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On a Nonuniformly Coercive Quasilinear $p(\cdot)$-Elliptic Neumann Problem with L^1 Data

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Abstract

This paper studies a quasilinear and nonuniformly coercive elliptic Neumann problem with right hand side in $L^1(\Omega)$ and boundary data in $L^1(\partial\Omega)$, in the context of Sobolev spaces with variable exponent. We analyze the existence and the uniqueness of renormalized solutions to this problem.

Keywords: variable exponent Sobolev spaces; Neumann boundary condition; renormalized solutions; uniqueness; quasilinear elliptic problem; nonuniformly coercive operator

MSC: 35J60; 46E35; 46E30

1. Introduction

Let Ω be a bounded open domain of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz continuous boundary $\partial\Omega$. In recent decades, extensive research has been dedicated to the analysis of nonlinear elliptic problems in the context of variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$. These spaces were systematically studied by Fan and Zhao in [1,2] (see also [3,4]). These spaces naturally arise in the modeling of physical phenomena characterized by nonstandard growth conditions, such as electrorheological fluids [5,6], where the viscosity depends on the magnitude of an external electric field. Applications also extend to image processing [7–9], particularly in edge detection and noise removal, by using variable diffusion models that adapt to the local image structure, and to nonlinear elasticity, where spatial heterogeneity of the material can be captured by allowing the growth exponent to vary [10–12]. Other significant physical models include porous media flow and phase transition processes [13]. These applications motivate the study of quasilinear elliptic problems involving variable exponent operators and lower-order terms with complex nonlinearities.

In this context, many works have been developed in the setting of Dirichlet problems and constant exponent spaces. For instance, Boccardo, Gallouët and Murat [14] studied a nonlinear elliptic problem involving a zero order term and a Lipschitz perturbation, proving existence and uniqueness of solutions for $1 < p \leq 2$, and illustrating the failure of uniqueness when $p > 2$. Ben Cheikh and Guibé [15] extended this study to nonlinear elliptic problems with mixed Dirichlet-Neumann boundary conditions and renormalized

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solutions in $W^{1,p}(\Omega)$, for $1 < p \leq N$, assuming only L^1-data. The work of Betta, Guibé and Mercaldo [16] addressed Neumann boundary-value problems of the form

$$\begin{cases} -\nabla \cdot \left((1+|\nabla u|^2)^{\frac{p-2}{2}} \nabla u + c(x)|u|^{p-2} \right) = f & \text{in } \Omega, \\ \left((1+|\nabla u|^2)^{\frac{p-2}{2}} \nabla u + c(x)|u|^{p-2} \right) \cdot \nu = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and established the existence of renormalized solutions under minimal assumptions on the data. The uniqueness of the renormalized solutions for the nonlinear problem (1) was shown in [17] under some compatibility conditions.

A particularly relevant reference is the work by Alvino, Boccardo, Ferone, Orsina and Trombetti [18], who studied the degenerate equation

$$\begin{cases} -\nabla \cdot \left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}} \right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and established existence and regularity results for weak solutions in the constant exponent case. However, their study is restricted to homogeneous Dirichlet boundary conditions, and the existence and uniqueness question remain open.

The present work builds upon and extends these studies by considering the following quasilinear elliptic equation with natural boundary condition

$$\begin{cases} -\nabla \cdot \left(\dfrac{a(x,|\nabla u|)\nabla u}{(1+|u|)^{\theta(x)}} \right) + |u|^{p(x)-2}u = f - \nabla \cdot F & \text{in } \Omega, \\ \left(\dfrac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}} \nabla u - F \right) \cdot \nu = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $p(\cdot)$ and $\theta(\cdot)$ are continuous variable exponents. The functions f and g are given in the spaces $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, whereas F belongs to a certain subspace of $L^{p'(\cdot)}(\Omega)^N$. The Carathéodory function $a(x,|\xi|)\xi$ leads to a Leray-Lions type operator in $W^{1,p(\cdot)}(\Omega)$. However, the elliptic operator does not lead to a uniform coercive operator in the same space since u may be unbounded. This degeneracy poses serious difficulties in applying classical existence theory. Problems involving natural and Fourier-type boundary conditions with coercive Leray-Lions type operators in the variable exponent framework have been considered, for instance, in these works [19–22].

Our contribution is threefold. First, we extend the framework of [18] to the case of variable exponent Sobolev space, which allows for more complex and physically realistic modelling. Second, we treat nonuniformly coercive operators under a Neumann boundary condition, going beyond the scope of existing results such as [19]. Third, we establish both the existence and the uniqueness of solutions for the problem above, using approximation, compactness and monotonicity methods. The combined complexity of all these aspects (nonlinearity, variable exponent Sobolev space, nonuniformly coercivity and Neumann boundary condition) makes this problem of particular interest. Moreover, the mathematical analysis of problem (3) described in this paper, has shown a special relation between the regularity of the function F with the variable exponents θ and p. This has led us to assume an extra regularity to F.

The outline of the paper is as follows. In Section 2 we recall some definitions and properties of variable exponent spaces. Section 3 describes the assumptions and some preliminary lemmas in order to deduce the existence of a renormalized solution for the problem (3). Section 4 is devoted to presenting the main existence result and develop its

proof. Finally, in Section 5 we show a uniqueness result. We end this presentation with some conclusions.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open and bounded domain. Its boundary is denoted by $\partial\Omega$ and it is assumed to be Lipschitz-continuous. The closure of Ω is denoted by $\bar{\Omega}$. We consider a function $p \in \mathcal{C}(\bar{\Omega})$ such that

$$2 < p_- = \min_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p_+ = \max_{x\in\bar{\Omega}} p(x). \tag{4}$$

The function $p' \in \mathcal{C}(\bar{\Omega})$ is the conjugate exponent of p, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ in $\bar{\Omega}$, and we also put $p'_+ = \max_{x\in\bar{\Omega}} p'(x)$.

We then introduce the so-called (convex) modular function $\rho_{p(\cdot)}$ given by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x.$$

The Lebesgue space with variable exponent, denoted by $L^{p(\cdot)}(\Omega)$, is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{ u : \Omega \mapsto \mathbb{R} \text{ measurable } / \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x < \infty \Big\}.$$

This space is endowed with the Luxemburg norm, namely

$$\|u\|_{L^{p}(\cdot)} = \inf \Big\{ \lambda > 0 \,/\, \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} \mathrm{d}x \leq 1 \Big\}.$$

Owing to (4) the space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ turns out to be a separable and reflexive Banach space. In fact, its dual space is given by $L^{p'(\cdot)}(\Omega)$. The following Hölder type inequality holds [2,23]

$$\Big| \int_\Omega uv\, \mathrm{d}x \Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_+}\Big) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \text{ for any } u \in L^{p(\cdot)}(\Omega) \text{ and } v \in L^{p'(\cdot)}(\Omega).$$

For any $A \subset \Omega$, χ_A stands for the characteristic function of A in Ω. It will be interesting to have an estimate of $\|\chi_A\|_{L^{p(\cdot)}(\Omega)}$ whenever A is measurable. The following result provides this estimate.

Lemma 1 ([23]). *Let A be any measurable subset of Ω. Then*

$$\|\chi_A\|_{L^{p(\cdot)}(\Omega)} \leq \max(\mathrm{meas}\,(A)^{1/p_-}, \mathrm{meas}\,(A)^{1/p_+}). \tag{5}$$

We define the Sobolev space with variable exponent by

$$W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) \,/\, |\nabla u| \in L^{p(\cdot)}(\Omega) \},$$

endowed with the norm

$$\|u\|_{1,p(\cdot)} = \inf \Big\{ \lambda > 0 \,/\, \int_\Omega \rho_{1,p(\cdot)} \Big(\frac{u(x)}{\lambda} \Big)\, \mathrm{d}x \leq 1 \Big\},$$

where $\rho_{1,p(\cdot)}$ is the first order (convex) modular function, namely,

$$\rho_{1,p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x + \int_\Omega |\nabla u(x)|^{p(x)}\, \mathrm{d}x,$$

and the space $\left(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ is a separable and reflexive Banach space.

The following result relates the Luxemburg norm with the corresponding convex modular function.

Lemma 2 ([1]). *Let $(u_n) \subset L^{p(\cdot)}(\Omega)$ and $u \in L^{p(\cdot)}(\Omega)$. Then,*

(a) $\|u\|_{L^{p(\cdot)}(\Omega)} \geq 1 \Rightarrow \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)}$;

(b) $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1 \Rightarrow \|u\|^{p_+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p_-}_{L^{p(\cdot)}(\Omega)}$;

(c) $\|u\|_{L^{p(\cdot)}(\Omega)} \leq 1$ *(resp.* $= 1; \geq 1$*)* $\Leftrightarrow \rho_{p(\cdot)}(u) \leq 1$ *(resp.* $= 1; \geq 1$*)*;

(d) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to 0$ *(resp.* $\to \infty$*)* $\Leftrightarrow \rho_{p(\cdot)}(u_n) \to 0 \quad (resp. \to \infty)$;

(e) $\rho_{p(\cdot)}\left(\dfrac{u}{\|u\|_{L^{p(\cdot)}(\Omega)}}\right) = 1.$

Lemma 3 ([24]). *Let $u \in W^{1,p(\cdot)}(\Omega)$, then,*

(a) $\|u\|_{1,p(\cdot)} \geq 1 \Rightarrow \|u\|^{p_-}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_+}_{1,p(\cdot)}$;

(b) $\|u\|_{1,p(\cdot)} \leq 1 \Rightarrow \|u\|^{p_+}_{1,p(\cdot)} \leq \rho_{1,p(\cdot)}(u) \leq \|u\|^{p_-}_{1,p(\cdot)}$;

(c) $\|u\|_{1,p(\cdot)} \leq 1$ *(respectively* $= 1; \geq 1$*)* $\Leftrightarrow \rho_{1,p(\cdot)}(u) \leq 1$ *(respectively* $= 1; \geq 1$*)*.

It will be interesting to introduce the embedding Sobolev exponents $p^*(\cdot)$ and $q^*(\cdot)$ given as follows.

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases} \quad \text{and} \quad q^*(x) = \begin{cases} \dfrac{(N-1)p(x)}{N - p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Lemma 4 ([25]). *Let $p \in \mathcal{C}(\bar{\Omega})$, with $p_- > 1$, and $q \in \mathcal{C}(\partial\Omega)$ such that*

$$1 \leq q(x) < q^*(x) \text{ for all } x \in \partial\Omega. \tag{6}$$

Then, $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ with compact embedding. In particular, the injection $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\partial\Omega)$ is compact.

Lemma 5 ([2,25]). *The following embeddings hold and are compact:*

- *If $p(x) < N$ in $\bar{\Omega}$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, for any $q \in \mathcal{C}(\bar{\Omega})$ such that $1 \leq q(x) < p^*(x)$ in $\bar{\Omega}$.*
- *If $p(x) = N$ in $\bar{\Omega}$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, for any $q \in \mathcal{C}(\bar{\Omega})$ such that $1 \leq q(x) < \infty$ in $\bar{\Omega}$.*
- *If $p(x) > N$ in $\bar{\Omega}$, then $W^{1,p(\cdot)}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$, for some $0 < \alpha < 1$.*

Definition 1. *Let $k > 0$, the truncation function at height k, denoted by $T_k : \mathbb{R} \mapsto \mathbb{R}$, is the function*

$$T_k(s) = \max\{-k, \min(s, k)\}.$$

We may introduce the space $\mathcal{T}^{1,p(\cdot)}(\Omega)$ defined by

$$\mathcal{T}^{1,p(\cdot)}(\Omega) := \{u : \Omega \mapsto \mathbb{R} \text{ is measurable and such that } T_k(u) \in W^{1,p(\cdot)}(\Omega) \text{ for each } k > 0\}.$$

Proposition 1 ([21,22,26]). *For any given function $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$, there exists a unique measurable function $v : \Omega \mapsto \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$ then $v \in L^{p(\cdot)}(\Omega)^N$ and also $v = \nabla u$ in the distributional sense.*

The next definition introduces the subspace $\mathcal{T}^{1,p(\cdot)}_{\mathrm{tr}}(\Omega)$ of $\mathcal{T}^{1,p(\cdot)}(\Omega)$.

Definition 2. *The space $\mathcal{T}^{1,p(\cdot)}_{\mathrm{tr}}(\Omega)$ consists of those functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ that can be approximated by a sequence of functions $(u_n) \subset W^{1,p(\cdot)}(\Omega)$ in the following sense.*

(a) $u_n \to u$ *a.e. in* Ω.
(b) $\nabla T_k(u_n) \to \nabla T_k(u)$ *in* $L^1(\Omega)^N$ *for each* $k > 0$.
(c) *There exists a measurable function* $U\colon \partial\Omega \mapsto \mathbb{R}$, *such that* $u_n|_{\partial\Omega} \to U$ *a.e. in* $\partial\Omega$.

The function U represents the boundary values of u, that is, the trace of u on $\partial\Omega$ in the sense given by this definition.

3. General Assumptions on the Elliptic Operator and Data

We go back to our nonlinear elliptic Neumann problem given in (3). Let A be the operator acting from $W^{1,p(\cdot)}(\Omega)$ into its dual $(W^{1,p(\cdot)}(\Omega))'$ given as

$$Au = -\nabla \cdot \Big(\frac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}} \nabla u \Big) + |u|^{p(x)-2}u.$$

We will consider the following assumptions on the exponents functions $\theta(x)$ and $p(x)$, the function $a(x,|\xi|)$, and the data f, g and F.

(A1) The exponents $p, \theta \in \mathcal{C}^0(\bar{\Omega})$ are such that p satisfies the conditions given in (4) and θ is such that

$$0 \le \theta(x) \le \theta_+ < p_- - 1 \text{ in } \bar{\Omega}, \qquad \theta_+ = \max_{x\in\bar{\Omega}} \theta(x), \tag{7}$$

$$1 + \theta_+ < (p_- - 1)^2. \tag{8}$$

(A2) $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $F \in L^{\tau(\cdot)}(\Omega)^N \cap L^{r_0'}(\Omega)^N$ where $\tau \in C(\bar{\Omega})$ is given by

$$\tau(x) = \frac{p(x)(p(x)-1)}{(p(x)-1)^2 - \theta(x)}, \tag{9}$$

whereas $r_0' = r_0/(r_0-1)$ and r_0 is introduced in (71).

(A3) The mapping $(x,\xi) \in \Omega \times \mathbb{R}^N \mapsto a(x,|\xi|)\xi$ is a Carathéodory function, (i.e., continuous with respect to $\xi \in \mathbb{R}^N$ for a.e. $x \in \Omega$ and measurable with respect to x in Ω for each $\xi \in \mathbb{R}^N$) and is such that (see [19,20,27])

$$|a(x,|\xi|)\xi| \le l(x) + \beta|\xi|^{p(x)-1} \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N, \tag{10}$$

where $l \in L^{p'(\cdot)}(\Omega)$ and $\beta > 0$;

$$\min\Big\{ a(x,|s|), a(x,|s|) + s\frac{\partial a(x,|s|)}{\partial s} \Big\} \ge a_3|s|^{p(x)-2} \text{ for a.e. } x \in \Omega \text{ and } s \in \mathbb{R}; \tag{11}$$

for some real constant $a_3 > 0$; and

$$s^2 a(x,s) \le p_+ \int_0^s \sigma a(x,\sigma)\, \mathrm{d}\sigma \text{ for all } s \ge 0 \text{ and for a.e. } x \in \Omega. \tag{12}$$

Remark 1. *Since $p(x) \ge p_- > 2$ in $\bar{\Omega}$, we have $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $N = 2$.*

Remark 2. *Notice that $p' \le \tau$ in $\bar{\Omega}$ so that $L^{\tau(\cdot)}(\Omega)^N \subset L^{p'(\cdot)}(\Omega)^N$, and that $p' = \tau$ in $\bar{\Omega}$ if and only if $\theta(x) = 0$ in $\bar{\Omega}$ which is the case when the operator A is uniformly coercive. Thus, the summability of F is related not only to the natural exponent $p(x)$ but to the function $\theta(x)$ as well. Consequently, the presence of the term $(1+|u(x)|)^{\theta(x)}$ in the denominator of the nonlinear*

divergence term is compensated by the assumption of more regularity to F, namely, $F \in L^{\tau(\cdot)}(\Omega)^N$ instead of just $F \in L^{p'(\cdot)}(\Omega)^N$. On the other hand, the extra regularity $F \in L^{r_0'}(\Omega)^N$ is requested in order to deal with the divergence term $\nabla \cdot F$ (see the paragrah Stage 5: Almost everywhere convergence of the gradients).

Now, we recall some lemmas which play an important role in the proof of our main result.

Lemma 6. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$. Let $(g_n) \subset L^r(\mathcal{O})$ and $g \in L^r(\mathcal{O})$ such that $\|g_n\|_r \leq C$ for some $r \in (1, \infty)$. If g_n converges to g a.e. in $\mathcal{O}$, then $g_n \rightharpoonup g$ weakly in $L^r(\mathcal{O})$.*

Lemma 7 ([19])**.** *Under the assumptions (10)–(12), there exist a positive constant C_1 such that, for all $\xi_1, \xi_2 \in \mathbb{R}^N \setminus \{(0,0)\}$, one has*

$$(a(x,|\xi_1|)\xi_1 - a(x,|\xi_2|)\xi_2) \cdot (\xi_1 - \xi_2) \geq C_1 4^{2-p_+} |\xi_1 - \xi_2|^{p(x)}. \tag{13}$$

Lemma 8 ([19])**.** *Let $p \in \mathcal{C}(\bar{\Omega})$ verifying the condition (4), and let $(u_n) \subset W^{1,p(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p(\cdot)}(\Omega)$ and*

$$\begin{aligned} &\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)\,\mathrm{d}x \\ &\quad + \int_\Omega (a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla u|)\nabla u) \cdot (\nabla u_n - \nabla u)\,\mathrm{d}x \to 0, \text{ as } n \to \infty, \end{aligned} \tag{14}$$

Then, under the assumptions (10)–(12), $u_n \to u$ strongly in $W^{1,p(\cdot)}(\Omega)$.

4. Main Results

Let $W_c^{1,\infty}(\mathbb{R})$ the subspace of $W^{1,\infty}(\mathbb{R})$ defined as

$$W_c^{1,\infty}(\mathbb{R}) = \{\zeta \in W^{1,\infty}(\mathbb{R}) / \text{support of } \zeta \text{ is compact}\}.$$

In the sequel, the support of ζ will be denoted by $\operatorname{supp}\zeta$.

Definition 3. *Let $u : \bar{\Omega} \mapsto \mathbb{R}$ be a measurable function. We say that u is a renormalized solution to problem (3), if the following conditions hold true.*

(R1) $T_k(u) \in W^{1,p(\cdot)}(\Omega)$ *for each* $k > 0$;

(R2) $|u|^{p(\cdot)-1} \in L^1(\Omega)$;

(R3) $$\lim_{h\to\infty} \frac{1}{h} \int_{\{|u|\leq h\}} \frac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}} |\nabla u|^2\,\mathrm{d}x = 0. \tag{15}$$

(R4) *For any $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for each $\zeta \in W_c^{1,\infty}(\mathbb{R})$ one has*

$$\begin{aligned} &\int_\Omega \frac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}} \nabla u \cdot [\zeta'(u)\nabla u\ \varphi + \zeta(u)\nabla\varphi]\,\mathrm{d}x + \int_\Omega |u|^{p(x)-2}u\zeta(u)\varphi\,\mathrm{d}x \\ &= \int_\Omega f(x)\zeta(u)\varphi\,\mathrm{d}x + \int_\Omega F \cdot [\zeta(u)\nabla\varphi + \zeta'(u)\nabla u\ \varphi]\,\mathrm{d}x + \int_{\partial\Omega} g(x)\zeta(u)\varphi\,\mathrm{d}\gamma. \end{aligned} \tag{16}$$

Remark 3. *According to Remark 1, if $N = 2$, any renormalized solution u is a weak solution with $u \in W^{1,p(\cdot)}(\Omega)$ and, in fact, the problem (16) is coercive in the standard sense. Thus, in this setting and under the previous assumptions, the concept of renormalized solution is useful for $N \geq 3$ where, in general, $u \notin L^\infty(\Omega)$.*

Remark 4. *Since the function ζ appearing in (16) has compact support, there exists $k > 0$ such that $\zeta = 0$ in $[-k, k]$. Thus, we may replace any occurrence of u in (16) by $T_k(u)$ without changing the value of this expression. In particular, owing to (R1), every term in (16) makes sense.*

The main existence result now follows.

Theorem 1. *Under the assumptions (4), (A1)–(A3), the quasilinear elliptic Neumann problem (3) admits at least one renormalized solution u. Moreover, $u \in W^{1,r_0}(\Omega)$, where $r_0 > 1$ is defined in (71).*

Proof of Theorem 1. The proof is split into six stages.

Stage 1: Introduction of the approximate problem

In the sequel, the letter C will stand for a generic nonnegative constant that may change from one occurrence to another. We put $f_n = T_n(f) \in L^\infty(\Omega)$ and $g_n = T_n(g) \in L^\infty(\partial\Omega)$. It is straightforward that $f_n \to f$ strongly in $L^1(\Omega)$ and $g_n \to g$ strongly in $L^1(\partial\Omega)$. The approximate problem now follows.

$$\begin{cases} -\nabla \cdot \left(\dfrac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}} \nabla u_n \right) + |u_n|^{p(x)-2}u_n = f_n - \nabla \cdot F & \text{in } \Omega, \\ \left(\dfrac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}} \nabla u_n - F \right) \cdot \nu = g_n & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Consider the operator $B_n \colon W^{1,p(\cdot)}(\Omega) \mapsto (W^{1,p(\cdot)}(\Omega))'$ given as

$$\begin{aligned} \langle B_n u, v \rangle = &\int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u)|)^{\theta(x)}} \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega |u|^{p(x)-2} uv \, \mathrm{d}x \\ &- \int_\Omega F \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} g_n v \, \mathrm{d}\gamma, \quad u, v \in W^{1,p(\cdot)}(\Omega). \end{aligned}$$

Lemma 9 ([28])**.** *B_n is a bounded and pseudo-monotone operator from $W^{1,p(\cdot)}(\Omega)$ to $(W^{1,p(\cdot)}(\Omega))'$. Furthermore, it is coercive, that is*

$$\lim_{\|v\|_{1,p(\cdot)} \to \infty} \frac{\langle B_n v, v \rangle}{\|v\|_{1,p(\cdot)}} = \infty.$$

Proof. Since the elliptic operator in (17) is uniformly coercive, along the proof of this lemma we will only make use of the regularity $F \in L^{p'(\cdot)}(\Omega)^N$. Also, we will not need the estimate (8) since (7) is enough.

Owing to Hölder's inequality and by using (12), we obtain, for $u \in W^{1,p(\cdot)}(\Omega)$,

$$\begin{aligned} \langle B_n u, u \rangle &= \int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u)|)^{\theta(x)}} \nabla u \cdot \nabla u \, \mathrm{d}x + \int_\Omega |u|^{p(x)-2} uu \, \mathrm{d}x - \int_\Omega F \cdot \nabla u \, \mathrm{d}x - \int_{\partial\Omega} g_n u \, \mathrm{d}\gamma \\ &\geq \int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u)|)^{\theta(x)}} |\nabla u|^2 \, \mathrm{d}x + \int_\Omega |u|^{p(x)} \, \mathrm{d}x - \int_\Omega |F||\nabla u| \, \mathrm{d}x - \int_{\partial\Omega} |g_n||u| \, \mathrm{d}\gamma \\ &\geq \alpha \int_\Omega \frac{|\nabla u|^{p(x)}}{(1+|T_n(u)|)^{\theta(x)}} \, \mathrm{d}x + \int_\Omega |u|^{p(x)} \, \mathrm{d}x - C_0 \|\nabla u\|_{L^{p(\cdot)}(\Omega)} - nC_1 \|u\|_{L^{p(\cdot)}(\partial\Omega)} \\ &\geq \min\left(\frac{\alpha}{(1+n)^{\theta_+}}, 1 \right) \left(\int_\Omega |\nabla u|^{p(x)} \, \mathrm{d}x + \int_\Omega |u|^{p(x)} \, \mathrm{d}x \right) - (1+n)C_2 \|u\|_{1,p(\cdot)} \\ &= \min\left(\frac{\alpha}{(1+n)^{\theta_+}}, 1 \right) \rho_{1,p(\cdot)}(u) - C_{3,n} \|u\|_{1,p(\cdot)} \\ &\geq \min\left(\frac{\alpha}{(1+n)^{\theta_+}}, 1 \right) \|u\|^{\Upsilon}_{1,p(\cdot)} - C_{3,n} \|u\|_{1,p(\cdot)}. \end{aligned} \tag{18}$$

with

$$\Upsilon = \begin{cases} p_- & \text{if } \|u\|_{1,p(\cdot)} \geq 1, \\ p_+ & \text{if } \|u\|_{1,p(\cdot)} < 1. \end{cases}$$

Then, in view of (18), since $p_- > 2$ we deduce that $\dfrac{\langle B_n v, v\rangle}{\|v\|_{1,p(\cdot)}} \to \infty$ as $\|v\|_{1,p(\cdot)} \to \infty$.

On the other hand, using (10) and Hölder's inequality, we obtain

$$\begin{aligned}
|\langle B_n u, v\rangle| &= \left| \int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u)|)^{\theta(x)}} \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega |u|^{p(x)-2} uv \, \mathrm{d}x - \int_\Omega F \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} g_n v \, \mathrm{d}\gamma \right| \\
&\leq \int_\Omega |a(x,|\nabla u|)| \nabla u| |\nabla v| \, \mathrm{d}x + \int_\Omega |u|^{p(x)-1} |v| \, \mathrm{d}x \\
&\quad + \int_\Omega |F| |\nabla v| \, \mathrm{d}x + \int_{\partial\Omega} |g_n| |v| \, \mathrm{d}\gamma \\
&\leq \|a(x,|\nabla u|)\nabla u\|_{L^{p'(\cdot)}(\Omega)} \|v\|_{L^{p(\cdot)}(\Omega)} + \||u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)} \|v\|_{L^{p(\cdot)}(\Omega)} \\
&\quad + C_4 \|\nabla v\|_{L^{p(\cdot)}(\Omega)} + nC_5 \|v\|_{L^1(\partial\Omega)} \\
&\leq C_6 (\|l\|_{L^{p'(\cdot)}(\Omega)} + \||\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)}) \|v\|_{L^{p(\cdot)}(\Omega)} \\
&\quad + \||u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)} \|v\|_{L^{p(\cdot)}(\Omega)} + nC_7 \|v\|_{1,p(\cdot)}.
\end{aligned}$$

Then, we get

$$|\langle B_n u, v\rangle| \leq C_8 \Big(n + \|l\|_{L^{p'(\cdot)}(\Omega)} + \||u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)} + \||\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega)} \Big) \|v\|_{1,p(\cdot)},$$

and thus the operator B_n is bounded. Now, we are going to prove that the operator B_n is pseudomonotone. We consider a sequence $(u_k) \subset W^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases}
u_k \rightharpoonup u \text{ weakly in } W^{1,p(\cdot)}(\Omega), \\
B_n u_k \rightharpoonup \chi \text{ weakly in } (W^{1,p(\cdot)}(\Omega))', \\
\limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle \text{ in } W^{1,p(\cdot)}(\Omega).
\end{cases} \tag{19}$$

We want to prove that $B_n u = \chi$ and $\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k \to \infty$. By using the growth condition (10), the sequence $\left(\dfrac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}} \nabla u_k \right)_k$ is bounded in $L^{p'(\cdot)}(\Omega)^N$, then there exists a measurable function $\Phi \in L^{p'(\cdot)}(\Omega)^N$ such that

$$\frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}} \nabla u_k \rightharpoonup \Phi \text{ weakly in } L^{p'(\cdot)}(\Omega)^N \text{ as } k \to \infty. \tag{20}$$

Having in mind the compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, then (u_k) converges to u strongly in $L^{p(\cdot)}(\Omega)$, and thus

$$|u_k|^{p(x)-2} u_k \to |u|^{p(x)-2} u \text{ strongly in } L^{p'(\cdot)}(\Omega) \text{ as } k \to \infty. \tag{21}$$

Additionally, since $g_n \in L^\infty(\partial\Omega)$, we have

$$\int_{\partial\Omega} g_n u_k \, \mathrm{d}\gamma \to \int_{\partial\Omega} g_n u \, \mathrm{d}\gamma \text{ as } k \to \infty. \tag{22}$$

Thanks to (19) and (20)–(22), we deduce

$$\begin{aligned}
\langle \chi, v\rangle = \lim_{k\to\infty} \Bigg\{ &\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}} \nabla u_k \cdot \nabla v \, \mathrm{d}x + \int_\Omega |u_k|^{p(x)-2} u_k v \, \mathrm{d}x \\
&- \int_\Omega F \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} g_n v \, \mathrm{d}\gamma \Bigg\} \\
= &\int_\Omega \Phi \cdot \nabla v \, \mathrm{d}x + \int_\Omega |u|^{p(x)-2} uv \, \mathrm{d}x - \int_\Omega F \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} g_n v \, \mathrm{d}\gamma.
\end{aligned} \tag{23}$$

According to (22) and using (23), we get

$$\begin{aligned}\langle \chi, u\rangle &\geq \limsup_{k\to\infty}\left\{\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u_k\cdot\nabla u_k\,\mathrm{d}x+\int_\Omega |u_k|^{p(x)}\,\mathrm{d}x\right.\\ &\qquad\left.-\int_\Omega F\cdot\nabla u_k\mathrm{d}x-\int_{\partial\Omega} g_n u_k\,\mathrm{d}\gamma\right\}\\ &=\limsup_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x+\int_\Omega |u|^{p(x)}\,\mathrm{d}x\\ &\qquad-\int_\Omega F\cdot\nabla u\,\mathrm{d}x-\int_{\partial\Omega} g_n u\,\mathrm{d}\gamma\\ &=\limsup_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x+\langle \chi, u\rangle-\int_\Omega \Phi\cdot\nabla u\,\mathrm{d}x.\end{aligned}$$

This yields

$$\limsup_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x\leq\int_\Omega \Phi\cdot\nabla u\,\mathrm{d}x. \tag{24}$$

From (12), we have

$$\int_\Omega\left(\frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u_k-\frac{a(x,|\nabla u|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u\right)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x\geq 0. \tag{25}$$

Then, we obtain

$$\begin{aligned}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x&\geq\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u_k\cdot\nabla u\,\mathrm{d}x\\ &\quad+\int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x.\end{aligned} \tag{26}$$

From (21), we may consider a subsequence of (u_k), still denoted in the same way, such that $u_k\to u$ a.e. in Ω. Then, it is straightforward to show that $\left(\frac{a(\cdot,|\nabla u|)}{(1+|T_n(u_k)|)^{\theta(\cdot)}}\nabla u\right)_k\subset L^{p'(\cdot)}(\Omega)^N$ converges strongly in $L^{p'(\cdot)}(\Omega)^N$. Since $\nabla u_k\rightharpoonup\nabla u$ weakly in $L^{p(\cdot)}(\Omega)^N$, we deduce that

$$\int_\Omega \frac{a(x,|\nabla u|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x\to 0\quad\text{as}\quad k\to\infty. \tag{27}$$

Consequently, from (20) and (26), we find

$$\liminf_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x\geq\int_\Omega \Phi\cdot\nabla u\,\mathrm{d}x. \tag{28}$$

Having in mind (24) together with (28), it yields

$$\lim_{k\to\infty}\int_\Omega \frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}|\nabla u_k|^2\,\mathrm{d}x=\int_\Omega \Phi\cdot\nabla u\,\mathrm{d}x. \tag{29}$$

□

By combining the results (20)–(22) and (29), we deduce that $\lim_{k\to\infty}\langle B_n u_k, u_k\rangle=\langle\chi,u\rangle$, and since the limit is the same for any subsequence, it is the whole sequence $(\langle B_n u_k, u_k\rangle)_k$ that converges to this limit.

Moreover, owing to (29), we obtain

$$\begin{aligned}&\limsup_{k\to\infty}\frac{1}{(1+n)^{\theta_+}}\int_\Omega\Big(a(x,|\nabla u_k|)\nabla u_k-a(x,|\nabla u|)\nabla u\Big)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x\\&\le\lim_{k\to\infty}\int_\Omega\left(\frac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u_k-\frac{a(x,|\nabla u|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u\right)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x=0.\end{aligned}\tag{30}$$

From (21) and (30), we obtain that

$$\begin{aligned}\lim_{k\to\infty}&\int_\Omega(a(x,|\nabla u_k|)\nabla u_k-a(x,|\nabla u|)\nabla u)\cdot(\nabla u_k-\nabla u)\,\mathrm{d}x\\&+\int_\Omega(|u_k|^{p(x)-2}u_k-|u|^{p(x)-2}u)(u_k-u)\,\mathrm{d}x=0.\end{aligned}\tag{31}$$

According to Lemma 8, and by using (31), it finally yields $u_k\to u$ strongly in $W^{1,p(\cdot)}(\Omega)$, and, up to a subsequence, we may also assume that $\nabla u_k\to\nabla u$ a.e. in Ω.

Consequently, by Lemma 6, we have $\dfrac{a(x,|\nabla u_k|)}{(1+|T_n(u_k)|)^{\theta(x)}}\nabla u_k\rightharpoonup\dfrac{a(x,|\nabla u|)}{(1+|T_n(u)|)^{\theta(x)}}\nabla u$ weakly in $L^{p'(\cdot)}(\Omega)^N$. This implies that $\chi=B_nu$ and then B_n is a pseudomonotone operator in $W^{1,p(\cdot)}(\Omega)$. Owing to [28] (Theorem 2.7, p. 180), the operator B_n is surjective onto $(W^{1,p(\cdot)}(\Omega))'$. Since $f_n\in L^\infty(\Omega)$, the mapping $\ell_n\colon W^{1,p(\cdot)}(\Omega)\mapsto\mathbb{R}$ given by $\langle\ell_n,v\rangle=\int_\Omega f_nv\,\mathrm{d}x$ is well defined and belongs to $(W^{1,p(\cdot)}(\Omega))'$. Consequently, there exists at least one element $u_n\in W^{1,p(\cdot)}(\Omega)$ such that $B_nu_n=\ell_n$, that is, for every $v\in W^{1,p(\cdot)}(\Omega)$, one has

$$\begin{cases}\displaystyle\int_\Omega\frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}})\nabla u_n\cdot\nabla v\,\mathrm{d}x+\int_\Omega|u_n|^{p(x)-2}u_nv\,\mathrm{d}x\\\qquad=\displaystyle\int_\Omega f_nv\,\mathrm{d}x+\int_{\partial\Omega}g_nv\,\mathrm{d}\gamma+\int_\Omega F\cdot\nabla v\,\mathrm{d}x.\end{cases}\tag{32}$$

This last expression is just the variational formulation of the approximate boundary value Neumann problem (17).

Stage 2: Weak convergence of the truncation sequence

By using $T_k(u_n)$ as a test function in (32), we obtain

$$\begin{aligned}\int_\Omega\frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}}\nabla u_n\cdot\nabla T_k(u_n)\,\mathrm{d}x&+\int_\Omega T_k(u_n)|u_n|^{p(x)-2}u_n\,\mathrm{d}x\\&=\int_\Omega f_nT_k(u_n)\,\mathrm{d}x+\int_{\partial\Omega}g_nT_k(u_n)\,\mathrm{d}\gamma+\int_\Omega F\cdot\nabla T_k(u_n)\,\mathrm{d}x.\end{aligned}\tag{33}$$

By using (12), we obtain

$$\begin{aligned}\alpha\int_\Omega\frac{|\nabla T_k(u_n)|^{p(x)}}{(1+|T_k(u_n)|)^{\theta(x)}}\,\mathrm{d}x&+\int_\Omega|T_k(u_n)||u_n|^{p(x)-1}\,\mathrm{d}x\\&\le\int_\Omega|f_n||T_k(u_n)|\,\mathrm{d}x+\int_{\partial\Omega}|g_n||T_k(u_n)|\,\mathrm{d}\gamma+\int_\Omega|F||\nabla T_k(u_n)|\,\mathrm{d}x\\&\le k\Big(\|f\|_{L^1(\Omega)}+\|g\|_{L^1(\partial\Omega)}\Big)+\int_\Omega|F||\nabla T_k(u_n)|\,\mathrm{d}x.\end{aligned}\tag{34}$$

As long as the last term of the inequality (34) is concerned, by applying Young's inequality, we get

$$\begin{aligned}\int_\Omega|\nabla T_k(u_n)||F|\,\mathrm{d}x&\le\frac{\alpha}{2}\int_\Omega\frac{|\nabla T_k(u_n)|^{p(x)}}{(1+|T_k(u_n)|)^{\theta(x)}}\,\mathrm{d}x+C_1\int_\Omega(1+|T_k(u_n)|)^{\frac{\theta(x)p'(x)}{p(x)}}|F|^{p'(x)}\,\mathrm{d}x\\&\le\frac{\alpha}{2}\int_\Omega\frac{|\nabla T_k(u_n)|^{p(x)}}{(1+|T_k(u_n)|)^{\theta(x)}}\,\mathrm{d}x+C_1(1+k)^{\frac{\theta_+}{p_--1}}\int_\Omega|F|^{p'(x)}\,\mathrm{d}x.\end{aligned}\tag{35}$$

Thanks to the fact that

$$\frac{\alpha}{2(1+k)^{\theta_+}}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx \le \frac{\alpha}{2}\int_\Omega \frac{1}{(1+|T_k(u_n)|)^{\theta(x)}}|\nabla T_k(u_n)|^{p(x)}\,dx. \tag{36}$$

Putting together the results (34)–(36), it follows that

$$\frac{\alpha}{2(1+k)^{\theta_+}}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_\Omega |u_n|^{p(x)-1}|T_k(u_n)|\,dx \le kC_2 + C_3(1+k)^{\frac{\theta_+}{p_--1}}. \tag{37}$$

Then, we deduce that

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx \le C_4(1+k)^{\theta_++1} + C_5(1+k)^{\frac{\theta_+p_-}{p_--1}} \tag{38}$$

and

$$\int_\Omega |T_k(u_n)|^{p(x)}\,dx \le kC_2 + C_3(1+k)^{\frac{\theta_+}{p_--1}}. \tag{39}$$

In view of (38) and (39), we get

$$\begin{aligned}&\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx + \int_\Omega |T_k(u_n)|^{p(x)}\,dx\\ &\quad\le C_2k + C_3(1+k)^{\frac{\theta_+}{p_--1}} + C_4(1+k)^{\theta_++1} + C_5(1+k)^{\frac{\theta_+p_-}{p_--1}}.\end{aligned} \tag{40}$$

From (40) and Lemma 3, it can be concluded that

$$\|T_k(u_n)\|_{1,p(\cdot)} \le C_6(k^{\frac{1}{p_-}} + (1+k)^{\frac{\theta_+}{p_-(p_--1)}} + (1+k)^{\frac{\theta_++1}{p_-}} + (1+k)^{\frac{\theta_+}{(p_--1)}}), \tag{41}$$

where the constant C_6 does not depend on n and k. Consequently, for each $k > 0$, $(T_k(u_n))_n \subset W^{1,p(\cdot)}(\Omega)$ is a bounded sequence. Therefore, for some subsequence, still denoted in the same way, and there is a measurable function $w_k \in W^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup w_k \text{ weakly in } W^{1,p(\cdot)}(\Omega),\\ T_k(u_n) \to w_k \text{ strongly in } L^1(\partial\Omega) \text{ and a.e. on } \partial\Omega,\\ T_k(u_n) \to w_k \text{ strongly in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{42}$$

The next step consists in showing that (u_n) is a Cauchy sequence in measure. To do so, according to the inequality (37) we have

$$k^{p_-}\,\mathrm{meas}\{|u_n| > k\} \le \int_\Omega |T_k(u_n)||u_n|^{p(x)-1}\,dx \le kC_2 + C_1(1+k)^{\frac{\theta_+}{p_--1}}. \tag{43}$$

Since $0 \le \theta_+ < p_- - 1$, this yields

$$\mathrm{meas}\{|u_n| > k\} \le \frac{C_2k + C_1(1+k)^{\frac{\theta_+}{p_--1}}}{k^{p_-}} \to 0 \text{ as } k \to \infty. \tag{44}$$

Let $\delta > 0$. Then, it is straightforward that

$$\mathrm{meas}\{|u_m - u_n| > \delta\} \le \mathrm{meas}\{|u_m| > k\} + \mathrm{meas}\{|u_n| > k\} + \mathrm{meas}\{|T_k(u_m) - T_k(u_n)| > \delta\}. \tag{45}$$

Now, for a given positive ε, by using (44) we may select $k = k(\varepsilon)$ big enough, so that

$$\mathrm{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3} \quad \text{and} \quad \mathrm{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3}. \tag{46}$$

In light of (42), $(T_k(u_n))_n$ forms a Cauchy sequence in measure, then for any $\varepsilon, \delta, k > 0$, we may find $n_0 = n_0(\varepsilon, \delta, k)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \text{ for all } m, n \geq n_0(\varepsilon, \delta, k). \tag{47}$$

From (46) and (47), we may deduce that for any $\varepsilon > 0$ and $\delta > 0$ and we can find $n_0 = n_0(\varepsilon, \delta)$ such that

$$\text{meas}\{|u_m - u_n| > \delta\} \leq \varepsilon \text{ for all } m, n > n_0(\varepsilon, \delta). \tag{48}$$

Therefore, (u_n) is a Cauchy sequence in measure. Consequently, we may extract a subsequence, still denoted in the same way, such that

$$u_n \to u \text{ almost everywhere in } \Omega. \tag{49}$$

Remark 5. *Owing to (49), we obtain* $\chi_{\{|u|>k\}} \leq \liminf_{n\to\infty} \chi_{\{|u_n|>k\}}$. *Then, by Fatou's lemma*

$$\begin{aligned}\text{meas}\left(\{|u| > k\}\right) &\leq \int_\Omega \liminf_{n\to\infty} \chi_{\{|u_n|>k\}}\, \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \chi_{\{|u_n|>k\}}\, \mathrm{d}x \\ &= \liminf_{n\to\infty} \text{meas}\left(\{|u_n| > k\}\right)\end{aligned}$$

and therefore, using (44)

$$\text{meas}\{|u| > k\} \leq \frac{C_2 k + C_1 (1+k)^{\frac{\theta_+}{p_- - 1}}}{k^{p_-}} \to 0 \text{ as } k \to \infty. \tag{50}$$

Finally, by using (42) and (49), we obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W^{1,p(\cdot)}(\Omega), \\ T_k(u_n) \to T_k(u) \text{ strongly in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega, \\ T_k(u_n) \to T_k(u) \text{ strongly in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{51}$$

Stage 3: Equiintegrability of the sequence $(|u_n|^{p(\cdot)-1})_n$

For $h > 1$, we choose $v = \frac{1}{h} T_h(u_n)$ as a test function in (17). It yields

$$\begin{aligned}&\frac{1}{h} \int_\Omega \frac{a(x, |\nabla u_n|)}{(1 + |T_n(u_n)|)^{\theta(x)}} \nabla u_n \cdot \nabla T_h(u_n)\, \mathrm{d}x + \int_\Omega |u_n|^{p(x)-2} u_n \frac{T_h(u_n)}{h}\, \mathrm{d}x \\ &\quad = \int_\Omega f_n \frac{T_h(u_n)}{h}\, \mathrm{d}x + \frac{1}{h} \int_\Omega F \cdot \nabla T_h(u_n)\, \mathrm{d}x + \int_{\partial\Omega} g_n \frac{T_h(u_n)}{h}\, \mathrm{d}\gamma.\end{aligned} \tag{52}$$

In view of (12), we obtain

$$\begin{aligned}&\frac{1}{2h} \int_\Omega \frac{a(x, |\nabla u_n|)}{(1 + |T_h(u_n)|)^{\theta(x)}} \nabla u_n \cdot \nabla T_h(u_n)\, \mathrm{d}x + \frac{\alpha}{2h} \int_\Omega \frac{|\nabla T_h(u_n)|^{p(x)}}{(1 + |T_h(u_n)|)^{\theta(x)}} \mathrm{d}x \\ &\qquad + \int_\Omega |u_n|^{p(x)-1} \frac{|T_h(u_n)|}{h} \mathrm{d}x \\ &\quad \leq \int_\Omega |f| \frac{|T_h(u_n)|}{h}\, \mathrm{d}x + \frac{1}{h} \int_\Omega |F| |\nabla T_h(u_n)|\, \mathrm{d}x + \int_{\partial\Omega} |g| \frac{|T_h(u_n)|}{h}\, \mathrm{d}\gamma.\end{aligned} \tag{53}$$

Having in mind that $f \in L^1(\Omega)$ and since $|T_h(u_n)|/h \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$ as $h \to \infty$, then we obtain

$$\lim_{h\to\infty} \int_\Omega |f| \frac{|T_h(u_n)|}{h}\, \mathrm{d}x = 0, \tag{54}$$

and similarly

$$\lim_{h\to\infty}\int_{\partial\Omega}|g|\frac{|T_h(u_n)|}{h}\,\mathrm{d}\gamma=0. \tag{55}$$

As long as the second integral of the right-hand side of the inequality (53) is concerned, by Young's inequality one has, for $h>1$,

$$\begin{aligned}\frac{1}{h}\int_\Omega |F||\nabla T_h(u_n)|\mathrm{d}x &\le \frac{\alpha}{2h}\int_\Omega \frac{|\nabla T_h(u_n)|^{p(x)}}{(1+|T_h(u_n)|)^{\theta(x)}}\mathrm{d}x+\frac{C_7}{h}\int_\Omega |F|^{p'(x)}(1+|T_h(u_n)|)^{\frac{\theta(x)p'(x)}{p(x)}}\mathrm{d}x\\ &\le \frac{\alpha}{2h}\int_\Omega \frac{|\nabla T_h(u_n)|^{p(x)}}{(1+|T_h(u_n)|)^{\theta(x)}}\,\mathrm{d}x+\frac{C_8}{h^{1-\frac{\theta_+}{p_--1}}}\int_\Omega |F|^{p'(x)}\,\mathrm{d}x.\end{aligned} \tag{56}$$

Consequently,

$$\begin{aligned}&\frac{1}{2h}\int_\Omega \frac{a(x,|\nabla u_n|)}{(1+|T_h(u_n)|)^{\theta(x)}}\nabla u_n\cdot\nabla T_h(u_n)\,\mathrm{d}x+\int_\Omega |u_n|^{p(x)-1}\frac{|T_h(u_n)|}{h}\,\mathrm{d}x\\ &\qquad\le \int_\Omega |f|\frac{|T_h(u_n)|}{h}\,\mathrm{d}x+\int_{\partial\Omega}|g|\frac{|T_h(u_n)|}{h}\,\mathrm{d}x+\frac{C_8}{h^{1-\frac{\theta_+}{p_--1}}}\int_\Omega |F|^{p'(x)}\,\mathrm{d}x,\end{aligned} \tag{57}$$

which yields

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\le h\}}\frac{a(x,|\nabla u_n|)}{(1+|T_h(u_n)|)^{\theta(x)}}|\nabla u_n|^2\,\mathrm{d}x=0, \tag{58}$$

then

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\le h\}}\frac{|\nabla u_n|^{p(x)}}{(1+|T_h(u_n)|)^{\theta(x)}}\,\mathrm{d}x=0, \tag{59}$$

and also

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p(x)-1}\,\mathrm{d}x=0. \tag{60}$$

Moreover, in view of (49) we get

$$\lim_{n\to\infty}|u_n|^{p(x)-1}=|u|^{p(x)-1}\quad\text{a.e. in }\Omega. \tag{61}$$

Now, let $E\subset\Omega$ be any measurable set and let $\varepsilon>0$. From, (60) there exists $h_0=h_0(\varepsilon)>1$ such that

$$\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p(x)-1}\,\mathrm{d}x\le\varepsilon\text{ for all }h\ge h_0.$$

Thus, there exists $n_0=n_0(\varepsilon)$ such that

$$\int_{\{|u_n|>h_0\}}|u_n|^{p(x)-1}\,\mathrm{d}x\le 2\varepsilon\text{ for all }n\ge n_0. \tag{62}$$

Since

$$\int_E|u_n|^{p(x)-1}\,\mathrm{d}x\le\int_{E\cap\{|u_n|\le h_0\}}|T_{h_0}(u_n)|^{p(x)-1}\,\mathrm{d}x+\int_{\{|u_n|>h_0\}}|u_n|^{p(x)-1}\,\mathrm{d}x, \tag{63}$$

the last term of (63) does not exceed 2ε for all $n\ge n_0$. On the other hand, for $\beta(\varepsilon)=\varepsilon/h_0^{p_+-1}$ it yields

$$\int_{E\cap\{|u_n|\le h_0\}}|T_{h_0}(u_n)|^{p(x)-1}\,\mathrm{d}x\le h_0^{p_+-1}\,\mathrm{meas}\,(E)\le\varepsilon\text{ whenever }\mathrm{meas}\,(E)\le\beta(\varepsilon). \tag{64}$$

Gathering the estimates (62)–(64), we have shown that

$$\int_E |u_n|^{p(x)-1}\,\mathrm{d}x \le 3\varepsilon \text{ as long as } \operatorname{meas}(E) \le \beta(\varepsilon). \tag{65}$$

Thus, in view of (61) and (65), and by applying Vitali's theorem, we finally deduce that

$$\lim_{n\to\infty} |u_n|^{p(x)-1} = |u|^{p(x)-1} \text{ strongly in } L^1(\Omega). \tag{66}$$

Stage 4: An estimate of (∇u_n) in $L^r(\Omega)^N$ for some $r > 1$

We start by establishing a certain uniform bound.

Lemma 10. *There exists a constant $C > 0$ such that*

$$\int_{\{k\le|u_n|<k+1\}} \frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}}|\nabla u_n|^2 \le C \quad \text{for all } k,n \in \mathbb{N}, \tag{67}$$

where C does not depend on n and k.

Proof. For $k \in \mathbb{N}$ we define the function $S_k \in W^{1,\infty}(\mathbb{R})$ as $S_k(s) = T_{k+1}(s) - T_k(s)$. Then, $|S_k(s)| \le 1$ in $\mathbb{R}$ and $\nabla S_k(u_n) = \nabla u_n \chi_{\{k\le|u_n|<k+1\}}$. Using $v = S_k(u_n)$ in (32) we obtain

$$\begin{aligned}\int_{\{k\le|u_n|<k+1\}} \frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}})|\nabla u_n|^2\,\mathrm{d}x + \int_\Omega |u_n|^{p(x)-2}u_n S_k(u_n)\,\mathrm{d}x \\ = \int_\Omega f_n S_k(u_n)\,\mathrm{d}x + \int_{\partial\Omega} g_n S_k(u_n)\,\mathrm{d}\gamma + \int_\Omega F\cdot\nabla S_k(u_n)\,\mathrm{d}x.\end{aligned}$$

Using the property (10) it yields

$$\begin{aligned}\int_{\{k\le|u_n|<k+1\}} \frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}})|\nabla u_n|^2\,\mathrm{d}x \le \int_\Omega |f|\,\mathrm{d}x + \int_{\partial\Omega}|g|\,\mathrm{d}\gamma \\ +C_1\int_\Omega |F|^{p'(x)}(1+|T_n(u_n)|)^{\theta(x)p'(x)/p(x)}\,\mathrm{d}x + \frac{\alpha}{2}\int_{\{k\le|u_n|<k+1\}}\frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\theta(x)}}\,\mathrm{d}x,\end{aligned}$$

whence

$$\begin{aligned}\int_{\{k\le|u_n|<k+1\}} \frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}})|\nabla u_n|^2\,\mathrm{d}x \le 2\int_\Omega |f|\,\mathrm{d}x + 2\int_{\partial\Omega}|g|\,\mathrm{d}\gamma \\ +C_2\int_\Omega |F|^{p'(x)}(1+|u_n|^{\theta(x)/(p(x)-1)})\,\mathrm{d}x.\end{aligned}$$

Now, since (u_n) is bounded in $L^{p(\cdot)-1}(\Omega)$, the sequence $|u_n|^{\theta(x)/(p(x)-1))}$ is bounded in $L^{s(\cdot)}(\Omega)$ where $s(x) = (p(x)-1)^2/\theta(x)$ if $\theta > 0$, $s(x) = +\infty$ if $\theta(x) = 0$. From (8) we have $s(x) > 1$ in $\bar{\Omega}$. The conjugate exponent of $s(x)$ is $s'(x) = (p(x)-1)^2/((p(x)-1)^2-\theta(x))$ and thus $s'(x)p'(x) = \tau(x)$ in $\bar{\Omega}$, where τ is defined in (9). Consequently, owing to (A2), $|F|^{p'(\cdot)} \in L^{s'(\cdot)}(\Omega)$. Then, owing to the Hölder inequality, we deduce

$$\int_\Omega |F|^{p'(x)}(1+|u_n|^{\theta(x)/(p(x)-1)})\mathrm{d}x \le C_3\||F|^{p'(\cdot)}\|_{L^{s'(\cdot)}(\Omega)}\|(1+|u_n|^{\theta(\cdot)/(p(\cdot)-1)}\|_{L^{s(\cdot)}(\Omega)} \le C_4.$$

This ends the proof of Lemma 10. □

Remark 6. *In the proof of the previous lemma, we have used a Hölder's inequality in $L^{s(\cdot)}(\Omega)$ where the exponent $s(\cdot)$ may attain the value $+\infty$. This inequality is still valid in this context (see Corollary 2.28 in [23]).*

Corollary 1. *There exists a constant $C > 0$, not depending on k and n, such that*

$$\int_{\{k\le|u_n|<k+1\}} \frac{|\nabla u_n|)^{p_-}}{(1+|T_n(u_n)|)^{\theta_+}} \le C \quad \text{for all } k,n \in \mathbb{N}. \tag{68}$$

Proof. From (11) and (67) we readily obtain

$$\int_{\{k\le|u_n|<k+1\}} \frac{|\nabla u_n|^{p(x)}}{(1+|T_n(u_n)|)^{\theta(x)}} \le C \quad \text{for all } k,n \in \mathbb{N}.$$

Since $|\nabla u_n|^{p(x)}/(1+|T_n(u_n)|)^{\theta(x)} \ge |\nabla u_n|^{p(x)}/(1+|T_n(u_n)|)^{\theta_+}$ we have $|\nabla u_n|/(1+|T_n(u_n)|)^{\theta_+/p(\cdot)} \in L^{p(\cdot)}(\Omega)$ and is bounded in $\{k \le |u_n| < k+1\}$ by the same constant C. Finally, from $|\nabla u_n|/(1+|T_n(u_n)|)^{\theta_+/p(\cdot)} \ge |\nabla u_n|/(1+|T_n(u_n)|)^{\theta_+/p_-}$, we have $|\nabla u_n|/(1+|T_n(u_n)|)^{\theta_+/p_-} \in L^{p(\cdot)}(\Omega)$ and is bounded in $\{k \le |u_n| < k+1\}$ by the same constant C. Owing to the continuous embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, we deduce the desired result. □

Lemma 11. *There exists $r > 1$ and a constant $C > 0$, not depending on n, such that*

$$\int_\Omega |\nabla u_n|^r \le C \quad \text{for all } n \ge 1. \tag{69}$$

Proof. For any $r > 0$, we can write

$$\int_\Omega |\nabla u_n|^r \,\mathrm{d}x = \int_\Omega \frac{|\nabla u_n|^r}{(1+|T_n(u_n)|)^{\theta_+ r/p_-}} (1+|T_n(u_n)|)^{\theta_+ r/p_-}$$
$$\le \sum_{k=0}^{\infty} \left(\int_{\{k\le|u_n|<k+1\}} \frac{|\nabla u_n|^{p_-}}{(1+|T_n(u_n)|)^{\theta_+}} \right)^{r/p_-} \left(\int_{\{k\le|u_n|<k+1\}} (1+|T_n(u_n)|)^{\theta_+ r/(p_- - r)} \right)^{(p_- - r)/p_-}$$

Let the numbers $a_{k,n}$ be defined as $a_{k,n} = \left(\int_{\{k\le|u_n|<k+1\}} (1+|T_n(u_n)|)^{\theta_+ r/(p_- - r)} \right)^{(p_- - r)/p_-}$. From Corollary 1 we obtain

$$\int_\Omega |\nabla u_n|^r \,\mathrm{d}x \le C_1 \sum_{k=0}^{\infty} a_{k,n}.$$

In order to estimate the right side of this last expression, we need the following result.

Lemma 12. *There exists $\delta > 0$ such that for $r = 1+\delta$ and $r_1 = (1+2\delta)/p_-$ one has*

$$\sum_{k=0}^{\infty} k^{r_1 p_-/(p_- - r)} a_{k,n}^{p_-/(p_- - r)} \le C_2, \tag{70}$$

where $C_2 > 0$ is a constant not depending on n. Moreover, we can take $\delta = \frac{(p_- - 1)^2 - \theta_+ - 1}{p_- + 1 + \theta_+}$. As a consequence, $\sum_{k=0}^{\infty} a_{k,n} \le C_3$, for some constant $C_3 > 0$ not depending on n.

Proof. According to the definition of the numbers $a_{k,n}$ we have

$$\sum_{k=0}^{\infty} k^{r_1 p_-/(p_- - r)} a_{k,n}^{p_-/(p_- - r)} = \sum_{k=0}^{\infty} k^{r_1 p_-/(p_- - r)} \int_{\{k\le|u_n|<k+1\}} (1+|T_n(u_n)|)^{\theta_+ r/(p_- - r)} \,\mathrm{d}x$$
$$\le C_1 \sum_{k=0}^{\infty} \int_{\{k\le|u_n|<k+1\}} |u_n|^{r_1 p_-/(p_- - r)} (1+|u_n|^{\theta_+ r/(p_- - r)}) \,\mathrm{d}x$$
$$\le C_1 \int_\Omega (|u_n|^{r_1 p_-/(p_- - r)} + |u_n|^{r_1 p_-/(p_- - r) + \theta_+ r/(p_- - r)}) \,\mathrm{d}x.$$

This last integral is uniformly bounded with respect to n if

$$\frac{r_1 p_-}{p_- - r} + \frac{\theta_+ r}{p_- - r} \leq p_- - 1,$$

which is equivalent to

$$r \leq \frac{(p_- - 1)p_- - r_1 p_-}{p_- - 1 + \theta_+}.$$

We search for $\delta > 0$ such that $r_1 = (1 + 2\delta)/p_-$ and $r = 1 + \delta$ fulfill the necessary requirements. Indeed, it will be enough to choose $\delta > 0$ such that

$$1 + \delta = \frac{(p_- - 1)p_- - (1 + 2\delta)}{p_- - 1 + \theta_+},$$

which yields $\delta = \frac{(p_- - 1)^2 - \theta_+ - 1}{p_- + 1 + \theta_+}$. This is a positive number because of (8). Thus, for this δ, by taking $r_1 = (1 + 2\delta)/p_-$ and $r = 1 + \delta$ we obtain (70).

Now we show the uniform bound $\sum_{k=0}^{\infty} a_{k,n} \leq C_3$. Indeed, by Hölder's inequality, we deduce

$$\sum_{k=0}^{\infty} a_{k,n} = \sum_{k=0}^{\infty} k^{r_1} a_{k,n} k^{-r_1} \leq \left(\sum_{k=0}^{\infty} k^{r_1 p_-/(p_- - r)} a_{k,n}^{p_-/(p_- - r)} \right)^{(p_- - r)/p_-} \left(\sum_{k=0}^{\infty} k^{-r_1 p_-/r} \right)^{r/p_-}$$

and therefore $\sum_{k=0}^{\infty} a_{k,n} \leq C_3$, C_3 not depending on n, because of (70) and $r_1 p_-/r = (1 + 2\delta)/(1 + \delta) > 1$. This ends the proof of Lemma 12 and then Lemma 11. □

We put $\delta_0 = \frac{(p_- - 1)^2 - \theta_+ - 1}{p_- + 1 + \theta_+}$ and define

$$r_0 = \min(1 + \delta_0, p_- - 1). \tag{71}$$

Then, $r_0 > 1$ and owing to (66) and the Lemmas 11 and 12, we obtain

$$(u_n) \text{ is bounded in } W^{1,r_0}(\Omega) \text{ and, in particular, } u \in W^{1,r_0}(\Omega), \tag{72}$$

where u is the same function appearing in the expressions (51) and (66).

Stage 5: Almost everywhere convergence of the gradients

We adapt the proof of Theorem 4.1 in [23] to the setting of variable exponent Sobolev spaces and to Neumann boundary conditions. For $\lambda \in \mathbb{R}$, $0 < \lambda < 1$ to be chosen later, we consider the quantity $J(n)$, $n \in \mathbb{N}$, defined as follows.

$$J(n) = \int_\Omega \left(\frac{(a(x, |\nabla u_n|)\nabla u_n - a(x, |\nabla u|)\nabla u) \cdot (\nabla u_n - \nabla u)}{(1 + |T_n(u_n)|)^{\theta(x)}} \right)^\lambda \mathrm{d}x.$$

Owing to the inequality of Lemma 7, $J(n)$ is well defined since the integrand is nonnegative (though it could be infinite). We write $\Omega = \{|u| > k\} \cup \{|u| \leq k\}$ so that $J(n) = J_1(k, n) + J_2(k, n)$ where

$$J_1(k, n) = \int_{\{|u| > k\}} \left(\frac{(a(x, |\nabla u_n|)\nabla u_n - a(x, |\nabla u|)\nabla u) \cdot (\nabla u_n - \nabla u)}{(1 + |T_n(u_n)|)^{\theta(x)}} \right)^\lambda \mathrm{d}x,$$

$$J_2(k, n) = \int_{\{|u| \leq k\}} \left(\frac{(a(x, |\nabla u_n|)\nabla u_n - a(x, |\nabla u|)\nabla u) \cdot (\nabla u_n - \nabla u)}{(1 + |T_n(u_n)|)^{\theta(x)}} \right)^\lambda \mathrm{d}x.$$

From (10) we readily obtain

$$J_1(k,n) \le C \int_{\{|u|>k\}} \left(1 + |\nabla u_n|^{\lambda p(x)} + |\nabla u|^{\lambda p(x)}\right) \mathrm{d}x.$$

In order to derive an estimate for this term, we will make use of the following result.

Lemma 13. *Let $A \subset \Omega$ be a measurable set. Let r_0 be the number defined in (71) and write $r_0 = 1 + \epsilon_0$. Let $\lambda = (1 + \frac{\epsilon_0}{2})\frac{1}{p_+}$ and $\tilde{r} = \frac{2+2\epsilon_0}{2+\epsilon_0}$. Then, there exists a constant $C > 0$, that does not depend on A, such that for all $w \in L^{1+\epsilon_0}(\Omega)$ one has*

$$\int_A |w|^{\lambda p(x)} \,\mathrm{d}x \le C \left(\int_A |w|^{1+\epsilon_0}\right)^{\frac{\hat{p}}{\tilde{r}p_+}} \mathrm{meas}\,(A)^{(1-\frac{1}{\tilde{r}})\frac{\hat{p}}{p_+}} \quad \textit{where } \hat{p} = \hat{p}(A,w) \in \{p_-, p_+\}.$$

Proof. In light of Lemma 2, we have

$$\int_A |w|^{\lambda p(x)} \,\mathrm{d}x \le \||w|^{\lambda}\|^{\hat{p}}_{L^{p(\cdot)}(A)} \le C\||w|^{\lambda}\|^{\hat{p}}_{L^{p_+}(A)} = C\left(\int_A |w|^{\lambda p_+} \,\mathrm{d}x\right)^{\hat{p}/p_+},$$

and applying Hölder's inequality to the functions $|w|^{\lambda p_+}$ and 1 with exponents $\tilde{r}$ and $\tilde{r}' = \tilde{r}/(\tilde{r}-1)$, respectively, we obtain the desire result. □

We use the previous result with $A = \{|u| > k\}$, first with $w = |\nabla u_n|$, then with $w = |\nabla u|$. Owing to (44), (50), (72), we readily obtain that

$$\lim_{k\to\infty} \limsup_{n\to\infty} J_1(k,n) = 0. \tag{73}$$

As long as the $J_2(k,n)$ is concerned, since $u = T_k(u)$ in the set $\{|u| \le k\}$, we have

$$J_2(k,n) = \int_{\{|u|\le k\}} \left(\frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot (\nabla u_n - \nabla T_k(u))}{(1+|T_n(u_n)|)^{\theta(x)}}\right)^{\lambda} \mathrm{d}x$$

$$\le \int_{\Omega} \left(\frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot (\nabla u_n - \nabla T_k(u))}{(1+|T_n(u_n)|)^{\theta(x)}}\right)^{\lambda} \mathrm{d}x = J_3(k,n).$$

Now we fix $j > 0$ and write $J_3(k,n) = J_4(j,k,n) + J_5(j,k,n)$ where

$$J_4(j,k,n) = \int_{A_{j,k,n}} \left(\frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot (\nabla u_n - \nabla T_k(u))}{(1+|T_n(u_n)|)^{\theta(x)}}\right)^{\lambda} \mathrm{d}x,$$

$$J_5(j,k,n) = \int_{A^{\complement}_{j,k,n}} \left(\frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot (\nabla u_n - \nabla T_k(u))}{(1+|T_n(u_n)|)^{\theta(x)}}\right)^{\lambda} \mathrm{d}x,$$

and, its turn, $A_{j,k,n} = \{|u_n - T_k(u)| > j\}$ and $A^{\complement}_{j,k,n} = \{|u_n - T_k(u)| \le j\}$.

For $J_4(j,k,n)$ we may estimate this term like it was done for $J_1(k,n)$. In this way, we would obtain an estimate like

$$J_4(j,k,n) \le C\,\mathrm{meas}\,(A_{j,k,n})^{(1-\frac{1}{\tilde{r}})\frac{\hat{p}}{p_+}}.$$

We can express $A_{j,k,n} = (A_{j,k,n} \cap B_{j,k}) \cup (A_{j,k,n} \cap B^{\complement}_{j,k})$ where $B_{j,k} = \{|u - T_k(u)| \ne j\}$ and $B^{\complement}_{j,k} = \{|u - T_k(u)| = j\}$. It is easy to check that meas $(A_{j,k,n} \cap B_{j,k}) \to$ meas $(\{|u -$

$T_k(u)| > j\}) \leq \text{meas}\,(\{|u| > k\})$ and $\text{meas}\,(A_{j,k,n} \cap B_{j,k}^{\complement}) \leq \text{meas}\,(B_{j,k}^{\complement}) = \text{meas}\,(\{|u| = k + j\}) \leq \text{meas}\,(\{|u| > k\})$. Consequently, thanks to (50), we have, for any $j > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} J_4(j,k,n) = 0. \tag{74}$$

As for $J_5(j,k,n)$ we have the identity $\nabla(u_n - T_k(u)) = \nabla T_j(u_n - T_k(u))$ in the set $A_{j,k,n}^{\complement}$. Hence,

$$J_5(j,k,n) = \int_{A_{j,k,n}^{\complement}} \left(\frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot \nabla T_j(u_n - T_k(u))}{(1 + |T_n(u_n)|)^{\theta(x)}} \right)^{\lambda} \mathrm{d}x.$$

Owing to Hölder's inequality it yields

$$J_5(j,k,n) \leq C \left(\int_{\Omega} \frac{(a(x,|\nabla u_n|)\nabla u_n - a(x,|\nabla T_k(u)|)\nabla T_k(u)) \cdot \nabla T_j(u_n - T_k(u))}{(1 + |T_n(u_n)|)^{\theta(x)}} \,\mathrm{d}x \right)^{\lambda}.$$

Consequently, $J_5(j,k,n) \leq C(J_6(j,k,n) - J_7(j,k,n))^{\lambda}$ where

$$J_6(j,k,n) = \int_{\Omega} \frac{a(x,|\nabla u_n|)\nabla u_n \cdot \nabla T_j(u_n - T_k(u))}{(1 + |T_n(u_n)|)^{\theta(x)}} \,\mathrm{d}x,$$

and

$$J_7(j,k,n) = \int_{\Omega} \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u) \cdot \nabla T_j(u_n - T_k(u))}{(1 + |T_n(u_n)|)^{\theta(x)}} \,\mathrm{d}x.$$

From (51) it is straigthforward that $\nabla T_j(u_n - T_k(u)) \to \nabla T_j(u - T_k(u))$ weakly in $L^{p(\cdot)}(\Omega)^N$ as $n \to \infty$ and also that

$$\frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1 + |T_n(u_n)|)^{\theta(x)}} \to \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u)}{(1 + |u|)^{\theta(x)}} \text{ strongly in } L^{p(\cdot)}(\Omega)^N \text{ as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} J_7(j,k,n) = \int_{\Omega} \frac{a(x,|\nabla T_k(u)|)\nabla T_k(u) \cdot \nabla T_j(u - T_k(u))}{(1 + |T_n(u)|)^{\theta(x)}} \,\mathrm{d}x = 0.$$

This last integral vanishes because $\nabla T_k(u) \cdot \nabla T_j(u - T_k(u)) = 0$ a.e. in Ω.

As long as $J_6(j,k,n)$ is concerned, we take $v = T_j(u_n - T_k(u))$ in the variational formulation of the approximate problem (32). Consequently,

$$\begin{aligned} J_6(j,k,n) = &\int_{\Omega} (f_n - |u_n|^{p(x)-2}u_n) T_j(u_n - T_k(u)) \,\mathrm{d}x + \int_{\Omega} g_n T_j(u_n - T_k(u)) \,\mathrm{d}\gamma \\ &+ \int_{\Omega} F \cdot \nabla T_j(u_n - T_k(u)) \,\mathrm{d}x. \end{aligned}$$

By dominated convergence it is easy to check that

$$\begin{aligned} \lim_{n\to\infty} &\int_{\Omega} (f_n - |u_n|^{p(x)-2}u_n) T_j(u_n - T_k(u)) \,\mathrm{d}x + \int_{\Omega} g_n T_j(u_n - T_k(u)) \,\mathrm{d}\gamma \\ &= \int_{\Omega} (f - |u|^{p(x)-2}u) T_j(u - T_k(u)) \,\mathrm{d}x + \int_{\Omega} g T_j(u - T_k(u)) \,\mathrm{d}\gamma \end{aligned}$$

and also

$$\lim_{k\to\infty} \int_{\Omega} (f - |u|^{p(x)-2}u) T_j(u - T_k(u)) \,\mathrm{d}x + \int_{\Omega} g T_j(u - T_k(u)) \,\mathrm{d}\gamma = 0.$$

It remains to study the term $\int_\Omega F\cdot\nabla T_j(u_n-T_k(u))\,\mathrm{d}x$. Firstly, $\int_\Omega F\cdot\nabla T_j(u_n-T_k(u))\,\mathrm{d}x\to\int_\Omega F\cdot\nabla T_j(u-T_k(u))\,\mathrm{d}x$ as $n\to\infty$ because, from (51), $\nabla T_j(u_n-T_k(u))\to\nabla T_j(u-T_k(u))$ weakly in $L^{p(\cdot)}(\Omega)$. Then, since

$$|\nabla T_j(u-T_k(u))|\le|\nabla u|+|\nabla T_k(u)|\le 2|\nabla u|\in L^{r_0}(\Omega)^N,$$

the sequence, $(T_j(u-T_k(u)))_{k\in\mathbb{N}}$ is bounded in $W^{1,r_0}(\Omega)^N$. Thus, up to a subsequence, $T_j(u-T_k(u))\to w_j$ weakly in $W^{1,r_0}(\Omega)^N$, strongly in $L^{r_0}(\Omega)^N$ and a.e. in Ω. But it is evident that $w_j=0$ for each $j>0$ because $T_j(u-T_k(u))\to 0$ a.e. in Ω. Consequently, $\nabla T_j(u-T_k(u))\to 0$ weakly in $L^{r_0}(\Omega)^N$ and, from the assumed regularity $F\in L^{r_0'}(\Omega)^N$ in (A2) we finally deduce that

$$\lim_{k\to\infty}\int_\Omega F\cdot\nabla T_j(u-T_k(u))\,\mathrm{d}x=0.$$

Gathering all these limit properties, we have shown that $J(n)\to 0$ as $n\to\infty$. Since the integrand defining $J(n)$ is nonnegative, we deduce that, for some subsequence, still denoted in the same way it is

$$\frac{(a(x,|\nabla u_n|)\nabla u_n-a(x,|\nabla u|)\nabla u)\cdot(\nabla u_n-\nabla u)}{(1+|T_n(u_n)|)^{\theta(x)}}\to 0\quad\text{a.e. in }\Omega.$$

Since meas $(\{|u|=+\infty\})=0$, and using (13) we readily derived the a.e. convergence of a subsequence of (∇u_n) to ∇u, that is

$$\nabla u_n\to\nabla u\quad\text{as } n\to\infty\quad\text{a.e. in }\Omega.\tag{75}$$

Stage 6: Passage to the limit

We take $\varphi\in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, $\zeta\in W^{1,\infty}_c(\mathbb{R})$, and let $M>0$ such that $\operatorname{supp}\zeta\subset[-M,M]$. Then, we put $\zeta(u_n)\varphi$ as a multiplier in (17). This yields,

$$\begin{aligned}&\int_\Omega\frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}}\nabla u_n\big(\zeta'(u_n)\nabla u_n\varphi+\zeta(u_n)\nabla\varphi\big)\,\mathrm{d}x+\int_\Omega|u_n|^{p(x)-2}u_n\zeta(u_n)\varphi\,\mathrm{d}x\\&\quad=\int_\Omega f_n(x)\zeta(u_n)\varphi\,\mathrm{d}x+\int_\Omega F\big(\zeta'(u_n)\nabla u_n\varphi+\zeta(u_n)\nabla\varphi\big)\,\mathrm{d}x+\int_{\partial\Omega}g_n\zeta(u_n)\varphi\,\mathrm{d}\gamma.\end{aligned}\tag{76}$$

We analyze each of the terms in (76). The first one is

$$\begin{aligned}&\int_\Omega\frac{a(x,|\nabla u_n|)}{(1+|T_n(u_n)|)^{\theta(x)}}\nabla u_n\big(\zeta'(u_n)\nabla u_n\varphi+\zeta(u_n)\nabla\varphi\big)\mathrm{d}x\\&\quad=\int_\Omega\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\big(\zeta'(T_M(u_n))\nabla T_M(u_n)\varphi+\zeta(T_M(u_n))\nabla\varphi\big)\mathrm{d}x\\&\quad=\int_\Omega\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\zeta'(T_M(u_n))\nabla T_M(u_n)\varphi\,\mathrm{d}x\\&\qquad+\int_\Omega\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\zeta(T_M(u_n))\nabla\varphi\,\mathrm{d}x.\end{aligned}$$

Thanks to (51) and (75), one has

$$\lim_{n\to\infty}\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)=\frac{a(x,|\nabla T_M(u)|)}{(1+|T_M(u)|)^{\theta(x)}}\nabla T_M(u)\quad\text{a.e. in }\Omega$$

and by using (10) we have $\left(\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\right)_n$ is bounded in $L^{p'(\cdot)}(\Omega)^N$. Hence, with the help of Lemma 6 we get

$$\frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n) \rightharpoonup \frac{a(x,|\nabla T_M(u)|)}{(1+|T_M(u)|)^{\theta(x)}}\nabla T_M(u)$$

weakly in $L^{p'(\cdot)}(\Omega)^N$ as $n \to \infty$. On one hand, we have $\zeta'(T_M(u_n))\nabla T_M(u_n)\varphi \to \zeta'(T_M(u))\nabla T_M(u)\varphi$ strongly in $L^{p(\cdot)}(\Omega)^N$. This yields

$$\begin{aligned}\int_\Omega \frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\zeta'(T_M(u_n))\nabla T_M(u_n)\varphi\,\mathrm{d}x \\ \to \int_\Omega \frac{a(x,|\nabla T_M(u)|)}{(1+|T_M(u)|)^{\theta(x)}}\nabla T_M(u)\zeta'(T_M(u))\nabla T_M(u)\varphi\,\mathrm{d}x \quad \text{as } n\to\infty,\end{aligned} \tag{77}$$

and considering that $\varphi\zeta(T_M(u_n))$ converges to $\varphi\zeta(T_M(u))$ strongly in $L^{p(\cdot)}(\Omega)^N$, we obtain

$$\begin{aligned}\int_\Omega \frac{a(x,|\nabla T_M(u_n)|)}{(1+|T_M(u_n)|)^{\theta(x)}}\nabla T_M(u_n)\nabla\varphi\zeta(T_M(u_n))\,\mathrm{d}x \\ \to \int_\Omega \frac{a(x,|\nabla T_M(u)|)}{(1+|T_M(u)|)^{\theta(x)}}\nabla T_M(u)\nabla\varphi\zeta(T_M(u))\,\mathrm{d}x \quad \text{as } n\to\infty,\end{aligned} \tag{78}$$

which implies that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \frac{a(x,|\nabla u_n|)}{(1+|u_n|)^{\theta(x)}}\nabla u_n\big(\zeta'(u_n)\nabla u_n\varphi+\zeta(u_n)\nabla\varphi\big)\mathrm{d}x \\ = \int_\Omega \frac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}}\nabla u\big(\zeta'(u)\nabla u\varphi+\zeta(u)\nabla\varphi\big)\mathrm{d}x.\end{aligned} \tag{79}$$

Inasmuch as F lies in $L^{p'(\cdot)}(\Omega)^N$ and that $\zeta(u_n)\nabla\varphi+\zeta'(u_n)\nabla u_n\varphi \rightharpoonup \zeta(u)\nabla\varphi+\zeta'(u)\nabla u\varphi$ weakly in $L^{p(\cdot)}(\Omega)^N$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega F\big[\zeta(u_n)\nabla\varphi+\zeta'(u_n)\nabla u_n\varphi\big]\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega F\big[\zeta(u_n)\nabla\varphi+\zeta'(u_n)\nabla T_M(u_n)\varphi\big]\,\mathrm{d}x \\ = \int_\Omega F\big[\zeta(u)\nabla\varphi+\zeta'(u)\nabla u\varphi\big]\mathrm{d}x.\end{aligned} \tag{80}$$

Having in mind (66) and thanks to the fact that $\zeta(u_n)\varphi \rightharpoonup \zeta(u)\varphi$ weakly-$*$ in $L^\infty(\Omega)$, we get

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p(x)-2}u_n\zeta(u_n)\varphi\,\mathrm{d}x = \int_\Omega |u|^{p(x)-2}u\zeta(u)\varphi\,\mathrm{d}x. \tag{81}$$

Similarly, as $\lim_{n\to\infty}\|f_n-f\|_{L^1(\Omega)}=0$, we obtain

$$\int_\Omega f_n\zeta(u_n)\varphi\,\mathrm{d}x \to \int_\Omega f\zeta(u)\varphi\,\mathrm{d}x \text{ as } n\to\infty. \tag{82}$$

Also, since $\lim_{n\to\infty}\|g_n-g\|_{L^1(\partial\Omega)}=0$ and $\lim_{n\to\infty}\zeta(u_n)\varphi=\zeta(u)\varphi$ weakly-$*$ in $L^\infty(\partial\Omega)$, it yields

$$\int_{\partial\Omega} g_n\zeta(u_n)\varphi\,\mathrm{d}\gamma \to \int_{\partial\Omega} g\zeta(u)\varphi\,\mathrm{d}\gamma \text{ as } n\to\infty. \tag{83}$$

Combining (76) and (79)–(83), we obtain, for any test function φ in the space $W^{1,p(\cdot)} \cap L^{\infty}(\Omega)$ and any $\zeta \in W_c^{1,\infty}(\mathbb{R})$,

$$\begin{aligned}\int_{\Omega} \frac{a(x,|\nabla u|)}{(1+|u|)^{\theta(x)}} \nabla u \big(\zeta(u)\nabla\varphi + \zeta'(u)\nabla u\varphi\big)\mathrm{d}x + \int_{\Omega} |u|^{p(x)-2}u\zeta(u)\varphi\,\mathrm{d}x \\ = \int_{\Omega} f\zeta(u)\varphi\,\mathrm{d}x + \int_{\Omega} F\big[\zeta(u)\nabla\varphi + \zeta'(u)\nabla u\varphi\big]\,\mathrm{d}x + \int_{\partial\Omega} g\zeta(u)\varphi\,\mathrm{d}\gamma.\end{aligned} \tag{84}$$

Now, the property (15) can be deduced from (84) by taking $\zeta = \zeta_k$, where the function ζ_k is $\zeta_k(s) = 1 - |T_{k+1}(s) - T_k(s)|$, $\varphi = \frac{T_h(u)}{h}$, and taking $k \to \infty$ then $h \to \infty$. This ends the proof of the existence theorem . □

5. A Uniqueness Result

This section is devoted to proving a uniqueness result of renormalized solution to the quasilinear problem (3) in the sense of Definition 3.

Theorem 2. *Assume the hypotheses (4) and (A1)–(A3). Then, the renormalized solution of the problem (3) is unique.*

Proof. Along this paragraph, C will denote different constants that do not depend on h, k or r.

Let u_1 and u_2 two renormalized solutions of (84). For $r > 0$, $h > 1$, $k > 2h+1$, we put $\zeta_h(s) = 1 - \frac{1}{h}|T_{2h}(s) - T_h(s)|$ so that $\operatorname{supp}\zeta_h \subset [-2h, 2h]$. We set $\varphi_1 = \zeta_h(u_1)\frac{T_r(T_k(u_1)-T_k(u_2))}{r}$ and $\varphi_2 = \zeta_h(u_2)\frac{T_r(T_k(u_1)-T_k(u_2))}{r}$ as admissible functions because $T_k(u_i) \in W^{1,p(\cdot)}(\Omega)$ for $i = 1,2$, $\zeta_h(u_1) = \zeta_h(T_{2h}(u_1)) \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$, $\zeta_h(u_2) = \zeta_h(T_{2h+1}(u_2)) \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$ and since T_r is a Lipschitz function, then $\frac{1}{r}(T_r(T_k(u_1) - T_k(u_2)) \in W^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$.

Then, by taking $\varphi = \zeta_h(u_1)\dfrac{T_r(T_k(u_1) - T_k(u_2))}{r} \in L^{\infty}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ as a multiplier in (84) we deduce

$$\begin{aligned}&\frac{1}{r}\int_{\Omega} \frac{a(x,|\nabla u_1|)\nabla u_1}{(1+|u_1|)^{\theta(x)}}\Big(\zeta_h'(u_1)\nabla u_1 T_r(T_k(u_1) - T_k(u_2)) + \zeta_h(u_1)\nabla T_r(T_k(u_1) - T_k(u_2))\Big)\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\Omega} |u_1|^{p(x)-2}u_1\zeta_h(u_1)T_r(T_k(u_1) - T_k(u_2))\mathrm{d}x \\ &= \frac{1}{r}\int_{\Omega} f(x)\zeta_h(u_1)T_r(T_k(u_1) - T_k(u_2))\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\Omega} F\Big(\zeta_h'(u_1)\nabla u_1 T_r(T_k(u_1) - T_k(u_2)) + \zeta_h(u_1)\nabla T_r(T_k(u_1) - T_k(u_2))\Big)\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\partial\Omega} g\zeta_h(u_1)T_r(T_k(u_1) - T_k(u_2))\mathrm{d}\gamma.\end{aligned} \tag{85}$$

Then, using $\varphi = \zeta_h(u_2)\dfrac{T_r(T_k(u_2) - T_k(u_1))}{r}$ as a multiplier in (84), we get

$$\begin{aligned}&\frac{1}{r}\int_{\Omega} \frac{a(x,|\nabla u_2|)\nabla u_2}{(1+|u_2|)^{\theta(x)}}\Big(\zeta_h'(u_2)\nabla u_2 T_r(T_k(u_2) - T_k(u_1)) + \zeta_h(u_2)\nabla T_r(T_k(u_2) - T_k(u_1))\Big)\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\Omega} |u_2|^{p(x)-2}u_2\zeta_h(u_2)T_r(T_k(u_2) - T_k(u_1))\mathrm{d}x \\ &= \frac{1}{r}\int_{\Omega} f(x)\zeta_h(u_2)T_r(T_k(u_2) - T_k(u_1))\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\Omega} F\Big(\zeta_h'(u_2)\nabla u_2 T_r(T_k(u_2) - T_k(u_1)) + \zeta_h(u_2)\nabla T_r(T_k(u_2) - T_k(u_1))\Big)\mathrm{d}x \\ &\qquad + \frac{1}{r}\int_{\partial\Omega} g\zeta_h(u_2)T_r(T_k(u_2) - T_k(u_1))\mathrm{d}\gamma.\end{aligned} \tag{86}$$

By adding (85) and (86), we get

$$A_{h,k,r} + B_{h,k,r} = C_{h,k,r} + D_{h,k,r} + E_{h,k,r}, \tag{87}$$

with

$$\begin{aligned} A_{h,k,r} &= \frac{1}{r}\int_\Omega \Big(\zeta_h'(u_1)\frac{a(x,|\nabla u_1|)\nabla u_1\nabla u_1}{(1+|u_1|)^{\theta(x)}} - \zeta_h'(u_2)\frac{a(x,|\nabla u_2|)\nabla u_2\nabla u_2}{(1+|u_2|)^{\theta(x)}}\Big) \\ &\quad \times T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ &\quad + \frac{1}{r}\int_\Omega \Big(\zeta_h(u_1)\frac{a(x,|\nabla u_1|)\nabla u_1}{(1+|u_1|)^{\theta(x)}} - \zeta_h(u_2)\frac{a(x,|\nabla u_2|)\nabla u_2}{(1+|u_2|)^{\theta(x)}}\Big) \\ &\quad \times \nabla T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ &= A^1_{h,k,r} + A^2_{h,k,r}, \end{aligned} \tag{88}$$

$$B_{h,k,r} = \frac{1}{r}\int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big)T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x, \tag{89}$$

$$C_{h,k,r} = \frac{1}{r}\int_\Omega f(\zeta_h(u_1) - \zeta_h(u_2))T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x, \tag{90}$$

$$\begin{aligned} D_{h,k,r} &= \frac{1}{r}\int_\Omega F(\zeta_h'(u_1)\nabla u_1 - \zeta_h'(u_2)\nabla u_2)T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ &\quad + \frac{1}{r}\int_\Omega F(\zeta_h(u_1) - \zeta_h(u_2))\nabla T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x, \end{aligned} \tag{91}$$

and

$$E_{h,k,r} = \frac{1}{r}\int_{\partial\Omega} g(\zeta_h(u_1) - \zeta_h(u_2))T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}\gamma. \tag{92}$$

Concerning the term $A^1_{h,k,r}$, one has

$$\begin{aligned} |A^1_{k,r,h}| &= \frac{1}{r}\Big|\int_\Omega (\zeta_h'(u_1)\frac{a(x,|\nabla u_1|)\nabla u_1}{(1+|u_1|)^{\theta(x)}}\cdot\nabla u_1 - \zeta_h'(u_2)\frac{a(x,|\nabla u_2|)\nabla u_2}{(1+|u_2|)^{\theta(x)}}\cdot\nabla u_2 T_r(T_k(u_1) - T_k(u_2))\Big| \\ &= \Big|\int_{\{|u_1|<2h\}} \zeta_h'(T_{2h}(u_1))\frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}}\cdot\nabla T_{2h}(u_1) \\ &\qquad \times\frac{T_r(T_k(T_{2h}(u_1) - T_k(T_{2h+1}(u_2)))}{r} \\ &\quad - \int_{\{|u_2|<2h+1\}} \zeta_h'(T_{2h+1}(u_2))\frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\cdot\nabla T_{2h+1}(u_2) \\ &\qquad \times\frac{T_r(T_k(T_{2h}(u_1) - T_k(T_{2h+1}(u_2)))}{r}\Big| \end{aligned} \tag{93}$$

Since

$$\zeta_h'(T_{2h}(u_1))\frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}}\cdot\nabla T_{2h}(u_1) \in L^1(\Omega)$$

and

$$\zeta_h'(T_{2h+1}(u_2))\frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\cdot\nabla T_{2h+1}(u_2) \in L^1(\Omega)$$

Also, we have

$$T_r(T_k(u_1) - T_k(u_2) \longrightarrow T_r(u_1 - u_2) \text{ a.e. in } \Omega \text{ as } k \to \infty$$

Then, by the dominated convergence theorem, we have

$$\lim_{k\to\infty} A^1_{h,k,r} = \Big| \int_{\{|u_1|<2h\}} \zeta_h'(T_{2h}(u_1)) \frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}} \cdot \nabla T_{2h}(u_1) \frac{T_r(T_{2h}(u_1)-T_{2h+1}(u_2))}{r}$$

$$- \int_{\{|u_2|<2h+1\}} \zeta_h'(T_{2h+1}(u_2)) \frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}} \cdot \nabla T_{2h+1}(u_2) \frac{T_r(T_{2h}(u_1)-T_{2h+1}(u_2))}{r} \Big|$$

Owing to

$$\lim_{r\to\infty} \frac{T_r(u_1-u_2)}{r} = \operatorname{sign}(u_1-u_2) \text{ a.e. in } \Omega,$$

and, in view of the dominated convergence theorem, we get

$$\lim_{r\to 0}\lim_{k\to\infty} |A^1_{k,r,h}| = \Big| \int_{\{|u_1|<2h\}} \zeta_h'(T_{2h}(u_1)) \frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}} \cdot \nabla T_{2h}(u_1) \operatorname{sign}(u_1-u_2)$$

$$- \int_{\{|u_2|<2h+1\}} \zeta_h'(T_{2h+1}(u_2)) \frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}} \cdot \nabla T_{2h+1}(u_2) \operatorname{sign}(u_1-u_2) \Big|$$

Now, we have $|\zeta_h'(T_{2h}(u_1))| \le \frac{1}{h}$ and $|\zeta_h'(T_{2h+1}(u_2))| < \frac{1}{h}$, then

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} |A^1_{k,r,h}| \le \lim_{h\to\infty} \frac{1}{h} \int_{\{|u_1|<2h\}} \frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}} \cdot \nabla T_{2h}(u_1)\,dx$$
$$+ \lim_{h\to\infty} \int_{\{|u_2|<2h+1\}} \frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}} \cdot \nabla T_{2h+1}(u_2)\,dx$$

Consequently, thanks to (15), we have

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} |A^1_{k,r,h}| = 0. \tag{94}$$

Let $H_{k,r,h} = \{0 < |T_k(u_1) - T_k(u_2)| \le r, |u_1| \le 2h, |u_2| \le 2h+1\}$ and $k > 2h+1$, then $T_k(T_{2h+1}) = T_{2h+1}$.

$$\begin{aligned}
A^2_{k,r,h} &= \frac{1}{r}\int_\Omega \Big(\zeta_h(u_1)\frac{a(x,|\nabla u_1|)\nabla u_1}{(1+|u_1|)^{\theta(x)}} - \zeta_h(u_2)\frac{a(x,|\nabla u_2|)\nabla u_2}{(1+|u_2|)^{\theta(x)}}\Big)\cdot \nabla T_r(T_k(u_1)-T_k(u_2))\,dx \\
&= \frac{1}{r}\int_{H_{k,r,h}} \Big(\zeta_h(T_{2h}(u_1))\frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}} \\
&\quad -\zeta_h(T_{2h+1}(u_2))\frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\Big) \\
&\quad \cdot\nabla T_r(T_k(T_{2h}(u_1)) - T_k(T_{2h+1}(u_2)))\,dx \\
&= \frac{1}{r}\int_{H_{k,r,h}} \zeta_h(T_{2h}(u_1))\Big(\frac{a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1)}{(1+|T_{2h}(u_1)|)^{\theta(x)}} - \frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\Big) \\
&\quad \cdot(\nabla T_k(T_{2h}(u_1))) - \nabla T_k(T_{2h+1}(u_2)))\,dx \\
&\quad +\frac{1}{r}\int_{H_{k,r,h}} (\zeta_h(T_{2h}(u_1)) - \zeta_h(T_{2h+1}(u_2)))\frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}} \\
&\quad \cdot(\nabla T_k(T_{2h}(u_1))) - \nabla T_k(T_{2h+1}(u_2)))\,dx \\
&\quad +\frac{1}{r}\int_{H_{k,r,h}} \zeta_h(T_{2h+1}(u_2))a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2) \\
&\quad \times\Big(\frac{1}{(1+|T_{2h}(u_1)|)^{\theta(x)}} - \frac{1}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\Big)\cdot(\nabla T_k(T_{2h}(u_1))) - \nabla T_k(T_{2h+1}(u_2)))\,dx \\
&= A^{2,1}_{k,r,h} + A^{2,2}_{k,r,h} + A^{2,3}_{k,r,h}
\end{aligned} \tag{95}$$

For $A^{2,1}_{k,r,h}$, since $k > 2h+1$, we have

$$A_{k,r,h}^{2,1} = \frac{1}{r}\int_{H_{k,r,h}} \frac{\zeta(T_{2h}(u_1))}{(1+|T_{2h}(u_1)|)^{\theta(x)}}(a(x,|\nabla T_{2h}(u_1)|)\nabla T_{2h}(u_1) - a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2))$$
$$\cdot(\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2))\,\mathrm{d}x$$

By the strict monotonicity of $a(x,|\xi|)\xi$, we obtain

$$A_{k,r,h}^{2,1} > 0 \tag{96}$$

For the term $A_{k,r,h}^{2,2}$,

$$A_{k,r,h}^{2,2} = \frac{1}{r}\int_{H_{k,r,h}} (\zeta_h(T_{2h}(u_1)) - \zeta_h(T_{2h+1}(u_2)))\frac{a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)}{(1+|T_{2h}(u_1)|)^{\theta(x)}}$$
$$\cdot(\nabla T_k(T_{2h}(u_1))) - \nabla T_k(T_{2h+1}(u_2)))\,\mathrm{d}x$$

In view of the Lipschitz continuity of ζ_h and the fact that $T_k(T_{2h+1}) = T_{2h+1}$, we get

$$|A_{k,r,h}^{2,2}| \le \frac{1}{h}\int_{\{|T_{2h}(u_1)-T_{2h+1}(u_2)|<r,|u_1|<2h,|u_2|<2h+1\}} \frac{a(x,|\nabla T_{2h+1}(u_2)|).|\nabla T_{2h+1}(u_2)|}{(1+|T_{2h}(u_1)|)^{\theta(x)}}$$
$$\times|\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2)|\,\mathrm{d}x$$

From (10) and Young's inequality we deduce

$$\begin{aligned}
&\frac{1}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|)||\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2)|\,\mathrm{d}x\\
&\le \frac{1}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} (|l(x)| + \beta|\nabla T_{2h+1}(u_2)|^{p(x)-1})|(|\nabla T_{2h}(u_1)| + |\nabla T_{2h+1}(u_2)|)|\,\mathrm{d}x\\
&\le \frac{1}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |l(x)|(|\nabla T_{2h}(u_1)| + |\nabla T_{2h+1}(u_2)|)\,\mathrm{d}x\\
&\quad + \frac{\beta}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |\nabla T_{2h+1}(u_2)|^{p(x)-1}(|\nabla T_{2h}(u_1)| + |\nabla T_{2h+1}(u_2)|)\,\mathrm{d}x,\\
&\le \frac{C}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |l(x)|^{p'(x)}\,\mathrm{d}x + \frac{C}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |\nabla T_{2h}(u_1)|^{p(x)}\,\mathrm{d}x\\
&\quad + \frac{C}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |\nabla T_{2h+1}(u_2)|^{p(x)}\,\mathrm{d}x.
\end{aligned} \tag{97}$$

In view of $l \in L^{p'(\cdot)}(\Omega)$, $|\nabla T_{2h}(u_1)|^{p(\cdot)}$, $|\nabla T_{2h+1}(u_2)|^{p(\cdot)} \in L^1(\Omega)$ this last estimate implies that, for $0 < r < r_0$,

$$\Big(|a(x,|\nabla T_{2h+1}(u_2)|)||\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2)|\Big)\chi_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} \in L^1(\Omega).$$

Moreover, we have

$$\Big(|a(x,|\nabla T_{2h+1}(u_2)|)||\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2)|\Big)$$
$$\times\chi_{\{0<|u_1-u_2|\le r_0,|u_1|\le 2h,|u_2|\le 2h+1\}} \to 0 \text{ a.e. in } \Omega,$$

as $r \to 0$, then by applying Lebesgue's dominated convergence theorem, it yields

$$\lim_{r\to 0}\frac{1}{h}\int_{\{0<|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|)||\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1) - \nabla T_{2h+1}(u_2)|\,\mathrm{d}x = 0.$$

Then, we get

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty}\left|A^{2,2}_{k,r,h}\right| = 0. \tag{98}$$

For the term $A^{2,3}_{k,r,h}$,

$$\begin{aligned}|A^{2,3}_{k,r,h}| &= \Big|\frac{1}{r}\int_{\{|T_{2h}(u_1)-T_{2h+1}(u_2)|<r,|u_1|\le 2h,|u_2|\le 2h+1\}} \zeta_h(T_{2h+1}(u_2))a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)\\ &\quad\cdot\Big(\frac{1}{(1+|T_{2h}(u_1)|)^{\theta(x)}}-\frac{1}{(1+|T_{2h+1}(u_2)|)^{\theta(x)}}\Big)(\nabla T_k(T_{2h}(u_1))-\nabla T_k(T_{2h+1}(u_2)))\,\mathrm{d}x\Big|.\end{aligned}$$

Since the continuous function $t\mapsto \dfrac{1}{(1+|t|)^{\theta(x)}}$ is a globally Lipschitz function, then for some constant $L>0$, which does not depend on $x\in\bar{\Omega}$, we have

$$\left|\frac{1}{(1+|s_1|)^{\theta(x)}}-\frac{1}{(1+|s_2|)^{\theta(x)}}\right| \le L|s_1-s_2| \text{ for all } s_1,s_2\in\mathbb{R}.$$

Thus, for $k>2h+1$, we can write

$$\begin{aligned}|A^{2,3}_{k,r,h}| &\le \frac{L}{r}\int_{\{|T_{2h}(u_1)-T_{2h+1}(u_2)|<r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|).\nabla T_{2h+1}(u_2)|\\ &\quad\times|T_{2h}(u_1)-T_{2h+1}(u_2)||\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2)|\,\mathrm{d}x\\ &\le L\int_{\{|T_{2h}(u_1)-T_{2h+1}(u_2)|<r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|).\nabla T_{2h+1}(u_2)|\\ &\quad\times|\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2)|\,\mathrm{d}x.\end{aligned}$$

By using Young's inequality we get

$$\begin{aligned}&\int_{\{|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1))-\nabla T_{2h+1}(u_2)|\,\mathrm{d}x\\ &\le C\int_{\{|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} ||a(x,|\nabla T_{2h+1}(u_2)|)|\nabla T_{2h+1}(u_2)|(|\nabla T_{2h}(u_1)|+|\nabla T_{2h+1}(u_2)|)\,\mathrm{d}x\\ &\le C\int_{\{|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)|^{p'(x)}\,\mathrm{d}x\\ &\quad +C\int_{\{|u_1-u_2|\le r,|u_1|\le 2h,|u_2|\le 2h+1\}} (|\nabla T_{2h}(u_1)|^{p(x)}+|\nabla T_{2h+1}(u_2)|^{p(x)})\,\mathrm{d}x.\end{aligned}$$

Since $|a(x,|\nabla T_{2h+1}(u_2)|)\nabla T_{2h+1}(u_2)|\in L^{p'(\cdot)}(\Omega)^N$ and $|\nabla T_{2h}(u_1)|,|\nabla T_{2h+1}(u_2)|\in L^{p(\cdot)}(\Omega)$, we get

$$\big(|a(x,|\nabla T_{2h+1}(u_2)|)|\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2)|\big)\chi_{\{0<|u_1-u_2|\le r_0,|u_1|\le 2h,|u_2|\le 2h+1\}}\in L^1(\Omega).$$

Thanks to $|a(x,|\nabla T_{2h+1}(u_2)||\nabla T_{2h+1}(u_2)||\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2)|)\chi_{\{0<|u_1-u_2|\le r,\ |u_1|\le 2h,|u_2|\le 2h+1\}}\to 0$ a.e. in Ω, as $r\to 0$.

By dominated convergence we then deduce that

$$\lim_{r\to 0}\int_{\{0<|u_1-u_2|\le r_0,|u_1|\le 2h,|u_2|\le 2h+1\}} |a(x,|\nabla T_{2h+1}(u_2)|)|\nabla T_{2h+1}(u_2)|(|\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2)|\,\mathrm{d}x = 0. \tag{99}$$

Consequently, we have

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} A^{2,3}_{k,r,h} = 0. \tag{100}$$

By combining (95) and (96)–(100), we conclude

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} A^2_{h,k,r} = 0. \tag{101}$$

For the term $B_{k,r,h}$, one has $|u_1|^{p(\cdot)-2}u_1$ and $|u_2|^{p(\cdot)-2}u_2$ belong to $L^1(\Omega)$, and since $T_r(T_k(u_1) - T_k(u_2)) \to T_r(u_1 - u_2)$ as $k \to \infty$, then by the dominated convergence theorem, we get that

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{r}\int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big) T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ = \frac{1}{r}\int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big) T_r(u_1 - u_2)\,\mathrm{d}x.\end{aligned}$$

Thanks to the fact that $\dfrac{T_r(u_1-u_2)}{r} \to \operatorname{sign}(u_1 - u_2)$ a.e.in Ω as $r \to 0$, Lebesgue's dominated convergence leads to

$$\begin{aligned}\lim_{r\to 0}\lim_{k\to\infty}\frac{1}{r}\int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big) T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ = \int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big)\operatorname{sign}(u_1 - u_2)\,\mathrm{d}x,\end{aligned} \tag{102}$$

and using again dominated convergence as $h \to \infty$, we obtain

$$\begin{aligned}\int_\Omega \Big(|u_1|^{p(x)-2}u_1\zeta_h(u_1) - |u_2|^{p(x)-2}u_2\zeta_h(u_2)\Big)\operatorname{sign}(u_1 - u_2)\,\mathrm{d}x \\ \to \int_\Omega \Big(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2\Big)\operatorname{sign}(u_1 - u_2)\,\mathrm{d}x \geq 0.\end{aligned} \tag{103}$$

For the term $C_{k,r,h}$, we have $f \in L^1(\Omega)$, then dominated convergence yields

$$\begin{aligned}\lim_{r\to 0}\lim_{k\to\infty}\frac{1}{r}\int_\Omega f(\zeta_h(u_1) - \zeta_h(u_2)) T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ = \int_\Omega f(\zeta_h(u_1) - \zeta_h(u_2))\operatorname{sign}(u_1 - u_2)\,\mathrm{d}x.\end{aligned}$$

Since $\zeta_h(u_1) \to 1$ (respectively, $\zeta_h(u_2) \to 1$) weak-$*$ in $L^\infty(\Omega)$ and $f \in L^1(\Omega)$, once again by dominated convergence, we get

$$\int_\Omega f(\zeta_h(u_1) - \zeta_h(u_2))\operatorname{sign}(u_1 - u_2)\,\mathrm{d}x \to 0, \quad \text{as } h \to \infty.$$

Therefore,

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} C_{k,r,h} = 0. \tag{104}$$

For the first term of $D_{k,r,h}$, we have

$$\begin{aligned}&\Big|\frac{1}{r}\int_\Omega F\cdot(\zeta_h'(u_1)\nabla u_1 - \zeta_h'(u_2)\nabla u_2) T_r(T_k(u_1) - T_k(u_2))\Big| \\ &= \Big|\frac{1}{r}\int_{\{|u_1|\leq 2h\}} F\cdot\zeta_h'(T_{2h}(u_1))\nabla T_{2h}(u_1) T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x \\ &-\frac{1}{r}\int_{\{|u_2|\leq 2h+1\}} F\cdot\zeta_h'(T_{2h+1}(u_2))\nabla T_{2h+1}(u_2) T_r(T_k(u_1) - T_k(u_2))\,\mathrm{d}x\Big|.\end{aligned}$$

Since $\frac{1}{r}F \cdot \zeta_h'(T_{2h}(u_1))\nabla T_{2h}(u_1)T_r(T_k(u_1) - T_k(u_2)) \in L^1(\Omega)$ and $\frac{1}{r}F \cdot \zeta_h'(T_{2h+1}(u_2))$ $\nabla T_{2h+1}(u_2)T_r(T_k(u_1) - T_k(u_2)) \in L^1(\Omega)$. Also, we have

$$T_r(T_k(u_1) - T_k(u_2)) \to T_r(u_1 - u_2) \text{ a.e. in } \Omega \text{ as } k \to \infty.$$

Then, it follows that

$$\begin{aligned}
&\lim_{k\to\infty} \Big| \frac{1}{r} \int_\Omega F \cdot (\zeta_h'(u_1)\nabla u_1 - \zeta_h'(u_2)\nabla u_2) T_r(T_k(u_1) - T_k(u_2)) \Big| \\
&= \Big| \frac{1}{r} \int_{\{|u_1| \le 2h\}} F \cdot \zeta_h'(T_{2h}(u_1))\nabla T_{2h}(u_1) T_r(u_1 - u_2)\, \mathrm{d}x \\
&- \frac{1}{r} \int_{\{|u_2| \le 2h+1\}} F \cdot \zeta_h'(T_{2h+1}(u_2))\nabla T_{2h+1}(u_2) T_r(u_1 - u_2)\, \mathrm{d}x \Big|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\lim_{r\to 0}\lim_{k\to\infty} \Big| \frac{1}{r} \int_\Omega F \cdot (\zeta_h'(u_1)\nabla u_1 - \zeta_h'(u_2)\nabla u_2) T_r(T_k(u_1) - T_k(u_2)) \Big| \\
&= \lim_{r\to 0} \Big| \frac{1}{r} \int_{\{|u_1| \le 2h\}} F \cdot \zeta_h'(T_{2h}(u_1))\nabla T_{2h}(u_1) T_r(u_1 - u_2)\, \mathrm{d}x \\
&- \frac{1}{r} \int_{\{|u_2| \le 2h+1\}} F \cdot \zeta_h'(T_{2h+1}(u_2))\nabla T_{2h+1}(u_2) T_r(u_1 - u_2)\, \mathrm{d}x \Big| \\
&= \Big| \int_{\{|u_1| \le 2h\}} F \cdot \zeta_h'(T_{2h}(u_1))\nabla T_{2h}(u_1) \operatorname{sgn}(u_1 - u_2)\, \mathrm{d}x \\
&- \int_{\{|u_2| \le 2h+1\}} F \cdot \zeta_h'(T_{2h+1}(u_2))\nabla T_{2h+1}(u_2) \operatorname{sgn}(u_1 - u_2)\, \mathrm{d}x \Big| \\
&= \frac{1}{h} \int_{\{|u_1| \le 2h\}} |F||\nabla T_{2h}(u_1)|\, \mathrm{d}x + \frac{1}{h} \int_{\{|u_2| \le 2h+1\}} |F||\nabla T_{2h+1}(u_2)|\, \mathrm{d}x.
\end{aligned} \tag{105}$$

Now, we investigate the right-hand side of (105). For the first term, we use Young's inequality and (11). This yields

$$\begin{aligned}
&\frac{1}{h} \int_{\{|u_1| \le 2h\}} |F||\nabla T_{2h}(u_1)|\, \mathrm{d}x \\
&\le \frac{C}{h} \int_{\{|u_1| \le 2h\}} |F|^{p'(x)} (1 + |T_{2h}(u_1)|)^{\frac{\theta(x)}{p(x)-1}}\, \mathrm{d}x + \frac{C}{h} \int_{\{|u_1| \le 2h\}} \frac{|\nabla u_1|^{p(x)}}{(1 + T_{2h}(u_1))^{\theta(x)}}\, \mathrm{d}x \\
&\le C \frac{(1+2h)^{\frac{\theta_+}{p_- - 1}}}{h} \int_\Omega |F|^{p'(x)}\, \mathrm{d}x + \frac{C}{h} \int_{|u_1| \le 2h} \frac{a(x, |\nabla u_1|)\nabla u_1}{(1 + |T_{2h}(u_1)|)^{\theta(x)}}\, \mathrm{d}x.
\end{aligned}$$

Since $F \in L^{p'(\cdot)}(\Omega)^N$, $\theta_+ < p_- - 1$ and owing to (15) we deduce

$$\lim_{h\to\infty} \frac{1}{h} \int_{\{|u_1| \le 2h\}} |F||\nabla T_{2h}(u_1)|\, \mathrm{d}x = 0. \tag{106}$$

By proceeding with the same arguments as in (106), we obtain that

$$\lim_{h\to\infty} \frac{1}{h} \int_{\{|u_2| \le 2h+1\}} |F||\nabla T_{2h+1}(u_2)|\, \mathrm{d}x = 0. \tag{107}$$

Combining (106) and (107) in (105), we conclude that

$$\lim_{h\to\infty}\lim_{r\to 0}\lim_{k\to\infty} \left| \frac{1}{r} \int_\Omega F \cdot (\zeta_h'(u_1)\nabla u_1 - \zeta_h'(u_2)\nabla u_2) T_r(T_k(u_1) - T_k(u_2))\, \mathrm{d}x \right| = 0. \tag{108}$$

For the second term of $D_{k,r,h}$, similarly as in (101), we get

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{r}\int_\Omega F\cdot(\zeta_h(u_1)-\zeta_h(u_2))\nabla T_r(T_k(u_1)-T_k(u_2))\,\mathrm{d}x\\ =\lim_{k\to\infty}\frac{1}{r}\int_{H_{r,h,k}} F\cdot(\zeta_h(u_1)-\zeta_h(u_2))(\nabla T_k(T_{2h}(u_1))-\nabla T_k(T_{2h+1}(u_2)))\,\mathrm{d}x,\\ =\frac{1}{r}\int_{H_{r,h}} F\cdot(\zeta_h(u_1)-\zeta_h(u_2))(\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2))\,\mathrm{d}x.\end{aligned}\tag{109}$$

Recalling that ζ_h is a Lipschitz continuous function with $\operatorname{supp}\zeta_h\subset[-2h,2h]$, we deduce that

$$\begin{aligned}\left|\frac{1}{r}\int_{H_{r,h}} F\cdot(\zeta_h(u_1)-\zeta_h(u_2))(\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2))\,\mathrm{d}x\right|\\ \le\frac{1}{h}\int_{H_{r,h}}|F||\nabla T_{2h}(u_1)|\,\mathrm{d}x+\frac{1}{h}\int_{H_{r,h}}|F||\nabla T_{2h+1}(u_2)|\,\mathrm{d}x.\end{aligned}\tag{110}$$

In the light of (106) and (107), we obtain

$$\lim_{h\to\infty}\left|\frac{1}{r}\int_\Omega F\cdot(\zeta_h(u_1)-\zeta_h(u_2))(\nabla T_{2h}(u_1)-\nabla T_{2h+1}(u_2))\,\mathrm{d}x\right|=0.\tag{111}$$

From (108) and (111), we deduce that

$$\lim_{h\to\infty}\lim_{r\to0}\lim_{k\to\infty}|D_{r,h,k}|=0.\tag{112}$$

Considering the term $E_{k,r,h}$, we have $g\in L^1(\partial\Omega)$. Thanks to the Lipschitz continuity of $\zeta(.)$ and to the continuity of the trace function, then $|g||\zeta_h(u_1)-\zeta_h(u_2)|\in L^1(\partial\Omega)$ and we have also

$$T_r(T_k(u_1)-T_k(u_2))\to T_r(u_1-u_2)\text{ a.e. in }\partial\Omega\text{ as }k\to\infty.$$

By using the dominated convergence theorem, we have

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{r}\int_{\partial\Omega} g(\zeta_h(u_1)-\zeta_h(u_2))T_r(T_k(u_1)-T_k(u_2))\,\mathrm{d}\gamma\\ =\frac{1}{r}\int_{\partial\Omega} g(\zeta_h(u_1)-\zeta_h(u_2))T_r(u_1-u_2)\,\mathrm{d}\gamma.\end{aligned}$$

Then,

$$\begin{aligned}\lim_{r\to0}\lim_{k\to\infty}\frac{1}{r}\int_{\partial\Omega} g(\zeta_h(u_1)-\zeta_h(u_2))T_r(T_k(u_1)-T_k(u_2))\,\mathrm{d}\gamma\\ =\int_{\partial\Omega} g(\zeta_h(u_1)-\zeta_h(u_2))\operatorname{sgn}(u_1-u_2)\,\mathrm{d}\gamma.\end{aligned}$$

Since $\zeta_h(u_1)\rightharpoonup 1$ (respectively, $\zeta_h(u_2)\to 1$) weak-$*$ in $L^\infty(\partial\Omega)$ and $g\in L^1(\partial\Omega)$, we get

$$\lim_{h\to\infty}\int_{\partial\Omega} g(\zeta_h(u_1)-\zeta_h(u_2))\operatorname{sgn}(u_1-u_2)\,\mathrm{d}\sigma=0,\tag{113}$$

one can conclude that

$$\lim_{h\to\infty}\lim_{r\to0}\lim_{k\to\infty}E_{k,r,h}=0.\tag{114}$$

Gathering (94), (101), (103), (104), (112) and (114), we deduce

$$\int_\Omega\left(|u_1|^{p(x)-2}u_1-|u_2|^{p(x)-2}u_2\right)\operatorname{sign}(u_1-u_2)\,\mathrm{d}x\le0,\tag{115}$$

Having in mind (103), we get

$$\int_\Omega \left(|u_1|^{p(x)-2}u_1 - |u_2|^{p(x)-2}u_2 \right) \operatorname{sign}(u_1 - u_2)\, \mathrm{d}x = 0. \tag{116}$$

Therefore, since for almost all $x \in \Omega$ the function $t \mapsto |t|^{p(x)-2}t$ is strictly monotone, then (116) implies that

$$u_1 = u_2 \quad \text{a.e. in } \Omega, \tag{117}$$

Furthermore, in view of Proposition 4, we also deduce

$$u_1 = u_2 \quad \text{a.e. on } \partial\Omega. \tag{118}$$

This completes the proof of Theorem 2. □

6. Conclusions

In this work, we have studied the existence and the uniqueness of a renormalized solution for the nonuniformly coercive Neumann problem (3) in the context of Sobolev spaces with variable exponent such that with $2 < p_- \leq p(x) \leq p_+ < \infty$. The exponent describing the nonuniform elliptic character is such that $0 \leq \theta(x) \leq \theta_+ < p_- - 1$ and $1 + \theta_+ < (p_- - 1)^2$. The data of this problem belongs to $L^1(\Omega)$ (right hand side of the partial differential equation) and to $L^1(\partial\Omega)$ (right hand side of the Neumann boundary condition).

An interesting situation correspond with the case in which the partial differential equation of (3) do not contains the zero order term $|u|^{p(\cdot)-2}u$. In this case, the existence/uniqueness of renormalized solutions is an open problem. The analysis described in this presentation has not considered the case $1 < p_- \leq 2$ even when $\theta = 0$ because in this case we have not found the good estimates to pass to the limit. Also, it is remarkable the extra regularity assumed on the function F since its natural space $L^{p'(\cdot)}(\Omega)$ is not enough in our approach.

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