perspective on how adaptation should be approached [7]. Traditional dynamic optimization assumes that solutions can be modified whenever environmental changes occur. In many practical settings, however, replacing a deployed solution may involve substantial operational, economic, computational, or organizational costs. Under such circumstances, continuously tracking the current optimum may not be desirable. Instead, the objective becomes identifying solutions that remain useful for extended periods despite environmental variation. This viewpoint was introduced by Yu et al. [7], further formalized by Jin et al. [8],

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and subsequently refined by Fu et al. [9,10], who established *average fitness* and *survival time* as fundamental measures of robustness. In particular, survival time quantifies the number of consecutive future environments in which a deployed solution remains above an acceptability threshold, providing a direct measure of solution durability. In this paper, we adopt the expected survival time as our primary metric because it provides a fundamental and analytically tractable summary of solution robustness. While alternative criteria like fixed-horizon survival probabilities, risk-sensitive measures, and explicit re-optimization costs are valuable for practical decision-making, they remain outside the scope of this foundational analysis.

Over the last decade, ROOT has evolved into a distinct research area within EDO [11,12]. A substantial body of work has focused on developing algorithms capable of anticipating future environmental changes and improving long-term solution quality. Representative approaches include prediction-based methods [7–10], learning-based models [13], surrogate-assisted optimization [14,15], reinforcement learning techniques [16], cost-aware optimization strategies [17,18], and data-driven adaptive frameworks. Recent reviews [11,12] portray a field that is both methodologically diverse and empirically active.

Despite the growing interest in ROOT, existing research exhibits two major theoretical gaps [11,12]. First, most dynamic models and benchmark environments focus on continuous search spaces, leaving discrete and combinatorial settings relatively unexplored despite their relevance in real-world applications [19–21]. Second, solution robustness is evaluated almost exclusively through empirical simulations [7–10]. As a result, fundamental questions regarding how survival time scales with problem dimension, environmental dynamics, acceptability thresholds, or initial solution quality remain largely unanswered. This work addresses these limitations by introducing a discrete ROOT framework designed to enable rigorous theoretical analysis.

Interestingly, this theoretical gap does not stem from a lack of analytical tools. During the last two decades, the theory of evolutionary computation has developed a rich collection of techniques for studying stochastic search processes on discrete landscapes. Drift analysis [22,23], fitness-level arguments [24,25], Markov-chain methods [26–28], and runtime analysis techniques [29,30] have enabled precise characterizations of optimization dynamics on canonical benchmark problems. These methods have provided fundamental insights into the behavior of evolutionary algorithms and guided the design of increasingly effective search procedures [31,32]. Furthermore, theoretical analyses have also been successfully extended to dynamic discrete optimization settings [33,34].

What remains largely unexplored is the application of these analytical tools to the central question posed by ROOT: given a deployed solution and a changing environment, how long can that solution be expected to remain acceptable before re-optimization becomes necessary? Unlike classical runtime analysis, which focuses on the time required to reach a target solution, the ROOT perspective naturally leads to a first-exit problem centered on the durability of already deployed solutions. Nevertheless, the mathematical machinery developed for evolutionary computation suggests that rigorous survival-time analyses may be attainable once suitable discrete ROOT models become available.

This observation motivates the present work. We introduce a discrete ROOT framework defined on the Hamming cube and study the survival-time behavior of solutions under stochastic environmental change. Our objective is not to propose a new optimization algorithm but rather to investigate the fundamental mechanisms governing solution durability. By adopting dynamic binary landscapes, we obtain a setting that is sufficiently simple to permit rigorous analysis while preserving the essential characteristics of robustness over time.

The contributions of this paper are fourfold.

1. We introduce a discrete formulation of ROOT on dynamic binary search spaces, contributing to an area that remains considerably less explored than its continuous counterpart.
2. We establish an initial theoretical framework for analyzing survival time as a first-exit probabilistic problem, addressing a major research gap identified in recent ROOT surveys.
3. We derive order-matching analytical bounds and phase transitions for gradual fitness erosion (dynamic ONEMAX), as well as exact geometric distributions for catastrophic failure (dynamic LEADINGONES), connecting ROOT with classical analytical tools from evolutionary computation.
4. We empirically evaluate these analytical predictions through Monte Carlo simulations, demonstrating consistency between our theoretical models and the observed stochastic behavior.

Beyond the specific results presented here, we view discrete ROOT as a promising avenue for bridging robustness-over-time research and the theoretical analysis of evolutionary computation. Analytical models of survival time may ultimately support the development of predictive robustness estimators, surrogate measures, survival-aware search strategies, and principled re-optimization policies. We therefore believe that establishing theoretical foundations for discrete ROOT constitutes an important step toward the long-term maturation of the field.

The remainder of the paper is organized as follows. Section 2 introduces the discrete ROOT framework and the required probabilistic tools. Section 3 presents the theoretical survival-time derivations for the ONEMAX and LEADINGONES landscapes. Section 4 empirically validates the theoretical bounds. Finally, Section 5 discusses the practical implications of our findings, assesses their limitations, and outlines future research directions.

2. Preliminaries

This section introduces the theoretical foundations required throughout the paper. We first formalize the discrete ROOT framework considered in this work, including the environmental dynamics, the notion of solution acceptability, and the survival-time objective. We then instantiate this framework on two canonical benchmark problems, dynamic ONEMAX and dynamic LEADINGONES, which capture fundamentally different failure mechanisms. Finally, we review the probabilistic tools used in the subsequent theoretical analysis.

2.1. The ROOT Framework

We follow the definitions and notation introduced by [10], which constitute the foundation of most subsequent work on Robust Optimization Over Time (ROOT). In this framework, solution quality is not measured by its fitness in the current environment alone, but by a robustness function F_t that evaluates its performance across future environments.

Let $\mathcal{X}$ denote the search space and let $t \in \mathbb{N}_0$ index a sequence of discrete macro-temporal periods, hereafter referred to as *environments*. At each environment t, an evolutionary algorithm determines a solution $x_t \in \mathcal{X}$. The overall objective is defined as:

$$\max \sum_{t=1}^{N} F_t(x_t), \tag{1}$$

where N denotes the number of environments considered.

Different robustness measures may be adopted depending on the application. In this work, we focus exclusively on the *survival time* criterion. Let $f_t : \mathcal{X} \to \mathbb{R}$ denote the stationary fitness function associated with environment t, and let δ be an acceptability threshold.

We assume that the deployed solution is acceptable at the deployment environment, i.e., $f_t(x) \geq \delta$. The survival time of a solution x deployed at environment t is defined as the first environment (counting the deployment environment as origin) at which the solution becomes unacceptable:

$$\tau_t(x) = \min\{v \geq 1 : f_{t+v}(x) < \delta\}. \tag{2}$$

Under this convention, the solution remains acceptable across $\tau_t(x)$ consecutive environments $t, t+1, \ldots, t+\tau_t(x)-1$, starting from the deployment environment t. Equivalently, $\tau_t(x)$ counts the total number of consecutive acceptable environments. For instance, a solution that is acceptable upon deployment at $t = 0$ but fails at the first environmental transition ($t = 1$) yields $\tau(x) = 1$, as it survives exclusively in the deployment environment.

Like Fu et al. [10], we assume no *time-linkage* between environments: the solution determined at one environment does not constrain the optimization problem in subsequent environments. Consequently, maximizing the cumulative robustness in Equation (1) is equivalent to independently maximizing the survival time F_t at each environment.

2.2. Environmental Dynamics on the Hamming Cube

We specifically focus on understanding the theoretical properties of survival time under a stochastic model of environmental change on the n-dimensional Hamming cube $\mathcal{X} = \{0,1\}^n$.

For algebraic convenience, the Hamming distance between two binary strings $x, y \in \{0,1\}^n$ is denoted by the Hamming weight of their bitwise exclusive-OR (XOR) displacement, $|x \oplus y|$. The landscape changes exclusively at the discrete transitions between environment t and $t+1$. The current environment is fully characterized by a *moving optimum* $c_t \in \mathcal{X}$.

The environmental dynamics are governed as follows: at each transition, exactly k bits (with $1 \leq k \leq n$) are chosen uniformly at random and flipped to yield the next optimum:

$$c_{t+1} = c_t \oplus B_t, \qquad B_t \sim \text{Uniform}\big(\{b \in \{0,1\}^n : \|b\|_1 = k\}\big). \tag{3}$$

Because the lack of time-linkage uncouples the optimization process across epochs, and the k-bit-flip dynamics are time-homogeneous, calculating the survival time at any arbitrary environment t is mathematically equivalent to analyzing a solution deployed at an initial environment $t = 0$.

We adopt the exact-k uniform bit-flip process as a foundational environmental model. This formulation isolates the change-severity parameter k while maintaining mathematical tractability. Natural extensions, such as stochastic change magnitudes K_t, independent Bernoulli mutations, correlated bit blocks, or non-stationary transitions, represent promising avenues for future research.

While the true objective of the ROOT framework is to maximize the actual survival time $\tau(x)$, the stochastic nature of the future environments implies that any theoretical analysis—and any rational algorithm operating under such uncertainty—must inherently focus on its expected value. Thus, without loss of generality, we hereafter drop the temporal deployment subindex and center our theoretical bounds and probabilistic analysis strictly on the expected survival time $\mathbb{E}[\tau(x)]$, where $\tau(x) = \min\{t \geq 1 : f_t(x) < \delta\}$.

Since the fitness functions considered below are integer-valued, we henceforth assume that the acceptability threshold δ is a positive integer; consequently, the tolerance radius $R := n - \delta$ is also a non-negative integer. This convention ensures that the events $\{f_t(x) \geq \delta\}$ and $\{|x \oplus c_t| \leq R\}$ coincide without any rounding ambiguity.

To study the theoretical limits of this framework, we formalize two fundamental discrete benchmarks that represent distinct failure dynamics: gradual erosion and catastrophic structural failure.

2.2.1. Dynamic OneMax (Gradual Erosion)

In this setting, the fitness function calculates the number of matching bits: $f_t(x) = n - |x \oplus c_t|$. A solution remains acceptable as long as $f_t(x) \geq \delta$, or equivalently, if $|x \oplus c_t| \leq R$, where $R := n - \delta$ represents the maximum allowable distance radius.

Since fitness depends on x solely through its distance to the moving optimum, we can track the system's evolution via the radial process $Y_t := |x \oplus c_t|$. This process constitutes a Markov chain on the state space $\{0, 1, \ldots, n\}$ with an initial distance state $Y_0 = \epsilon_0 = |x \oplus c_0|$, effectively simplifying the expected survival time to the expectation of the first hitting time past the acceptable radius:

$$\tau(x) = \min\{t \geq 1 : Y_t > R\}. \tag{4}$$

2.2.2. Dynamic LeadingOnes (Structural Fragility)

Here, the fitness function counts the length of the longest contiguous prefix that perfectly matches the optimum:

$$f_t(x) = \max\{i \in \{0, \ldots, n\} : x_j = c_{t,j} \text{ for all } 1 \leq j \leq i\}. \tag{5}$$

Consequently, a solution is acceptable ($f_t(x) \geq \delta$) if and only if its first δ bits identically match the first δ bits of the moving optimum. Unlike the aggregate distance metric in OneMax, the violation of a single specific bit within this crucial prefix instantly triggers acceptability failure, regardless of the quality of the remaining bits.

2.3. One-Step Displacement Statistics

The analysis of the survival time $\tau(x)$ rests on translating the one-step expected changes of the radial Markov chain $\{Y_t\}_{t \geq 0}$ into an expected first-exit time. While the drift theorems and martingale tools employed are standard in the runtime-analysis toolkit of evolutionary computation [29,30], Lemma 1 provides, to our knowledge, the first explicit closed-form computation of the k-bit flip drift dynamics on the Hamming cube tailored to the ROOT framework.

Lemma 1 (Drift and variance of the Hamming distance)**.** *Under the k-bit-flip dynamics, conditional on* $Y_t = w$,

$$\mathbb{E}[Y_{t+1} - Y_t \mid Y_t = w] = \mu(w) := k \cdot \frac{n - 2w}{n}, \tag{6}$$

$$\mathrm{Var}[Y_{t+1} - Y_t \mid Y_t = w] = \sigma^2(w) := 4k \frac{w}{n}\Big(1 - \frac{w}{n}\Big)\frac{n-k}{n-1}. \tag{7}$$

Moreover, $|Y_{t+1} - Y_t| \leq k$ *almost surely.*

Proof. Let $z = x \oplus c_t$; then $|z| = w$. The next optimum is $c_{t+1} = c_t \oplus B_t$ with B_t uniform over k-subsets. Then $Y_{t+1} = |z \oplus B_t|$. Let a be the number of 1-bits of z that are flipped; a follows a hypergeometric distribution:

$$\mathbb{P}(a = j) = \frac{\binom{w}{j}\binom{n-w}{k-j}}{\binom{n}{k}}.$$

The new distance is $w' = w - a + (k - a) = w + k - 2a$, so the increment is $\Delta = k - 2a$. Hence

$$\mathbb{E}[\Delta] = k - 2\mathbb{E}[a] = k - 2\frac{kw}{n} = k\left(1 - \frac{2w}{n}\right).$$

The variance: $\mathrm{Var}(\Delta) = 4\mathrm{Var}(a)$, and for the hypergeometric distribution

$$\mathrm{Var}(a) = k\frac{w}{n}\left(1 - \frac{w}{n}\right)\frac{n-k}{n-1}.$$

Substituting gives $\sigma^2(w)$. The bound $|\Delta| \leq k$ follows from $0 \leq a \leq k$. □

Remark 1 (Geometric interpretation and continuous analogy)**.** *Equation (6) represents the geometric core of our analysis. The drift $\mu(w)$ is strictly positive when $w < n/2$ (pushing the dynamic optimum away from the deployed solution, thereby increasing the distance) and negative when $w > n/2$ (pulling the optimum back toward it).*

It vanishes entirely at $w = n/2$, which corresponds to the mean Hamming radius under the uniform stationary distribution. This radius represents an equilibrium point rather than an absorbing state, as stochastic fluctuations persist around it. This behavior provides a discrete analog to the radial displacement of an isotropic random walk away from a reference point. Since our demanding acceptability threshold requires $\delta > n/2$, the entire survival analysis operates in the region $w < n/2$ where $\mu(w) > 0$.

Remark 2 (Hypergeometric vs. Binomial change models)**.** *The environmental mutation model defined in our framework forces an exact k-bit flip, leading to the hypergeometric distribution of the disagreement increment. An alternative approach found in the discrete dynamic optimization literature involves independent bit-flip perturbations with a fixed probability p [34], which yields a binomial increment with drift $p(n - 2w)$. While both models asymptotically converge when $k = pn$ as $n \to \infty$, the hypergeometric choice is preferred here as it enforces a sharp, invariant magnitude-of-change parameter k, isolating the impact of environmental severity on survival time.*

2.4. Drift Theorems

To establish rigorous bounds on the exit (survival) time τ, we map our radial process to additive drift variants. In the classical runtime analysis of EAs, additive drift theorems are conventionally framed to bound the first hitting time to a target state, such as reaching the global optimum at zero [22]. However, since our survival analysis investigates the time it takes for an environmental process to escape a safe region and cross an unacceptable threshold going upwards, we require drift formulations tailored to positive drift and upper bounds.

Rather than invoking the standard target-hitting drift theorems, we establish our bounds by direct application of the *Optional Stopping Theorem* [35] for sub/supermartingales. The following lemmas provide the custom theoretical templates for the upper and lower bounds of our survival time process.

Lemma 2 (Upper bound via positive drift)**.** *Let $\{Y_t\}_{t\geq 0}$ be a stochastic process adapted to a filtration $\{\mathcal{F}_t\}$, with $Y_0 = y_0$, and let $\tau = \min\{t : Y_t \geq b\}$ be the first exit time past a threshold $b > y_0$. If there exists a constant $\mu_{\min} > 0$ such that $\mathbb{E}[Y_{t+1} - Y_t \mid \mathcal{F}_t] \geq \mu_{\min}$ whenever $Y_t < b$, and the step size is bounded such that $|Y_{t+1} - Y_t| \leq K$ almost surely, then:*

$$\mathbb{E}[\tau] \leq \frac{b - y_0 + K}{\mu_{\min}}. \tag{8}$$

Proof. Define the compensated process $M_t = Y_t - \mu_{\min} t$. It follows from the drift condition that $\mathbb{E}[M_{t+1} \mid \mathcal{F}_t] \geq M_t$, meaning that M_t is a submartingale. Applying the Optional Stopping Theorem at the stopping time τ (which is almost surely finite with finite expectation) yields $\mathbb{E}[M_\tau] \geq M_0$, which expands to $\mathbb{E}[Y_\tau] - \mu_{\min}\mathbb{E}[\tau] \geq y_0$. For the finite-state irreducible Markov chains analyzed in this work, the presence of an accessible exit boundary guarantees that $\mathbb{P}(\tau < \infty) = 1$ and $\mathbb{E}[\tau] < \infty$, ensuring the validity of the Optional Stopping Theorem. Since the maximum step size is bounded by K, the process at the stopping time satisfies $Y_\tau \leq b + K$. Rearranging the terms, we obtain $\mu_{\min}\mathbb{E}[\tau] \leq b + K - y_0$, which completes the proof. □

Lemma 3 (Lower bound via positive drift)**.** *Under the same hypotheses as Lemma 2, but assuming instead that the expected drift is upper-bounded by a constant $\mu_{\max} > 0$ such that $\mathbb{E}[Y_{t+1} - Y_t \mid \mathcal{F}_t] \leq \mu_{\max}$ whenever $Y_t < b$, then:*

$$\mathbb{E}[\tau] \geq \frac{b - y_0}{\mu_{\max}}. \tag{9}$$

Proof. Define the process $N_t = Y_t - \mu_{\max} t$. The upper bound on the drift implies that $\mathbb{E}[N_{t+1} \mid \mathcal{F}_t] \leq N_t$, establishing that N_t is a supermartingale. By the Optional Stopping Theorem, we have $\mathbb{E}[N_\tau] \leq N_0$, which implies $\mathbb{E}[Y_\tau] - \mu_{\max}\mathbb{E}[\tau] \leq y_0$. As in Lemma 2, the stopping time is almost surely finite and has finite expectation for the finite-state irreducible chains with an escape boundary considered here. Since the process must cross or touch the threshold to trigger the stopping time, we have $Y_\tau \geq b$. Substituting this into the inequality yields $b - \mu_{\max}\mathbb{E}[\tau] \leq y_0$. Rearranging for $\mathbb{E}[\tau]$ completes the proof. □

3. Survival Time Analysis on Benchmark Landscapes

In this section, we apply the drift foundations and martingale tools established in Section 2 to analyze the expected survival time $\mathbb{E}[\tau]$ under k-bit-flip environmental dynamics. We investigate two canonical benchmark landscapes that represent polar extremes in structural vulnerability: OneMax, where we derive order-matching analytical bounds alongside phase-transition scaling regimes, and LeadingOnes, where we obtain an exact closed-form geometric distribution.

Before detailing the formal derivations, Figure 1 provides a structural roadmap of our theoretical framework, mapping the logical flow and proof dependencies among results. As illustrated, the foundational lemmas from Section 2 serve as building blocks for the main bounds and scaling theorems developed throughout this section.

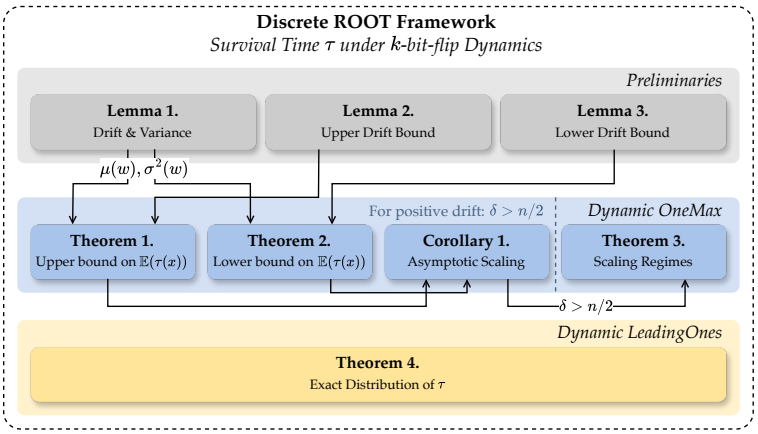


Figure 1. Schematic overview of logical dependencies among theoretical results. Foundational drift lemmas (Section 2) establish the upper bound (Theorem 1), lower bound (Theorem 2), asymptotic scaling (Corollary 1), regime transitions (Theorem 3), and exact prefix failure dynamics (Theorem 4).

3.1. Bounds and Phase Transitions on Dynamic OneMax

We consider the natural case where the threshold is demanding: $\delta > n/2$, which implies $R = n - \delta < n/2$. Then the drift $\mu(w) = k(n - 2w)/n$ is strictly positive for all $0 \le w \le R$.

Theorem 1 (Upper bound). *Let x be a deployed solution with initial distance $\epsilon_0 \le R$, $R = n - \delta$, $\delta > n/2$. Under the k-bit-flip dynamics,*

$$\mathbb{E}[\tau(x)] \;\le\; \frac{(R + 1 - \epsilon_0 + k)\, n}{k\,(2\delta - n)}.$$

Proof. For any $w \le R$,

$$\mu(w) = k\frac{n - 2w}{n} \ge k\frac{n - 2R}{n} = k\frac{2\delta - n}{n} =: \mu_{\min} > 0.$$

The step size is bounded by k. Apply Lemma 2 with $b = R + 1$, $K = k$ to get

$$\mathbb{E}[\tau] \le \frac{(R + 1) - \epsilon_0 + k}{\mu_{\min}} = \frac{(n - \delta + 1 - \epsilon_0 + k)\, n}{k(2\delta - n)}.$$

□

Theorem 2 (Lower bound). *Under the same conditions as Theorem 1,*

$$\mathbb{E}[\tau(x)] \;\ge\; \frac{n - \delta + 1 - \epsilon_0}{k}.$$

Proof. The maximum drift in $[0, R]$ is $\mu_{\max} = \max_{w \le R} \mu(w) = \mu(0) = k$. Using Lemma 3 with $b = R + 1$ yields

$$\mathbb{E}[\tau] \ge \frac{R + 1 - \epsilon_0}{\mu_{\max}} = \frac{n - \delta + 1 - \epsilon_0}{k}.$$

□

Corollary 1 (Asymptotic Scaling). *Assume that $\delta = \alpha n$ for a constant $\alpha \in (1/2, 1)$, with $k = o(n)$ and $R - \epsilon_0 = \Theta(n)$. The expected survival time on dynamic* OneMax *satisfies*

$$\mathbb{E}[\tau] = \Theta\!\left(\frac{n - \delta}{k}\right) \quad \text{as } n \to \infty.$$

Proof. We establish asymptotic equivalence by bounding $\mathbb{E}[\tau]$ from both sides. For the lower bound, Theorem 2 yields $\mathbb{E}[\tau] \ge (n - \delta + 1 - \epsilon_0)/k$. Since $R - \epsilon_0 = \Theta(n)$, the numerator grows proportionally with $n - \delta$, establishing $\mathbb{E}[\tau] = \Omega((n - \delta)/k)$.

For the upper bound, substituting $\delta = \alpha n$ into the denominator of Theorem 1 gives $k(2\delta - n) = kn(2\alpha - 1)$. The linear factor n in the denominator cancels the corresponding factor in the numerator, yielding

$$\mathbb{E}[\tau] \le \frac{(n - \delta + 1 - \epsilon_0 + k)\, n}{k\, n\, (2\alpha - 1)} = \mathcal{O}\!\left(\frac{n - \delta}{k}\right),$$

where α, k, and ϵ_0 act as constant factors relative to $n - \delta$. Matching the upper and lower orders establishes the Θ scaling. □

Theorem 3 (Scaling Regimes of Survival Time)**.** *Let τ denote the survival time of a solution on dynamic* OneMax *under k-bit-flip dynamics with $k = o(n)$. For any fixed constant $\epsilon > 0$, the expected survival time $\mathbb{E}[\tau]$ exhibits the following distinct scaling regimes depending on the acceptability threshold δ:*

1. *Positive drift ($\delta \geq (1/2 + \epsilon)n$): $\mathbb{E}[\tau] = \Theta\big((n - \delta)/k\big)$.*
2. *Negative drift ($\delta \leq (1/2 - \epsilon)n$): $\mathbb{E}[\tau]$ grows superpolynomially with respect to n.*

Proof. The positive-drift scaling follows directly from Corollary 1. Under negative drift with linear separation ϵn from the center, standard hitting-time bounds for reversible Markov chains establish that crossing against the directional drift requires superpolynomial expected time (detailed in Appendix A). □

Remark 3 (The Diffusive Boundary ($\delta \approx n/2$))**.** *At the equilibrium boundary ($\delta = n/2$), the expected directional drift vanishes. In this regime, survival is driven primarily by stochastic variance rather than systematic drift. Analytical approximations treating this boundary as a zero-drift random walk suggest an expected scaling of $\mathbb{E}[\tau] = \Theta(n^2/k)$, an order of magnitude that aligns consistently with empirical observations.*

Remark 4 (Contextualizing the Phase Transitions)**.** *This structural transition is a purely discrete phenomenon, noticeably absent in unbounded continuous settings where the radial drift remains strictly negative (directed toward the origin) for any threshold given a sufficiently large dimension. In our discrete framework, only the positive drift scenario ($\delta > n/2$) poses a realistic, non-trivial challenge for robust optimization, as deploying solutions with an expected quality equal to or worse than random guessing ($\delta \leq n/2$) provides no practical engineering value. The diffusive boundary is reported separately in Remark 3 since it is best interpreted as a heuristic analytical approximation rather than as part of the rigorous scaling theorem. Table 1 summarizes these regimes.*

Table 1. Asymptotic scaling regimes of $\mathbb{E}[\tau]$ on dynamic OneMax for large n. The rigorous conditions mirror those of Theorem 3 for the separated drift regimes; the diffusive boundary and full-randomization entries are heuristic observations rather than proven scalings.

Regime	Condition	$\mathbb{E}[\tau]$ (Order of Magnitude)
Positive drift	$\delta \geq (1/2 + \epsilon)n$	$\Theta\big((n - \delta)/k\big)$
Diffusive boundary	$\delta \approx n/2$	$\Theta\big(n^2/k\big)$
Negative drift	$\delta \leq (1/2 - \epsilon)n$	Superpolynomial/Exponential
Full randomization	$k = \Theta(n)$	$\Theta(1)$ (conjectured)

The diffusive-boundary and full-randomization rows are heuristic observations, not proven scalings.

3.2. Catastrophic Failure on Dynamic LeadingOnes

While dynamic OneMax captures scenarios where solutions undergo gradual, aggregate erosion of robustness (analyzed via martingale drift techniques), dynamic LeadingOnes embodies systems prone to catastrophic structural failure. In this problem, acceptability is lost at the first environmental transition that flips any bit of the required prefix. This first-exit structure bypasses additive drift analysis and yields an exact geometric survival-time distribution.

Theorem 4 (Exact survival time on dynamic LEADINGONES)**.** *Let x be a deployed solution such that it initially agrees with the optimum c_0 on the first δ bits (hence $f_0(x) \geq \delta$). Under the k-bit-flip dynamics, the survival time τ_{LO} follows a geometric distribution on $\{1, 2, 3, \dots\}$ with success parameter (failure probability)*

$$p = 1 - \frac{\binom{n-\delta}{k}}{\binom{n}{k}}, \tag{10}$$

and therefore the expected survival time is given by:

$$\mathbb{E}[\tau_{LO}] = \left(1 - \frac{\binom{n-\delta}{k}}{\binom{n}{k}}\right)^{-1}. \tag{11}$$

Proof. At each environmental transition, the set of flipped bits B_t is a uniformly random k-subset of the indices $\{1, \dots, n\}$. According to the acceptability threshold established in Section 2.2.2, the solution becomes unacceptable exactly when B_t contains at least one index from the vital prefix $\{1, \dots, \delta\}$, as the mutation breaks the prefix match.

The environmental transitions are independent. The probability that no bit from the required prefix is flipped during a transition is exactly the ratio of ways to choose all k bits from the remaining $n - \delta$ positions over the total ways to choose k bits. Thus, the probability of surviving a single step is $\binom{n-\delta}{k}/\binom{n}{k}$. The number of steps until the first failure event (where at least one prefix bit is flipped) is geometrically distributed on the positive integers with the failure probability $p = 1 - \binom{n-\delta}{k}/\binom{n}{k}$, directly yielding the expectation. □

The exact geometric distribution follows directly from the memoryless property induced by independent environmental transitions: conditional on survival up to environment t, the failure probability at transition $t + 1$ remains strictly invariant at p. If environmental changes were temporally correlated, this conditional failure probability would depend on historical states, breaking the geometric memoryless structure.

To contextualize this catastrophic failure against the gradual erosion observed in ONEMAX, we analyze its asymptotic behavior. For large n and $k\delta \ll n$, the expected survival time is well approximated by the inverse of the first-step prefix hit probability:

$$\mathbb{E}[\tau_{\mathrm{LO}}] \approx \frac{n}{k\delta}. \tag{12}$$

This hyperbolic dependency strictly contrasts with the linear scaling $\Theta((n - \delta)/k)$ established for ONEMAX. It demonstrates that for demanding acceptability thresholds δ, LEADINGONES is inherently far more sensitive to environmental changes, as survival time degrades inversely with the required prefix length.

4. Empirical Validation of Theoretical Bounds

We empirically evaluate the survival-time predictions derived in Section 3 under the k-bit-flip environmental dynamics introduced in Section 2.1. Rather than evaluating full evolutionary algorithms—whose tracking dynamics would introduce confounding optimization effects—our experiments isolate the exact probabilistic mechanisms governing solution durability. Deploying a fixed initial solution ($\epsilon_0 = 0$) eliminates algorithmic variance and isolates the unadulterated survival time τ across 100 independent Monte Carlo runs per parameter configuration.

4.1. Experimental Design and Parameter Settings

Uncertainty in empirical estimates is quantified using 95% confidence intervals computed as $\widehat{\mathbb{E}}[\tau] \pm t_{0.975,99}\,\mathrm{SE}$, where SE denotes the standard error of the sample mean and $t_{0.975,99} \approx 1.984$ is the Student-t critical value for 99 degrees of freedom. This Monte Carlo budget ensures high statistical precision while maintaining computational feasibility across large dimensional sweeps. Reproducibility is guaranteed by using NumPy's PCG64 bit generator with deterministic seed tuples derived from each $(\text{landscape}, n, k, \delta)$ configuration. Table 2 summarizes the parameter settings for the conducted empirical validation.

Table 2. Parameter settings for empirical validation.

Parameter	Values
Problem dimension n	$\{50, 100, 200, 500, 1000, 2000, 5000, 10000\}$
Environmental change k	$\{1, 5, 10\}$
Acceptability threshold δ (OneMax)	$0.5n,\ 0.6n,\ 0.75n,\ 0.9n$
Acceptability threshold δ (LeadingOnes)	$0.25n,\ 0.5n,\ 0.75n,\ 0.9n$
Independent runs	100
Measured quantity	Survival time τ

4.2. OneMax *Dynamics: Comparison of Bounds and Asymptotic Scaling*

Figure 2 compares the empirical mean survival time $\widehat{\mathbb{E}}[\tau]$ with the analytical bounds derived in Theorems 1 and 2. In the diffusive column ($\delta/n = 0.5$), bounds are omitted as Theorems 1 and 2 strictly require positive drift ($\delta > n/2$); nevertheless, the empirical trajectories remain consistent with the quadratic scaling $\Theta(n^2/k)$ discussed in Remark 3.

Across all positive-drift configurations ($\delta/n > 0.5$), the empirical mean remains tightly bounded between the theoretical curves. As established in Corollary 1, these drift-based bounds share the same asymptotic order $\Theta((n-\delta)/k)$. Note that for proportional thresholds $\delta = \alpha n$ with $\alpha \in (1/2, 1)$, the limiting ratio U/L approaches $1/(2\alpha - 1)$ rather than unity; hence, we interpret the analytical bounds as order-matching envelopes rather than ratio-one asymptotes. Table 3 details representative numerical values, standard errors, and the precise relative position of the empirical mean within the bound gap.

Table 3. Empirical mean survival time $\widehat{\mathbb{E}}[\tau]$ and standard error (SE) on dynamic OneMax for $k = 5$, $\epsilon_0 = 0$, and 100 independent runs (PCG64, configuration seeds as in Section 4). Bounds L and U are those of Theorems 2 and 1; $(\widehat{\mathbb{E}}[\tau] - L)/(U - L)$ locates the mean in the bound gap. The subset complements Figure 2.

n	δ/n	$\widehat{\mathbb{E}}[\tau]$	SE	L	U	$\frac{\widehat{\mathbb{E}}[\tau] - L}{U - L}$
100	0.60	15.96	0.41	8.20	46.00	0.21
	0.75	7.56	0.13	5.20	12.40	0.33
	0.90	3.05	0.03	2.20	4.00	0.47
1000	0.60	163.84	1.70	80.20	406.00	0.26
	0.75	70.18	0.42	50.20	102.40	0.38
	0.90	23.09	0.11	20.20	26.50	0.46
10,000	0.60	1604.53	4.24	800.20	4006.00	0.25
	0.75	692.60	1.31	500.20	1002.40	0.38
	0.90	223.78	0.36	200.20	251.50	0.46

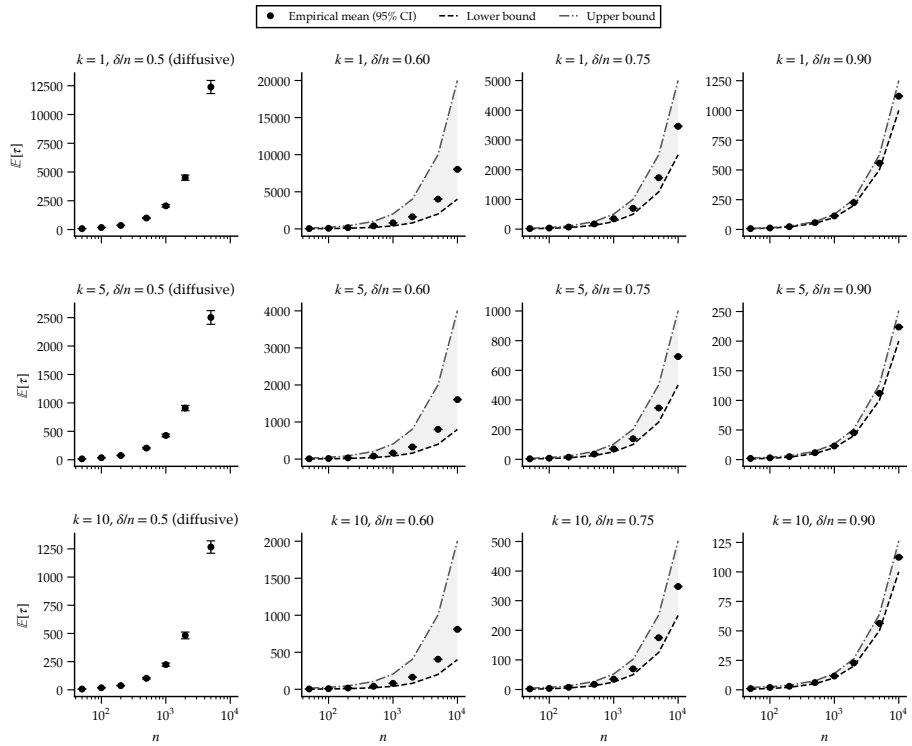


Figure 2. Empirical mean survival time on dynamic OneMax (black points, 95% CI) versus analytical lower (dashed) and upper (dash-dotted) bounds, for $\epsilon_0 = 0$. Columns correspond to $\delta/n \in \{0.5, 0.6, 0.75, 0.9\}$ and rows to $k \in \{1, 5, 10\}$. The diffusive column ($\delta/n = 0.5$) omits bounds because Theorems 2 and 1 require $\delta > n/2$; the gray band marks the region between bounds in the positive-drift regime, and the diffusive column is interpreted through Remark 3.

To validate the asymptotic scaling, Figure 3 shows the normalized survival time $\widehat{\mathbb{E}}[\tau]/((n-\delta)/k)$ for $k = 5$. As n increases, each curve converges smoothly toward a horizontal asymptote, confirming the predicted $\Theta((n-\delta)/k)$ scaling law. The dashed horizontal lines correspond to the fluid-limit constants derived in Appendix A.3:

$$A(r) = \frac{1}{2r} \ln\left(\frac{1}{1-2r}\right), \qquad r = \frac{n-\delta}{n}.$$

The close agreement between empirical data and these fluid-limit constants confirms that our deterministic approximation accurately captures the reduction in radial drift as solutions recede from the optimum.

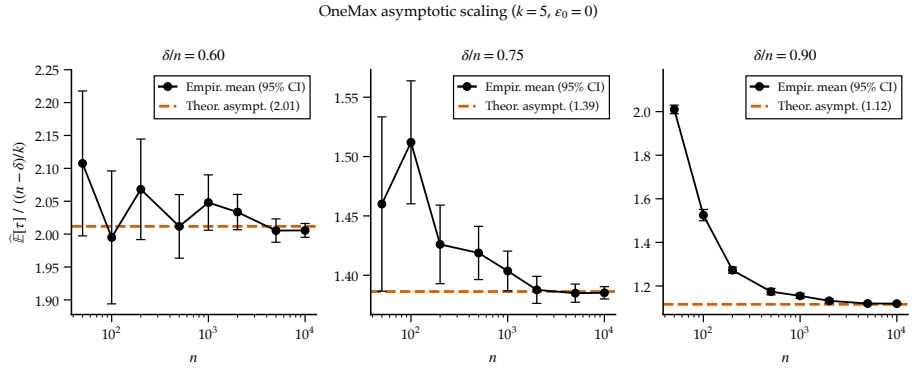


Figure 3. Normalized survival time $\widehat{\mathbb{E}}[\tau]/((n-\delta)/k)$ on dynamic OneMax versus n for $k = 5$, $\epsilon_0 = 0$, and $\delta/n \in \{0.6, 0.75, 0.9\}$. Solid curves: empirical means (95% CI); red dashed horizontals: approximate asymptotic constants from the fluid-limit approximation capturing the non-constant drift.

4.3. LeadingOnes *Dynamics: Exact Geometric Distribution*

Unlike the gradual degradation observed in OneMax, fitness erosion in LeadingOnes is structural and memoryless. A deployed solution remains acceptable if and only if its length-δ prefix remains untouched during an environmental change; the very first flip within this prefix triggers an immediate first-exit event.

Figure 4 shows the Empirical Cumulative Distribution Function (ECDF) of survival times across 12 parameter combinations ($k \in \{1, 5, 10\}$ and $\delta/n \in \{0.25, 0.5, 0.75, 0.9\}$ for $n = 100$). Across all panels, the empirical step functions closely match the theoretical geometric CDF from Theorem 4. Table 4 reports sample means, standard errors, exact expectations, and relative errors, demonstrating close agreement across parameter settings.

Figure 4 illustrates three operational regimes:

- Extended Survival Regime (Low k, Small δ/n): For mild environmental changes ($k = 1$) and moderate thresholds ($\delta/n \leq 0.5$), the failure probability p per step remains small. The distribution exhibits a wide support with a heavy right tail ($\tau \gg 10$).
- Discretized Step Regime (Intermediate k and δ/n): As environmental severity k increases or robustness requirements tighten, p increases substantially. The CDF support contracts rapidly to a few discrete steps.
- High First-Step Failure Regime (High k, Large δ/n): Under severe changes ($k = 10$) and strict thresholds ($\delta/n \geq 0.5$), the single-step failure probability approaches unity ($p \approx 1$). The distribution becomes near-degenerate, with nearly all deployed solutions failing at the very first transition ($t = 1$).

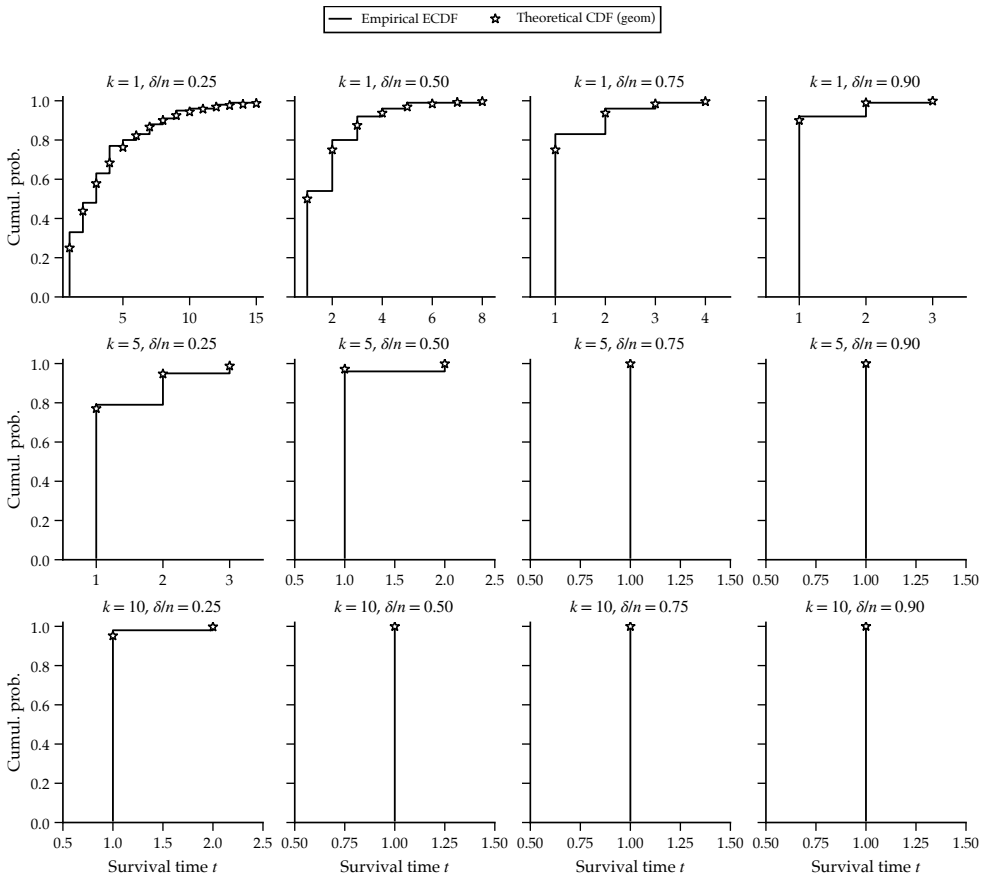


Figure 4. Empirical CDF (step function) and exact geometric CDF (markers) of survival time on dynamic LeadingOnes for $n = 100$. Columns correspond to $\delta/n \in \{0.25, 0.5, 0.75, 0.9\}$ and rows to $k \in \{1, 5, 10\}$ (100 independent runs per panel). Theoretical curves utilize the memoryless failure probability $p = 1 - \binom{n-\delta}{k}/\binom{n}{k}$.

Table 4. Empirical mean survival time $\widehat{\mathbb{E}}[\tau]$ and standard error (SE) on dynamic LEADINGONES for $n = 100$ and 100 independent runs (same randomization protocol as Section 4). The exact expectation $\mathbb{E}[\tau_{\mathrm{LO}}]$ follows Theorem 4; the relative error is $|\widehat{\mathbb{E}}[\tau] - \mathbb{E}[\tau_{\mathrm{LO}}]|/\mathbb{E}[\tau_{\mathrm{LO}}]$. The table complements Figure 4.

k	δ/n	$\widehat{\mathbb{E}}[\tau]$	SE	$\mathbb{E}[\tau_{\mathrm{LO}}]$	Rel. Err.
1	0.25	3.54	0.31	4.00	0.115
	0.50	1.81	0.12	2.00	0.095
	0.75	1.22	0.05	1.33	0.085
	0.90	1.09	0.03	1.11	0.019
5	0.25	1.26	0.05	1.30	0.029
	0.50	1.04	0.02	1.03	0.011
	0.75	1.00	0.00	1.00	0.001
	0.90	1.00	0.00	1.00	0.000
10	0.25	1.02	0.01	1.05	0.029
	0.50	1.00	0.00	1.00	0.001
	0.75	1.00	0.00	1.00	0.000
	0.90	1.00	0.00	1.00	0.000

5. Concluding Remarks

In this paper, we advanced the theoretical foundations of Robust Optimization Over Time (ROOT) by analyzing solution survival time in discrete search spaces. While previous ROOT research has predominantly focused on continuous domains and empirical evaluations [11,12], our work introduces a formal probabilistic framework for solution durability. By modeling survival time as a first-exit problem under k-bit-flip dynamics, we bridge ROOT with classical tools from runtime analysis and stochastic processes. We derive order-matching analytical bounds and phase transitions for dynamic ONEMAX, capturing the mechanisms of gradual fitness erosion. Furthermore, we establish that dynamic LEADINGONES induces an exact geometric survival distribution. Overall, these results show that solution durability can be analyzed with mathematical rigor rather than treated purely as an empirical outcome. We note that while expected survival time $\mathbb{E}[\tau]$ serves as an analytically tractable primary metric, other criteria—such as fixed-horizon survival probabilities, quantiles, risk-sensitive objectives, or expected re-optimization costs—offer complementary decision-theoretic insights depending on application requirements.

The theoretical nature of this study implies certain structural limitations. The benchmark landscapes and uniform k-bit-flip dynamics represent simplified abstractions of complex real-world environments, which often exhibit non-uniform mutations, combinatorial constraints, or correlated dynamics. Consequently, quantitative predictions should be interpreted as baseline performance guarantees. Furthermore, canonical unimodal models like dynamic ONEMAX do not inherently feature a trade-off between initial fitness and durability; reducing initial distance simultaneously maximizes current quality and the margin to the acceptability threshold. Observing scenarios where a lower-quality solution exhibits superior durability requires richer problem structures, such as heterogeneous landscapes, non-stationary transitions, or explicit switching costs. We naturally view these extensions as promising directions for future research.

Several avenues for further investigation follow from this work. Extending this first-exit framework to continuous search spaces will require diffusion approximations or continuous-time martingale methods. In discrete settings, analyzing structured combinatorial problems—such as dynamic routing or subset selection under constraints—will enhance practical applicability. Finally, incorporating survival-time bounds directly into

search operators could enable evolutionary algorithms to proactively balance short-term quality against long-term durability.

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Abbreviations

The following abbreviations are used in this manuscript:

ROOT	Robust Optimization Over Time
EDO	Evolutionary Dynamic Optimization
CDF	Cumulative Distribution Function
ECDF	Empirical Cumulative Distribution Function
ODE	Ordinary Differential Equation

Appendix A. Deferred Proofs and Asymptotic Derivations

In this appendix, we outline the analytical derivations for the diffusive ($\delta = n/2$) and negative-drift ($\delta < n/2$) regimes on dynamic OneMax, as well as the fluid-limit derivation of the asymptotic constant for the positive-drift regime ($\delta > n/2$).

Appendix A.1. Diffusive Regime ($\delta = n/2$)

When the acceptability threshold sits exactly at the midpoint, the maximum allowable distance is $R = n/2$. For any state w, the directional drift $\mu(w) = k(1 - 2w/n)$ vanishes at $w = n/2$. The following argument provides a heuristic analytical justification for the quadratic scaling of $\mathbb{E}[\tau]$, treating the radial distance near the boundary as an unbiased random walk.

Near the equilibrium boundary $w \approx n/2$, the process Y_t behaves as a martingale. To approximate the expected exit time, we apply the Optional Stopping Theorem to the compensated quadratic process:

$$M_t := Y_t^2 - \sum_{s=0}^{t-1} \sigma^2(Y_s),$$

where $\sigma^2(Y_s) := \mathrm{Var}[Y_{s+1} - Y_s \mid Y_s]$. Since the directional drift is negligible near $n/2$, M_t is approximately a martingale. Applying optional stopping at τ yields:

$$\mathbb{E}[Y_\tau^2] = Y_0^2 + \mathbb{E}\left[\sum_{s=0}^{\tau-1} \sigma^2(Y_s)\right].$$

For states near $n/2$, Lemma 1 gives $\sigma^2(Y_s) \approx 4k \cdot \frac{1}{4} \cdot \frac{n-k}{n-1} \to k$ as $n \to \infty$, implying $\mathbb{E}[\sum_{s=0}^{\tau-1} \sigma^2(Y_s)] \approx k\,\mathbb{E}[\tau]$. Since exit occurs at $Y_\tau \approx n/2+1$, we obtain $(n/2)^2 \approx \epsilon_0^2 + k\mathbb{E}[\tau]$. Solving for $\mathbb{E}[\tau]$ yields:

$$\mathbb{E}[\tau] \approx \frac{(n/2)^2 - \epsilon_0^2}{k}, \tag{A1}$$

which indicates the quadratic scaling $\mathbb{E}[\tau] = \Theta(n^2/k)$ for any constant initial distance ϵ_0.

Appendix A.2. Negative Drift Regime ($\delta < n/2$)

When the threshold requires fewer than half the bits to match, the failure boundary lies beyond the hypercube midpoint ($R > n/2$). In this domain, the drift $\mu(w)$ is strictly negative for $w > n/2$, directing the process back toward the center and away from the exit boundary.

Under a strict linear separation condition $\delta \leq (1/2 - \epsilon)n$ for a fixed constant $\epsilon > 0$, reaching the exit boundary $R > n/2$ requires traversing against a constant directional drift. Standard hitting-time results for reversible Markov chains on hypercubes establish that crossing an opposing drift to an extreme tail set requires an expected time $\mathbb{E}[\tau]$ that grows superpolynomially (and exponentially) with respect to n.

Appendix A.3. Fluid-Limit Approximation of the Asymptotic Constant

For the positive-drift regime ($\delta > n/2$), let $r = (n-\delta)/n \in (0, 1/2)$ denote the normalized tolerance radius. Corollary 1 establishes that $\mathbb{E}[\tau] = \Theta(nr/k)$. To approximate the multiplicative leading constant, we analyze the deterministic fluid limit of the Markov chain as $n \to \infty$.

Let $x_t = Y_t/n \in [0, r]$ denote the normalized distance at step t. From Lemma 1, the expected normalized displacement is:

$$\mathbb{E}[x_{t+1} - x_t \mid x_t = x] = \frac{k}{n}(1 - 2x).$$

Since displacement variance scales as $\mathcal{O}(1/n^2)$, stochastic fluctuations vanish in the macroscopic limit. Rescaling time as $s = tk/n$ yields the ordinary differential equation (ODE):

$$\frac{dx}{ds} = 1 - 2x.$$

For an optimal initial deployment ($x(0) = 0$), integrating this ODE up to the normalized boundary $x(s^*) = r$ gives:

$$s^* = \int_0^r \frac{1}{1-2x}\,dx = \frac{1}{2}\ln\left(\frac{1}{1-2r}\right).$$

Converting back to discrete time, the expected survival time is approximated by:

$$\mathbb{E}[\tau] \approx \frac{n}{k}s^* = \frac{n}{2k}\ln\left(\frac{1}{1-2r}\right).$$

Normalizing by the baseline linear factor $(n-\delta)/k = nr/k$, we obtain the asymptotic stabilization constant A:

$$A = \frac{\mathbb{E}[\tau]}{nr/k} \approx \frac{1}{r}s^* = \frac{1}{2r}\ln\left(\frac{1}{1-2r}\right),$$

which matches the horizontal asymptotes observed empirically in Figure 3.

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