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Characterization of the Boundedness of a Generalized Fractional Maximal Operator in Weighted Lorentz Spaces for Certain Parameter Values

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Abstract

The object of this study is generalized fractional-maximal operators acting in weighted Lorentz spaces. The goal of this paper is to obtain boundedness criteria for a generalized fractional-maximal operator between weighted Lorentz spaces and to describe these criteria in terms of weighted integral inequalities. To achieve this goal, methods from the theory of function rearrangements, estimates for nonincreasing rearrangements of operators, and methods for studying weighted inequalities for Hardy-type operators and supremal operators are used. Sharp estimates are obtained for a nonincreasing rearrangement of a generalized fractional-maximal operator. An equivalence is established between the boundedness of this operator in weighted Lorentz spaces and the boundedness of the corresponding supremal operator. Based on reduction theorems, the problem is reduced to the study of weighted integral inequalities of Hardy type. Necessary and sufficient conditions for the boundedness of the generalized fractional maximal operator from $\Lambda^1(v)$ to $\Lambda^q(w)$ in the case $p = 1, 0 < q < \infty$ and from $\Lambda^p(v)$ to $\Lambda^1(w)$ in the case $q = 1 < p < \infty$.

Keywords: generalized fractional-maximal operator; nonincreasing rearrangement; Hardy operator; supremal operator; weighted inequalities; reduction theorems; weighted Lorentz spaces

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1. Introduction

The classical Hardy–Littlewood maximal operator and its fractional analog are widely used in function theory, potential theory, and in the study of various classes of equations in mathematical physics [1–4]. One of the most important areas of research is the study of the mapping properties of fractional-maximal operators in Lebesgue, Lorentz, Morrey, Orlicz, and other rearrangement-invariant spaces. The works of E. M. Stein and G. Weiss [1], C. Bennett and R. Sharpley [2], and L. Grafakos [3] laid the foundations of the modern theory of maximal operators and function spaces. A. Cianchi, R. Kerman, B. Opic, and L. Pick [4] proved a sharp estimate for the nonincreasing rearrangement of a fractional-maximal operator, which made it possible to obtain criteria for its boundedness between classical Lorentz spaces. In the works of A. Gogatishvili, V. D. Stepanov [5], A. Gogatishvili, L. Pick [6], A. Gogatishvili, L. Pick, and B. Opic [7], and M. Křepela [8], reduction theorems

were developed and boundedness criteria for Hardy operators and supremal operators on cones of monotone functions were obtained.

In recent years, the theory of generalized fractional-maximal operators has been actively developing. Significant results in this direction were obtained by R. Mustafayev and N. Bilgiçli [9], who studied generalized fractional-maximal functions in Lorentz spaces, and by D. Edmunds and B. Opic [10], who studied the boundedness of a fractional-maximal operator between different Lorentz spaces.

The present work is closely related to research [11–13]. In [11], cones generated by nonincreasing rearrangements of generalized fractional maximal functions were introduced and investigated; their embedding properties into rearrangement-invariant spaces were studied, and inclusion relations for cones generated by generalized Riesz potentials were established. In [12], a sharp pointwise estimate for the nonincreasing rearrangement of a generalized fractional maximal function was obtained in terms of the rearrangement of the original function. This estimate serves as a fundamental preliminary result for the present paper. In [13], the study of cones of monotone functions generated by generalized fractional maximal operators was further developed. Equivalent characterizations and embedding relations for these cones were established, and their optimal embeddings into rearrangement invariant spaces were investigated.

In contrast to [11–13], this paper focuses primarily on the boundedness of the generalized fractional maximal operator in weighted Lorentz spaces. Using the rearrangement obtained in [12], we reduce the original multidimensional boundedness problem to weight inequalities for one-dimensional supremal operators. We investigate these inequalities by applying reduction theorems for Hardy-type operators and supremal operators.

The classical fractional maximal operator is defined by

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{B(x,r)} |f(y)|dy, \quad 0 \le \alpha < n,$$

where $B(x,r)$ is a ball with the center at the point $x \in \mathbb{R}^n$ and radius r, and $f \in \mathfrak{M}(\mathbb{R}^n)$ is a locally integrable function.

In this paper, we consider a generalized fractional-maximal operator of the form

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} |f(y)|dy,$$

where $f \in \mathfrak{M}(\mathbb{R}^n)$ is a locally integrable function, and $\Phi : (0,\infty) \to (0,\infty)$ is a nonincreasing function that determines a generalized fractional maximal operator [11,13]. In the case of $\Phi(r) = r^{\alpha-n}$, the operator M_Φ coincides with the classical fractional maximal operator M_α.

We give a characterization of the boundedness of generalized fractional maximal functions M_Φ between weighted Lorentz spaces. We show that this problem is equivalent to the boundedness of a supremal operator involving a Hardy operator

$$T_\Phi(f)(t) = \operatorname*{ess\,sup}_{\tau\in(t,\infty)} \Phi(\tau^{\frac{1}{n}}) \int_0^\tau f^*(s)ds$$

in the weighted Lebesgue space restricted to the cone of non-negative decreasing functions [12].

Suppose f is a locally integrable function on $\mathbb{R}^n$. Then its nonincreasing rearrangement f^* is given by

$$f^*(t) = \inf\{\lambda > 0; |\{x \in \mathbb{R}^n; |f(x)| > \lambda\}| \le t\}, \quad t \in (0,\infty),$$

$$f^{**}(s) = \frac{1}{s} \int_0^s f^*(u)du.$$

It is known that the function f^{**} also non-increases on $(0,\infty)$.

It is easy to see that

$$(T_\Phi f)(t) = \operatorname*{ess\,sup}_{t<\tau<\infty} \Phi(\tau^{1/n})\tau f^{**}(\tau), \qquad t > 0. \tag{1}$$

Quite many familiar function spaces can be defined using the nonincreasing rearrangement of a function. One of the most important classes of such spaces is the so-called classical Lorentz spaces.

Let $p \in (0,\infty)$ and let v be a weight, that is, a non-negative measurable and a.e. finite function on $(0,\infty)$. Then the classical Lorentz space $\Lambda^p(v)$ is defined as the set of all measurable functions f whose nonincreasing rearrangement satisfies the condition

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{1/p} < \infty.$$

Remark 1. *Let $1 < p < \infty$, and let v be a weight function on $(0,\infty)$. In general, the functional $\|f\|_{\Lambda^p(v)}$ does not necessarily define the norm of a Banach function space for an arbitrary weight v. In the present paper, the space $\Lambda^p(v)$ is initially understood as the class of measurable functions for which the indicated functional is finite. If a Banach space structure is required later on, the corresponding additional conditions on the weight v will be explicitly stated.*

Our goal is to obtain necessary and sufficient conditions for the boundedness of the generalized fractional maximal operator in Lorentz spaces

$$M_\Phi : \Lambda^p(v) \longrightarrow \Lambda^q(w),$$

that is, to characterize the inequality

$$\left(\int_0^\infty [(M_\Phi f)^*(t)]^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{\frac{1}{p}} \tag{2}$$

for the cases:

(i) $1 = p \le q < \infty$, $M_\Phi : \Lambda^1(v) \longrightarrow \Lambda^q(w)$;
(ii) $q = 1 < p < \infty$, $M_\Phi : \Lambda^p(v) \longrightarrow \Lambda^1(w)$.

For the classical fractional-maximal operator, the boundedness of $M_\alpha(f)$ from $\Lambda^p(v)$ to $\Lambda^q(w)$ for case $1 < p \le q < \infty$, was given in [4].

2. Preliminaries

As usual, here and below, by $A \lesssim B$ and $A \gtrsim B$ we mean that $A \le CB$ and $CA \ge B$, respectively, where C is a positive constant independent of appropriate quantities involved in the expressions A and B. We will also write $A \approx B$ when both $A \lesssim B$ and $A \gtrsim B$ are satisfied.

Throughout this work, we will denote positive constants by C, C_1, C_2, which may, generally speaking, vary in different places.

Definition 1 ([12]). *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called quasi-decreasing (quasi-increasing) if there exists $C > 1$, such that*

$$f(t_2) \le Cf(t_1) \text{ if } t_1 < t_2,$$
$$\big(f(t_1) \le Cf(t_2) \text{ if } t_1 < t_2\big).$$

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$, and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^\downarrow(0,\infty)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are nonincreasing on $(0,\infty)$. Similarly, let $\mathfrak{M}(\mathbb{R}^n)$ denote the set of all Lebesgue-measurable functions on $\mathbb{R}^n$, and let $\mathfrak{M}^+(\mathbb{R}^n)$ denote its subset consisting of all non-negative functions on $\mathbb{R}^n$.

Definition 2 ([11]). *Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. We say that a function $\Phi : (0;R) \to (0;\infty)$ belongs to the class $A_n(R)$ if:*

(1) Φ is decreasing and continuous on $(0;R)$;
(2) The function $\Phi(r)r^n$ is quasi-increasing on $(0,R)$.

For example, $\Phi(r) = r^{\alpha-n} \in A_n(\infty)$, $0 < \alpha < n$. Another example of a function from the A_n class is the following function of non-power-law decay:

$$\Phi(\rho) = \rho^{\alpha-n}\Big(ln\frac{eR}{\rho}\Big)^{\beta}, \text{ if } R \in \mathbb{R}_+ \text{ and } \begin{cases} \alpha = 0,\ \beta < 0, \\ 0 < \alpha < n,\ 0 < \beta < R, \\ \alpha = n,\ \beta > 0. \end{cases}$$

In the study of the generalized fractional maximal operator M_Φ, we assume that $\Phi \in A_n(\infty)$. This assumption is used in estimates based on the rearrangement of functions, which allow us to reduce the original multidimensional problem to weighted inequalities for one-dimensional supremal operators.

We use the following estimate from [12] for a nonincreasing rearrangement of a generalized fractional-maximal operator.

Theorem 1 ([12]). *Let $\Phi \in A_n(\infty)$. Then there exists a positive constant C_0, depending only on Φ and n, such that*

$$(M_\Phi f)^*(t) \le C_0 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}})f^{**}(s), \quad t \in (0,\infty), \tag{3}$$

for every locally integrable function on $\mathbb{R}^n$. This inequality is sharp in the sense that for every $\varphi \in \mathfrak{M}^{\downarrow}(0,\infty)$ there exists a function f such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\Phi f)^*(t) \ge C_1 \sup_{t<s<\infty} s\Phi(s^{\frac{1}{n}})f^{**}(s), \quad t \in (0,\infty), \tag{4}$$

where C_1 is a positive constant which depends only on Φ and n.

For the case of the classical fractional maximal function, such a theorem was proved in [4].

3. Main Results

The following result is a reduction principle based on a rearrangement estimate for the generalized fractional maximal operator obtained in [12]. It allows one to reduce the study of the boundedness of the operator M_Φ between weighted Lorentz spaces to the study of the corresponding one-dimensional supremal operator.

Theorem 2. *Let $0 < p, q < \infty$, and let v, w be weight functions on $(0,\infty)$ and $\varphi(t) := \Phi(t^{1/n})$. Let $\Phi \in A_n(\infty)$. Inequality (2) holds for all $f \in \mathfrak{M}(\mathbb{R}^n)$, if and only if the following inequality*

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} \varphi(s)sf^{**}(s)\right)^q w(t)dt\right)^{\frac{1}{q}} \le C_2\left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{\frac{1}{p}}. \tag{5}$$

Remark 2. *For $\Phi(r) = r^{\alpha-n}$, $0 < \alpha < n$, hence, for the fractional maximal operator M_α, Theorems 1 and 2 were proved in [5].*

Proof of Theorem 2. Step 1. (5) ⇒ (2). Using (1) and (3), the inequality (5) can be rewritten as

$$\left(\int_0^\infty [(M_\Phi f)^*(t)]^q w(t)dt\right)^{1/q} \le C_0\left(\int_0^\infty [(T_\varphi f)(t)]^q w(t)dt\right)^{1/q} \le C_0 C_2\left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{\frac{1}{p}}.$$

That is why we have (2) with $C \approx C_0 C_2$.

Step 2. (2) ⇒ (5). Let (2) hold. Now suppose that (2) is satisfied. Let us take an arbitrary function $g \in \mathfrak{M}^{\downarrow}(0,\infty)$. By Theorem 1, there is a function $f \in \mathfrak{M}^+(\mathbb{R}^n)$ such that $f^* = g$ almost everywhere on$(0,\infty)$, and at the same time there exists a constant $C_1 > 0$, depending only on n and Φ, such that

$$(M_\Phi f)^*(t) \ge C_1 \sup_{t<s<\infty} s\Phi(s^{1/n})g^*(s), \qquad t > 0. \tag{6}$$

Hence,

$$\sup_{t<s<\infty} \varphi(s)\, s f^{**}(s) \le \frac{1}{C_1}(M_\Phi f)^*(t).$$

Therefore, raising q to the power, integrating with weight w we obtain

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} s\varphi(s) f^{**}(s)\right)^q w(t)dt\right)^{\frac{1}{q}} \le \frac{1}{C_1}\left(\int_0^\infty [(M_\Phi f)^*(t)]^q w(t)dt\right)^{\frac{1}{q}}.$$

From this, using (2), we have obtained (5)

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} s\varphi(s) f^{**}(s)\right)^q w(t)dt\right)^{\frac{1}{q}} \le C_1^{-1}\left(\int_0^\infty [f^*(t)]^p v(t)dt\right)^{\frac{1}{p}}.$$

Step 1 implies that the constant in (2) is at most C_0 times the constant in (5). Step 2 implies the converse: the constant in (5) is at most C_1^{-1} times the constant in (2). Therefore, the optimal constants in these two inequalities are equivalent. □

Thus, Theorem 2 is subsequently used as an intermediate reduction result. The main task is to characterize the resulting one-dimensional weighted inequality, which depends on the relationship between the parameters p and q and the corresponding conditions on the weight functions.

We denote $g = f^*$. Then we have

$$tf^{**}(t) = \int_0^t g(s)ds.$$

Therefore, (5) can be rewritten as follows:

$$\left(\int_0^\infty \left(\sup_{t\le\tau<\infty} \varphi(\tau)\int_0^\tau g(s)\,ds\right)^q w(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty g(t)^p v(t)\,dt\right)^{1/p}, \tag{7}$$

where $g \in \mathfrak{M}^{\downarrow}(0,\infty)$ and $\varphi(\tau) := \Phi(\tau^{1/n})$.

Thus, the problem is reduced to studying the inequality (7) on the cone of nonincreasing functions.

Let a positive monotone operator $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ satisfy the following conditions [5]:

(i) $T(\lambda f) = \lambda T f$ for all $\lambda > 0$ and $f \in \mathfrak{M}^+(0,\infty)$;
(ii) If $f(x) \le \psi(x)$ almost everywhere on $(0,\infty)$, then $Tf(x) \le c\,T\psi(x)$ almost everywhere on $(0,\infty)$, where the constant $c > 0$ is independent of f and ψ;

(iii) $T(f+\lambda\mathbf{1}) \le c\big(Tf+\lambda T\mathbf{1}\big)$ for all $f \in \mathfrak{M}^+(0,\infty)$ and $\lambda > 0$, where the constant $c > 0$ is independent of f and λ, and $\mathbf{1}$ denotes the function identically equal to one on $(0,\infty)$.

We denote

$$V(t) = \int_0^t v(s)\,ds, \qquad t > 0.$$

Theorem 3 ([5], Theorem 3.1)**.** *Let $0 < q \le \infty$, $1 \le p < \infty$, and v, w be weight functions on $(0,\infty)$. Let $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ be a positive operator. Then:*

(a) The condition

$$\left(\int_0^\infty (Tf(t))^q w(t)dt\right)^{1/q} \leqslant C\left(\int_0^\infty (f(t))^p v(t)dt\right)^{1/p}, \quad f \in \mathfrak{M}^{\downarrow}(0,\infty) \tag{8}$$

implies the inequality

$$\left(\int_0^\infty \left(T\left(\int_x^\infty h\right)\right)^q w\right)^{1/q} \leqslant C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{1/p}, \quad h \in \mathfrak{M}^+(0,\infty); \tag{9}$$

(b) If $V(\infty) = \infty$ and if T is a monotone operator satisfying conditions (i)–(ii), then the condition (9) holds for all $h \in \mathfrak{M}^+(0,\infty)$ and is sufficient for inequality (8) to hold on the cone $\mathfrak{M}^{\downarrow}(0,\infty)$.

(c) If $0 < V(\infty) < \infty$, then sufficient condition for (8) to hold on $\mathfrak{M}^{\downarrow}(0,\infty)$ is that both (9) and

$$\left(\int_0^\infty (T\mathbf{1})^q w\right)^{1/q} \leqslant C\left(\int_0^\infty v\right)^{1/p} \tag{10}$$

hold in the case when T satisfies the conditions (i)–(iii).

Let us also present the following equivalent inequalities to inequality (7):

Theorem 4. *Let $0 < q \le \infty$, $1 \le p < \infty$, and v, w be weight functions on $(0,\infty)$, $\varphi(t) := \Phi(t^{1/n})$ and $\Phi \in A_n$. Then:*

(a) Inequality

$$\left(\int_0^\infty \left(\sup_{t\le\tau<\infty} \varphi(\tau)\int_0^\tau \left(\int_s^\infty h(y)dy\right)ds\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h(t)^p V(t)^p v(t)^{1-p}dt\right)^{\frac{1}{p}} \tag{11}$$

for any $h \in \mathfrak{M}^+(0,\infty)$, is a necessary condition for inequality (7) for all $g \in \mathfrak{M}^{\downarrow}(0,\infty)$.

(b) If $V(\infty) = \infty$, then condition (11) for all $h \in \mathfrak{M}^+(0,\infty)$ is sufficient for inequality (7) to hold on cone $\mathfrak{M}^{\downarrow}(0,\infty)$.

(c) If $0 < V(\infty) < \infty$, a sufficient condition for (7) for all $g \in \mathfrak{M}^{\downarrow}(0,\infty)$ is that both (11) for all $h \in \mathfrak{M}^+(0,\infty)$ and

$$\left(\int_0^\infty \left(\sup_{t\le\tau<\infty} \varphi(\tau)\tau\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(t)dt\right)^{\frac{1}{p}} \tag{12}$$

hold, then inequality (11) is sufficient for inequality (7) or all $g \in \mathfrak{M}^{\downarrow}(0,\infty)$.

Proof of Theorem 4. To prove the Theorem 4, we will use Theorem 3 for the operator

$$(T_\Phi g)(t) = \sup_{t<\tau<\infty} \varphi(\tau)\int_0^\tau g(s)\,ds.$$

(a) Follows from inequality (8), since the operator $(T_\Phi f)$ is a positive operator.

(b) According to Theorem 3 for positive monotone operator T inequality

$$\left(\int_0^\infty (Tf(t))^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f(t))^p v(t)dt\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^{\downarrow}(0,\infty) \tag{13}$$

is equivalent to inequality

$$\left(\int_0^\infty \left(T\left(\int_x^\infty h\right)\right)^q w\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+. \tag{14}$$

for all $h \in \mathfrak{M}^+(0,\infty)$ if $V(\infty) = \infty$ and (i)–(ii).

For the supremal operator $(T_\Phi f)$, let us verify the conditions (i)–(iii).

For condition (i), for any $\lambda > 0$, we have

$$\begin{aligned} T_\Phi(\lambda g)(t) &= \sup_{t<\tau<\infty} \varphi(\tau) \int_0^\tau \lambda g(s)\, ds \\ &= \lambda T_\Phi g(t). \end{aligned}$$

Consequently, condition (i) is satisfied.

To verify condition (ii), suppose that $f_1, f_2 \in \mathfrak{M}^+(0,\infty)$ and $f_1(x) \le f_2(x)$ for almost all $x \in \mathbb{R}_+$. Then, for any $\tau > 0$,

$$\int_0^\tau f_1(s)\, ds \le \int_0^\tau f_2(s)\, ds.$$

Consequently,

$$\begin{aligned} T_\Phi f_1(t) &= \sup_{t<\tau<\infty} \varphi(\tau) \int_0^\tau f_1(s)\, ds \\ &\le \sup_{t<\tau<\infty} \varphi(\tau) \int_0^\tau f_2(s)\, ds \\ &= T_\Phi f_2(t). \end{aligned}$$

Thus, condition (ii) holds.

Check the condition (iii). For any $f \in \mathfrak{M}^+(0,\infty)$ and $\lambda > 0$, we have

$$\begin{aligned} T_\Phi(f + \lambda \mathbf{1})(t) &= \sup_{t<\tau<\infty} \varphi(\tau) \int_0^\tau \big(f(s) + \lambda\big)\, ds \\ &= \sup_{t<\tau<\infty} \left(\varphi(\tau) \int_0^\tau f(s)\, ds + \lambda\tau\varphi(\tau)\right) \\ &\le \sup_{t<\tau<\infty} \varphi(\tau) \int_0^\tau f(s)\, ds + \lambda \sup_{t<\tau<\infty} \tau\varphi(\tau) \\ &= T_\Phi f(t) + \lambda T_\Phi \mathbf{1}(t). \end{aligned}$$

Consequently, condition (iii) also holds. Thus, the operator T_Φ satisfies conditions (i)–(iii) from Theorem 3.

Furthermore, the left-hand side of condition (10) takes the form

$$\left(\int_0^\infty (T_\Phi 1(t))^q w(t) dt\right)^{\frac{1}{q}} = C\left(\int_0^\infty \left(\sup_{t<\tau<\infty} \varphi(\tau)\tau\right)^q w(t) dt\right)^{\frac{1}{q}}.$$

Therefore, condition (11) takes the form (13). And for the supremal operator $(T_\Phi f)(t)$, the left-hand side of (14) takes the following form:

$$T\left(\int_t^\infty h(y) dy\right)(t) = \sup_{\tau\in(t,\infty)} \varphi(\tau) \int_0^\tau \left(\int_s^\infty h(y) dy\right) ds.$$

Therefore, by applying Theorem 3 to the supremal operator $(T_\Phi f)(t)$, we obtain Theorem 4. □

Let us present some well-known results.

Theorem 5 ([6], Theorem 3.1)**.** *Assume that $0 < p \le 1$, $0 < q < \infty$, and let u, v, w be weights on $(0,\infty)$. Assume further that u is such that $u(s)/s$ is nonincreasing on $(0,\infty)$. Then the following two statements are equivalent:*

(*i*) $\left(\int_0^\infty \left(\sup_{t\leqslant s<\infty} u(s)f^{**}(s)\right)^q w(t)\mathrm{d}t\right)^{1/q} \lesssim \left(\int_0^\infty f^*(t)^p v(t)\mathrm{d}t\right)^{1/p}$;

(*ii*) $\left(\int_0^\infty \left(\sup_{t\leqslant s<\infty} \frac{u(s)^p}{s^p}\int_0^s y^{p-1}f^*(y)^p\,\mathrm{d}y\right)^{q/p} w(t)\mathrm{d}t\right)^{1/q} \lesssim \left(\int_0^\infty f^*(t)^p v(t)\mathrm{d}t\right)^{1/p}$.

Theorem 6. *Let $0<q<\infty$, $0<p\leq 1$. Assume that $\Phi\in A_n$, and let v and w be weight functions on $(0,\infty)$. Set*

$$\varphi(t)=\Phi(t^{1/n}),\qquad u(t)=t\varphi(t),\qquad V(t)=\int_0^t v(s)\,ds.$$

Then inequality (5) holds for all $f\in\mathfrak{M}^{\downarrow}(0,\infty)$ if and only if, for all $h\in\mathfrak{M}^{+}(0,\infty)$ the following inequality is valid:

$$\left(\int_0^\infty\left(\sup_{t\leq\tau<\infty}\varphi(\tau)^p\int_0^\tau s^{p-1}\left(\int_s^\infty h(y)dy\right)ds\right)^{\frac{q}{p}}w(t)dt\right)^{\frac{1}{q}}\leq C\left(\int_0^\infty h(t)V(t)dt\right)^{\frac{1}{p}}. \tag{15}$$

Proof of Theorem 6. The proof follows from Theorem 5 (cases (i) and (ii)) of [6]. □

Theorem 7 ([14])**.** *Let $1\leq q<\infty$, and let u,v, and w be weights on $(0,\infty)$. Let φ be a U-quasiconcave function on $(0,\infty)$. Then there exists a constant $C>0$ such that inequality*

$$\left(\int_0^\infty\left(\sup_{t<s<\infty}\frac{1}{\varphi(s)}\int_0^s\left(\int_\tau^\infty h(y)dy\right)u(\tau)d\tau\right)^q w(t)dt\right)^{\frac{1}{q}}\leq \leq C\int_0^\infty h(s)v(s)ds$$

holds for all $h\in\mathfrak{M}^{+}(0,\infty)$ if and only if the following condition is satisfied:

$$C_1:=\sup_{t\in(0,\infty)}\left(\int_0^\infty\frac{w(s)ds}{(\varphi(s)+\varphi(t))^q}\right)^{\frac{1}{q}}\left(\sup_{s\in(0,\infty)}\frac{U(t)U(s)}{(U(s)+U(t))v(s)}\right)<\infty.$$

The following result is the main result of this work.

Theorem 8. *Let $1\leq p<\infty$, $0<q<\infty$, and let v,w be weights on $(0,\infty)$. $\Phi\in A_n$, then:*

(*i*) *If $1=p\leq q<\infty$, then the generalized fractional maximal operator is bounded from $\Lambda^1(v)$ to $\Lambda^q(w)$ if and only if*

$$\mathbf{C}_1:=\sup_{t\in(0,\infty)}\left(\int_0^\infty\frac{w(s)}{(\varphi(s)+\varphi(t))^q}ds\right)^{\frac{1}{q}}\left(\operatorname*{ess\,sup}_{s\in(0,\infty)}\frac{ts}{(s+t)V(s)}\right)<\infty;$$

(*ii*) *If $q=1<p<\infty$, $p'=\frac{p}{p-1}$, then the generalized fractional maximal operator is bounded from $\Lambda^p(v)$ to $\Lambda^1(w)$ if and only if*

$$\mathbf{C}_2:=\left(\int_0^\infty\left(\int_0^\infty\frac{w(s)}{\varphi(s)+\varphi(t)}ds\right)^{p'-1}\varphi^{1-p'}(t)\right.$$
$$\left.\times\operatorname*{ess\,sup}_{s\in(t,\infty)}\varphi^{p'}(s)\left(\int_0^\infty\frac{\tau^{pp'}s^{pp'}}{s^{pp'}+\tau^{pp'}}V^{-p'}(\tau)v(\tau)d\tau\right)w(t)dt\right)^{\frac{1}{p'}}<\infty;$$

(*iii*) *If $q<p=1$, $r=\frac{pq}{p-q}$, then*

$$\mathbf{C}_3:=\left(\int_0^\infty\left(\int_0^\infty\frac{w(s)}{(\varphi(s)+\varphi(t))^q}ds\right)^{\frac{r}{q}-1}\varphi^{q-r}(t)\right.$$
$$\left.\times\operatorname*{ess\,sup}_{s\in(t,\infty)}\varphi^{r}(s)\left(\operatorname*{ess\,sup}_{\tau\in(0,\infty)}\frac{\tau s}{(s+\tau)V(\tau)}\right)^r w(t)dt\right)^{\frac{1}{r}}<\infty.$$

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Proof of Theorem 8. (i) Let $1 = p \le q < \infty$. The proof of case (i) follows by using Theorems 2, 6 and 7. Indeed, according to Theorem 2, inequality (2) is equivalent to inequality (5) and is equivalent to inequality (7) for all $g \in \mathfrak{M}^{\downarrow}(0,\infty)$.

Applying Theorem 6 for $p = 1$, $u(s) = s\varphi(s)$, inequality (7) for all $g \in \mathfrak{M}^{\downarrow}(0,\infty)$ is equivalent to inequality

$$\left(\int_0^\infty \left(\sup_{t\le\tau<\infty} \varphi(\tau)\int_0^\tau \left(\int_s^\infty h(y)dy\right)ds\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\int_0^\infty h(t)V(t)dt \tag{16}$$

for all $h \in \mathfrak{M}^{+}(0,\infty)$. Furthermore, according to the inequality from Theorem 7, inequality (16) holds if and only if

$$\mathbf{C}_1 := \sup_{t\in(0,\infty)} \left(\int_0^\infty \frac{w(s)}{(\varphi(s)+\varphi(t))^q}ds\right)^{\frac{1}{q}} \left(\operatorname*{ess\,sup}_{s\in(0,\infty)} \frac{ts}{(s+t)V(s)}\right) < \infty.$$

Furthermore, the condition $\mathbf{C}_1 < \infty$ is necessary and sufficient for inequality (2) in the case $1 = p \le q < \infty$,

$$M_\Phi : \Lambda^1(v) \longrightarrow \Lambda^q(w).$$

(ii) Let $q = 1 < p < \infty$. The proof of case (ii) follows from Theorem 2, Theorem 4 for the case $V(\infty) = \infty$ and $0 < V(\infty) < \infty$ for (12) and [15] (Theorem 1).

Indeed, according to Theorem 2, for the case $V(\infty) = \infty$, the inequality (5) is equivalent to the inequality (11) for all $h \in \mathfrak{M}^{+}(0,\infty)$. According to [15] (Theorem 1), the inequality (11) holds for all $h \in \mathfrak{M}^{+}(0,\infty)$ if and only if $\mathbf{C}_2 < \infty$.

According to Theorem 4 for the case $0 < V(\infty) < \infty$ and under condition (12), the inequality (5) holds if and only if (11) holds for all $h \in \mathfrak{M}^{+}(0,\infty)$.

According to Theorem 7, inequality (11) holds if and only if $\mathbf{C}_2 < \infty$. Therefore, $\mathbf{C}_2 < \infty$ is necessary and sufficient for inequality (2) in the case $q = 1 < p < \infty$,

$$M_\Phi : \Lambda^p(v) \longrightarrow \Lambda^1(w).$$

Theorem 8 is proved. □

4. Conclusions

Maximal operators play an important role in harmonic analysis, function theory, and operator theory. This paper provides a complete description of the boundedness of the generalized fractional-maximal operator M_Φ in weighted Lorentz spaces. The main result reduces the multidimensional problem to one-dimensional weighted inequalities for Hardy-type operators. This allows us to use a developed reduction theory and obtain precise boundedness criteria for the boundedness of the generalized fractional maximal operator in Lorentz spaces.

The results obtained in this work can serve as a theoretical foundation for further research in the fields of probability theory and machine learning. In probability theory, weighted Lorentz spaces provide a flexible framework for studying the distribution of measurable functions—particularly in cases where large values of a function or its non-uniform integrability play a significant role. Thus, boundedness criteria for generalized fractional maximal operators may prove useful in deriving estimates for certain classes of random functions and their maximal transformations.

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