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Lower Bounds on Expressions Depending on the Functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$: Part 4

Stoyan Dimitrov [1,*] **and Meglena Lazarova** [2,3]

[1] Department of Mathematical Analysis and Differential Equations, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Blvd. St. Kliment Ohridski 8, 1000 Sofia, Bulgaria
[2] Department of Mathematical Modelling and Numerical Methods, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, 8 "Kliment Ohridski" Blvd., 1000 Sofia, Bulgaria; meglena.laz@tu-sofia.bg
[3] Department of Mathematics, Informatics and Physics, Prof. Dr. Assen Zlatarov State University of Burgas, 1 Prof. Yakimov Blvd., 8000 Burgas, Bulgaria
* Correspondence: sdimitrov@tu-sofia.bg or xyzstoyan@gmail.com

Abstract

In this article, we establish four lower-bound inequalities for rational expressions involving the Euler totient function $\varphi(n)$, the Dedekind psi function $\psi(n)$, and the sum-of-divisors function $\sigma(n)$. The inequalities concern rational expressions involving sums of cubes, products of these arithmetic functions, and sums of pairs of these functions. The lower bounds are given explicitly as rational functions of n and are attained when n is prime. Furthermore, we formulate four conjectures that extend the established inequalities to arbitrary positive integer powers k.

Keywords: arithmetic functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$; lower bounds

MSC: 11A25

1. Introduction

Throughout this paper, the letter p always denotes a prime number. Let

$$n = \prod_{i=1}^{k} p_i^{a_i}, \qquad n \geq 2,$$

be the prime factorization of n. We denote by $\Omega(n)$ the total number of prime divisors of n, counted according to their multiplicities. Thus

$$\Omega(n) = \sum_{i=1}^{k} a_i,$$

with the convention $\Omega(1) = 0$. The Euler totient function $\varphi(n)$ is defined as the number of positive integers not exceeding n that are coprime to n. Its well-known product representation is

$$\varphi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1),$$

and $\varphi(1) = 1$. Another arithmetic function of interest is the Dedekind psi function, given by

$$\psi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i + 1),$$

together with the normalization $\psi(1) = 1$. We shall also make use of the divisor-sum function $\sigma(n)$, defined by

$$\sigma(n) = \sum_{d|n} d.$$

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Using the prime factorization of n, it can be expressed as

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

while $\sigma(1) = 1$.

The study of lower estimates involving the arithmetic functions $\varphi(n)$, $\psi(n)$, and $\sigma(n)$ was initiated by a result of Atanassov [1], who proved that for every integer $n \geq 2$,

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1.$$

This theorem stimulated further research on inequalities containing these classical arithmetic functions. Among the first developments in this direction are the estimates, valid for $n \geq 2$,

$$\varphi^k(n) + \psi^k(n) + \sigma^k(n) \geq (n-1)^k + 2(n+1)^k,$$

which were established for $k = 2, 3, 4$ in [2,3]. Subsequently, Sándor and Gryszka [4] extended the validity of this inequality to all integers $k \geq 2$. A second family of inequalities, valid for $n \geq 2$, concerns pairwise products of the functions. Specifically,

$$\varphi^k(n)\psi^k(n) + \varphi^k(n)\sigma^k(n) + \sigma^k(n)\psi^k(n) \geq 2(n^2-1)^k + (n+1)^{2k}$$

was proved for $k = 1, 2$ in [2,3], while the general case $k \geq 2$ was later obtained in [4]. Another related result states that, for every $n \geq 2$,

$$\begin{aligned}&\varphi^k(n)\big(\psi(n) + \sigma(n)\big) + \psi^k(n)\big(\varphi(n) + \sigma(n)\big) + \sigma^k(n)\big(\varphi(n) + \psi(n)\big)\\ &\geq 2(n+1)[(n-1)^k + 2n(n+1)^{k-1}].\end{aligned}$$

This inequality was established for $k = 2, 3$ in [3] and later generalized to all $k \geq 2$ by Sándor and Gryszka [4]. An extensive survey of the known results and related developments may be found in [5]. Very recently, Dimitrov [6] established lower bounds for rational expressions involving the functions $\varphi(n)$, $\psi(n)$, and $\sigma(n)$. Motivated by Atanassov's result [1] and using the method of mathematical induction, we establish lower bounds for rational expressions involving the functions $\varphi(n)$, $\psi(n)$, and $\sigma(n)$. The main difficulties arise in the proofs of Theorems 3 and 4, where, for the first time, we need to employ analytical methods.

2. Auxiliary Lemmas

We recall the following lemma, which will be used in the proofs of Theorem 1 and Theorem 2.

Lemma 1 ([7], Proposition 5)**.** *For $n \geq 2$, we have*

$$\frac{6}{\pi^2} < \frac{\varphi(n)\,\psi(n)}{n^2} < 1.$$

3. Main Results

Theorem 1. *For every integer $n \geq 2$, we have*

$$\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} \geq \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1}. \tag{1}$$

Proof. Consider several cases.

Case 1. $\Omega(n) = 1$. Bearing in mind that n is a prime number, we write

$$\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} = \frac{(n-1)^3 + 2(n+1)^3}{(n-1)(n+1)^2} = \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1}.$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. We have

$$\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} = \frac{(p-1)^3(q-1)^3 + 2(p+1)^3(q+1)^3}{(p-1)(q-1)(p+1)^2(q+1)^2} > \frac{3p^3q^3 + 3p^2q^2 + 9pq + 1}{p^3q^3 + p^2q^2 - pq - 1} = \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1},$$

where the inequality is equivalent to the following expression being positive:

$$4(p^2q + pq^2 - pq - 1)\big(3p^4q^3 + p^4q^2 + 3p^3q^4 + 5p^3q^3 + 7p^3q^2 + p^3q + p^2q^4 + 7p^2q^3 + 9p^2q^2 + 5p^2q + 2p^2 + pq^3 + 5pq^2 + 9pq + p + 2q^2 + q + 1\big) > 0\,.$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} = \frac{p^3(p-1)^3 + p^3(p+1)^3 + (p^2+p+1)^3}{p^2(p^2-1)(p^2+p+1)} > \frac{3p^6 + 3p^4 + 9p^2 + 1}{p^6 + p^4 - p^2 - 1} = \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1},$$

where the inequality follows from the positivity of the polynomial

$$12p^8 + 10p^7 + 21p^6 + 11p^5 + 15p^4 + 12p^3 + 7p^2 + 3p + 1 > 0\,.$$

Now we assume that (1) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using that

$$\varphi(n) \leq n - 1 < n + 1 \leq \psi(n) \leq \sigma(n) \tag{2}$$

together with straightforward calculations, we obtain

$$\begin{aligned}
\frac{\varphi^3(np) + \psi^3(np) + \sigma^3(np)}{\varphi(np)\psi(np)\sigma(np)} &= \frac{(p-1)^3\varphi^3(n) + (p+1)^3\psi^3(n) + (p+1)^3\sigma^3(n)}{(p-1)(p+1)^2\varphi(n)\psi(n)\sigma(n)} \\
&= \frac{(p+1)^3\big(\varphi^3(n) + \psi^3(n) + \sigma^3(n)\big) - (6p^2+2)\varphi^3(n)}{(p-1)(p+1)^2\varphi(n)\psi(n)\sigma(n)} \\
&= \frac{p+1}{p-1}\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} - \frac{(6p^2+2)\varphi^2(n)}{(p-1)(p+1)^2\psi(n)\sigma(n)} \\
&\geq \frac{p+1}{p-1}\frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1} - \frac{(6p^2+2)(n-1)^2}{(p-1)(p+1)^2(n+1)^2} \\
&\geq \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1} > \frac{3n^3p^3 + 3n^2p^2 + 9np + 1}{n^3p^3 + n^2p^2 - np - 1}\,.
\end{aligned}$$

Case B. $p \mid n$. We have

$$\varphi(np) = p\varphi(n)\,, \qquad \psi(np) = p\psi(n)\,, \qquad \sigma(np) > p\sigma(n)\,. \tag{3}$$

It is easy to see that (2), (3) and the inequality $\sigma(n)/\varphi(n)\psi(n) \geq (n+1)/(n-1)$ imply

$$\frac{\sigma^3(np) - p^3\sigma^3(n)}{p^2\varphi(n)\psi(n)\sigma(np)} \geq \left(1 - \frac{p\sigma(n)}{\sigma(np)}\right)\frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1}$$

which, together with (3), yields

$$\begin{aligned}
\frac{\varphi^3(np) + \psi^3(np) + \sigma^3(np)}{\varphi(np)\psi(np)\sigma(np)} &= \frac{p\sigma(n)}{\sigma(np)}\frac{\varphi^3(n) + \psi^3(n) + \sigma^3(n)}{\varphi(n)\psi(n)\sigma(n)} + \frac{\sigma^3(np) - p^3\sigma^3(n)}{p^2\varphi(n)\psi(n)\sigma(np)} \\
&\geq \frac{p\sigma(n)}{\sigma(np)}\frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1} + \frac{\sigma^3(np) - p^3\sigma^3(n)}{p^2\varphi(n)\psi(n)\sigma(np)} \\
&\geq \frac{3n^3 + 3n^2 + 9n + 1}{n^3 + n^2 - n - 1} > \frac{3n^3p^3 + 3n^2p^2 + 9np + 1}{n^3p^3 + n^2p^2 - np - 1}\,.
\end{aligned}$$

This completes the proof of Theorem 1. □

Theorem 2. *For every integer $n \geq 2$, we have*

$$\frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} \geq \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1}. \tag{4}$$

Proof. A direct verification shows that the theorem holds for $2 \leq n \leq 9$. Henceforth, we may assume that $n \geq 10$. Consider several cases.

Case 1. $\Omega(n) = 1$. Taking into account that n is a prime number, we have

$$\frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} = \frac{2(n^2-1)^3 + (n+1)^6}{(n-1)^2(n+1)^4} = \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1}.$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. We write

$$\begin{aligned}\frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} &= \frac{2(p^2-1)^3(q^2-1)^3 + (p+1)^6(q+1)^6}{(p-1)^2(q-1)^2(p+1)^4(q+1)^4} \\ &> \frac{3p^3q^3 - 3p^2q^2 + 9pq - 1}{p^3q^3 - p^2q^2 - pq + 1} = \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1},\end{aligned}$$

where the inequality is equivalent to the positivity of the following expression, which can be written as a sum of positive differences by grouping its terms as follows:

$$\begin{aligned}&4(p+1)^3(q+1)^3(p^2q + pq^2 - pq - 1) \\ &\times [(p^4q^3 - p^4q^2) + (p^3q^4 - p^2q^4) + (p^4q^3 + p^3q^3 - p^3q^2 - p^2q^3) \\ &+ (7p^2q^2 + p^4q^3 + 2p^3q^4 - 3p^2q - 2p^2 - 3pq^2 - 2pq) \\ &+ (p^3q + pq^3 + p + q - 2q^2 - pq - 1)] > 0.\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}&\frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} \\ &= \frac{p^6(p^2-1)^3 + p^3(p+1)^3(p^2+p+1)^3 + p^3(p-1)^3(p^2+p+1)^3}{p^4(p^2-1)^2(p^2+p+1)^2} \\ &> \frac{3p^6 - 3p^4 + 9p^2 - 1}{p^6 - p^4 - p^2 + 1} = \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1},\end{aligned}$$

where the inequality is equivalent to the following expression being positive:

$$(p-1)^2(p+1)^2\left(12p^8 + 38p^7 + 63p^6 + 68p^5 + 65p^4 + 50p^3 + 37p^2 + 20p + 7\right) > 0.$$

Now we assume that (4) is true for every natural number $n \geq 2$ with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. From (2) and straightforward calculations, we get

$$\begin{aligned}&\frac{\varphi^3(np)\psi^3(np) + \psi^3(np)\sigma^3(np) + \varphi^3(np)\sigma^3(np)}{\varphi^2(np)\psi^2(np)\sigma^2(np)} \\ &= \frac{(p^2-1)^3\varphi^3(n)\psi^3(n) + (p+1)^6\psi^3(n)\sigma^3(n) + (p^2-1)^3\varphi^3(n)\sigma^3(n)}{(p-1)^2(p+1)^4\varphi^2(n)\psi^2(n)\sigma^2(n)} \\ &= \frac{(p^2-1)^3\left(\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)\right) + \left((p+1)^6 - (p^2-1)^3\right)\psi^3(n)\sigma^3(n)}{(p-1)^2(p+1)^4\varphi^2(n)\psi^2(n)\sigma^2(n)} \\ &= \frac{p^2-1}{(p+1)^2}\frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} + \frac{(6p^2+2)\psi(n)\sigma(n)}{(p-1)^2(p+1)\varphi^2(n)} \\ &\geq \frac{p^2-1}{(p+1)^2}\frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1} + \frac{(6p^2+2)(n+1)^2}{(p+1)(p-1)^2(n-1)^2} \\ &\geq \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1} > \frac{3n^3p^3 - 3n^2p^2 + 9np - 1}{n^3p^3 - n^2p^2 - np + 1}.\end{aligned}$$

Case B. $p \mid n$. By (3), we find

$$\begin{aligned}
&\frac{\varphi^3(np)\psi^3(np) + \psi^3(np)\sigma^3(np) + \varphi^3(np)\sigma^3(np)}{\varphi^2(np)\psi^2(np)\sigma^2(np)} \\
&= \frac{p^2\sigma^2(n)}{\sigma^2(np)} \frac{\varphi^3(n)\psi^3(n) + \psi^3(n)\sigma^3(n) + \varphi^3(n)\sigma^3(n)}{\varphi^2(n)\psi^2(n)\sigma^2(n)} \\
&+ \frac{\big(\sigma^3(np) - p^3\sigma^3(n)\big)\big(\varphi^3(n) + \psi^3(n)\big)}{p\varphi^2(n)\psi^2(n)\sigma^2(np)} \\
&\geq \frac{p^2\sigma^2(n)}{\sigma^2(np)} \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1} + \frac{\big(\sigma^3(np) - p^3\sigma^3(n)\big)\big(\varphi^3(n) + \psi^3(n)\big)}{p\varphi^2(n)\psi^2(n)\sigma^2(np)}. \qquad (5)
\end{aligned}$$

It remains to prove that

$$\frac{\big(\sigma^3(np) - p^3\sigma^3(n)\big)\big(\varphi^3(n) + \psi^3(n)\big)}{p\varphi^2(n)\psi^2(n)\sigma^2(np)} \geq \left(1 - \frac{p^2\sigma^2(n)}{\sigma^2(np)}\right) \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1}. \qquad (6)$$

Using (3), we reduce (6) to

$$\begin{aligned}
&\big(\sigma^2(np) + p\sigma(n)\sigma(np) + p^2\sigma^2(n)\big)\big(\varphi^3(n) + \psi^3(n)\big)(n^3 - n^2 - n + 1) \\
&\geq (3n^3 - 3n^2 + 9n - 1)\big(\sigma(np) + p\sigma(n)\big)p\varphi^2(n)\psi^2(n). \qquad (7)
\end{aligned}$$

From (3), we have

$$\sigma^2(np) + p\sigma(n)\sigma(np) + p^2\sigma^2(n) > \frac{3}{2}p\sigma(n)\big(\sigma(np) + p\sigma(n)\big) \qquad (8)$$

On the other hand by AM–GM inequality, we have

$$\varphi^3(n) + \psi^3(n) \geq 2\sqrt{\varphi^3(n)\psi^3(n)} \qquad (9)$$

In view of (7)–(9), we deduce

$$3\sigma(n)(n^3 - n^2 - n + 1) \geq (3n^3 - 3n^2 + 9n - 1)\sqrt{\varphi(n)\psi(n)}. \qquad (10)$$

Now (2), (10) and the right-hand side of the inequality in Lemma 1 lead to

$$3(n+1)(n^3 - n^2 - n + 1) \geq n(3n^3 - 3n^2 + 9n - 1)$$

or

$$3n^3 - 15n^2 + n + 3 \geq 0$$

which holds for $n \geq 5$. Combining (5) and (6), we derive

$$\begin{aligned}
&\frac{\varphi^3(np)\psi^3(np) + \psi^3(np)\sigma^3(np) + \varphi^3(np)\sigma^3(np)}{\varphi^2(np)\psi^2(np)\sigma^2(np)} \\
&\geq \frac{3n^3 - 3n^2 + 9n - 1}{n^3 - n^2 - n + 1} > \frac{3n^3p^3 - 3n^2p^2 + 9np - 1}{n^3p^3 - n^2p^2 - np + 1}.
\end{aligned}$$

This completes the proof of Theorem 2. □

Theorem 3. *For every integer $n \geq 2$, we have*

$$\frac{\big(\varphi(n) + \psi(n)\big)^3 + \big(\psi(n) + \sigma(n)\big)^3 + \big(\varphi(n) + \sigma(n)\big)^3}{\big(\varphi(n) + \psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big(\varphi(n) + \sigma(n)\big)} \geq \frac{3n^3 + 3n^2 + 3n + 1}{n^3 + n^2}. \qquad (11)$$

Proof. Consider several cases.

Case 1. $\Omega(n) = 1$. Bearing in mind that n is a prime number, we write

$$\frac{\big(\varphi(n) + \psi(n)\big)^3 + \big(\psi(n) + \sigma(n)\big)^3 + \big(\varphi(n) + \sigma(n)\big)^3}{\big(\varphi(n) + \psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big(\varphi(n) + \sigma(n)\big)} = \frac{2(2n)^3 + (2n+2)^3}{(2n)^2(2n+2)} = \frac{3n^3 + 3n^2 + 3n + 1}{n^3 + n^2}.$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}&\frac{\big(\varphi(n) + \psi(n)\big)^3 + \big(\psi(n) + \sigma(n)\big)^3 + \big(\varphi(n) + \sigma(n)\big)^3}{\big(\varphi(n) + \psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big(\varphi(n) + \sigma(n)\big)}\\ &= \frac{2\big((p-1)(q-1) + (p+1)(q+1)\big)^3 + 2^3(p+1)^3(q+1)^3}{2\big((p-1)(q-1) + (p+1)(q+1)\big)^2(p+1)(q+1)}\\ &> \frac{3p^3q^3 + 3p^2q^2 + 3pq + 1}{p^3q^3 + p^2q^2} = \frac{3n^3 + 3n^2 + 3n + 1}{n^3 + n^2},\end{aligned}$$

where the inequality is equivalent to the following expression being positive:

$$\begin{aligned}&(pq+1)(p^2q + pq^2 - pq - 1)\\ &\times (3p^3q^2 + p^3q + 3p^2q^3 + 5p^2q^2 + 4p^2q + pq^3 + 4pq^2 + 4pq + p + q + 1) > 0.\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. We obtain

$$\begin{aligned}&\frac{\big(\varphi(n) + \psi(n)\big)^3 + \big(\psi(n) + \sigma(n)\big)^3 + \big(\varphi(n) + \sigma(n)\big)^3}{\big(\varphi(n) + \psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big(\varphi(n) + \sigma(n)\big)}\\ &= \frac{\big(p(p-1) + p(p+1)\big)^3 + \big(p(p+1) + p^2 + p + 1\big)^3 + \big(p(p-1) + p^2 + p + 1\big)^3}{\big(p(p-1) + p(p+1)\big)\big(p(p+1) + p^2 + p + 1\big)\big(p(p-1) + p^2 + p + 1\big)}\\ &= \frac{12p^6 + 12p^5 + 24p^4 + 16p^3 + 12p^2 + 3p + 1}{4p^6 + 4p^5 + 4p^4 + 2p^3 + p^2}\\ &> \frac{3p^6 + 3p^4 + 3p^2 + 1}{p^6 + p^4} = \frac{3n^3 + 3n^2 + 3n + 1}{n^3 + n^2},\end{aligned}$$

where the inequality is equivalent to the positivity of the following expression, which can be written as a sum of positive differences by grouping its terms as follows:

$$6p^4(2p^4 - 1) + p^3(10p^4 - 7) + 3p^2(3p^4 - 2) + (p^5 - 2p - 1) > 0.$$

Now we assume that (11) is true for every positive integer n with $\Omega(n) = m$ for some positive integer $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. We have

$$\begin{aligned}&\frac{\big(\varphi(np) + \psi(np)\big)^3 + \big(\psi(np) + \sigma(np)\big)^3 + \big(\varphi(np) + \sigma(np)\big)^3}{\big(\varphi(np) + \psi(np)\big)\big(\psi(np) + \sigma(np)\big)\big(\varphi(np) + \sigma(np)\big)}\\ &= \frac{\big((p-1)\varphi(n) + (p+1)\psi(n)\big)^3 + (p+1)^3\big(\psi(n) + \sigma(n)\big)^3 + \big((p-1)\varphi(n) + (p+1)\sigma(n)\big)^3}{(p+1)\big((p-1)\varphi(n) + (p+1)\psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big((p-1)\varphi(n) + (p+1)\sigma(n)\big)}.\end{aligned} \tag{12}$$

It remains to show that

$$\begin{aligned}&\frac{\big((p-1)\varphi(n) + (p+1)\psi(n)\big)^3 + (p+1)^3\big(\psi(n) + \sigma(n)\big)^3 + \big((p-1)\varphi(n) + (p+1)\sigma(n)\big)^3}{(p+1)\big((p-1)\varphi(n) + (p+1)\psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big((p-1)\varphi(n) + (p+1)\sigma(n)\big)}\\ &\geq \frac{\big(\varphi(n) + \psi(n)\big)^3 + \big(\psi(n) + \sigma(n)\big)^3 + \big(\varphi(n) + \sigma(n)\big)^3}{\big(\varphi(n) + \psi(n)\big)\big(\psi(n) + \sigma(n)\big)\big(\varphi(n) + \sigma(n)\big)}.\end{aligned} \tag{13}$$

To prove (13), we bring the difference between its left-hand side and right-hand side to a common denominator. Since all factors in the denominator are positive, the desired inequality is equivalent to the nonnegativity of the resulting numerator. After cancelling a positive common factor and taking into account (2), this numerator can be expressed as a sum of positive terms and nonnegative differences, from which its nonnegativity follows:

$$
\begin{aligned}
&\big(4p^2\varphi(n)\psi^5(n) - 4p^2\varphi^5(n)\psi(n)\big) + \big(4p^2\varphi^2(n)\psi^3(n)\sigma(n) - 4p^2\varphi^5(n)\sigma(n)\big)\\
&+ \big(8p^2\varphi^2(n)\psi^4(n) - 8p^2\varphi^4(n)\psi^2(n)\big) + \big(16p^2\varphi(n)\psi^3(n)\sigma^2(n) - 16p^2\varphi^4(n)\psi(n)\sigma(n)\big)\\
&+ \big(8p^2\varphi^2(n)\sigma^4(n) - 8p^2\varphi^4(n)\sigma^2(n)\big) + \big(4p^2\varphi^2(n)\psi^3(n)\sigma(n) - 4p^2\varphi^3(n)\psi^2(n)\sigma(n)\big)\\
&+ \big(8p^2\varphi(n)\psi^4(n)\sigma(n) - 8p^2\varphi^3(n)\psi^2(n)\sigma(n)\big) + \big(8p^2\varphi(n)\psi(n)\sigma^4(n) - 8p^2\varphi^3(n)\psi^2(n)\sigma(n)\big)\\
&+ \big(4p^2\varphi(n)\sigma^5(n) - 4p^2\varphi^3(n)\psi^2(n)\sigma(n)\big)\\
&+ \big(16p^2\varphi(n)\psi^2(n)\sigma^3(n) + 8p^2\varphi^2(n)\psi(n)\sigma^3(n) - 24p^2\varphi^3(n)\psi(n)\sigma^2(n)\big)\\
&+ 8p\varphi^5(n)\psi(n) + 8p\varphi^5(n)\sigma(n) + 8p\varphi^4(n)\psi^2(n) + 16p\varphi^4(n)\psi(n)\sigma(n)\\
&+ 8p\varphi^4(n)\sigma^2(n) + 8p\varphi^2(n)\psi^4(n) + 8p\varphi^2(n)\psi^3(n)\sigma(n) + 8p\varphi^2(n)\psi(n)\sigma^3(n)\\
&+ 8p\varphi^2(n)\sigma^4(n) + 8p\varphi(n)\psi^5(n) + 16p\varphi(n)\psi^4(n)\sigma(n) + 32p\varphi(n)\psi^3(n)\sigma^2(n)\\
&+ 32p\varphi(n)\psi^2(n)\sigma^3(n) + 16p\varphi(n)\psi(n)\sigma^4(n) + 8p\varphi(n)\sigma^5(n)\\
&+ \big(8\varphi^3(n)\psi^2(n)\sigma(n) - 4\varphi^5(n)\psi(n) - 4\varphi^5(n)\sigma(n)\big)\\
&+ 8\varphi^3(n)\psi(n)\sigma^2(n) + 4\varphi(n)\psi^5(n) + 8\varphi(n)\psi^4(n)\sigma(n) + 16\varphi(n)\psi^3(n)\sigma^2(n)\\
&+ 16\varphi(n)\psi^2(n)\sigma^3(n) + 8\varphi(n)\psi(n)\sigma^4(n) + 4\varphi(n)\sigma^5(n) \geq 0\,.
\end{aligned}
$$

Now (12) and (13) give us

$$
\begin{aligned}
&\frac{\big(\varphi(np)+\psi(np)\big)^3 + \big(\psi(np)+\sigma(np)\big)^3 + \big(\varphi(np)+\sigma(np)\big)^3}{\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)}\\
&\geq \frac{3n^3+3n^2+3n+1}{n^3+n^2} > \frac{3n^3p^3+3n^2p^2+3np+1}{n^3p^3+n^2p^2}\,.
\end{aligned}
$$

Case B. $p \mid n$. From (3), we have

$$
\begin{aligned}
&\frac{\big(\varphi(np)+\psi(np)\big)^3 + \big(\psi(np)+\sigma(np)\big)^3 + \big(\varphi(np)+\sigma(np)\big)^3}{\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)}\\
&= \frac{p^2\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}{\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)} \frac{\big(\varphi(n)+\psi(n)\big)^3 + \big(\psi(n)+\sigma(n)\big)^3 + \big(\varphi(n)+\sigma(n)\big)^3}{\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}\\
&+ \frac{\big(p\psi(n)+\sigma(np)^3\big) + \big(p\varphi(n)+\sigma(np)\big)^3 - p^3\big(\psi(n)+\sigma(n)\big)^3 - p^3\big(\varphi(n)+\sigma(n)\big)^3}{p\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\\
&\geq \frac{p^2\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}{\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)} \frac{3n^3+3n^2+3n+1}{n^3+n^2}\\
&+ \frac{\big(p\psi(n)+\sigma(np)^3\big) + \big(p\varphi(n)+\sigma(np)\big)^3 - p^3\big(\psi(n)+\sigma(n)\big)^3 - p^3\big(\varphi(n)+\sigma(n)\big)^3}{p\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\,.
\end{aligned}
\tag{14}
$$

It remains to prove that

$$
\begin{aligned}
&\frac{\big(p\psi(n)+\sigma(np)^3\big) + \big(p\varphi(n)+\sigma(np)\big)^3 - p^3\big(\psi(n)+\sigma(n)\big)^3 - p^3\big(\varphi(n)+\sigma(n)\big)^3}{p\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\\
&\geq \left(1 - \frac{p^2\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}{\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\right)\frac{3n^3+3n^2+3n+1}{n^3+n^2}\,.
\end{aligned}
\tag{15}
$$

By (3), we reduce (15) to

$$(n^3+n^2)\Big(\big(p\varphi(n)+\sigma(np)\big)^2+\big(p\varphi(n)+\sigma(np)\big)p\big(\varphi(n)+\sigma(n)\big)+p^2\big(\varphi(n)+\sigma(n)\big)^2 + \big(p\psi(n)+\sigma(np)\big)^2+\big(p\psi(n)+\sigma(np)\big)p\big(\psi(n)+\sigma(n)\big)+p^2\big(\psi(n)+\sigma(n)\big)^2\Big) - (3n^3+3n^2+3n+1)p\big(\varphi(n)+\psi(n)\big)\big(p\varphi(n)+p\psi(n)+p\sigma(n)+\sigma(np)\big)\geq 0\,. \tag{16}$$

Put

$$x=\varphi(n)\,,\qquad y=\psi(n)\,,\qquad z=\sigma(n)\,,\qquad t=\sigma(np)\,. \tag{17}$$

Taking into account (2), (3), (16) and (17), we conclude that it suffices to prove that the function

$$f(t)=(n^3+n^2)\Big((px+t)^2+(px+t)p(x+z)+p^2(x+z)^2 + (py+t)^2+(py+t)p(y+z)+p^2(y+z)^2\Big) - (3n^3+3n^2+3n+1)p(x+y)(px+py+pz+t)$$

is positive under the assumptions

$$x\leq n-1<n+1\leq y\leq z\,,\qquad t\geq pz\,. \tag{18}$$

We write the function $f(t)$ in the form

$$f(t)=2n^2(n+1)(p^2z^2+ptz+t^2)-(3n+1)p^2(x+y)^2 - (3n+1)p(x+y)(pz+t)-6n^2(n+1)p^2xy\,. \tag{19}$$

We have

$$\frac{\partial f(t)}{\partial t}=2n^2(n+1)(2t+pz)-(3n+1)p(x+y)\,.$$

For $t\geq pz$, we find

$$\frac{\partial f(t)}{\partial t}\geq p\big(6n^2(n+1)z-(3n+1)(x+y)\big)\,.$$

It follows from (18) that

$$x+y\leq(n-1)+z\,. \tag{20}$$

Therefore

$$6n^2(n+1)z-(3n+1)(x+y)\geq 6n^2(n+1)z-(3n+1)(z+n-1)>0\,.$$

Consequently, $f(t)$ is increasing for $t\geq pz$, and hence

$$f(t)\geq f(pz)\,. \tag{21}$$

In view of (18)–(21), we deduce

$$f(t)\geq p^2\Big(6n^2(n+1)(z^2-xy)-(3n+1)(x+y)^2-(6n+2)(x+y)z\Big) \geq p^2\Big(6n^2(n+1)\big(z^2-(n-1)z\big)-(3n+1)(z+n-1)^2 - (6n+2)(z+n-1)z\Big)\,. \tag{22}$$

Set

$$z=n+1+u\,,\qquad u\geq 0\,. \tag{23}$$

Now (22) and (23) imply

$$f(t)\geq(6n^3+6n^2-9n-3)u^2+(6n^4+24n^3-12n^2-16n-2)u + 4n(n-1)(3n^2+3n+1)>0\,.$$

Finally (14) and (15) yield

$$\frac{\big(\varphi(np)+\psi(np)\big)^3+\big(\psi(np)+\sigma(np)\big)^3+\big(\varphi(np)+\sigma(np)\big)^3}{\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)}$$
$$\geq \frac{3n^3+3n^2+3n+1}{n^3+n^2} > \frac{3n^3p^3+3n^2p^2+3np+1}{n^3p^3+n^2p^2}.$$

This completes the proof of Theorem 3. □

Theorem 4. *For every integer $n \geq 2$, we have*

$$\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}$$
$$\geq \frac{3n^3-3n^2+3n-1}{n^3-n^2}. \tag{24}$$

Proof. Consider several cases.

Case 1. $\Omega(n)=1$. Taking into account that n is a prime number, we obtain

$$\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}$$
$$= \frac{(n-1)^3(2n+2)^3+2(n+1)^3(2n)^3}{(n-1)(n+1)^2(2n)^2(2n+2)} = \frac{3n^3-3n^2+3n-1}{n^3-n^2}.$$

Case 2. $\Omega(n)=2$, $n=pq$, where p and q are distinct primes. Then

$$\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}$$
$$= \frac{\big((p-1)(q-1)+(p+1)(q+1)\big)^3+4(p-1)^3(q-1)^3}{(p-1)(q-1)\big((p-1)(q-1)+(p+1)(q+1)\big)^2}$$
$$> \frac{3p^3q^3-3p^2q^2+3pq-1}{p^3q^3-p^2q^2} = \frac{3n^3-3n^2+3n-1}{n^3-n^2},$$

where the inequality is equivalent to the positivity of the following expression, which can be written as a sum of positive differences by grouping its terms as follows:

$$(p^2q+pq^2-pq-1)$$
$$\times\,[(p^4q^3-p^4q^2)+(2p^4q^3-p^3q^2)+(3p^3q^4-p^2q^4)+(p^3q^3-p^2q^3)$$
$$+\,(7p^2q^2-3p^2q-3pq^2)+p^3q+pq^3+pq+p+q-1]>0.$$

Case 3. $\Omega(n)=2$, $n=p^2$, where p is a prime. We derive

$$\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}$$
$$= \frac{p^3(p-1)^3(2p^2+2p+1)^3+p^3(p+1)^3(2p^2+1)^3+8p^6(p^2+p+1)^3}{2p^4(p^2-1)(2p^2+2p+1)(2p^2+1)(p^2+p+1)}$$
$$= \frac{12p^8+24p^7+36p^6+44p^5+48p^4+33p^3+16p^2+3p}{4p^8+8p^7+8p^6+2p^5-5p^4-7p^3-6p^2-3p-1}$$
$$> \frac{3p^6-3p^4+3p^2-1}{p^6-p^4} = \frac{3n^3-3n^2+3n-1}{n^3-n^2},$$

where the inequality is equivalent to the following expression being positive:

$$(p^2-1)\big(12p^{10}+38p^9+51p^8+30p^7+2p^6-10p^5-6p^4+p^3+4p^2+3p+1\big)>0.$$

Now we assume that (24) is true for every positive integer n with $\Omega(n) = m$ for some positive integer $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Put

$$\begin{aligned} F(n,p) &= (p-1)^3\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3 + \psi^3(n)\big((p-1)\varphi(n)+(p+1)\sigma(n)\big)^3 \\ &\quad + \sigma^3(n)\big((p-1)\varphi(n)+(p+1)\psi(n)\big)^3, \end{aligned} \tag{25}$$

$$\begin{aligned} G(n,p) &= (p-1)\varphi(n)\psi(n)\sigma(n)\big((p-1)\varphi(n)+(p+1)\psi(n)\big)\big(\psi(n)+\sigma(n)\big) \\ &\quad \times \big((p-1)\varphi(n)+(p+1)\sigma(n)\big). \end{aligned} \tag{26}$$

Using (25) and (26), we find

$$\begin{aligned} &\frac{\varphi^3(np)\big(\psi(np)+\sigma(np)\big)^3 + \psi^3(np)\big(\varphi(np)+\sigma(np)\big)^3 + \sigma^3(np)\big(\varphi(np)+\psi(np)\big)^3}{\varphi(np)\psi(np)\sigma(np)\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)} \\ &= \frac{F(n,p)}{G(n,p)}. \end{aligned} \tag{27}$$

It remains to show that

$$\frac{F(n,p)}{G(n,p)} \geq \frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3 + \psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3 + \sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}. \tag{28}$$

To establish (28), we write the difference between its left-hand side and right-hand side over a common denominator. As all denominator factors are positive, it suffices to show that the resulting numerator is nonnegative. Upon cancelling a positive common factor and using (2), we can rewrite the numerator as a sum of positive terms and nonnegative differences, thereby establishing the desired inequality:

$$\begin{aligned} &\big(4p^2\varphi^2(n)\psi^4(n)\sigma^3(n) - 4p^2\varphi^4(n)\psi^5(n)\big) + \big(8p^2\varphi^2(n)\psi^4(n)\sigma^3(n) - 8p^2\varphi^4(n)\psi^4(n)\sigma(n)\big) \\ &+ \big(12p^2\varphi^2(n)\psi^4(n)\sigma^3(n) + 4p^2\varphi(n)\psi^5(n)\sigma^3(n) - 16p^2\varphi^4(n)\psi^3(n)\sigma^2(n)\big) \\ &+ \big(4p^2\varphi(n)\psi^5(n)\sigma^3(n) + 12p^2\varphi^2(n)\psi^3(n)\sigma^4(n) - 16p^2\varphi^4(n)\psi^2(n)\sigma^3(n)\big) \\ &+ \big(12p^2\varphi^2(n)\psi^3(n)\sigma^4(n) - 8p^2\varphi^4(n)\psi(n)\sigma^4(n) - 4p^2\varphi^4(n)\sigma^5(n)\big) \\ &+ \big(16p^2\varphi(n)\psi^4(n)\sigma^4(n) - 8p^2\varphi^3(n)\psi^5(n)\sigma(n) - 8p^2\varphi^3(n)\psi^4(n)\sigma^2(n)\big) \\ &+ \big(4p^2\psi^5(n)\sigma^4(n) + 4p^2\psi^4(n)\sigma^5(n) - 8p^2\varphi^3(n)\psi^2(n)\sigma^4(n)\big) \\ &+ \big(8p^2\varphi(n)\psi^3(n)\sigma^5(n) - 8p^2\varphi^3(n)\psi(n)\sigma^5(n)\big) \\ &+ \big(8p\varphi^4(n)\psi^5(n) - 4\varphi^4(n)\psi^5(n)\big) + \big(16p\varphi^4(n)\psi^4(n)\sigma(n) - 8\varphi^4(n)\psi^4(n)\sigma(n)\big) \\ &+ \big(32p\varphi^4(n)\psi^3(n)\sigma^2(n) - 16\varphi^4(n)\psi^3(n)\sigma^2(n)\big) \\ &+ \big(32p\varphi^4(n)\psi^2(n)\sigma^3(n) - 16\varphi^4(n)\psi^2(n)\sigma^3(n)\big) \\ &+ \big(16p\varphi^4(n)\psi(n)\sigma^4(n) - 8\varphi^4(n)\psi(n)\sigma^4(n)\big) + 8p\varphi^3(n)\psi^5(n)\sigma(n) \\ &+ 8p\varphi^3(n)\psi^4(n)\sigma^2(n) + 8p\varphi^3(n)\psi^2(n)\sigma^4(n) + 8p\varphi^3(n)\psi(n)\sigma^5(n) \\ &+ \big(8p\varphi(n)\psi^5(n)\sigma^3(n) - 8\varphi^2(n)\psi^4(n)\sigma^3(n)\big) + 16p\varphi(n)\psi^4(n)\sigma^4(n) \\ &+ 4\psi^5(n)\sigma^4(n) + 8p\varphi(n)\psi^3(n)\sigma^5(n) + 8p\varphi^4(n)\sigma^5(n) + 8p\psi^4(n)\sigma^5(n) \\ &+ \big(8p\psi^5(n)\sigma^4(n) - 8\varphi^2(n)\psi^3(n)\sigma^4(n)\big) + \big(4\psi^4(n)\sigma^5(n) - 4\varphi^4(n)\sigma^5(n)\big) \geq 0. \end{aligned}$$

Now (27) and (28) lead to

$$\begin{aligned} &\frac{\varphi^3(np)\big(\psi(np)+\sigma(np)\big)^3 + \psi^3(np)\big(\varphi(np)+\sigma(np)\big)^3 + \sigma^3(np)\big(\varphi(np)+\psi(np)\big)^3}{\varphi(np)\psi(np)\sigma(np)\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)} \\ &\geq \frac{3n^3-3n^2+3n-1}{n^3-n^2} > \frac{3n^3p^3-3n^2p^2+3np-1}{n^3p^3-n^2p^2}. \end{aligned}$$

Case B. $p \mid n$. By (3), we get

$$\begin{aligned}
&\frac{\varphi^3(np)\big(\psi(np)+\sigma(np)\big)^3+\psi^3(np)\big(\varphi(np)+\sigma(np)\big)^3+\sigma^3(np)\big(\varphi(np)+\psi(np)\big)^3}{\varphi(np)\psi(np)\sigma(np)\big(\varphi(np)+\psi(np)\big)\big(\psi(np)+\sigma(np)\big)\big(\varphi(np)+\sigma(np)\big)}\\
&=\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}\\
&+\frac{\varphi^3(n)\big(p\psi(n)+\sigma(np)\big)^3+\psi^3(n)\big(p\varphi(n)+\sigma(np)\big)^3+\sigma^3(np)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(np)\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\\
&-\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}\\
&\geq\frac{3n^3-3n^2+3n-1}{n^3-n^2}+\\
&+\frac{\varphi^3(n)\big(p\psi(n)+\sigma(np)\big)^3+\psi^3(n)\big(p\varphi(n)+\sigma(np)\big)^3+\sigma^3(np)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(np)\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\\
&-\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}\,. \qquad (29)
\end{aligned}$$

It remains to prove that

$$\begin{aligned}
&\frac{\varphi^3(n)\big(p\psi(n)+\sigma(np)\big)^3+\psi^3(n)\big(p\varphi(n)+\sigma(np)\big)^3+\sigma^3(np)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(np)\big(\varphi(n)+\psi(n)\big)\big(p\psi(n)+\sigma(np)\big)\big(p\varphi(n)+\sigma(np)\big)}\\
&-\frac{\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3}{\varphi(n)\psi(n)\sigma(n)\big(\varphi(n)+\psi(n)\big)\big(\psi(n)+\sigma(n)\big)\big(\varphi(n)+\sigma(n)\big)}\geq 0\,, \qquad (30)
\end{aligned}$$

which is equivalent to the inequality

$$\begin{aligned}
&\sigma(n)\big(\varphi(n)+\sigma(n)\big)\big(\psi(n)+\sigma(n)\big)\\
&\times\Big(\varphi^3(n)\big(p\psi(n)+\sigma(np)\big)^3+\psi^3(n)\big(p\varphi(n)+\sigma(np)\big)^3+\sigma^3(np)\big(\varphi(n)+\psi(n)\big)^3\Big)\\
&-\sigma(np)\big(p\varphi(n)+\sigma(np)\big)\big(p\psi(n)+\sigma(np)\big)\\
&\times\Big(\varphi^3(n)\big(\psi(n)+\sigma(n)\big)^3+\psi^3(n)\big(\varphi(n)+\sigma(n)\big)^3+\sigma^3(n)\big(\varphi(n)+\psi(n)\big)^3\Big)\geq 0\,. \qquad (31)
\end{aligned}$$

Bearing in mind (2), (3), (17) and (31), we conclude that it suffices to show that the function

$$\begin{aligned}
g(t)&=z(x+z)(y+z)\Big(x^3(py+t)^3+y^3(px+t)^3+t^3(x+y)^3\Big)\\
&\quad-t(px+t)(py+t)\Big(x^3(y+z)^3+y^3(x+z)^3+z^3(x+y)^3\Big)
\end{aligned}$$

is nonnegative under the assumptions (18). We first note that

$$g(pz)=0\,. \qquad (32)$$

We shall now prove that g is strictly increasing on $[pz,\infty)$. Differentiating three times gives us

$$\begin{aligned}
g'(t)&=z(x+z)(y+z)\big(3x^3(py+t)^2+3y^3(px+t)^2+3t^2(x+y)^3\big)\\
&\quad-(3t^2+2tpx+2tpy+p^2xy)\big(x^3(y+z)^3+y^3(x+z)^3+z^3(x+y)^3\big)\,. \qquad (33)
\end{aligned}$$

$$\begin{aligned}
g''(t)&=6z(x+z)(y+z)\big(x^3(py+t)+y^3(px+t)+t(x+y)^3\big)\\
&\quad-(6t+2px+2py)\big(x^3(y+z)^3+y^3(x+z)^3+z^3(x+y)^3\big)\,. \qquad (34)
\end{aligned}$$

$$g'''(t)=12\big(x^4yz+x^4z^2-x^3y^3+x^3yz^2+3x^2y^2z^2+xy^4z+xy^3z^2+y^4z^2\big)\,. \qquad (35)$$

From (18), it follows that

$$x^3yz^2\geq x^3y^3\,,$$

and all the remaining terms in (35) are positive. Hence

$$g'''(t) > 0\,. \tag{36}$$

Consequently, g'' is strictly increasing. Now put $t = pz$ in (34). Direct simplification implies

$$\begin{aligned} g''(pz) = 4p[&(3x^4yz^2 - x^4y^3) + (2x^3yz^3 - x^3y^4) + (6x^2y^2z^3 - 6x^3y^3z) \\ &+ 2x^4z^3 + 3xy^4z^2 + 2xy^3z^3 + 2y^4z^3]\,. \end{aligned}$$

It is easy to see that (18) implies that the differences enclosed in parentheses in the expression for $g''(pz)$ satisfy

$$3x^4yz^2 - x^4y^3 > 0\,, \quad 2x^3yz^3 - x^3y^4 > 0\,, \quad 6x^2y^2z^3 - 6x^3y^3z \geq 0\,.$$

Thus

$$g''(pz) > 0\,.$$

Together with (36), this leads to

$$g''(t) > 0\,. \tag{37}$$

Therefore g' is strictly increasing on $[pz, \infty)$. Again, setting $t = pz$ in (33), we deduce

$$\begin{aligned} g'(pz) = 2p^2[&(2x^4yz^3 - 2x^4y^3z) + (x^4z^4 - x^4y^4) + (2xy^4z^3 - 2x^3y^4z) \\ &+ (x^3yz^4 + 3x^2y^2z^4 + xy^3z^4 + y^4z^4 - 6x^3y^3z^2)]\,. \end{aligned}$$

It follows immediately from (18) that all the differences enclosed in parentheses in the expression for $g'(pz)$ are nonnegative. For example the difference $x^3yz^4 + 3x^2y^2z^4 + xy^3z^4 + y^4z^4 - 6x^3y^3z^2$ is nonnegative because (18) yields

$$x^3yz^4 \geq x^3y^3z^2\,, \quad 3x^2y^2z^4 \geq 3x^3y^3z^2\,, \quad xy^3z^4 \geq x^3y^3z^2\,, \quad y^4z^4 \geq x^3y^3z^2\,.$$

Consequently

$$g'(pz) \geq 0\,,$$

which together with (37) gives us

$$g'(t) \geq 0\,.$$

Thus g is increasing on $[pz, \infty)$. Using (32), we obtain

$$g(t) \geq g(pz) = 0 \quad \text{for} \quad t > pz.$$

Therefore

$$g(t) \geq 0 \quad \text{for} \quad t > pz.$$

Finally (29) and (30) imply

$$\begin{aligned} &\frac{\varphi^3(np)\big(\psi(np) + \sigma(np)\big)^3 + \psi^3(np)\big(\varphi(np) + \sigma(np)\big)^3 + \sigma^3(np)\big(\varphi(np) + \psi(np)\big)^3}{\varphi(np)\psi(np)\sigma(np)\big(\varphi(np) + \psi(np)\big)\big(\psi(np) + \sigma(np)\big)\big(\varphi(np) + \sigma(np)\big)} \\ &\geq \frac{3n^3 - 3n^2 + 3n - 1}{n^3 - n^2} > \frac{3n^3p^3 - 3n^2p^2 + 3np - 1}{n^3p^3 - n^2p^2}\,. \end{aligned}$$

This completes the proof of Theorem 4. □

4. Conjectures

Based on numerical checks for prime values of n, we formulate the following four conjectures, which extend the inequalities established in Theorems 1–4 to arbitrary positive integer powers k. For $k = 1$, these conjectures reduce to Theorems 1–4, respectively.

Conjecture 1. *For every integer $k \geq 1$ and $n \geq 2$, we have*

$$\frac{\varphi^{3k}(n) + \psi^{3k}(n) + \sigma^{3k}(n)}{\varphi^k(n)\psi^k(n)\sigma^k(n)} \geq \frac{2(n+1)^{3k} + (n-1)^{3k}}{(n+1)^{2k}(n-1)^k}.$$

Conjecture 2. *For every integer $k \geq 1$ and $n \geq 2$, we have*

$$\frac{\varphi^{3k}(n)\psi^{3k}(n) + \psi^{3k}(n)\sigma^{3k}(n) + \varphi^{3k}(n)\sigma^{3k}(n)}{\varphi^{2k}(n)\psi^{2k}(n)\sigma^{2k}(n)} \geq \frac{2(n^2-1)^{3k} + (n+1)^{6k}}{(n-1)^{2k}(n+1)^{4k}}.$$

Conjecture 3. *For every integer $k \geq 1$ and $n \geq 2$, we have*

$$\frac{\big(\varphi(n) + \psi(n)\big)^{3k} + \big(\psi(n) + \sigma(n)\big)^{3k} + \big(\varphi(n) + \sigma(n)\big)^{3k}}{\big(\varphi(n) + \psi(n)\big)^{k}\big(\psi(n) + \sigma(n)\big)^{k}\big(\varphi(n) + \sigma(n)\big)^{k}} \geq \frac{2n^{3k} + (n+1)^{3k}}{n^{2k}(n+1)^k}.$$

Conjecture 4. *For every integer $k \geq 1$ and $n \geq 2$, we have*

$$\frac{\varphi^{3k}(n)\big(\psi(n) + \sigma(n)\big)^{3k} + \psi^{3k}(n)\big(\varphi(n) + \sigma(n)\big)^{3k} + \sigma^{3k}(n)\big(\varphi(n) + \psi(n)\big)^{3k}}{\varphi^k(n)\psi^k(n)\sigma^k(n)\big(\varphi(n) + \psi(n)\big)^{k}\big(\psi(n) + \sigma(n)\big)^{k}\big(\varphi(n) + \sigma(n)\big)^{k}}$$
$$\geq \frac{2n^{3k} + (n-1)^{3k}}{n^{2k}(n-1)^k}.$$

5. Conclusions

In this paper, we have established four lower bounds involving the Euler totient function $\varphi(n)$, the Dedekind psi function $\psi(n)$, and the sum-of-divisors function $\sigma(n)$ for integers $n \geq 2$. The results provide explicit rational lower bounds for rational expressions involving these functions and their combinations, including sums of cubes, products, and sums of pairs of functions. The four inequalities highlight relationships between these classical arithmetic functions and yield bounds expressed in terms of rational functions of n. The approach relies on inequalities involving $\varphi(n)$, $\psi(n)$, and $\sigma(n)$, together with algebraic estimates and mathematical induction, to derive the stated results. The four conjectures formulated in the preceding section suggest possible extensions of the main results to arbitrary positive integer powers k. Proving these conjectures remains an open problem and provides a natural direction for future research.

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References

1. Atanassov, K. Note on φ, ψ and σ-functions. Part 6. *Notes Number Theory Discret. Math.* **2013**, *19*, 22–24.
2. Dimitrov, S.I. Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$. *Notes Number Theory Discret. Math.* **2023**, *29*, 713–716. [CrossRef]
3. Dimitrov, S.I. Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$, II. *Notes Number Theory Discret. Math.* **2024**, *30*, 547–556. [CrossRef]
4. Sándor, J.; Gryszka, K. On certain inequalities for $\varphi(n)$, $\psi(n)$ and $\sigma(n)$ and related functions, III. *Notes Number Theory Discret. Math.* **2025**, *31*, 361–369.
5. Dimitrov, S.I. Inequalities involving arithmetic functions. *Lith. Math. J.* **2024**, *64*, 421–452. [CrossRef]
6. Dimitrov, S.I. Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$, III. *arXiv* **2026**, arXiv:2606.12484.
7. Sándor, J.; Gryszka, K. Extreme values of the Dedekind Ψ function. *J. Comb. Number Theory* **2011**, *3*, 33–38.

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