mathematics

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Admissible E-Value Merging Under Uncertain Input Validity

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Abstract

Suppose that a nonnegative random vector contains valid e-variables on an unknown member of a prescribed family of coordinate sets. The remaining coordinates are unrestricted, and arbitrary dependence is allowed. We characterize admissible merging rules under pointwise domination among all finite-valued nonnegative Borel rules. Every valid rule is dominated by a minimum of normalized affine functions, with one certificate supported on each possible valid set. For the convex hull of these certificates, validity is equivalent to intersection with specified coordinate faces, whereas domination reverses set inclusion. Admissibility holds exactly when every vertex is the unique point of intersection with at least one required face. This condition has an equivalent formulation in terms of inclusion-minimal coverage by the vertex supports. It yields a finite deletion procedure producing an admissible envelope that dominates the starting valid envelope, with strict improvement whenever a vertex is deleted. Every admissible rule is continuous, concave, and coordinate-wise nondecreasing. The number of vertices is bounded by the number of inclusion-minimal validity sets, and this bound is sharp. Algebraic examples include asymmetric mergers and lower trimmed means. We also establish the existence of an admissible maximizer of expected logarithmic evidence under bounded alternatives and identify a sufficient condition for zero optimal growth.

Keywords: e-values; merging functions; admissibility; partial conjunction; arbitrary dependence; convex geometry; robust inference

MSC: 62G10; 62C15; 52B05

1. Introduction

An e-variable is a nonnegative statistic with expectation at most one under every distribution in a specified null hypothesis. Its observed value is an e-value. The expectation constraint permits averaging without a dependence assumption. The usual distribution-free guarantee for a weighted average requires valid inputs wherever the weights are positive. If only an unspecified subset of the inputs is known to be valid, the output must remain an e-variable for every allowed location of that subset.

The expectation-based treatment of evidence is developed by Vovk and Wang [1], Shafer [2], and Grünwald, de Heide, and Koolen [3]. Universal inference [4] provides finite-sample-valid evidence through likelihood constructions. Ramdas et al. [5] review the relation between e-values, test martingales, and safe inference. We take the input statistics as given and study the class of merging rules permitted by uncertainty about their validity.

For ordinary merging, Wang [6] proves that every valid monotone rule is dominated by a weighted arithmetic average that may assign weight to the constant one. Clerico [7] gives a geometric proof and removes the monotonicity assumption. These results characterize

Check for updates

Academic Editor: Fernando Rojas

Received: 24 September 2026
Revised: 24 October 2026
Accepted: 27 October 2026
Published: 28 October 2026

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merging when all designated inputs are valid. Here, a union of coordinate moment classes replaces the single collection of moment constraints. An affine certificate may depend on the hypothesized valid set, but its weights remain fixed with respect to the observations.

Partial conjunction testing provides a related statistical formulation. Benjamini and Heller [8] study evidence that a specified minimum number of component hypotheses are false. Wang and Owen [9] develop admissibility results for partial-conjunction procedures based on p-values. Merging p-values under arbitrary dependence also has an admissibility theory [10]. These results concern tail-probability constraints, whereas e-value validity is an expectation constraint. Subset minima of e-values already appear in the discovery constructions of Vovk and Wang [11], including the arithmetic-mean construction that produces lower trimmed means.

Our main result characterizes admissible scalar mergers for an arbitrary prescribed family of possible valid sets. A polytope of nonnegative weights represents the merger. Each possible valid set specifies a coordinate face that the polytope must meet. Admissibility holds exactly when every vertex is the sole point at which the polytope meets at least one required face. We call such a face a private face of the vertex. The criterion has an equivalent support-cover formulation and gives a finite procedure for removing unnecessary affine terms.

The ordering used here compares the numerical values of scalar mergers at every nonnegative input vector. This differs from the risk and power comparisons of classical decision theory [12,13]. Pointwise domination enlarges or preserves every threshold rejection region, but admissibility of a scalar e-value does not establish admissibility of a test at one fixed significance level. The complete-class and admissibility results for false discovery proportions in [14] concern multiple-testing procedures, whereas the present analysis concerns scalar merging rules.

Table 1 summarizes the closest results and their relation to the present analysis. The affine domination argument and the use of subset minima are inherited ingredients. The result developed here is the equivalence between scalar admissibility, private coordinate faces, and minimal support coverage under uncertain coordinate validity.

Table 1. Comparison with related theoretical results.

Source	Validity Model	Comparison or Output	Relevant Conclusion
Wang [6]	All designated inputs satisfy e-variable moment constraints; arbitrary dependence	Pointwise comparison of monotone scalar mergers	Normalized affine rules characterize ordinary admissible merging
Clerico [7]	Ordinary e-merging under arbitrary dependence	Borel scalar mergers without monotonicity	Affine domination follows from a concave-envelope argument
Wang and Owen [9]	Partial-conjunction p-value model under the assumptions of their results	Partial-conjunction testing procedures	Admissibility results for subset-based p-value constructions
Vovk and Wang [11]	E-value-based inference under arbitrary dependence	Confidence and discovery guarantees	Subset minima provide valid constructions
Present analysis	Union of coordinate moment classes indexed by $\mathcal{A}$	Pointwise comparison of all finite-valued nonnegative Borel scalar mergers	Private-face characterization, minimal support covers, and finite admissible completion

The additional difficulty is that a certificate supported on one possible valid set need not be valid under another. Theorem 1 assembles set-specific certificates into a common lower envelope and handles the possibly non-Borel supremum over unrestricted coordinates. Theorem 2 then identifies which certificates are indispensable for validity across the entire

union of law classes. Thus the contribution is the characterization of admissible lower envelopes for this union model, rather than a new ordinary affine domination theorem.

Section 2 states the model and its statistical interpretation. Section 3 proves the affine representation, including the measurability argument for nonmonotone rules. Section 4 relates validity and domination to convex sets. Section 5 gives the admissibility characterization and a sharp bound on the number of vertices. The deletion procedure and algebraic examples appear in Sections 6 and 7. Section 8 studies the existence of an optimal rule under an expected-log-evidence criterion and states a question for further work.

2. Model, Notation, and Statistical Interpretation

2.1. Uncertain Validity Sets

Fix $K \geq 1$ and write $[K] = \{1, \dots, K\}$. Let $\mathcal{A}$ be a nonempty family of nonempty subsets of $[K]$. All input vectors have finite coordinates. For a Borel probability law P on $[0, \infty)^K$, let $E = (E_1, \dots, E_K)$ denote the coordinate vector. Define

$$\mathcal{P}_S = \{P : \mathbb{E}_P E_i \leq 1 \text{ for every } i \in S\}, \qquad \mathcal{P}_{\mathcal{A}} = \bigcup_{S \in \mathcal{A}} \mathcal{P}_S. \tag{1}$$

No restriction is placed on dependence. Coordinates outside the selected set may have infinite expectations, although they are finite almost surely. Under each law, at least one fixed set $S \in \mathcal{A}$ must satisfy the moment constraints. This set may differ between laws; it is not selected separately at each realized outcome.

Equivalently, membership in $\mathcal{P}_{\mathcal{A}}$ means that there exists $S = S(P) \in \mathcal{A}$ for which all the stated expectations are at most one. It does not mean that a set $S(E)$ can be chosen after observing E.

Definition 1. *A finite-valued Borel function $F : [0, \infty)^K \to [0, \infty)$ is $\mathcal{A}$-valid if*

$$\mathbb{E}_P F(E) \leq 1 \qquad \text{for every } P \in \mathcal{P}_{\mathcal{A}}.$$

Write $\mathcal{U}_{\mathcal{A}}$ for this class. A member F is admissible if $G \in \mathcal{U}_{\mathcal{A}}$ and $G(e) \geq F(e)$ for every e imply $G = F$ everywhere.

Strict domination means weak inequality at every input and strict inequality at at least one input. The definition concerns deterministic scalar rules and does not impose symmetry, monotonicity, or continuity.

When $\mathcal{A} = \{[K]\}$, the model reduces to ordinary merging with all inputs valid. Tolerance of up to r invalid inputs, where $0 \leq r < K$, corresponds to

$$\mathcal{A} = \{S \subseteq [K] : |S| = m\}, \qquad m = K - r. \tag{2}$$

An invalid coordinate is one without a moment guarantee; no contamination distribution is specified.

For example, three evidence sources under a common null, with a guarantee that at most one lacks calibration, give $\mathcal{A} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. If source 1 is guaranteed valid and at least one of sources 2 and 3 is also valid, use $\mathcal{A} = \{\{1,2\}, \{1,3\}\}$. These guarantees must be justified by the construction or external knowledge of the sources and specified before inspecting their evidence values; the model does not infer reliability from large observed e-values.

Proposition 1 (Removing redundant validity sets)**.** *Let $\mathcal{A}_{\min}$ consist of the inclusion-minimal members of $\mathcal{A}$. Then*

$$\mathcal{P}_{\mathcal{A}_{\min}} = \mathcal{P}_{\mathcal{A}}, \qquad \mathcal{U}_{\mathcal{A}_{\min}} = \mathcal{U}_{\mathcal{A}}.$$

Proof. If $T \subseteq S$, then $\mathcal{P}_S \subseteq \mathcal{P}_T$. Every member of the finite family $\mathcal{A}$ contains an inclusion-minimal member. Taking unions proves equality of the law classes. The valid-rule classes are therefore equal. □

2.2. Weights and Coordinate Faces

Define

$$D = \left\{ w \in [0,\infty)^K : \sum_{i=1}^{K} w_i \le 1 \right\}, \qquad D_S = \{ w \in D : w_i = 0 \text{ for } i \notin S \}. \tag{3}$$

For $w \in D$, let

$$L_w(e) = 1 + w^\top (e - \mathbf{1}) = 1 - \sum_i w_i + \sum_i w_i e_i. \tag{4}$$

The unused weight multiplies the constant one.

This represents retaining part of the weight in neutral evidence: $w = 0$ gives the constant rule one, while $\sum_i w_i = 1$ gives an ordinary weighted average. Allowing the constant term is necessary to include the full affine complete class and is not an additional assumption on the input distribution. Each L_w is nonnegative on the input domain and satisfies $L_w(\mathbf{1}) = 1$. If $w \in D_S$, then $\mathbb{E}_P L_w(E) \le 1$ for every $P \in \mathcal{P}_S$. Table 2 summarizes the notation used throughout the paper.

Table 2. Notation used throughout the paper.

Symbol	Meaning
$\mathbf{1}$	Vector with every coordinate equal to one
$\mathcal{A}$	Family of possible valid coordinate sets
$\mathcal{A}_{\min}$	Inclusion-minimal members of $\mathcal{A}$
$\mathcal{P}_\mathcal{A}, \mathcal{U}_\mathcal{A}$	Allowed laws and valid Borel mergers
D, D_S	Weight simplex and its face supported on S
L_w	Normalized affine function in (4)
$\mathrm{conv}(V)$, $\mathrm{vert}(C)$	Convex hull and vertex set
$\mathrm{supp}(w)$	$\{i : w_i > 0\}$
H_C	Minimum of L_w over $w \in C$
$e_{(1)} \le \cdots \le e_{(K)}$	Increasing order statistics

2.3. Testing Interpretation and Scope

For $0 < \alpha < 1$, Markov's inequality gives

$$\sup_{P \in \mathcal{P}_\mathcal{A}} P\{F(E) \ge \alpha^{-1}\} \le \alpha, \qquad F \in \mathcal{U}_\mathcal{A}. \tag{5}$$

Suppose that E_i is an e-variable whenever a component null H_i holds. The family in (2) then gives a test of the union null that at least m component nulls hold. Rejection supports the conclusion that at least $K - m + 1$ component nulls are false. This is the partial-conjunction interpretation. The same law class can also describe evidence sources whose validity under a common null is uncertain.

Closed testing [15] and Holm's procedure [16] address simultaneous decisions. False discovery rate control [17,18] uses a different error criterion. The e-BH procedure [19] obtains FDR control without assumptions on dependence among the input e-values, and e-values can also provide unnormalized weights in multiple testing [20]. Admissibility of a scalar building block does not by itself establish admissibility of a procedure using that building block.

The broader distinction between identifying promising findings and quantifying their uncertainty also arises in high-dimensional genomics. Fan et al. [21] review frequentist

and Bayesian inferential tools for sparse cancer-genomics models, including confidence or credible intervals and false discovery rates. This provides an applied context for the need to distinguish evidence summaries from the guarantees of the resulting inferential procedure.

More explicitly, if $G \geq F$, then for any alternative law Q and fixed level α,

$$Q\{G(E) \geq \alpha^{-1}\} - Q\{F(E) \geq \alpha^{-1}\} = Q\{F(E) < \alpha^{-1} \leq G(E)\}. \tag{6}$$

Power therefore cannot decrease but increases strictly only if the added rejection region has positive Q-probability. Strict numerical improvement may occur without crossing the chosen threshold. Moreover, admissibility within $\mathcal{U}_{\mathcal{A}}$ does not compare the thresholded rule with every possible level-α test. Section 7 gives an exact illustration.

All results below concern the single vector in (1). Validity after optional stopping requires a process-level formulation, as in confidence sequences [22] and sequential e-value merging [23].

3. Affine Certificates for All Borel Rules

Wang [6] established the ordinary affine domination theorem under monotonicity. Clerico [7] removed this assumption using a concave-envelope argument. For the present analysis, we state and prove a finite-support formulation that does not require measurability of the function being dominated. This formulation is needed because taking a supremum over unconstrained coordinates need not preserve Borel measurability. The proof uses finite-support mixtures, concave envelopes, and a supporting hyperplane at an interior point [24,25].

Lemma 1 (Finite-support affine bound)**.** *Let $d \geq 1$ and let $g : [0,\infty)^d \to [0,\infty)$ be finite, without a measurability assumption. Suppose that every finite choice of $x_j \in [0,\infty)^d$ and $p_j \geq 0$ satisfies*

$$\sum_{j=1}^{n} p_j = 1, \quad \sum_{j=1}^{n} p_j x_j \leq \mathbf{1} \quad \Longrightarrow \quad \sum_{j=1}^{n} p_j g(x_j) \leq 1. \tag{7}$$

Then there is $w \in [0,\infty)^d$ with $\sum_i w_i \leq 1$ such that

$$g(x) \leq 1 + w^{\top}(x - \mathbf{1}) \qquad \text{for every } x \in [0,\infty)^d. \tag{8}$$

Proof. Define

$$h(x) = \sup\left\{\sum_{j=1}^{n} p_j g(x_j) : n \geq 1,\ p_j \geq 0,\ \sum_j p_j = 1,\ \sum_j p_j x_j = x\right\}.$$

A one-point mixture shows that $h \geq g$.

For any mixture with barycenter x, put $b = \max(1, x_1, \ldots, x_d)$. Give this mixture probability $1/b$ and give the remaining probability to zero. The resulting coordinate means are at most one. By (7),

$$\frac{1}{b}\sum_j p_j g(x_j) + \left(1 - \frac{1}{b}\right) g(0) \leq 1.$$

Since $g(0) \geq 0$, the original mixture has expected g-value at most b. Taking the supremum gives

$$0 \leq h(x) \leq b, \qquad h(\mathbf{1}) \leq 1.$$

Combining two finite mixtures and approximating their suprema proves that h is concave. It is finite on the open positive orthant, so it has a supergradient at the interior point $\mathbf{1}$. Thus, for some $w \in \mathbb{R}^d$,

$$h(y) \leq h(\mathbf{1}) + w^{\top}(y - \mathbf{1}), \qquad y \in (0,\infty)^d. \tag{9}$$

The inequality also holds on the boundary. For arbitrary $x \in [0,\infty)^d$ and $0 < \varepsilon < 1$, set $y = (1-\varepsilon)x + \varepsilon\mathbf{1}$. Concavity and (9) give

$$(1-\varepsilon)h(x) + \varepsilon h(\mathbf{1}) \le h(y) \le h(\mathbf{1}) + (1-\varepsilon)w^\top(x-\mathbf{1}).$$

Canceling terms and dividing by $1-\varepsilon$ proves the same affine bound at x.

This affine bound is nonnegative on the entire nonnegative orthant because it dominates h. Sending any one coordinate to infinity gives $w_i \ge 0$. Evaluating at zero gives

$$\sum_i w_i \le h(\mathbf{1}) \le 1.$$

Finally, replacing $h(\mathbf{1})$ in the intercept by one yields (8). □

Theorem 1 (Complete domination representation)**.** *A finite-valued nonnegative Borel rule F belongs to $\mathcal{U}_\mathcal{A}$ if and only if there are fixed vectors $w^S \in D_S$, one for each $S \in \mathcal{A}$, such that*

$$F(e) \le \min_{S\in\mathcal{A}} L_{w^S}(e) \qquad \text{for every } e \in [0,\infty)^K. \tag{10}$$

Proof. Suppose first that the vectors exist. For $P \in \mathcal{P}_\mathcal{A}$, choose $S \in \mathcal{A}$ such that $P \in \mathcal{P}_S$. Then

$$\mathbb{E}_P F(E) \le \mathbb{E}_P L_{w^S}(E) = 1 - \sum_{i\in S} w_i^S + \sum_{i\in S} w_i^S \mathbb{E}_P E_i \le 1.$$

Only coordinates in S enter this expectation.

For necessity, fix $S \in \mathcal{A}$ and define

$$g_S(x) = \sup_{z\in[0,\infty)^{S^c}} F(x_S, z_{S^c}), \qquad x \in [0,\infty)^S.$$

Here (x_S, z_{S^c}) denotes the full vector obtained by inserting the indicated coordinates. If $S = [K]$, take $g_S = F$.

The function g_S is finite. To see this, put $b = \max(1, \max_{i\in S} x_i)$ and assign probability $1/b$ to (x_S, z_{S^c}) and the remaining probability to the all-zero vector. This law belongs to $\mathcal{P}_S$. Nonnegativity and validity imply $F(x_S, z_{S^c}) \le b$, uniformly in z.

Consider a finite mixture of x_j with coordinate means at most one. For $\eta > 0$, choose z_j at each atom so that

$$F((x_j)_S, (z_j)_{S^c}) \ge g_S(x_j) - \eta.$$

The resulting law on the full vectors belongs to $\mathcal{P}_S$, and hence

$$\sum_j p_j g_S(x_j) \le \sum_j p_j F((x_j)_S, (z_j)_{S^c}) + \eta \le 1 + \eta.$$

Letting $\eta \downarrow 0$ verifies the hypothesis of Lemma 1. Only finitely many vectors z_j are chosen for each finite mixture; no measurable map $x \mapsto z(x)$ is constructed and no integral of g_S under a general law is taken. This is sufficient because the lemma requires only the finite inequalities in (7).

Extend the weight vector supplied by that lemma by zeros outside S. The resulting $w^S \in D_S$ satisfies

$$F(e) \le g_S(e_S) \le L_{w^S}(e).$$

Repeating the argument for every $S \in \mathcal{A}$ proves (10). □

The finite choices at the mixture atoms avoid any need for measurable selection. The theorem places each valid rule below a valid, normalized, concave, piecewise-affine envelope.

4. Geometry of Validity and Domination

For a nonempty compact convex set $C \subseteq D$, define

$$H_C(e) = \min_{w \in C} L_w(e) = 1 + \min_{w \in C} w^\top (e - \mathbf{1}). \tag{11}$$

The minimum exists. The function is nonnegative, concave, and coordinatewise nondecreasing. It is also continuous because

$$|H_C(e) - H_C(f)| \le \|e - f\|_\infty.$$

Indeed, every $w \in D$ satisfies $\|w\|_1 \le 1$. All such envelopes satisfy $H_C(\mathbf{1}) = 1$.

Lemma 2 (Validity as face intersection)**.** *For a nonempty compact convex set $C \subseteq D$,*

$$H_C \in \mathcal{U}_\mathcal{A} \quad \Longleftrightarrow \quad C \cap D_S \neq \emptyset \quad \textit{for every } S \in \mathcal{A}. \tag{12}$$

Proof. If $w \in C \cap D_S$, then $H_C \le L_w$, and L_w has expectation at most one under every law in $\mathcal{P}_S$. Applying this for each S proves sufficiency.

Conversely, suppose that $C \cap D_S = \emptyset$ for some $S \in \mathcal{A}$. Compactness and nonnegative coordinates give

$$\delta = \min_{w \in C} \sum_{i \notin S} w_i > 0.$$

A zero minimum would produce a point of $C \cap D_S$.

Take the deterministic vector with $e_i = 1$ on S and $e_i = t > 1$ outside S. Its law belongs to $\mathcal{P}_S$, but

$$H_C(e) = 1 + (t - 1)\delta > 1.$$

This contradicts validity. □

Lemma 3 (Domination reverses containment)**.** *For nonempty compact convex sets $B, C \subseteq D$,*

$$H_B \ge H_C \textit{ on } [0, \infty)^K \quad \Longleftrightarrow \quad B \subseteq C. \tag{13}$$

In particular, $H_B = H_C$ if and only if $B = C$.

Proof. If $B \subseteq C$, minimizing over B gives a value at least as large as minimizing over C.

For the converse, fix $u \in \mathbb{R}^K$. For sufficiently small $\varepsilon > 0$, the vector $\mathbf{1} + \varepsilon u$ is nonnegative. Domination at this input gives

$$\min_{w \in B} w^\top u \ge \min_{w \in C} w^\top u.$$

If $b \in B \setminus C$, strict separation of a point from a compact convex set supplies a direction u such that

$$b^\top u < \min_{w \in C} w^\top u,$$

which is a contradiction. Equality of the envelopes gives both inclusions. □

Normalization at $\mathbf{1}$ allows inputs in its neighborhood to identify every direction in weight space. This explains why the nonnegative input domain suffices for the containment result.

https://doi.org/10.3390/math14213933

5. The Admissibility Characterization

Definition 2. *Let $C \subseteq D$ be a nonempty polytope meeting every D_S, $S \in \mathcal{A}$. A required face D_S is private to a vertex v of C if*

$$C \cap D_S = \{v\}.$$

Theorem 2 (Private-face criterion)**.** *Let $C \subseteq D$ be a nonempty polytope satisfying (12). Then H_C is admissible in $\mathcal{U}_\mathcal{A}$ if and only if each vertex $v \in \mathrm{vert}(C)$ has a private face. Every admissible member of $\mathcal{U}_\mathcal{A}$ has such a representation.*

A private face makes its vertex indispensable: every smaller valid weight set must still meet that face and hence must retain that vertex. Without a private face, every required face containing the vertex also contains another vertex. Its affine certificate can therefore be removed while preserving validity; removing a term from a minimum raises or preserves the reported evidence.

Proof. Write $V = \mathrm{vert}(C)$. Nonnegativity of the coordinates gives

$$C \cap D_S = \mathrm{conv}(V \cap D_S). \tag{14}$$

To verify this identity, express a point of $C \cap D_S$ as a convex combination of vertices. Its coordinates outside S vanish, so every vertex receiving positive weight must also have zero coordinates outside S. The reverse inclusion follows from convexity.

Suppose that a vertex v has no private face. If V were a singleton, every required face would be private to its sole vertex. Thus $|V| \geq 2$. Put

$$B = \mathrm{conv}(V \setminus \{v\}).$$

Every required face still contains a vertex. A face containing v must contain another vertex by (14), while a face not containing v is unaffected. Lemma 2 therefore makes H_B valid.

Since v is a vertex, $B \subsetneq C$. Lemma 3 gives $H_B \geq H_C$, with strict inequality somewhere. Hence H_C is inadmissible.

Now suppose every vertex has a private face, and let $G \in \mathcal{U}_\mathcal{A}$ dominate H_C. Theorem 1 supplies a polytope $B \subseteq D$ meeting every required face and satisfying

$$H_C \leq G \leq H_B.$$

By Lemma 3, $B \subseteq C$. For each vertex v, choose S_v with $C \cap D_{S_v} = \{v\}$. The set B must meet D_{S_v}, so $v \in B$. It follows that $C \subseteq B$, and hence $B = C$. The displayed inequalities imply $G = H_C$.

Finally, Theorem 1 places any admissible F below a valid envelope generated by finitely many certificates. Admissibility forces equality, and the preceding argument gives the private-face criterion. □

Corollary 1 (Shape and size)**.** *Every admissible member of $\mathcal{U}_\mathcal{A}$ is continuous, concave, coordinatewise nondecreasing, and normalized to one at* **1**. *Its representation as H_C, with C a nonempty compact convex subset of D, is unique. This set is a polytope satisfying*

$$|\mathrm{vert}(C)| \leq |\mathcal{A}_{\min}| \leq |\mathcal{A}|.$$

Proof. The shape properties follow from (11), and uniqueness follows from Lemma 3. Distinct vertices cannot share a private face. Choosing one private face for each vertex therefore gives an injection into the family of required faces. Applying Proposition 1 yields the bound using $\mathcal{A}_{\min}$. □

Equivalent Support Condition

For $v \in D$, write $\mathrm{supp}(v) = \{i : v_i > 0\}$. A support covers a set $S \in \mathcal{A}$ when it is contained in S. Coverage here concerns the family of possible valid sets, rather than the union of the coordinate indices.

Corollary 2 (Minimal support covers)**.** *A rule is admissible in $\mathcal{U}_{\mathcal{A}}$ if and only if it can be written as*

$$F(e) = \min_{v \in V} L_v(e) \tag{15}$$

for a finite nonempty set of distinct vectors $V \subseteq D$ satisfying:

1. *Every $S \in \mathcal{A}$ contains $\mathrm{supp}(v)$ for at least one $v \in V$.*
2. *For each $v \in V$, some $S_v \in \mathcal{A}$ contains $\mathrm{supp}(v)$ and contains none of the supports of the other vectors in V.*

Proof. For sufficiency, let $C = \mathrm{conv}(V)$. The first condition gives validity. Under the second condition, a convex combination belonging to D_{S_v} can assign positive weight only to v: every other generator has a positive coordinate outside S_v. Thus

$$C \cap D_{S_v} = \{v\}.$$

This also proves that v is a vertex, since it cannot be a convex combination of the other generators. Theorem 2 applies.

For necessity, take the vertex set of the polytope in Theorem 2 and use (14). □

The cover is inclusion-minimal: deleting any member destroys coverage. Minimum cardinality is not required. The corollary asserts the existence of a reduced representation. An arbitrary generating list may contain duplicate or nonvertex terms that leave its minimum unchanged.

Corollary 3 (Sharpness of the vertex bound)**.** *For every family $\mathcal{A}$ in the model, there is an admissible rule whose unique weight polytope has exactly $|\mathcal{A}_{\min}|$ vertices.*

Proof. For each $S \in \mathcal{A}_{\min}$, choose $w^S \in D$ with $\mathrm{supp}(w^S) = S$. For example, set

$$w_i^S = \begin{cases} |S|^{-1}, & i \in S, \\ 0, & i \notin S. \end{cases}$$

Every member of $\mathcal{A}$ contains a member of $\mathcal{A}_{\min}$, so these supports cover $\mathcal{A}$.

The family $\mathcal{A}_{\min}$ is an antichain. Therefore, if $S, T \in \mathcal{A}_{\min}$ and $S \neq T$, then $\mathrm{supp}(w^T) = T$ is not contained in S. Each S is a private witness for w^S. Corollary 2 proves admissibility and shows that every w^S is a vertex of their convex hull. There are exactly $|\mathcal{A}_{\min}|$ such vertices. □

Corollary 4 (Constant and ordinary rules)**.** *If zero belongs to the vertex set of an admissible representation, the rule is identically one. For $\mathcal{A} = \{[K]\}$, the admissible rules are exactly L_w, $w \in D$.*

Proof. The support of zero is empty and is contained in every valid set. No other vertex could then have a private witness. Thus zero must be the sole vertex.

When $\mathcal{A} = \{[K]\}$, the only required face is D. The private-face criterion holds exactly when the polytope is a singleton. □

The second statement recovers the ordinary affine class [6,7]. A general family of possible valid sets permits several indispensable affine pieces.

6. A Finite Procedure for Admissible Improvement

Theorem 3 (Finite admissible completion)**.** *Every $F \in \mathcal{U}_{\mathcal{A}}$ is dominated by an admissible member of $\mathcal{U}_{\mathcal{A}}$.*

More precisely, suppose that $C \subseteq D$ is a nonempty polytope satisfying

$$F \le H_C, \qquad C \cap D_S \neq \emptyset \quad \text{for every } S \in \mathcal{A}.$$

Starting from C, repeatedly delete a vertex without a private face and take the convex hull of the remaining vertices. The procedure terminates at a polytope C_* for which

$$F \le H_C \le H_{C_*},$$

and H_{C_*} is admissible. Each deletion produces strict pointwise domination of the preceding envelope.

Proof. Theorem 1 supplies a starting polytope with the stated properties for every $F \in \mathcal{U}_{\mathcal{A}}$.

At any stage, the necessity proof of Theorem 2 shows that deleting a vertex without a private face preserves intersection with every required face. The resulting envelope is valid. The new convex hull is a proper subset of the preceding hull, so Lemma 3 gives weak domination everywhere and strict inequality somewhere.

The number of vertices decreases at each deletion. A valid singleton has a private face and cannot be deleted. The procedure therefore terminates, and all remaining vertices have private faces. Theorem 2 proves admissibility of the final envelope. □

For an explicitly listed family $\mathcal{A}$ and a current vertex set V, define the incidence counts

$$n_S = |\{v \in V : \operatorname{supp}(v) \subseteq S\}|, \qquad S \in \mathcal{A}. \tag{16}$$

A vertex has a private witness exactly when it occurs in some row with $n_S = 1$. Deleting a vertex with no such row leaves all counts positive.

A subset of the vertex set remains a vertex set for its own convex hull: a convex representation contradicting this fact would also contradict extremality in the original hull. Thus, after reducing the initial generating list to its vertices, subsequent deletions can be tracked through the incidence counts.

The starting envelope must be valid. Merely dominating a valid rule is insufficient. For example, take $K = 2$ and $\mathcal{A} = \{\{1\}, \{2\}\}$. The rule $\min(e_1, e_2)$ is valid and is dominated by e_1, but e_1 is not valid for this family.

Theorem 1 proves the existence of certificates without providing an efficient method for extracting them from an arbitrary Borel rule. The deletion procedure assumes that the certificates and required incidences are available. For an arbitrary Borel input rule, finding w^S involves the constraints $L_{w^S}(e) \ge F(e)$ for every e; Theorem 1 is an existence result, not a finite oracle for checking these constraints. In explicit constructions, the weights can instead be supplied directly, as in Section 7. A nonvertex generator can be recognized by a finite linear feasibility problem expressing it as a convex combination of the other generators.

For $N = |\mathcal{A}_{\min}|$ listed sets and n initial vertices, the incidence table can be built with $O(KnN)$ coordinate checks and stored in $O(nN)$ space. A direct implementation that rescans the table after each deletion uses at most $O(n^2 N)$ further operations. These bounds exclude certificate extraction and initial vertex reduction. They describe a finite procedure for moderate explicitly supplied families, without asserting scalability for arbitrary implicit families.

Some structured rules avoid enumeration. When all m-subsets are possible, the lower trimmed mean in Section 7 is evaluated by sorting in $O(K \log K)$ time, despite its $\binom{K}{m}$ certificates. Another construction is $\min_{j \in J} e_j$ for a fixed set J of size $K - m + 1$. Every m-subset meets J, and $([K] \setminus J) \cup \{j\}$ is a private witness for each $j \in J$; hence Corollary 2 proves

admissibility without listing all m-subsets. Neither shortcut applies to general asymmetric weights without additional structure.

The completion is not canonical: different deletion orders can produce incomparable admissible improvements. Theorem 3 guarantees an admissible dominating rule, not a unique final rule or an optimum for a specified alternative. A fixed deletion order ensures reproducibility. If an alternative law Q is specified in advance, the expected-log criterion in Section 8 can rank the available completions; optimizing over these completions is a smaller problem than optimizing over all of $\mathcal{U}_{\mathcal{A}}$.

7. Algebraic Examples

7.1. An Asymmetric Admissible Merger

Let $K = 3$ and let $\mathcal{A}$ consist of the three pairs. For any fixed $0 < a < 1$, define

$$F_a(e) = \min\{e_1,\ ae_2 + (1-a)e_3\}. \tag{17}$$

The two weights have supports $\{1\}$ and $\{2,3\}$. The first covers $\{1,2\}$ and $\{1,3\}$; the second covers $\{2,3\}$. The sets $\{1,2\}$ and $\{2,3\}$ are private witnesses for the respective weights.

Corollary 2 proves admissibility against all competitors in $\mathcal{U}_{\mathcal{A}}$. The numerical choice of a changes the output but does not affect the support argument.

7.2. Deleting an Unnecessary Certificate

For the same family, consider

$$F(e) = \min\{e_1, e_2, (e_1+e_3)/2\}, \qquad G(e) = \min\{e_1, e_2\}. \tag{18}$$

The supports $\{1\}$ and $\{2\}$ already cover every pair. The third support has no private witness.

Both rules are valid, and $G \geq F$ everywhere. At $e = (4,4,0)$,

$$F(e) = 2, \qquad G(e) = 4.$$

The rule G is admissible because $\{1,3\}$ and $\{2,3\}$ are private witnesses. The displayed vector is a pointwise comparison, not a deterministic null law.

For a fixed-level comparison, retain these F, G and set $\alpha = 0.05$. Under an alternative law Q_+, let E take the values $(30,30,0)$, $(60,60,0)$, and $(2,2,0)$ with probabilities $1/2$, $1/4$, and $1/4$, respectively. The corresponding (F,G) values are $(15,30)$, $(30,60)$, and $(1,2)$. At threshold 20, the rejection probabilities are therefore

$$Q_+\{F(E) \geq 20\} = \tfrac{1}{4}, \qquad Q_+\{G(E) \geq 20\} = \tfrac{3}{4}.$$

By contrast, let Q_0 assign probability $1/2$ to each of $(60,60,0)$ and $(2,2,0)$. Both rejection probabilities then equal $1/2$, although $G(E) > F(E)$ at both support points. Both laws lie outside $\mathcal{P}_{\mathcal{A}}$, since the first two coordinate means exceed one and only the third has mean at most one. These exact probabilities illustrate (6); they are not simulation estimates.

The result of deletion need not be unique. Starting from $\min(e_1,e_2,e_3)$, any coordinate weight can be deleted, leaving the admissible minimum of the other two coordinates. The resulting three rules are pairwise incomparable. For example, $\min(e_1,e_2)$ is smaller than $\min(e_1,e_3)$ at $(3,0,3)$ and larger at $(3,3,0)$.

Figure 1 illustrates the removal of an affine certificate without a private face.

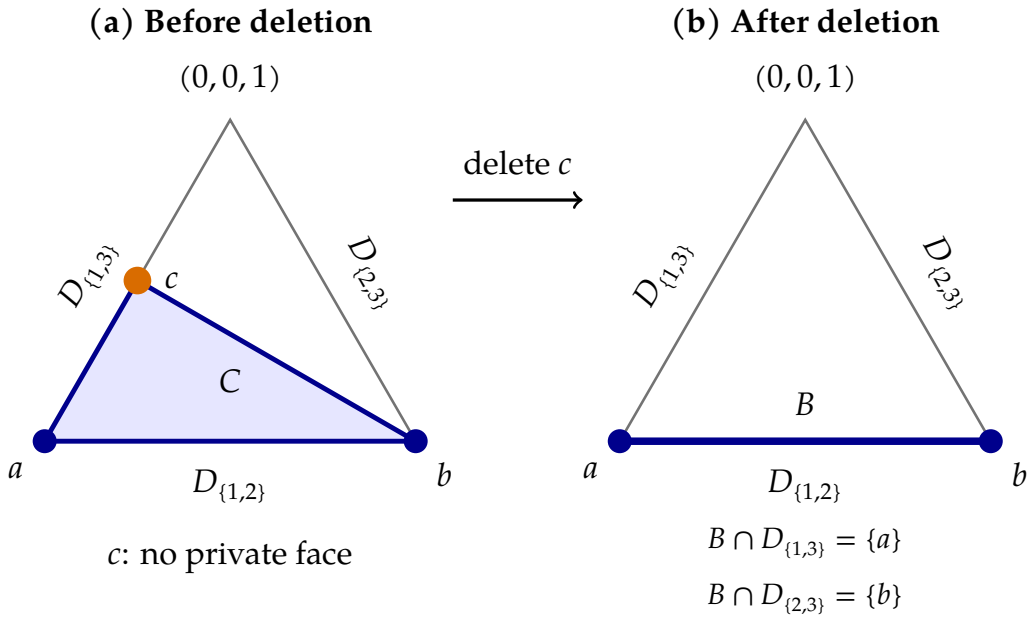


Figure 1. Vertex deletion for $K = 3$ when every pair is a possible valid set. Here $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $c = (1/2, 0, 1/2)$. The outer triangle is the normalized slice $\sum_i w_i = 1$ of D; its edges are the intersections of this slice with the indicated coordinate faces. In panel (**a**), $C = \text{conv}\{a, b, c\}$ meets all three required faces, but c has no private face: $C \cap D_{\{1,3\}} = [a, c]$. Deleting c gives the segment B in panel (**b**), where a and b have private faces $D_{\{1,3\}}$ and $D_{\{2,3\}}$, respectively. Hence $G = H_B$ is admissible and dominates $F = H_C$. The improvement is strict, for example, at $e = (4, 4, 0)$, where $F(e) = 2$ and $G(e) = 4$. Blue dots mark the retained vertices, and the orange dot marks the deleted vertex. The blue shading indicates the initial polytope, the blue segment indicates the remaining polytope, and the gray lines show the boundary of the normalized simplex.

7.3. Full-Support Certificates on an Antichain

Suppose $\mathcal{A}$ is an antichain: no member is properly contained in another. For each $S \in \mathcal{A}$, choose $w^S \in D$ whose support is exactly S. Then

$$F(e) = \min_{S \in \mathcal{A}} \left\{ 1 - \sum_i w_i^S + \sum_i w_i^S e_i \right\} \tag{19}$$

is admissible.

Indeed, each S is a private witness for its own weight. This includes every family of distinct equal-size sets. Positive constant terms are permitted because $\sum_i w_i^S$ may be less than one. The construction also attains the vertex bound in Corollary 1.

7.4. Lower Trimmed Means

Under (2), choose $w_i^S = \lambda/m$ for $i \in S$ and zero otherwise. For $0 < \lambda \leq 1$, the resulting rule is

$$T_\lambda(e) = 1 - \lambda + \frac{\lambda}{m} \sum_{j=1}^{m} e_{(j)}. \tag{20}$$

The minimum sum over an m-subset is the sum of the m smallest inputs.

The preceding support argument proves admissibility in the full class $\mathcal{U}_\mathcal{A}$. At $\lambda = 0$, the rule is the admissible constant one. The formula and its subset-minimum validity argument are established constructions, not new merging formulas; see the arithmetic-mean discovery construction in Section 6 of [11]. The conclusion here is their admissibility for the union model (2) against all finite-valued nonnegative Borel competitors, including asymmetric and nonmonotone ones. For $\lambda > 0$, every m-subset is the full support of its own certificate

and is a private witness because no other distinct m-subset is contained in it. This proves the stated admissibility directly from the general characterization.

Proposition 2 (No greatest valid merger)**.** *For every nonempty family $\mathcal{A}$ in the model, $\mathcal{U}_{\mathcal{A}}$ has no greatest member under pointwise ordering.*

Proof. The constant one is admissible by Corollary 4. A greatest valid rule would dominate it and hence equal it.

However,

$$M(e) = \min_{1 \le i \le K} e_i$$

is valid. For any $P \in \mathcal{P}_{\mathcal{A}}$, choose a valid set $S \in \mathcal{A}$ and an index $i \in S$. Then $M(E) \le E_i$, so $\mathbb{E}_P M(E) \le 1$. At $e = \mathbf{21}$, one has $M(e) = 2 > 1$, which contradicts the existence of a greatest rule. □

8. Further Development: Selecting an Admissible Rule

Admissibility leaves a choice among incomparable rules. Let Q be a specified law supported on $[0, M]^K$ for some finite $M > 0$. Define

$$\Gamma_{\mathcal{A}}(Q) = \sup_{F \in \mathcal{U}_{\mathcal{A}}} \mathbb{E}_Q \log F(E), \qquad \log 0 = -\infty. \tag{21}$$

Expected logarithmic evidence is related to the growth-rate criterion in safe testing [3]. Bounded support removes integrability issues at infinity, while allowing the objective to equal $-\infty$ for individual rules.

Specifically, it provides the common finite upper bound $\log \max(1, M)$ for all certificate envelopes, which is used in the Fatou argument below to obtain upper semicontinuity. It does not bound the evidence away from zero. Bounded support is a sufficient assumption for this proof, rather than a claim that existence always fails for unbounded alternatives.

Proposition 3 (Existence of an admissible maximizer)**.** *Under the bounded-support assumption, $\Gamma_{\mathcal{A}}(Q)$ is finite and is attained by an admissible envelope satisfying Theorem 2.*

Proof. Theorem 1 and monotonicity of the logarithm reduce the optimization to tuples

$$w = (w^S)_{S \in \mathcal{A}} \in \prod_{S \in \mathcal{A}} D_S$$

and their envelopes

$$F_w(e) = \min_{S \in \mathcal{A}} L_{w^S}(e).$$

The parameter space is compact. For each e, the map $w \mapsto F_w(e)$ is continuous.

Put $B = \max(1, M)$. On the support of Q, every affine certificate and its envelope satisfy $0 \le F_w(e) \le B$. For a sequence $w_n \to w$, Fatou's lemma applied to the nonnegative extended-real functions

$$\log B - \log F_{w_n}(E)$$

gives

$$\mathbb{E}_Q \log F_w(E) \ge \limsup_{n \to \infty} \mathbb{E}_Q \log F_{w_n}(E).$$

The objective is therefore upper semicontinuous on a compact parameter space and attains its maximum.

The all-zero tuple produces $F_w \equiv 1$, with objective zero. The uniform upper bound $\log B$ shows that the maximum is finite. Finally, Theorem 3 produces an admissible envelope dominating any maximizing envelope. Its objective cannot decrease and hence remains maximal. □

8.1. Choosing Among Different Completions

Return to the three admissible completions from Section 7:

$$G_{12}(e) = \min(e_1, e_2), \quad G_{13}(e) = \min(e_1, e_3), \quad G_{23}(e) = \min(e_2, e_3).$$

Specify an alternative Q_p with mass p at $u = (4, 4, 1)$ and mass $1 - p$ at $v = (4, 1, 4)$, where $0 < p < 1$. Its coordinate means are $4, 1 + 3p$, and $4 - 3p$, so it lies outside the union null. The evidence values at (u, v) are $(4, 1)$ for G_{12}, $(1, 4)$ for G_{13}, and $(1, 1)$ for G_{23}. Consequently,

$$\mathbb{E}_{Q_p} \log G_{12} = p \log 4, \qquad \mathbb{E}_{Q_p} \log G_{13} = (1 - p) \log 4, \qquad \mathbb{E}_{Q_p} \log G_{23} = 0. \tag{22}$$

For example, at $p = 3/4$, these are approximately 1.040, 0.347, and 0. Thus G_{12} is preferred among these three completions when $p > 1/2$, G_{13} when $p < 1/2$, and the first two tie when $p = 1/2$. This ranks the completions of the given initial envelope; it does not assert that one of them maximizes the objective over all valid mergers.

The law used for this design comparison is specified before observing the vector to be merged. Choosing the rule with the largest realized evidence after inspecting that vector is a different operation and is not justified by validity of each fixed rule.

For a specified finite-support law $Q = \sum_{\ell=1}^{L} q_\ell \delta_{x^\ell}$ with positive masses, the full certificate optimization can be written explicitly as

$$\begin{aligned} \text{maximize} \quad & \sum_{\ell=1}^{L} q_\ell \log t_\ell \\ \text{subject to} \quad & w^S \in D_S \quad (S \in \mathcal{A}), \\ & 0 \le t_\ell \le L_{w^S}(x^\ell) \quad (\ell = 1, \dots, L,\ S \in \mathcal{A}). \end{aligned} \tag{23}$$

The logarithm is interpreted as in (21). The constant rule makes a finite objective feasible, so an optimizer has positive t_ℓ at all positive-mass atoms. For fixed weights, the objective is maximized by setting each t_ℓ to the corresponding envelope value. Hence this finite-dimensional concave optimization has the same optimum as Proposition 3; admissible completion of an optimizing envelope preserves that optimum. The formulation still requires the supplied support of Q and the validity sets and does not resolve the general enumeration problem.

8.2. A Sufficient Condition for Zero Optimal Growth

The candidate law Q need not lie outside the null class. The following result identifies a class of cases where the constant rule is already optimal.

Proposition 4 (Zero growth under a null law)**.** *If $Q \in \mathcal{P}_{\mathcal{A}}$ and Q has bounded support, then*

$$\Gamma_{\mathcal{A}}(Q) = 0.$$

In particular, suppose $\mathcal{A} = \{S \subseteq [K] : |S| = m\}$ and

$$Q(E_i = b_i) = p_i, \qquad Q(E_i = 0) = 1 - p_i,$$

where $b_i > 1$ and $0 < p_i < 1$. If at least m coordinates satisfy $p_i b_i \le 1$, then $\Gamma_{\mathcal{A}}(Q) = 0$. Independence is not needed for this conclusion.

Proof. For every $F \in \mathcal{U}_{\mathcal{A}}$, validity under Q gives $\mathbb{E}_Q F(E) \le 1$. Jensen's inequality yields

$$\mathbb{E}_Q \log F(E) \le \log \mathbb{E}_Q F(E) \le 0.$$

The inequality remains valid when the logarithmic expectation is $-\infty$. It can also be obtained by applying Jensen's inequality to $F(E) + \varepsilon$ and letting $\varepsilon \downarrow 0$. The constant rule $F \equiv 1$ attains zero.

In the two-point model, $\mathbb{E}_Q E_i = p_i b_i$. If at least m of these means are at most one, there is an m-subset S with $Q \in \mathcal{D}_S$. Thus $Q \in \mathcal{D}_{\mathcal{A}}$, and the first part applies. □

The condition in Proposition 4 is sufficient. It does not characterize every law with zero optimal growth, and its failure alone does not establish positive optimal growth.

8.3. Question for Further Study

Problem 1. *Let $\mathcal{A} = \{S \subseteq [K] : |S| = m\}$. Suppose that under Q the coordinates are independent and*

$$Q(E_i = b_i) = p_i, \qquad Q(E_i = 0) = 1 - p_i, \qquad b_i > 1, \quad 0 < p_i < 1.$$

Determine the parameter region in which $\Gamma_{\mathcal{A}}(Q) > 0$. Within that region, characterize how the optimal minimal support covers change with $(p_i, b_i)_{i=1}^K$.

Can an optimal admissible representation be identified without enumerating every m-subset when m grows with K?

The model allows nonexchangeable alternatives because the coordinate parameters may differ. For an explicitly enumerated family and a finite-support law, the certificate formulation is a finite-dimensional concave maximization problem over a convex parameter set. Indeed, at each input, F_w is a nonnegative concave function of the weights, and the extended logarithm is increasing and concave. This formulation establishes a route to optimization but does not characterize the support transitions or eliminate the enumeration cost. Problem 1 is left unresolved here.

A further extension would introduce conditional moment constraints and allow the reliable set to vary over time. That requires a sequential model of the kind studied in [23]. The fixed-set quantifier in (1) cannot simply be replaced by an outcome-dependent choice. Simultaneous discovery statements likewise require additional structure, as in closed-testing approaches to exploratory inference [26].

9. Conclusions

Under uncertain coordinate validity, each possible valid set supplies an affine certificate. The convex hull of these certificates encodes both validity and pointwise comparison. Admissibility holds precisely when every vertex is forced by a private coordinate face.

This characterization gives a finite admissible-completion procedure and a sharp bound on the number of affine pieces. It also shows that continuity, concavity, and monotonicity are consequences of admissibility within the full Borel class.

Selecting a rule requires an additional criterion. Expected logarithmic evidence admits an admissible maximizer under bounded support, but the constant rule is optimal for every candidate law belonging to the null class. Characterizing positive growth and the associated optimal support configurations is the next question posed by this framework.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

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