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Moment Structure and Approximation Properties of Univariate and Bivariate Durrmeyer–Beta-Type Operators

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Abstract

We study a Durrmeyer–Beta-type positive linear operator on the half-line, constructed from rising-factorial weights and a normalized beta-prime kernel. Exact raw moments through degree four and the corresponding central moments are derived under the required moment-existence conditions. These identities yield local approximation estimates, weighted Korovkin convergence, and a weighted $O(n^{-1})$ rate for functions with bounded first and second derivatives. A tensor-product extension provides bivariate convergence and error bounds involving partial moduli of continuity and coordinatewise regularity. Comparisons with related Durrmeyer constructions clarify the operator’s structure, while a matched-index comparison with a Dunkl–Beta operator examines its numerical accuracy. The numerical examples include smooth, oscillatory, and nonsmooth bounded functions, with an additional experiment on $[0, 20]$. We also investigate reconstruction from noisy observations using 30 signal–noise realizations at each of five Gaussian noise levels. The comparison includes a Gaussian kernel smoother, an ARIMA model, and an LSTM reconstruction network. The Gaussian smoother gives the best mean accuracy at the lowest noise level, whereas D_{80} gives the lowest mean normalized errors and the highest mean R^2 at the four higher levels. These experiments illustrate the approximation and smoothing behavior of the operator under the stated computational settings.

Keywords: Durrmeyer–Beta operators; positive linear operators; beta-prime kernels; central moments; weighted approximation; bivariate approximation; signal reconstruction; kernel smoothing



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1. Introduction

Positive linear operators provide constructive methods for approximating functions on both compact and unbounded domains. Bernstein’s probabilistic proof of the Weierstrass approximation theorem is a classical starting point in this direction [1]. On the half-line, Szász–Mirakjan-type operators replace the binomial structure of the Bernstein polynomials with Poisson-type weights. Lupaş subsequently introduced a family of positive linear oper-

ators based on rising factorials [2], and Agratini investigated a linear positive specialization of this family [3].

Several Kantorovich, beta-type, and fractional modifications have subsequently been developed. Representative contributions can be found in [4–21]. These studies motivate the construction considered here, which combines rising-factorial weights with a beta-prime kernel on $[0, \infty)$.

Recent constructions also employ special polynomials and modified approximation nodes. Rao et al. [22] studied positive linear operators combining adjoint Bernoulli polynomials with gamma kernels, including a bivariate extension. Their Bernstein operators with shifted knots provide local, weighted, and statistical approximation results [23]. Another development uses Appell polynomials to construct Szász–Durrmeyer–Chlodowsky operators [24]. These studies provide further examples of how the choice of generating polynomials, nodes, and integral kernels determines the moment structure of an approximation process.

A closely related contribution is the generalized Lupaş–Durrmeyer construction studied by Qasim et al. [25]. Their operators use beta-type basis functions determined by an auxiliary function μ . The corresponding analysis includes moment identities, weighted approximation results, local estimates, and a Voronovskaya-type theorem.

The operator studied here has a different weight–kernel structure. It combines the rising-factorial weights

$$p_{n,k}(z) = \frac{2^{-nz}(nz)_k}{2^k k!}$$

with the normalized beta-prime kernel

$$q_{n,k}(t) = \frac{t^k}{B(k+1, n)(1+t)^{k+n+1}}.$$

Here, $n > 0$ is the approximation parameter, $z \geq 0$ is the point at which the approximation is evaluated, and $k = 0, 1, 2, \dots$ is the summation index. The notation $(nz)_k$ denotes the rising factorial, and B denotes the beta function; both are defined in Section 2. The weights $p_{n,k}(z)$ determine the contribution of each integral average, while $q_{n,k}(t)$ is the density used to form that average over the integration variable $t \geq 0$.

Although the two families share certain beta-type features, no single admissible parameter choice in the construction of Qasim et al. simultaneously reproduces both the weights $p_{n,k}(z)$ and the kernels $q_{n,k}(t)$. Therefore, the present operator is not obtained as a direct special case of that generalized Lupaş–Durrmeyer family. This structural distinction is discussed further in Section 2.

Another related direction concerns Durrmeyer sampling-type operators. These operators combine generalized sampling series with integral averages generated by discrete and continuous kernels [26,27]. Multivariate and weighted extensions have been studied on $\mathbb{R}^d$, where d denotes the number of variables [28]. More recent direct and inverse results relate their convergence rates to Sobolev regularity and generalized Lipschitz classes [29].

Both constructions combine a discrete sum with integral averages. The sampling operators considered in these works act on the real line or its higher-dimensional counterpart, and their sums run over integer lattice points. The present operator acts on the half-line $[0, \infty)$ and sums over the nonnegative integers $k = 0, 1, 2, \dots$. Further differences in the averaging kernels and function spaces are discussed in Section Relation to Other Durrmeyer-Type Constructions.

The purpose of this study is to establish the approximation behavior of this particular weight–kernel pairing from its moment structure. The main contributions are as follows:

1. Exact raw moments through degree four are derived together with the conditions on n required for their existence;
2. The first four central moments are calculated explicitly, including their compact-uniform asymptotic limits and the estimate $\mu_{n,4}(z) = O(n^{-2})$ on compact intervals;
3. Local approximation estimates, Hölder–Lipschitz bounds, weighted Korovkin convergence, and an $O(n^{-1})$ weighted rate are obtained from these moment identities;
4. A tensor-product bivariate extension is constructed, and its compact-uniform, partial-modulus, and coordinatewise Lipschitz estimates are established; and
5. Numerical experiments examine approximation of smooth and nonsmooth functions, including a wider-interval example on $[0, 20]$, and assess noisy-signal reconstruction through 30 realizations at each noise level, with a Gaussian kernel smoother, ARIMA, and an LSTM reconstruction network as reference procedures.

The remainder of the paper is organized as follows. Section 2 introduces the Durrmeyer–Beta operator, specifies its natural domain, compares it with related constructions, and derives its raw moments. Section 3 establishes the central-moment identities and their asymptotic behavior. Sections 4 and 5 present the local and weighted approximation results, respectively. Section 6 develops the bivariate tensor-product extension and proves the associated approximation estimates. Section 7 reports the univariate and bivariate numerical illustrations. Section 8 presents an exploratory comparison using synthetic signals contaminated by Gaussian noise. Finally, Section 9 summarizes the main conclusions and identifies directions for further study.

The Python 3.13 code used to reproduce the numerical experiments, figures, and reported metrics is publicly available at <https://github.com/metinzontul/Durrmeyer-Beta-Approximation> (accessed on 16 September 2026).

2. Construction, Natural Domain, and Raw Moments of the Durrmeyer–Beta Operator

One of the fundamental positive linear approximation processes on the half-line is the Szász–Mirakjan operator [30,31]. For $f \in C_B[0, \infty)$ and $n \in \mathbb{N}$, it is defined by

$$S_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} e^{-nx}, \quad x \geq 0. \quad (1)$$

The restriction to bounded continuous functions ensures that the series in (1) is absolutely convergent. More general function classes can be considered under suitable growth conditions, but continuity alone on the unbounded interval is not sufficient.

A different family of positive linear operators on the half-line, based on rising factorials, was introduced by Lupaş [2] and subsequently studied by Agratini [3]. In the notation used here, this family is given by

$$L_n(h; z) = (1-a)^{nz} \sum_{k=0}^{\infty} \frac{(nz)_k}{k!} a^k h\left(\frac{k}{n}\right), \quad z \geq 0, \quad 0 \leq a < 1, \quad (2)$$

where h is a bounded function on $[0, \infty)$.

The normalization of the weights follows from the generalized binomial expansion [32]

$$(1-a)^{-\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1, \quad (3)$$

where

$$(\alpha)_0 = 1$$

and

$$(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1), \quad k \geq 1.$$

Taking $\alpha = nz$ in (3) gives

$$(1-a)^{nz} \sum_{k=0}^{\infty} \frac{(nz)_k}{k!} a^k = 1.$$

Thus, the coefficients in (2) form a normalized system of nonnegative weights. Choosing $a = 1/2$ in (2) yields the modified Lupaş–Agratini operator

$$L_n^M(h; z) = 2^{-nz} \sum_{k=0}^{\infty} \frac{(nz)_k}{2^k k!} h\left(\frac{k}{n}\right). \quad (4)$$

For later use, put

$$p_{n,k}(z) = \frac{2^{-nz} (nz)_k}{2^k k!}, \quad k \geq 0. \quad (5)$$

It follows from (3) that

$$p_{n,k}(z) \geq 0, \quad \sum_{k=0}^{\infty} p_{n,k}(z) = 1. \quad (6)$$

The Durrmeyer approach replaces point evaluations by suitable integral averages; its origin can be traced to the construction introduced by Durrmeyer [33]. Related integral modifications with beta-type kernels on unbounded intervals have subsequently been considered in several settings [12,14,25]. Motivated by this framework, we combine the discrete weights in (5) with a normalized beta-prime kernel.

For $k \geq 0$ and $n > 0$, define

$$q_{n,k}(t) = \frac{1}{B(k+1, n)} \frac{t^k}{(1+t)^{k+n+1}}, \quad t \geq 0, \quad (7)$$

where

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a, b > 0. \quad (8)$$

The beta and gamma function identities used here are standard; see [32].

The substitution $u = \frac{t}{1+t}$, $t = \frac{u}{1-u}$, $dt = \frac{du}{(1-u)^2}$ gives

$$\begin{aligned} \int_0^{\infty} \frac{t^k}{(1+t)^{k+n+1}} dt &= \int_0^1 u^k (1-u)^{n-1} du \\ &= B(k+1, n). \end{aligned}$$

Consequently,

$$\int_0^{\infty} q_{n,k}(t) dt = 1, \quad n > 0. \quad (9)$$

We now define the Durrmeyer–Beta-type operator by

$$\begin{aligned} D_n(f; z) &= \sum_{k=0}^{\infty} p_{n,k}(z) \int_0^{\infty} q_{n,k}(t) f(t) dt \\ &= \sum_{k=0}^{\infty} p_{n,k}(z) \frac{1}{B(k+1, n)} \int_0^{\infty} \frac{t^k}{(1+t)^{k+n+1}} f(t) dt, \end{aligned} \quad z \geq 0, \quad n > 0. \quad (10)$$

This definition is motivated by the Durrmeyer and beta-type constructions cited above, but the particular pairing of the weights $p_{n,k}(z)$ with the kernels $q_{n,k}(t)$ in (10) is the operator studied in the present work.

For an integer $j \geq 0$, the j th moment of the beta-prime kernel is

$$\begin{aligned} \int_0^\infty t^j q_{n,k}(t) dt &= \frac{1}{B(k+1, n)} \int_0^\infty \frac{t^{k+j}}{(1+t)^{k+n+1}} dt \\ &= \frac{B(k+j+1, n-j)}{B(k+1, n)}, \quad n > j. \end{aligned} \quad (11)$$

The condition $n > j$ is necessary for the integral to be finite. Indeed, as $t \rightarrow \infty$, its integrand is of order t^{j-n-1} , which is integrable at infinity precisely when $n > j$. Thus, the kernel is normalized for every $n > 0$, whereas its j th raw moment exists only for $n > j$.

We next specify the natural domain of the operator. Let $\mathcal{B}([0, \infty))$ denote the Borel σ -algebra on $[0, \infty)$. We follow the standard measure-theoretic convention that the integral of a nonnegative Borel measurable function is first interpreted as an extended nonnegative value in $[0, \infty]$; see ([34], Chapters 1–2). Accordingly, if f is nonnegative and Borel measurable, the right-hand side of (10) is well defined as an element of $[0, \infty]$.

For each fixed $n > 0$, define

$$\begin{aligned} \mathcal{A}_n := \{f : [0, \infty) \rightarrow \mathbb{R} : f \text{ is } \mathcal{B}([0, \infty))\text{-measurable, and} \\ D_n(|f|; z) < \infty \text{ for every } z \geq 0\}. \end{aligned} \quad (12)$$

For $f \in \mathcal{A}_n$, let

$$f^+(t) = \max\{f(t), 0\}, \quad f^-(t) = \max\{-f(t), 0\}.$$

Since $0 \leq f^+(t) \leq |f(t)|$, $0 \leq f^-(t) \leq |f(t)|$, the positivity of the operator gives $D_n(f^+; z) < \infty$, $D_n(f^-; z) < \infty$.

Therefore,

$$D_n(f; z) = D_n(f^+; z) - D_n(f^-; z) \quad (13)$$

is finite and well defined for every $f \in \mathcal{A}_n$ and $z \geq 0$.

The normalizations in (6) and (9) imply $D_n(1; z) = 1$.

If $f \in C_B[0, \infty)$, then

$$D_n(|f|; z) \leq \|f\|_\infty D_n(1; z) = \|f\|_\infty.$$

Hence,

$$C_B[0, \infty) \subset \mathcal{A}_n, \quad n > 0. \quad (14)$$

More generally, suppose that f is continuous and satisfies

$$|f(t)| \leq C(1 + t^2), \quad t \geq 0,$$

for some constant $C > 0$. Then

$$D_n(|f|; z) \leq C(D_n(1; z) + D_n(t^2; z)),$$

which is finite whenever $n > 2$. Therefore, every continuous function of at most quadratic growth belongs to $\mathcal{A}_n$ for $n > 2$.

Finally, for an integer $j \geq 0$, Formula (11) shows that

$$e_j(t) = t^j \in \mathcal{A}_n \iff n > j. \quad (15)$$

In particular, the first, second, third, and fourth raw moments require $n > 1$, $n > 2$, $n > 3$, and $n > 4$, respectively.

Remark 1. The operator (10) is related to, but is not a special case of, the generalized Lupaş–Durrmeyer operators of Qasim et al. [25]. Choosing $m = n$ and $\mu(y) = y$ makes their discrete weights coincide with $p_{n,k}(z)$, but their beta kernel $b_{n,k-1}$ differs from $q_{n,k}$. Conversely, the choice $m = n - 1$ with the index shift $j = k + 1$ gives $b_{n-1,k}(t) = q_{n,k}(t)$, but the corresponding discrete weights do not coincide. The difference is also evident at the origin:

$$D_{m,\mu}^*(f;0) = f(0), \quad D_n(f;0) = n \int_0^\infty \frac{f(t)}{(1+t)^{n+1}} dt.$$

For example, $D_n(t;0) = 1/(n-1) \neq 0$ for $n > 1$. Moreover, the unstarred operator of Qasim et al. does not preserve constants, whereas $D_n(1;z) = 1$. Hence neither operator introduced by Qasim et al. reduces to (10) through these parameter choices.

Relation to Other Durrmeyer-Type Constructions

The term “Durrmeyer-type” covers several constructions that share an integral-averaging principle but may otherwise have substantially different structures. The classical Bernstein–Durrmeyer operator is defined on $[0, 1]$ through a finite sum of Bernstein basis functions and integral means generated by the same polynomial basis. The generalized family studied by Abel et al. [35] retains this finite polynomial structure and includes both the classical and genuine Bernstein–Durrmeyer operators. In contrast, the operator D_n considered here acts on $[0, \infty)$, contains an infinite series, and combines rising-factorial weights with beta-prime densities.

A different implementation of the Durrmeyer principle is provided by the Durrmeyer sampling-type operators. Their standard univariate form is

$$(\mathcal{D}_w^{\varphi,\psi} f)(x) = \sum_{k \in \mathbb{Z}} \varphi(wx - k) w \int_{\mathbb{R}} \psi(wu - k) f(u) du, \quad x \in \mathbb{R}, \quad w > 0, \quad (16)$$

where φ and ψ are, respectively, discrete and continuous generating kernels. This family was introduced by applying the Durrmeyer integral modification to generalized sampling series [26]. Its asymptotic behavior and quantitative Voronovskaya-type formulas have subsequently been studied through algebraic kernel moments and K -functionals [36].

The operators $\mathcal{D}_w^{\varphi,\psi}$ have also been examined in Orlicz and L^p spaces. The available results include pointwise and uniform convergence, modular convergence, and quantitative estimates in terms of moduli of smoothness [27,37]. Further developments address their regularization properties in L^p spaces [38] and establish direct and inverse theorems characterizing generalized Lipschitz classes through the rate of convergence [29].

The present operator is as described in (10). Therefore, D_n and $\mathcal{D}_w^{\varphi,\psi}$ share a sum–integral structure and replace point evaluations by integral averages. Their kernel architectures, however, are different. In a Durrmeyer sampling-type operator, the factors $\varphi(wx - k)$ and $w\psi(wu - k)$ are translations and dilations of fixed generating kernels, and the summation is taken over the full integer lattice. For D_n , the summation is taken over $\mathbb{N}_0$, the outer weights depend jointly on n , k , and z , and the beta-prime density $q_{n,k}$ depends on both n and k . Consequently, $q_{n,k}$ does not have the fixed translation–dilation form used in the standard sampling framework. Thus, D_n is not obtained from the standard Durrmeyer sampling construction by a direct choice of fixed kernels φ and ψ .

The distinction remains in the multivariate setting. Multivariate Durrmeyer sampling operators are constructed over the lattice $\mathbb{Z}^d$ and have been studied in Orlicz and weighted function spaces [28,39]. The bivariate operator introduced in the present paper is instead the

tensor product of two half-line operators and acts on $[0, \infty)^2$. Nevertheless, both approaches use multivariate integral averaging and lead to convergence results and estimates in terms of suitable moduli of continuity.

Table 1 summarizes the principal similarities and differences.

Table 1. Structural and approximation-theoretic comparison of related Durrmeyer-type constructions.

Construction	Domain and Indices	Defining Structure	Principal Approximation Results
Bernstein–Durrmeyer [35]	$[0, 1]$; finite index set	Bernstein polynomial weights and integral means generated by Bernstein basis functions	Asymptotic expansions for classical, generalized, and genuine Bernstein–Durrmeyer operators
Durrmeyer sampling [27,29]	$\mathbb{R}; k \in \mathbb{Z}$	Translations and dilations of fixed discrete and continuous kernels	Pointwise, uniform, modular, and L^p convergence; quantitative, direct, and inverse estimates
Multivariate Durrmeyer sampling [28,39]	$\mathbb{R}^d; k \in \mathbb{Z}^d$	Multidimensional lattice-based sampling kernels and integral means	Orlicz and weighted approximation, multivariate moduli, and Voronovskaya-type results
Present operator D_n	$[0, \infty); k \in \mathbb{N}_0$	Rising-factorial weights and k - and n -dependent beta-prime densities	Exact moments, local and weighted estimates, Korovkin convergence, and a tensor-product bivariate extension

The approximation results obtained for these families are therefore complementary rather than directly ordered. Their approximation parameters, domains, kernel assumptions, and underlying norms are different.

Lemma 1. Let $e_j(t) = t^j$. A direct calculation from (4) gives, for every $n \in \mathbb{N}$ and $z \geq 0$,

$$\begin{aligned}
 L_n^M(e_0; z) &= 1, \\
 L_n^M(e_1; z) &= z, \\
 L_n^M(e_2; z) &= z^2 + \frac{2z}{n}, \\
 L_n^M(e_3; z) &= z^3 + \frac{6z^2}{n} + \frac{6z}{n^2}.
 \end{aligned} \tag{17}$$

Proof. For L_n^M write

$$p_{n,k}(z) = \frac{2^{-nz}(nz)_k}{2^k k!}, \quad G(s) = \sum_{k=0}^{\infty} p_{n,k}(z) s^k = (2-s)^{-nz}.$$

Thus $G(1) = 1$ and $G'(1) = nz$, so that $L_n^M(e_0; z) = 1$ and $L_n^M(e_1; z) = G'(1)/n = z$. Furthermore, $G''(1) = nz(nz+1)$, $G'''(1) = nz(nz+1)(nz+2)$. Consequently,

$$\sum_{k \geq 0} k^2 p_{n,k}(z) = n^2 z^2 + 2nz, \quad \sum_{k \geq 0} k^3 p_{n,k}(z) = n^3 z^3 + 6n^2 z^2 + 6nz.$$

Dividing these identities by n^2 and n^3 , respectively, gives the stated formulas for $L_n^M(e_2; z)$ and $L_n^M(e_3; z)$. $\square$

Lemma 2. Let $e_j(t) = t^j$ for $j = 0, 1, 2, 3, 4$. Here t is the integration variable, whereas $z \geq 0$ is the point at which the operator is evaluated. The operator D_n defined in (10) satisfies

$$\begin{aligned}
D_n(e_0; z) &= 1, & n > 0, \\
D_n(e_1; z) &= \frac{nz + 1}{n - 1}, & n > 1, \\
D_n(e_2; z) &= \frac{n^2 z^2 + 5nz + 2}{(n - 1)(n - 2)}, & n > 2, \\
D_n(e_3; z) &= \frac{n^3 z^3 + 12n^2 z^2 + 29nz + 6}{(n - 1)(n - 2)(n - 3)}, & n > 3, \\
D_n(e_4; z) &= \frac{n^4 z^4 + 22n^3 z^3 + 131n^2 z^2 + 206nz + 24}{(n - 1)(n - 2)(n - 3)(n - 4)}, & n > 4.
\end{aligned} \tag{18}$$

Proof. We first calculate the sums involving the weights of D_n , and then evaluate the beta integrals.

Step 1: The weights and their sums: Let us write $p_{n,k}(z) = \frac{2^{-nz}(nz)_k}{2^k k!}$, $k = 0, 1, 2, \dots$. The notation $(nz)_k$ means the rising factorial:

$$(nz)_0 = 1, \quad (nz)_k = (nz)(nz + 1) \cdots (nz + k - 1) \quad (k \geq 1).$$

For fixed $n > 0$ and $z \geq 0$, introduce the auxiliary variable s and the power series

$$G(s) = \sum_{k=0}^{\infty} p_{n,k}(z) s^k = 2^{-nz} \left(1 - \frac{s}{2}\right)^{-nz} = (2 - s)^{-nz}, \quad |s| < 2.$$

Thus $\sum_{k=0}^{\infty} p_{n,k}(z) = G(1) = 1$. Since $s = 1$ lies inside the interval of convergence, the series can be differentiated term by term there. This gives

$$\begin{aligned}
\sum_{k=0}^{\infty} k p_{n,k}(z) &= G'(1) = nz, \\
\sum_{k=0}^{\infty} k(k-1) p_{n,k}(z) &= G''(1) = nz(nz + 1), \\
\sum_{k=0}^{\infty} k(k-1)(k-2) p_{n,k}(z) &= G'''(1) = nz(nz + 1)(nz + 2), \\
\sum_{k=0}^{\infty} k(k-1)(k-2)(k-3) p_{n,k}(z) &= G^{(4)}(1) \\
&= nz(nz + 1)(nz + 2)(nz + 3).
\end{aligned}$$

The elementary identities

$$\begin{aligned}
k^2 &= k(k-1) + k, \\
k^3 &= k(k-1)(k-2) + 3k(k-1) + k, \\
k^4 &= k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k
\end{aligned}$$

therefore imply

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 p_{n,k}(z) &= n^2 z^2 + 2nz, \\
\sum_{k=0}^{\infty} k^3 p_{n,k}(z) &= n^3 z^3 + 6n^2 z^2 + 6nz, \\
\sum_{k=0}^{\infty} k^4 p_{n,k}(z) &= n^4 z^4 + 12n^3 z^3 + 36n^2 z^2 + 26nz.
\end{aligned}$$

Step 2: The beta integral: For an integer $j \geq 1$ and $n > j$, we have

$$\begin{aligned} \frac{1}{B(k+1, n)} \int_0^\infty \frac{t^{k+j}}{(1+t)^{k+n+1}} dt &= \frac{B(k+j+1, n-j)}{B(k+1, n)} \\ &= \frac{\Gamma(k+j+1)\Gamma(n-j)}{\Gamma(k+1)\Gamma(n)} \\ &= \frac{(k+1)(k+2)\cdots(k+j)}{(n-1)(n-2)\cdots(n-j)}. \end{aligned}$$

The last equality follows from $\Gamma(x+1) = x\Gamma(x)$. For $j = 0$, the integral divided by $B(k+1, n)$ equals 1.

Step 3: The five raw moments: For the constant function, $D_n(e_0; z) = \sum_{k=0}^\infty p_{n,k}(z) = 1$. For $e_1(t) = t$,

$$D_n(e_1; z) = \frac{1}{n-1} \sum_{k=0}^\infty (k+1)p_{n,k}(z) = \frac{nz+1}{n-1}.$$

For $e_2(t) = t^2$, use $(k+1)(k+2) = k^2 + 3k + 2$ to obtain

$$\begin{aligned} D_n(e_2; z) &= \frac{1}{(n-1)(n-2)} \sum_{k=0}^\infty (k^2 + 3k + 2)p_{n,k}(z) \\ &= \frac{(n^2z^2 + 2nz) + 3nz + 2}{(n-1)(n-2)} \\ &= \frac{n^2z^2 + 5nz + 2}{(n-1)(n-2)}. \end{aligned}$$

For $e_3(t) = t^3$, the identity

$$(k+1)(k+2)(k+3) = k^3 + 6k^2 + 11k + 6$$

gives

$$\begin{aligned} (n-1)(n-2)(n-3)D_n(e_3; z) &= (n^3z^3 + 6n^2z^2 + 6nz) + 6(n^2z^2 + 2nz) + 11nz + 6 \\ &= n^3z^3 + 12n^2z^2 + 29nz + 6. \end{aligned}$$

Finally, for $e_4(t) = t^4$, use

$$(k+1)(k+2)(k+3)(k+4) = k^4 + 10k^3 + 35k^2 + 50k + 24.$$

It follows that

$$\begin{aligned} (n-1)(n-2)(n-3)(n-4)D_n(e_4; z) &= (n^4z^4 + 12n^3z^3 + 36n^2z^2 + 26nz) \\ &\quad + 10(n^3z^3 + 6n^2z^2 + 6nz) + 35(n^2z^2 + 2nz) + 50nz + 24 \\ &= n^4z^4 + 22n^3z^3 + 131n^2z^2 + 206nz + 24. \end{aligned}$$

Dividing the last two identities by their respective factors gives the third and fourth raw moments.

All calculations include $z = 0$: in that case $p_{n,0}(0) = 1$ and $p_{n,k}(0) = 0$ for $k \geq 1$. $\square$

3. Central Moments

Lemma 3. Let $z \geq 0$ and $n > 4$. For $r = 1, 2, 3, 4$, define

$$\mu_{n,r}(z) = D_n((t-z)^r; z). \quad (19)$$

Here, r denotes the moment order, and z is held fixed while the operator acts on the function $t \mapsto (t-z)^r$. The first central moment measures the mean displacement from z , while the second measures the mean squared distance from z . Then

$$\begin{aligned} \mu_{n,1}(z) &= \frac{z+1}{n-1}, \\ \mu_{n,2}(z) &= \frac{(n+2)z^2 + (3n+4)z + 2}{(n-1)(n-2)}, \\ \mu_{n,3}(z) &= \frac{(7n+6)z^3 + (30n+18)z^2 + (23n+18)z + 6}{(n-1)(n-2)(n-3)}, \\ \mu_{n,4}(z) &= \frac{(3n^2+46n+24)z^4 + (18n^2+256n+96)z^3}{(n-1)(n-2)(n-3)(n-4)} \\ &\quad + \frac{(27n^2+380n+144)z^2 + (182n+96)z + 24}{(n-1)(n-2)(n-3)(n-4)}. \end{aligned} \quad (20)$$

Individually, the first, second, and third identities require only $n > 1$, $n > 2$, and $n > 3$, respectively.

For every fixed $R > 0$, the following limits hold uniformly for $0 \leq z \leq R$:

$$\begin{aligned} \lim_{n \rightarrow \infty} n\mu_{n,1}(z) &= z+1, \\ \lim_{n \rightarrow \infty} n\mu_{n,2}(z) &= z^2+3z, \\ \lim_{n \rightarrow \infty} n^2\mu_{n,3}(z) &= 7z^3+30z^2+23z, \\ \lim_{n \rightarrow \infty} n^2\mu_{n,4}(z) &= 3z^4+18z^3+27z^2. \end{aligned} \quad (21)$$

In particular, $\mu_{n,4}(z) = O(n^{-2})$ uniformly on $[0, R]$. This means that a constant $C_R > 0$, independent of n and z , can be chosen so that

$$0 \leq \mu_{n,4}(z) \leq \frac{C_R}{n^2} \quad (n \geq 8, 0 \leq z \leq R). \quad (22)$$

The constant is allowed to depend on R .

Proof. We use the raw moments from Lemma 2. Since z is fixed, linearity allows us to expand $(t-z)^r$ first and then apply D_n to each power of t separately.

Step 1: The first central moment: Using $t-z = t-z \cdot 1$ and $D_n(1; z) = 1$, we obtain

$$\begin{aligned} \mu_{n,1}(z) &= D_n(t; z) - zD_n(1; z) \\ &= \frac{nz+1}{n-1} - z \\ &= \frac{nz+1-z(n-1)}{n-1} = \frac{z+1}{n-1}. \end{aligned}$$

Step 2: The second central moment: The expansion $(t - z)^2 = t^2 - 2zt + z^2$ gives

$$\begin{aligned}\mu_{n,2}(z) &= D_n(t^2; z) - 2zD_n(t; z) + z^2D_n(1; z) \\ &= \frac{n^2z^2 + 5nz + 2}{(n-1)(n-2)} - \frac{2z(nz+1)}{n-1} + z^2.\end{aligned}$$

Multiplying by the common denominator yields

$$\begin{aligned}(n-1)(n-2)\mu_{n,2}(z) &= n^2z^2 + 5nz + 2 - 2z(nz+1)(n-2) + z^2(n-1)(n-2) \\ &= n^2z^2 + 5nz + 2 - 2n^2z^2 + 4nz^2 - 2nz + 4z \\ &\quad + n^2z^2 - 3nz^2 + 2z^2 \\ &= (n+2)z^2 + (3n+4)z + 2.\end{aligned}$$

Dividing by $(n-1)(n-2)$ proves the second formula.

Step 3: The third central moment: Since $(t - z)^3 = t^3 - 3zt^2 + 3z^2t - z^3$,

$$\begin{aligned}\mu_{n,3}(z) &= D_n(t^3; z) - 3zD_n(t^2; z) + 3z^2D_n(t; z) - z^3 \\ &= \frac{n^3z^3 + 12n^2z^2 + 29nz + 6}{(n-1)(n-2)(n-3)} \\ &\quad - \frac{3z(n^2z^2 + 5nz + 2)}{(n-1)(n-2)} + \frac{3z^2(nz+1)}{n-1} - z^3.\end{aligned}$$

Hence

$$\begin{aligned}(n-1)(n-2)(n-3)\mu_{n,3}(z) &= n^3z^3 + 12n^2z^2 + 29nz + 6 \\ &\quad - 3z(n-3)(n^2z^2 + 5nz + 2) \\ &\quad + 3z^2(n-2)(n-3)(nz+1) \\ &\quad - z^3(n-1)(n-2)(n-3).\end{aligned}$$

The last three terms expand as follows:

$$\begin{aligned}-3z(n-3)(n^2z^2 + 5nz + 2) &= (-3n^3 + 9n^2)z^3 \\ &\quad + (-15n^2 + 45n)z^2 + (-6n + 18)z, \\ 3z^2(n-2)(n-3)(nz+1) &= (3n^3 - 15n^2 + 18n)z^3 \\ &\quad + (3n^2 - 15n + 18)z^2, \\ -z^3(n-1)(n-2)(n-3) &= (-n^3 + 6n^2 - 11n + 6)z^3.\end{aligned}$$

Adding these expressions gives

$$\begin{aligned}(n-1)(n-2)(n-3)\mu_{n,3}(z) &= (7n+6)z^3 + (30n+18)z^2 + (23n+18)z + 6.\end{aligned}$$

Division by the common denominator proves the third formula.

Step 4: The fourth central moment: Expanding $(t - z)^4$, we have

$$\begin{aligned}\mu_{n,4}(z) &= D_n(t^4; z) - 4zD_n(t^3; z) + 6z^2D_n(t^2; z) \\ &\quad - 4z^3D_n(t; z) + z^4.\end{aligned}$$

Substitute the four raw moments and multiply by $(n-1)(n-2)(n-3)(n-4)$. This gives

$$\begin{aligned} & (n-1)(n-2)(n-3)(n-4)\mu_{n,4}(z) \\ &= n^4 z^4 + 22n^3 z^3 + 131n^2 z^2 + 206nz + 24 \\ & \quad - 4z(n-4)(n^3 z^3 + 12n^2 z^2 + 29nz + 6) \\ & \quad + 6z^2(n-3)(n-4)(n^2 z^2 + 5nz + 2) \\ & \quad - 4z^3(n-2)(n-3)(n-4)(nz + 1) \\ & \quad + z^4(n-1)(n-2)(n-3)(n-4). \end{aligned}$$

To see the cancellations explicitly, let us expand the last four terms:

$$\begin{aligned} & -4z(n-4)(n^3 z^3 + 12n^2 z^2 + 29nz + 6) \\ &= (-4n^4 + 16n^3)z^4 + (-48n^3 + 192n^2)z^3 \\ & \quad + (-116n^2 + 464n)z^2 + (-24n + 96)z, \\ & 6z^2(n-3)(n-4)(n^2 z^2 + 5nz + 2) \\ &= (6n^4 - 42n^3 + 72n^2)z^4 + (30n^3 - 210n^2 + 360n)z^3 \\ & \quad + (12n^2 - 84n + 144)z^2, \\ & -4z^3(n-2)(n-3)(n-4)(nz + 1) \\ &= (-4n^4 + 36n^3 - 104n^2 + 96n)z^4 \\ & \quad + (-4n^3 + 36n^2 - 104n + 96)z^3, \\ & z^4(n-1)(n-2)(n-3)(n-4) \\ &= (n^4 - 10n^3 + 35n^2 - 50n + 24)z^4. \end{aligned}$$

After adding these expressions to the first term, all terms containing n^4 or n^3 cancel. The remaining numerator is

$$\begin{aligned} & (3n^2 + 46n + 24)z^4 + (18n^2 + 256n + 96)z^3 \\ & \quad + (27n^2 + 380n + 144)z^2 + (182n + 96)z + 24. \end{aligned}$$

Dividing by $(n-1)(n-2)(n-3)(n-4)$ proves the fourth formula.

Step 5: The limits and the meaning of the order estimate: The first three normalized moments can be written as

$$\begin{aligned} n\mu_{n,1}(z) &= \frac{z+1}{1-1/n}, \\ n\mu_{n,2}(z) &= \frac{(1+2/n)z^2 + (3+4/n)z + 2/n}{(1-1/n)(1-2/n)}, \\ n^2\mu_{n,3}(z) &= \frac{(7+6/n)z^3 + (30+18/n)z^2 + (23+18/n)z + 6/n}{(1-1/n)(1-2/n)(1-3/n)}. \end{aligned}$$

Letting $n \rightarrow \infty$ gives their stated limits. For the fourth moment, first multiply the formula by n^2 . Then divide both the numerator and denominator by n^4 to obtain

$$\begin{aligned} n^2\mu_{n,4}(z) &= \frac{(3+46/n+24/n^2)z^4 + (18+256/n+96/n^2)z^3}{(1-1/n)(1-2/n)(1-3/n)(1-4/n)} \\ & \quad + \frac{(27+380/n+144/n^2)z^2 + (182/n+96/n^2)z + 24/n^2}{(1-1/n)(1-2/n)(1-3/n)(1-4/n)}. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} n^2 \mu_{n,4}(z) = 3z^4 + 18z^3 + 27z^2 = 3z^2(z+3)^2.$$

On $[0, R]$, all powers of z in these expressions are bounded by constants depending only on R . The denominators are independent of z and tend to 1. Hence all four limits are uniform on $[0, R]$.

For an explicit bound, note that $n \geq 8$ implies

$$(1 - 1/n)(1 - 2/n)(1 - 3/n)(1 - 4/n) \geq \frac{1}{16}.$$

Using $0 \leq z \leq R$ and $1/n, 1/n^2 \leq 1$ in the preceding formula gives

$$0 \leq n^2 \mu_{n,4}(z) \leq 16(73R^4 + 370R^3 + 551R^2 + 278R + 24).$$

Thus one may take

$$C_R = 16(73R^4 + 370R^3 + 551R^2 + 278R + 24),$$

which proves the claimed $O(n^{-2})$ estimate with a constant independent of n and $z \in [0, R]$. $\square$

Remark 2 (Interpretation of the integral kernel). *The operator in (10) averages f over the whole half-line with a beta-prime kernel. All claims about convergence rates below follow from the explicit moments and the stated function-space assumptions.*

Quantitative Comparison with a Dunkl–Beta Operator

We compare the present operator with the Dunkl Szász–Schurer–Beta operator of Raiz et al. [12], matching its effective index $m + p$ with n . Denote the resulting operator by S_n^ν , where $\nu \geq 0$. Define

$$\theta_k = \begin{cases} 0, & k \text{ even}, \\ 1, & k \text{ odd}, \end{cases} \quad (23)$$

and define

$$\gamma_\nu(0) = 1, \quad \gamma_\nu(k) = (k + 2\nu\theta_k)\gamma_\nu(k-1), \quad k \geq 1.$$

Put

$$a_k = k + 2\nu\theta_k + 1, \quad k \geq 0.$$

In particular, $a_0 = 1$.

Writing

$$e_\nu(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_\nu(k)}, \quad (24)$$

the operator is

$$\begin{aligned} S_n^\nu(f; z) &= \frac{1}{e_\nu(nz)} \sum_{k=0}^{\infty} \frac{(nz)^k}{\gamma_\nu(k)} \frac{1}{B(a_k, n)} \\ &\quad \times \int_0^\infty \frac{t^{a_k-1} f(t)}{(1+t)^{a_k+n}} dt. \end{aligned} \quad (25)$$

Both constructions use normalized beta-prime integral means, but their outer weights and first beta-prime shape parameters differ.

For $n > 2$, put

$$r_{n,\nu}(z) = \frac{e_\nu(-nz)}{e_\nu(nz)}. \quad (26)$$

Direct calculation gives

$$\begin{aligned} \mathcal{S}_n^\nu(1; z) &= 1, \\ \mathcal{S}_n^\nu(t; z) &= \frac{nz + 1}{n - 1}, \\ \mathcal{S}_n^\nu(t^2; z) &= \frac{n^2 z^2 + 4nz + 2\nu n z r_{n,\nu}(z) + 2}{(n - 1)(n - 2)}. \end{aligned} \quad (27)$$

Indeed, the first two conditional moments of the beta-prime kernel $\frac{a_k}{n-1}, \frac{a_k(a_k+1)}{(n-1)(n-2)}$.

Set $A_k = a_k - 1$, $w_k(x) = \frac{x^k}{\gamma_\nu(k)e_\nu(x)}$. The recurrence for γ_ν yields $\sum_{k=0}^\infty w_k(x)A_k = x$. Since $A_{k+1} = A_k + 1 + 2\nu(-1)^k$, an index shift also gives

$$\begin{aligned} \sum_{k=0}^\infty w_k(x)A_k^2 &= x \sum_{k=0}^\infty w_k(x)A_{k+1} \\ &= x^2 + x + 2\nu x \frac{e_\nu(-x)}{e_\nu(x)}. \end{aligned}$$

Substituting $x = nz$ and using $a_k(a_k + 1) = A_k^2 + 3A_k + 2$ proves the raw-moment formulas.

Consequently, the central moments satisfy

$$\begin{aligned} \tilde{\mu}_{n,1}^\nu(z) &:= \mathcal{S}_n^\nu(t - z; z) = \frac{z + 1}{n - 1}, \\ \tilde{\mu}_{n,2}^\nu(z) &:= \mathcal{S}_n^\nu((t - z)^2; z) \\ &= \frac{(n + 2)z^2 + (2n + 4)z + 2\nu n z r_{n,\nu}(z) + 2}{(n - 1)(n - 2)}. \end{aligned} \quad (28)$$

Thus, the first central moments coincide, while

$$\mu_{n,2}(z) - \tilde{\mu}_{n,2}^\nu(z) = \frac{nz(1 - 2\nu r_{n,\nu}(z))}{(n - 1)(n - 2)}. \quad (29)$$

The positivity of $\gamma_\nu(k)$ implies $|e_\nu(-x)| \leq e_\nu(x)$, $x \geq 0$, so $|r_{n,\nu}(z)| \leq 1$. Therefore, $\mu_{n,2}(z) \geq \tilde{\mu}_{n,2}^\nu(z)$, $0 \leq \nu \leq \frac{1}{2}$. For $\nu = 0.3$, in particular,

$$\mu_{n,2}(z) - \tilde{\mu}_{n,2}^{0.3}(z) \geq \frac{0.4nz}{(n - 1)(n - 2)}. \quad (30)$$

For $f \in C^2[0, \infty)$ with bounded first and second derivatives, Taylor's formula gives

$$\begin{aligned} |D_n(f; z) - f(z)| &\leq \frac{z + 1}{n - 1} \|f'\|_\infty + \frac{\mu_{n,2}(z)}{2} \|f''\|_\infty, \\ |\mathcal{S}_n^\nu(f; z) - f(z)| &\leq \frac{z + 1}{n - 1} \|f'\|_\infty + \frac{\tilde{\mu}_{n,2}^\nu(z)}{2} \|f''\|_\infty. \end{aligned} \quad (31)$$

For fixed ν and $R > 0$, both bounds are $O(n^{-1})$ uniformly on $[0, R]$. For $0 \leq \nu \leq 1/2$, the second-derivative coefficient in the Dunkl bound is no larger.

The ordering of these bounds does not determine the actual error ordering for every function. For the test function t^2 , however, positivity and the common first raw moment imply that both operators overestimate z^2 . Hence

$$|D_n(t^2; z) - z^2| - |\mathcal{S}_n^\nu(t^2; z) - z^2| \\ = \frac{nz(1 - 2\nu r_{n,\nu}(z))}{(n-1)(n-2)}.$$

This difference is strictly positive when $0 \leq \nu < 1/2$ and $z > 0$.

The comparison establishes a common first-order scale for the smooth-function bounds, without a uniform advantage in the error constants for D_n . The contribution of the present construction concerns its specific weight–kernel pairing, the explicit higher-order moments, and the associated tensor-product approximation results.

4. Local Approximation

Local approximation estimates for related integral operators have recently been obtained through central moments and moduli of smoothness. Srivastava et al. [40] developed such estimates for general-Appell-based Szász–Durrmeyer operators. Aslan [41] investigated local approximation and weighted-modulus estimates for α -Chlodowsky–Durrmeyer operators. In this section, we use the explicit central moments of D_n to establish the corresponding local bounds for the weight–kernel pairing considered here.

We use the standard notation for spaces of bounded continuous functions and moduli of continuity employed in approximation theory; see, for example, Refs. [42,43]. Put

$$C_B[0, \infty) = \{f : [0, \infty) \rightarrow \mathbb{R} : f \text{ is bounded and continuous}\}, \\ \|f\|_\infty = \sup_{z \geq 0} |f(z)|. \quad (32)$$

For $f \in C_B[0, \infty)$ and $\delta > 0$, its first-order global modulus of continuity is

$$\omega(f; \delta) = \sup_{\substack{x, y \geq 0 \\ |x-y| \leq \delta}} |f(x) - f(y)|. \quad (33)$$

Because $[0, \infty)$ is unbounded, membership in $C_B[0, \infty)$ does not by itself imply

$$\lim_{\delta \rightarrow 0+} \omega(f; \delta) = 0.$$

This limit holds when f is uniformly continuous on $[0, \infty)$. Accordingly, whenever convergence is deduced from the modulus of continuity, we assume

$$f \in C_{BU}[0, \infty) := \{f \in C_B[0, \infty) : f \text{ is uniformly continuous}\}. \quad (34)$$

Theorem 1 (First-order modulus estimate). *Let $f \in C_B[0, \infty)$, $z \geq 0$, and $n > 2$. Then*

$$|D_n(f; z) - f(z)| \leq 2\omega\left(f; \sqrt{\mu_{n,2}(z)}\right), \quad (35)$$

where $\mu_{n,2}$ is given in Lemma 3.

Proof. For every $\delta > 0$ and $t, z \geq 0$,

$$|f(t) - f(z)| \leq \left(1 + \frac{|t-z|}{\delta}\right) \omega(f; \delta).$$

Applying the positive operator $D_n(\cdot; z)$ and using $D_n(1; z) = 1$ gives

$$|D_n(f; z) - f(z)| \leq \omega(f; \delta) \left(1 + \frac{D_n(|t - z|; z)}{\delta} \right).$$

Cauchy–Schwarz and $D_n(1; z) = 1$ imply

$$D_n(|t - z|; z) \leq \sqrt{D_n((t - z)^2; z)} = \sqrt{\mu_{n,2}(z)}.$$

Choosing $\delta = \sqrt{\mu_{n,2}(z)}$ proves the claim. $\square$

Theorem 2 (Hölder–Lipschitz estimate). *Let $n > 2$, $z \geq 0$, $M > 0$, and $0 < \gamma \leq 1$. Define*

$$\text{Lip}_M(\gamma) = \{f \in C_B[0, \infty) : |f(x) - f(y)| \leq M|x - y|^\gamma\}. \quad (36)$$

If $f \in \text{Lip}_M(\gamma)$, then

$$|D_n(f; z) - f(z)| \leq M \mu_{n,2}(z)^{\gamma/2}. \quad (37)$$

For every $R > 0$, this estimate is uniform on $0 \leq z \leq R$ and is $O(n^{-\gamma/2})$ as $n \rightarrow \infty$.

Proof. The Lipschitz condition and positivity give

$$|D_n(f; z) - f(z)| \leq M D_n(|t - z|^\gamma; z).$$

Hölder’s inequality with exponents $2/\gamma$ and $2/(2 - \gamma)$ yields

$$D_n(|t - z|^\gamma; z) \leq D_n((t - z)^2; z)^{\gamma/2} D_n(1; z)^{(2-\gamma)/2} = \mu_{n,2}(z)^{\gamma/2}.$$

The compact-uniform rate follows from Lemma 3. $\square$

Theorem 3 (Second-order modulus estimate). *Let $n > 2$, $z \geq 0$, and $f \in C_B[0, \infty)$. Define*

$$\omega_2(f; \delta) = \sup_{\substack{x \geq 0 \\ 0 < h \leq \delta}} |f(x + 2h) - 2f(x + h) + f(x)|. \quad (38)$$

Then, for an absolute constant $C > 0$,

$$|D_n(f; z) - f(z)| \leq C \omega_2\left(f; \sqrt{\mu_{n,2}(z)}\right) + \omega(f; \mu_{n,1}(z)). \quad (39)$$

Consequently, on every compact interval $[0, R]$,

$$|D_n(f; z) - f(z)| \leq C \omega_2\left(f; \sqrt{C_R/n}\right) + \omega(f; C_R/n)$$

for all sufficiently large n .

Proof. Set

$$\eta_n(z) = D_n(e_1; z) = \frac{nz + 1}{n - 1} = z + \mu_{n,1}(z). \quad (40)$$

Let $C_B^2[0, \infty)$ denote the functions in $C_B[0, \infty)$ with bounded continuous second derivative, and let $g \in C_B^2[0, \infty)$. Taylor’s formula in the form

$$g(t) = g(z) + (t - z)g'(z) + (t - z)^2 \int_0^1 (1 - s)g''(z + s(t - z)) ds$$

shows that

$$|D_n(g; z) - g(\eta_n(z))| \leq \frac{1}{2} \|g''\|_\infty (\mu_{n,2}(z) + \mu_{n,1}(z)^2).$$

Since $\mu_{n,1}(z)^2 \leq \mu_{n,2}(z)$ by Cauchy–Schwarz,

$$|D_n(g; z) - g(\eta_n(z))| \leq \|g''\|_\infty \mu_{n,2}(z).$$

For arbitrary g we therefore have

$$|D_n(f; z) - f(z)| \leq 2\|f - g\|_\infty + \mu_{n,2}(z)\|g''\|_\infty + \omega(f; \mu_{n,1}(z)).$$

Taking the infimum over g gives

$$|D_n(f; z) - f(z)| \leq 2K_2\left(f; \frac{\mu_{n,2}(z)}{2}\right) + \omega(f; \mu_{n,1}(z)),$$

where

$$K_2(f; \delta) = \inf_{g \in C_B^2[0, \infty)} \{\|f - g\|_\infty + \delta\|g''\|_\infty\}. \quad (41)$$

The standard equivalence between K_2 and ω_2 proves the result. $\square$

5. Weighted Approximation

Define

$$\begin{aligned} B_2[0, \infty) &= \left\{ f : \|f\|_2 := \sup_{z \geq 0} \frac{|f(z)|}{1+z^2} < \infty \right\}, \\ C_2[0, \infty) &= B_2[0, \infty) \cap C[0, \infty), \\ C_2^*[0, \infty) &= \left\{ f \in C_2[0, \infty) : \lim_{z \rightarrow \infty} \frac{f(z)}{1+z^2} \text{ exists and is finite} \right\}. \end{aligned} \quad (42)$$

Theorem 4 (Weighted Korovkin convergence). *For every $f \in C_2^*[0, \infty)$ and $n > 2$,*

$$\lim_{n \rightarrow \infty} \|D_n f - f\|_2 = 0. \quad (43)$$

Proof. Put $\rho(t) = 1 + t^2$. Positivity gives

$$|D_n(f; z)| \leq \|f\|_2 D_n(\rho; z), \quad f \in C_2[0, \infty).$$

By Lemma 2,

$$\frac{D_n(\rho; z)}{1+z^2} \leq 1 + \frac{n^2 + \frac{5}{2}n + 2}{(n-1)(n-2)}, \quad z \geq 0.$$

Consequently, D_n maps $C_2[0, \infty)$ into $B_2[0, \infty)$, and these operators are uniformly bounded for $n \geq 3$. The weighted Korovkin theorem reduces the verification to $e_0(z) = 1$, $e_1(z) = z$, and $e_2(z) = z^2$. Lemma 2 gives $D_n(e_0; z) - e_0(z) = 0$, $D_n(e_1; z) - e_1(z) = \frac{z+1}{n-1}$, and $D_n(e_2; z) - e_2(z) = \frac{(3n-2)z^2 + 5nz + 2}{(n-1)(n-2)}$. Consequently,

$$\|D_n(e_1) - e_1\|_2 \leq \frac{2}{n-1}, \quad \|D_n(e_2) - e_2\|_2 \leq \frac{11n}{2(n-1)(n-2)},$$

which both tend to zero. $\square$

Theorem 5. *Let*

$$f \in C_2[0, \infty) \cap C^2[0, \infty),$$

and suppose that f' and f'' are bounded. Then, for $n > 2$,

$$\|D_n f - f\|_2 \leq \frac{1 + \sqrt{2}}{2(n-1)} \|f'\|_\infty + \frac{5n+12}{4(n-1)(n-2)} \|f''\|_\infty. \quad (44)$$

Consequently,

$$\|D_n f - f\|_2 = O(n^{-1}) \quad (n \rightarrow \infty).$$

Proof. Recall that

$$\|g\|_2 = \sup_{z \geq 0} \frac{|g(z)|}{1+z^2}.$$

Fix $z \geq 0$. Taylor's formula with integral remainder gives, for every $t \geq 0$,

$$f(t) = f(z) + (t-z)f'(z) + R_z(t),$$

where

$$R_z(t) = (t-z)^2 \int_0^1 (1-s)f''(z+s(t-z)) ds.$$

Since f'' is bounded,

$$\begin{aligned} |R_z(t)| &\leq (t-z)^2 \|f''\|_\infty \int_0^1 (1-s) ds \\ &= \frac{1}{2} \|f''\|_\infty (t-z)^2. \end{aligned}$$

Using the linearity of D_n and the identity $D_n(1; z) = 1$, we obtain

$$\begin{aligned} D_n(f; z) - f(z) &= f'(z)D_n(t-z; z) + D_n(R_z; z) \\ &= f'(z)\mu_{n,1}(z) + D_n(R_z; z). \end{aligned}$$

Because D_n is positive,

$$|D_n(R_z; z)| \leq D_n(|R_z|; z).$$

Therefore,

$$\begin{aligned} |D_n(f; z) - f(z)| &\leq |f'(z)|\mu_{n,1}(z) + D_n(|R_z|; z) \\ &\leq \|f'\|_\infty \mu_{n,1}(z) + \frac{1}{2} \|f''\|_\infty D_n((t-z)^2; z) \\ &= \|f'\|_\infty \mu_{n,1}(z) + \frac{1}{2} \|f''\|_\infty \mu_{n,2}(z). \end{aligned}$$

We next estimate the two central moments in the weighted norm.

For the first central moment,

$$\mu_{n,1}(z) = \frac{z+1}{n-1}.$$

Consider

$$h(z) = \frac{z+1}{1+z^2}, \quad z \geq 0.$$

Its derivative is

$$h'(z) = \frac{1-2z-z^2}{(1+z^2)^2}.$$

The only nonnegative critical point is $z = \sqrt{2} - 1$. Since $h(0) = 1$, $\lim_{z \rightarrow \infty} h(z) = 0$, this critical point gives the global maximum. Hence,

$$\sup_{z \geq 0} \frac{z+1}{1+z^2} = h(\sqrt{2}-1) = \frac{1+\sqrt{2}}{2}.$$

It follows that

$$\sup_{z \geq 0} \frac{\mu_{n,1}(z)}{1+z^2} = \frac{1+\sqrt{2}}{2(n-1)}.$$

For the second central moment, we use

$$\mu_{n,2}(z) = \frac{(n+2)z^2 + (3n+4)z + 2}{(n-1)(n-2)}.$$

Therefore,

$$\frac{\mu_{n,2}(z)}{1+z^2} = \frac{1}{(n-1)(n-2)} \left[(n+2) \frac{z^2}{1+z^2} + (3n+4) \frac{z}{1+z^2} + \frac{2}{1+z^2} \right].$$

For $z \geq 0$,

$$\frac{z^2}{1+z^2} \leq 1, \quad \frac{z}{1+z^2} \leq \frac{1}{2}, \quad \frac{1}{1+z^2} \leq 1.$$

Thus,

$$\begin{aligned} \frac{\mu_{n,2}(z)}{1+z^2} &\leq \frac{(n+2) + \frac{1}{2}(3n+4) + 2}{(n-1)(n-2)} \\ &= \frac{5n+12}{2(n-1)(n-2)}. \end{aligned}$$

Dividing the pointwise error estimate by $1+z^2$ and taking the supremum over $z \geq 0$, we obtain

$$\|D_n f - f\|_2 \leq \frac{1+\sqrt{2}}{2(n-1)} \|f'\|_\infty + \frac{5n+12}{4(n-1)(n-2)} \|f''\|_\infty.$$

It remains to justify the order estimate explicitly. For $n \geq 4$, $\frac{1}{n-1} \leq \frac{2}{n}$, and hence $\frac{1+\sqrt{2}}{2(n-1)} \leq \frac{1+\sqrt{2}}{n}$. Moreover, for $n \geq 4$,

$$5n+12 \leq 8n, \quad n-1 \geq \frac{n}{2}, \quad n-2 \geq \frac{n}{2}.$$

Consequently,

$$\begin{aligned} \frac{5n+12}{4(n-1)(n-2)} &\leq \frac{8n}{4\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)} \\ &= \frac{8}{n}. \end{aligned}$$

Therefore, for every $n \geq 4$,

$$\|D_n f - f\|_2 \leq \frac{(1+\sqrt{2})\|f'\|_\infty + 8\|f''\|_\infty}{n}. \quad (45)$$

Setting

$$C_f = (1+\sqrt{2})\|f'\|_\infty + 8\|f''\|_\infty,$$

where C_f is independent of n , gives

$$\|D_n f - f\|_2 \leq \frac{C_f}{n}.$$

This proves

$$\|D_n f - f\|_2 = O(n^{-1}).$$

□

6. Bivariate Extension of the Durrmeyer–Beta Operators

The tensor-product construction adopted below belongs to the broader literature on bivariate Durrmeyer-type approximation. In particular, Karateke et al. [44] studied Stancu-type bivariate Jakimovski–Leviatan–Durrmeyer operators and established Korovkin-type, modulus-of-continuity, and weighted approximation results. The present section develops the bivariate extension associated specifically with the weight–kernel pairing introduced in Equation (10).

Let $I = [0, \infty)^2$ and let $C_B(I)$ be the space of bounded continuous functions on I with the supremum norm. Put

$$q_{n,k}(t) = \frac{t^k}{B(k+1, n)(1+t)^{k+n+1}}.$$

For $n_1, n_2 > 0$, define

$$\begin{aligned} D_{n_1, n_2}(f; z_1, z_2) &= \sum_{k_1, k_2 \geq 0} p_{n_1, k_1}(z_1) p_{n_2, k_2}(z_2) \\ &\times \int_0^\infty \int_0^\infty q_{n_1, k_1}(t_1) q_{n_2, k_2}(t_2) f(t_1, t_2) dt_1 dt_2, \end{aligned} \quad (46)$$

where $p_{n,k}$ is the weight from Lemma 2. The operator is the tensor product of the two univariate operators.

For fixed $n_1, n_2 > 0$, let

$$\mathcal{A}_{n_1, n_2} = \{f : I \rightarrow \mathbb{R} \text{ Borel measurable} : D_{n_1, n_2}(|f|; z_1, z_2) < \infty \text{ for every } (z_1, z_2) \in I\}. \quad (47)$$

Then $C_B(I) \subset \mathcal{A}_{n_1, n_2}$ for $n_1, n_2 > 0$. Moreover, every continuous function satisfying

$$|f(t_1, t_2)| \leq C(1 + t_1^2 + t_2^2)$$

belongs to $\mathcal{A}_{n_1, n_2}$ whenever $n_1, n_2 > 2$. Consequently, the polynomial test functions used below are admissible under these restrictions.

Lemma 4 (Bivariate raw moments). *Let $e_{i,j}(t_1, t_2) = t_1^i t_2^j$. Assume $n_1, n_2 > 2$. Then*

$$\begin{aligned} D_{n_1, n_2}(e_{0,0}; z_1, z_2) &= 1, \\ D_{n_1, n_2}(e_{1,0}; z_1, z_2) &= \frac{n_1 z_1 + 1}{n_1 - 1}, \\ D_{n_1, n_2}(e_{0,1}; z_1, z_2) &= \frac{n_2 z_2 + 1}{n_2 - 1}, \\ D_{n_1, n_2}(e_{2,0}; z_1, z_2) &= \frac{n_1^2 z_1^2 + 5n_1 z_1 + 2}{(n_1 - 1)(n_1 - 2)}, \\ D_{n_1, n_2}(e_{0,2}; z_1, z_2) &= \frac{n_2^2 z_2^2 + 5n_2 z_2 + 2}{(n_2 - 1)(n_2 - 2)}, \\ D_{n_1, n_2}(e_{1,1}; z_1, z_2) &= \frac{(n_1 z_1 + 1)(n_2 z_2 + 1)}{(n_1 - 1)(n_2 - 1)}. \end{aligned} \quad (48)$$

Proof. For the univariate monomials, let us write $e_r(t) = t^r$, $r = 0, 1, 2$.

We first show explicitly how the tensor-product structure acts on the bivariate monomial

$$e_{i,j}(t_1, t_2) = e_i(t_1)e_j(t_2) = t_1^i t_2^j.$$

By the definition of the bivariate operator,

$$\begin{aligned} D_{n_1, n_2}(e_{i,j}; z_1, z_2) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_{n_1, k_1}(z_1) p_{n_2, k_2}(z_2) \\ &\quad \times \frac{1}{B(k_1+1, n_1)B(k_2+1, n_2)} \int_0^{\infty} \int_0^{\infty} \frac{t_1^{k_1+i} t_2^{k_2+j}}{(1+t_1)^{k_1+n_1+1} (1+t_2)^{k_2+n_2+1}} dt_1 dt_2. \end{aligned}$$

The integrand is a product of a function of t_1 and a function of t_2 . Hence,

$$\begin{aligned} &\int_0^{\infty} \int_0^{\infty} \frac{t_1^{k_1+i} t_2^{k_2+j}}{(1+t_1)^{k_1+n_1+1} (1+t_2)^{k_2+n_2+1}} dt_1 dt_2 \\ &= \left(\int_0^{\infty} \frac{t_1^{k_1+i}}{(1+t_1)^{k_1+n_1+1}} dt_1 \right) \left(\int_0^{\infty} \frac{t_2^{k_2+j}}{(1+t_2)^{k_2+n_2+1}} dt_2 \right). \end{aligned}$$

Moreover,

$$\int_0^{\infty} \frac{t^{k+r}}{(1+t)^{k+n+1}} dt = B(k+r+1, n-r), \quad n > r.$$

Thus, all the integrals occurring in the present lemma are finite because $i, j \leq 2$ and $n_1, n_2 > 2$. Because all summands and integrands are nonnegative, and the counting measures on $\mathbb{N}_0$ and the Lebesgue measures on $[0, \infty)$ are σ -finite, Tonelli's theorem [34] allows the sums and integrals to be rearranged and grouped according to the two coordinates. Consequently,

$$\begin{aligned} D_{n_1, n_2}(e_{i,j}; z_1, z_2) &= \left[\sum_{k_1=0}^{\infty} p_{n_1, k_1}(z_1) \frac{1}{B(k_1+1, n_1)} \int_0^{\infty} \frac{t_1^{k_1+i}}{(1+t_1)^{k_1+n_1+1}} dt_1 \right] \\ &\quad \times \left[\sum_{k_2=0}^{\infty} p_{n_2, k_2}(z_2) \frac{1}{B(k_2+1, n_2)} \int_0^{\infty} \frac{t_2^{k_2+j}}{(1+t_2)^{k_2+n_2+1}} dt_2 \right] \\ &= D_{n_1}(e_i; z_1) D_{n_2}(e_j; z_2). \end{aligned}$$

We now use the univariate raw moments from Lemma 2:

$$\begin{aligned} D_n(e_0; z) &= 1, \\ D_n(e_1; z) &= \frac{nz+1}{n-1}, \\ D_n(e_2; z) &= \frac{n^2 z^2 + 5nz + 2}{(n-1)(n-2)}. \end{aligned}$$

Applying the product identity to each test function gives

$$\begin{aligned}
 D_{n_1, n_2}(e_{0,0}; z_1, z_2) &= D_{n_1}(e_0; z_1) D_{n_2}(e_0; z_2) = 1, \\
 D_{n_1, n_2}(e_{1,0}; z_1, z_2) &= D_{n_1}(e_1; z_1) D_{n_2}(e_0; z_2) \\
 &= \frac{n_1 z_1 + 1}{n_1 - 1}, \\
 D_{n_1, n_2}(e_{0,1}; z_1, z_2) &= D_{n_1}(e_0; z_1) D_{n_2}(e_1; z_2) \\
 &= \frac{n_2 z_2 + 1}{n_2 - 1}, \\
 D_{n_1, n_2}(e_{2,0}; z_1, z_2) &= D_{n_1}(e_2; z_1) D_{n_2}(e_0; z_2) \\
 &= \frac{n_1^2 z_1^2 + 5n_1 z_1 + 2}{(n_1 - 1)(n_1 - 2)}, \\
 D_{n_1, n_2}(e_{0,2}; z_1, z_2) &= D_{n_1}(e_0; z_1) D_{n_2}(e_2; z_2) \\
 &= \frac{n_2^2 z_2^2 + 5n_2 z_2 + 2}{(n_2 - 1)(n_2 - 2)}, \\
 D_{n_1, n_2}(e_{1,1}; z_1, z_2) &= D_{n_1}(e_1; z_1) D_{n_2}(e_1; z_2) \\
 &= \frac{(n_1 z_1 + 1)(n_2 z_2 + 1)}{(n_1 - 1)(n_2 - 1)}.
 \end{aligned}$$

This proves all the stated identities. $\square$

Bivariate Central Moments and Approximation Estimates

The central moments obtained in Lemma 5 describe the dispersion of the bivariate operator in the two coordinate directions. To express the resulting approximation estimates, we introduce separate moduli of continuity for the first and second variables.

Since $I = [0, \infty)^2$ is unbounded, a bounded continuous function on I need not be uniformly continuous. We therefore consider the subspace

$$C_{BU}(I) = \{f \in C_B(I) : f \text{ is uniformly continuous on } I\},$$

where uniform continuity is understood with respect to the Euclidean distance on I .

Definition 1. For $f \in C_{BU}(I)$ and $\delta > 0$, the partial moduli of continuity with respect to the first and second variables are defined by

$$\begin{aligned}
 \omega^{(1)}(f; \delta) &= \sup_{\substack{z_1, t_1, z_2 \geq 0 \\ |t_1 - z_1| \leq \delta}} |f(t_1, z_2) - f(z_1, z_2)|, \\
 \omega^{(2)}(f; \delta) &= \sup_{\substack{z_1, z_2, t_2 \geq 0 \\ |t_2 - z_2| \leq \delta}} |f(z_1, t_2) - f(z_1, z_2)|.
 \end{aligned} \tag{49}$$

The quantity $\omega^{(1)}(f; \delta)$ measures the variation in f in the first coordinate while the second coordinate is held fixed. Similarly, $\omega^{(2)}(f; \delta)$ measures its variation in the second coordinate. Since f is uniformly continuous on I , we have $\lim_{\delta \rightarrow 0+} \omega^{(i)}(f; \delta) = 0$, $i = 1, 2$.

Indeed, let $\varepsilon > 0$. Uniform continuity provides an $\eta > 0$ such that $\|(t_1, t_2) - (z_1, z_2)\|_2 < \eta$ implies $|f(t_1, t_2) - f(z_1, z_2)| < \varepsilon$. If only the first coordinate varies and $|t_1 - z_1| < \eta$, then

$$\|(t_1, z_2) - (z_1, z_2)\|_2 = |t_1 - z_1| < \eta.$$

It follows that $\omega^{(1)}(f; \delta) \leq \varepsilon$ for $0 < \delta < \eta$. The same argument, with only the second coordinate varying, gives $\omega^{(2)}(f; \delta) \leq \varepsilon$ for $0 < \delta < \eta$. This proves the stated limits.

The following approximation theorem combines these partial moduli with the coordinatewise second central moments $\mu_{n_1, n_2}^{(2,0)}(z_1, z_2)$ and $\mu_{n_1, n_2}^{(0,2)}(z_1, z_2)$.

Lemma 5 (Bivariate central moments). *For $n_1, n_2 > 2$, set*

$$\mu_{n_1, n_2}^{(r,s)}(z_1, z_2) = D_{n_1, n_2}((t_1 - z_1)^r (t_2 - z_2)^s; z_1, z_2). \quad (50)$$

Then

$$\begin{aligned} \mu_{n_1, n_2}^{(1,0)}(z_1, z_2) &= \frac{z_1 + 1}{n_1 - 1}, \\ \mu_{n_1, n_2}^{(0,1)}(z_1, z_2) &= \frac{z_2 + 1}{n_2 - 1}, \\ \mu_{n_1, n_2}^{(2,0)}(z_1, z_2) &= \frac{(n_1 + 2)z_1^2 + (3n_1 + 4)z_1 + 2}{(n_1 - 1)(n_1 - 2)}, \\ \mu_{n_1, n_2}^{(0,2)}(z_1, z_2) &= \frac{(n_2 + 2)z_2^2 + (3n_2 + 4)z_2 + 2}{(n_2 - 1)(n_2 - 2)}, \\ \mu_{n_1, n_2}^{(1,1)}(z_1, z_2) &= \frac{(z_1 + 1)(z_2 + 1)}{(n_1 - 1)(n_2 - 1)}. \end{aligned} \quad (51)$$

Proof. For convenience, define the zeroth univariate central moment by

$$\mu_{n,0}(z) := D_n(1; z) = 1,$$

and recall that, for $r = 1, 2$,

$$\mu_{n,r}(z) := D_n((t - z)^r; z).$$

We first justify the tensor-product decomposition for the centered monomials. Since $|t - z| \leq t + z$ and $|t - z|^2 \leq 2t^2 + 2z^2$, the positivity of D_n gives

$$\begin{aligned} D_n(|t - z|; z) &\leq D_n(t; z) + zD_n(1; z), \\ D_n(|t - z|^2; z) &\leq 2D_n(t^2; z) + 2z^2D_n(1; z). \end{aligned}$$

The raw moments on the right-hand sides are finite for $n > 2$. Consequently, all centered monomials used below are absolutely integrable with respect to the corresponding Durrmeyer–Beta kernels.

Because the bivariate kernel and its weights have a tensor-product structure, Tonelli’s theorem applied to the absolute value, followed by Fubini’s theorem [34], gives

$$\begin{aligned} D_{n_1, n_2}((t_1 - z_1)^i (t_2 - z_2)^j; z_1, z_2) \\ = D_{n_1}((t_1 - z_1)^i; z_1) D_{n_2}((t_2 - z_2)^j; z_2) \\ = \mu_{n_1, i}(z_1) \mu_{n_2, j}(z_2) \end{aligned} \quad (52)$$

for the index pairs $(i, j) \in \{(1, 0), (0, 1), (2, 0), (0, 2), (1, 1)\}$.

By Lemma 3,

$$\begin{aligned} \mu_{n,0}(z) &= 1, \\ \mu_{n,1}(z) &= \frac{z + 1}{n - 1}, \\ \mu_{n,2}(z) &= \frac{(n + 2)z^2 + (3n + 4)z + 2}{(n - 1)(n - 2)}. \end{aligned}$$

We now evaluate the five required moments separately. First,

$$\begin{aligned}\mu_{n_1, n_2}^{(1,0)}(z_1, z_2) &= \mu_{n_1,1}(z_1)\mu_{n_2,0}(z_2) \\ &= \frac{z_1 + 1}{n_1 - 1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mu_{n_1, n_2}^{(0,1)}(z_1, z_2) &= \mu_{n_1,0}(z_1)\mu_{n_2,1}(z_2) \\ &= \frac{z_2 + 1}{n_2 - 1}.\end{aligned}$$

For the pure second-order moments, we obtain

$$\begin{aligned}\mu_{n_1, n_2}^{(2,0)}(z_1, z_2) &= \mu_{n_1,2}(z_1)\mu_{n_2,0}(z_2) \\ &= \frac{(n_1 + 2)z_1^2 + (3n_1 + 4)z_1 + 2}{(n_1 - 1)(n_1 - 2)}\end{aligned}$$

and

$$\begin{aligned}\mu_{n_1, n_2}^{(0,2)}(z_1, z_2) &= \mu_{n_1,0}(z_1)\mu_{n_2,2}(z_2) \\ &= \frac{(n_2 + 2)z_2^2 + (3n_2 + 4)z_2 + 2}{(n_2 - 1)(n_2 - 2)}.\end{aligned}$$

Finally, the mixed central moment is the product of the two first central moments:

$$\begin{aligned}\mu_{n_1, n_2}^{(1,1)}(z_1, z_2) &= \mu_{n_1,1}(z_1)\mu_{n_2,1}(z_2) \\ &= \frac{z_1 + 1}{n_1 - 1} \frac{z_2 + 1}{n_2 - 1} \\ &= \frac{(z_1 + 1)(z_2 + 1)}{(n_1 - 1)(n_2 - 1)}.\end{aligned}$$

This proves all the stated identities. $\square$

Theorem 6 (Compact-uniform bivariate convergence). *Let $f \in C_B(I)$ and let $K \subset I$ be compact. Then*

$$\lim_{n_1, n_2 \rightarrow \infty} \sup_{(z_1, z_2) \in K} |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| = 0. \quad (53)$$

Proof. Let $M = \|f\|_\infty$. Since K is compact, there is an $R > 0$ such that $K \subset [0, R]^2$. Given $\varepsilon > 0$, the function f is uniformly continuous on $[0, R + 1]^2$. Hence, there exists $0 < \delta \leq 1$ such that

$$|f(t_1, t_2) - f(z_1, z_2)| < \varepsilon$$

whenever $(z_1, z_2) \in K$ and $(t_1 - z_1)^2 + (t_2 - z_2)^2 < \delta^2$.

For arbitrary $(t_1, t_2) \in I$ and $(z_1, z_2) \in K$, it follows that

$$|f(t_1, t_2) - f(z_1, z_2)| \leq \varepsilon + \frac{2M}{\delta^2} ((t_1 - z_1)^2 + (t_2 - z_2)^2).$$

Using positivity and $D_{n_1, n_2}(1; z_1, z_2) = 1$, we obtain

$$\begin{aligned} & |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \\ & \leq \varepsilon + \frac{2M}{\delta^2} \left(\mu_{n_1, n_2}^{(2,0)}(z_1, z_2) + \mu_{n_1, n_2}^{(0,2)}(z_1, z_2) \right). \end{aligned}$$

Lemma 5 shows that the two central moments converge to zero uniformly on K as $n_1, n_2 \rightarrow \infty$. Since ε is arbitrary, the assertion follows. $\square$

Theorem 7 (Bivariate modulus estimate). *Let $n_1, n_2 > 2$, $(z_1, z_2) \in I$, and $f \in C_{BU}(I)$. Then*

$$\begin{aligned} |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| & \leq 2\omega^{(1)}\left(f; \sqrt{\mu_{n_1, n_2}^{(2,0)}(z_1, z_2)}\right) \\ & + 2\omega^{(2)}\left(f; \sqrt{\mu_{n_1, n_2}^{(0,2)}(z_1, z_2)}\right). \end{aligned} \quad (54)$$

Proof. Let $\delta_1, \delta_2 > 0$. The triangle inequality gives

$$|f(t_1, t_2) - f(z_1, z_2)| \leq |f(t_1, t_2) - f(z_1, t_2)| + |f(z_1, t_2) - f(z_1, z_2)|.$$

By the standard property of the partial moduli,

$$\begin{aligned} |f(t_1, t_2) - f(z_1, t_2)| & \leq \left(1 + \frac{|t_1 - z_1|}{\delta_1}\right) \omega^{(1)}(f; \delta_1), \\ |f(z_1, t_2) - f(z_1, z_2)| & \leq \left(1 + \frac{|t_2 - z_2|}{\delta_2}\right) \omega^{(2)}(f; \delta_2). \end{aligned}$$

Positivity, linearity, and preservation of constants therefore imply

$$\begin{aligned} & |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \\ & \leq \omega^{(1)}(f; \delta_1) \left(1 + \frac{D_{n_1, n_2}(|t_1 - z_1|; z_1, z_2)}{\delta_1}\right) \\ & + \omega^{(2)}(f; \delta_2) \left(1 + \frac{D_{n_1, n_2}(|t_2 - z_2|; z_1, z_2)}{\delta_2}\right). \end{aligned}$$

Cauchy–Schwarz gives

$$\begin{aligned} D_{n_1, n_2}(|t_1 - z_1|; z_1, z_2) & \leq \left(\mu_{n_1, n_2}^{(2,0)}(z_1, z_2)\right)^{1/2}, \\ D_{n_1, n_2}(|t_2 - z_2|; z_1, z_2) & \leq \left(\mu_{n_1, n_2}^{(0,2)}(z_1, z_2)\right)^{1/2}. \end{aligned}$$

Taking

$$\delta_1 = \sqrt{\mu_{n_1, n_2}^{(2,0)}(z_1, z_2)}, \quad \delta_2 = \sqrt{\mu_{n_1, n_2}^{(0,2)}(z_1, z_2)}$$

proves the stated inequality. $\square$

Definition 2. For $0 < \alpha, \beta \leq 1$, write $f \in \text{Lip}_M^{(\alpha, \beta)}$ if $f \in C_B(I)$ and

$$|f(t_1, t_2) - f(z_1, z_2)| \leq M|t_1 - z_1|^\alpha + M|t_2 - z_2|^\beta \quad (55)$$

for all $(t_1, t_2), (z_1, z_2) \in I$.

Theorem 8 (Bivariate Lipschitz estimate). Let $n_1, n_2 > 2$, $(z_1, z_2) \in I$, and $0 < \alpha, \beta \leq 1$. If $f \in \text{Lip}_M^{(\alpha, \beta)}$, then

$$|D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \leq M \left(\mu_{n_1, n_2}^{(2,0)}(z_1, z_2) \right)^{\alpha/2} + M \left(\mu_{n_1, n_2}^{(0,2)}(z_1, z_2) \right)^{\beta/2}. \quad (56)$$

Proof. Fix $(z_1, z_2) \in I$. Since the bivariate operator is positive, linear, and preserves constants, we have $D_{n_1, n_2}(1; z_1, z_2) = 1$ and therefore

$$\begin{aligned} & |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \\ &= |D_{n_1, n_2}(f(t_1, t_2) - f(z_1, z_2); z_1, z_2)| \\ &\leq D_{n_1, n_2}(|f(t_1, t_2) - f(z_1, z_2)|; z_1, z_2). \end{aligned}$$

Because $f \in \text{Lip}_M^{(\alpha, \beta)}$, its defining inequality gives

$$|f(t_1, t_2) - f(z_1, z_2)| \leq M|t_1 - z_1|^\alpha + M|t_2 - z_2|^\beta.$$

Using the positivity and linearity of the operator once more, we obtain

$$\begin{aligned} & |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \\ &\leq MD_{n_1, n_2}(|t_1 - z_1|^\alpha; z_1, z_2) \\ &\quad + MD_{n_1, n_2}(|t_2 - z_2|^\beta; z_1, z_2). \end{aligned} \quad (57)$$

We estimate the two terms separately. For the first term, take the conjugate Hölder exponents $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$. Indeed, $\frac{1}{p} + \frac{1}{q} = \frac{\alpha}{2} + \frac{2-\alpha}{2} = 1$. Applying Hölder's inequality to the positive normalized functional D_{n_1, n_2} gives

$$\begin{aligned} & D_{n_1, n_2}(|t_1 - z_1|^\alpha; z_1, z_2) \\ &\leq \left[D_{n_1, n_2} \left((|t_1 - z_1|^\alpha)^p; z_1, z_2 \right) \right]^{1/p} \\ &\quad \times \left[D_{n_1, n_2}(1^q; z_1, z_2) \right]^{1/q}. \end{aligned}$$

Since $\alpha p = 2$, $1/p = \alpha/2$, and $D_{n_1, n_2}(1; z_1, z_2) = 1$, this becomes

$$\begin{aligned} & D_{n_1, n_2}(|t_1 - z_1|^\alpha; z_1, z_2) \\ &\leq \left[D_{n_1, n_2} \left((t_1 - z_1)^2; z_1, z_2 \right) \right]^{\alpha/2} \\ &= \left(\mu_{n_1, n_2}^{(2,0)}(z_1, z_2) \right)^{\alpha/2}. \end{aligned}$$

For the second coordinate, choose $p = \frac{2}{\beta}$, $q = \frac{2}{2-\beta}$. The same argument yields

$$\begin{aligned} & D_{n_1, n_2}(|t_2 - z_2|^\beta; z_1, z_2) \\ &\leq \left[D_{n_1, n_2} \left((t_2 - z_2)^2; z_1, z_2 \right) \right]^{\beta/2} \\ &= \left(\mu_{n_1, n_2}^{(0,2)}(z_1, z_2) \right)^{\beta/2}. \end{aligned}$$

Substituting these two estimates into (57), we conclude that

$$\begin{aligned} & |D_{n_1, n_2}(f; z_1, z_2) - f(z_1, z_2)| \\ & \leq M \left(\mu_{n_1, n_2}^{(2,0)}(z_1, z_2) \right)^{\alpha/2} \\ & \quad + M \left(\mu_{n_1, n_2}^{(0,2)}(z_1, z_2) \right)^{\beta/2}. \end{aligned}$$

This proves the stated estimate. $\square$

7. Numerical Illustrations

7.1. Univariate Numerical Examples

The univariate experiments are conducted on the compact interval $K_1 = [0, 2]$. We consider

$$f(z) = \frac{\pi z}{1 + z^2}, \quad g(z) = \frac{\cos(\pi z)}{1 + z^2}, \quad z \geq 0. \quad (58)$$

Both functions are continuous and bounded on $[0, \infty)$ and therefore belong to $\mathcal{A}_n$ for every fixed $n > 0$. Indeed, for $h \in \{f, g\}$,

$$D_n(|h|; z) \leq \|h\|_\infty D_n(1; z) = \|h\|_\infty < \infty.$$

The function f is nonnegative and unimodal on K_1 , whereas g is a smooth, damped oscillatory function that changes sign.

Since both functions have bounded first derivatives on $[0, \infty)$, they belong to suitable classes $\text{Lip}_{M_h}(1)$. Theorem 2 therefore gives

$$|D_n(h; z) - h(z)| \leq M_h \sqrt{\mu_{n,2}(z)}, \quad h \in \{f, g\}.$$

Because $\mu_{n,2}(z) = O(n^{-1})$ uniformly on compact intervals, this implies an $O(n^{-1/2})$ compact-uniform bound for these examples. A sharper estimate follows from the additional smoothness of these functions. Both have bounded first and second derivatives, so Theorem 5 gives a weighted $O(n^{-1})$ estimate. Consequently, for every fixed $R > 0$,

$$\sup_{0 \leq z \leq R} |D_n(h; z) - h(z)| \leq (1 + R^2) \|D_n h - h\|_2 = O(n^{-1}), \quad h \in \{f, g\}. \quad (59)$$

For numerical evaluation, the substitution $x = \frac{t}{1+t}$, $t = \frac{x}{1-x}$ transforms the inner Beta-prime integral into

$$\begin{aligned} A_{n,k}^{(h)} &:= \frac{1}{B(k+1, n)} \int_0^\infty \frac{t^k h(t)}{(1+t)^{k+n+1}} dt \\ &= \frac{1}{B(k+1, n)} \int_0^1 x^k (1-x)^{n-1} h\left(\frac{x}{1-x}\right) dx. \end{aligned} \quad (60)$$

The transformed integrals were evaluated by adaptive quadrature with absolute and relative tolerances 10^{-12} and 10^{-11} , respectively. The outer series was truncated so that its omitted weight mass was below 10^{-14} for every $z \in K_1$.

The computations were performed for $n \in \{5, 10, 20, 40, 80\}$ on a uniform grid of $N_g = 1001$ points. In Figures 1–4, orange, gold, green, blue, and purple represent $n = 5, 10, 20, 40$, and 80 , respectively; the black curves represent the target functions.

7.1.1. First Test Function

This example considers $f(z) = \pi z / (1 + z^2)$ on $[0, 2]$. This function increases up to its maximum at $z = 1$ and then decreases. Figure 1 compares the target function with its Durrmeyer–Beta approximations for five values of n . As n increases, the curves capture the peak and the overall shape of f more closely.

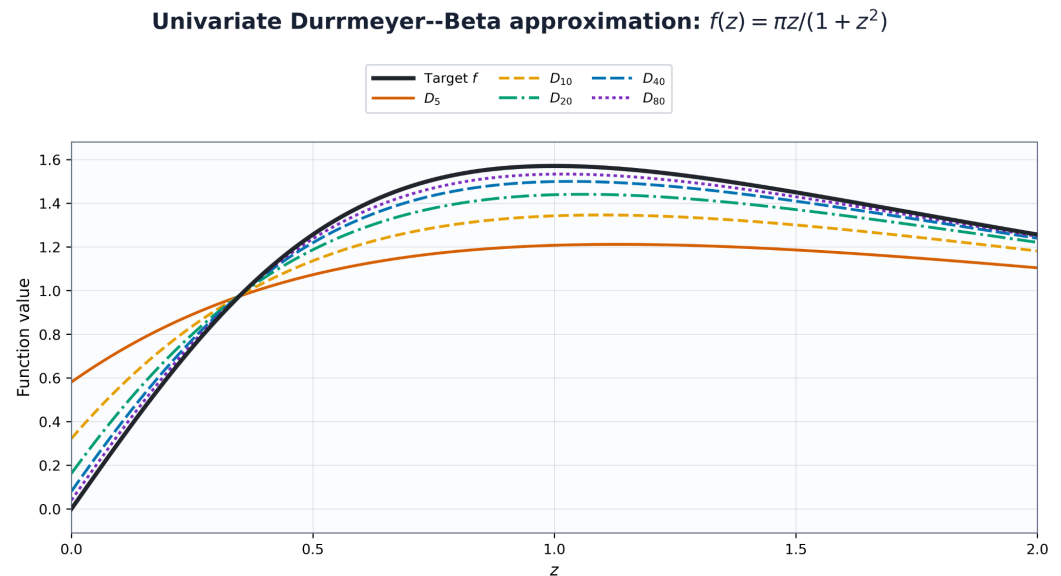


Figure 1. The target function $f(z) = \pi z / (1 + z^2)$ and its Durrmeyer–Beta approximations on $[0, 2]$ for $n \in \{5, 10, 20, 40, 80\}$.

Figure 2 shows the pointwise absolute errors for the approximations in Figure 1. The grid-based maximum error decreases from 0.5809 at $n = 5$ to 0.0400 at $n = 80$.

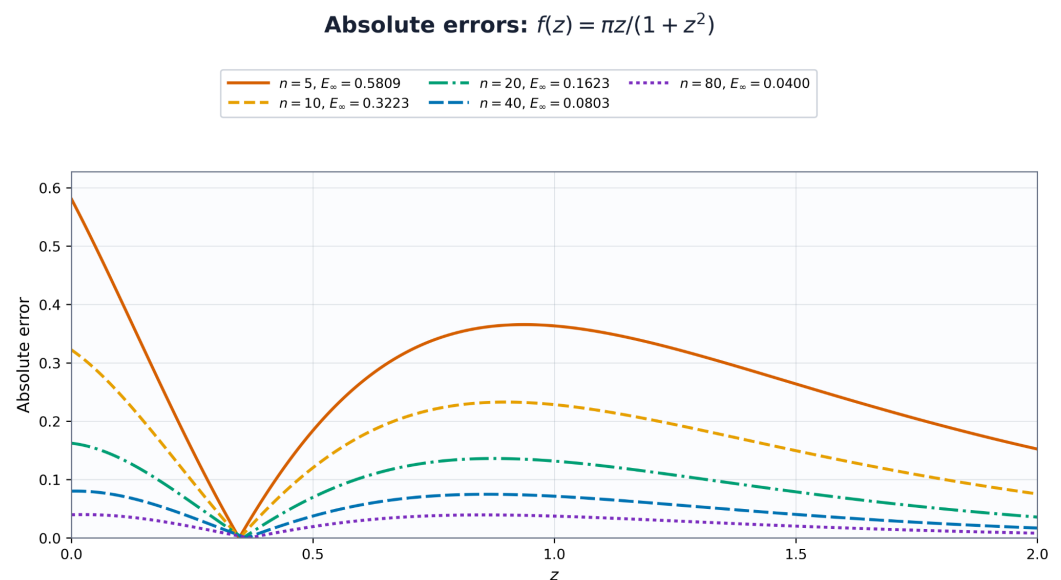


Figure 2. Pointwise absolute errors $|D_n(f; z) - f(z)|$ on $[0, 2]$ for $n \in \{5, 10, 20, 40, 80\}$. The legend reports the grid-based maximum error E_∞ for each n .

7.1.2. Second Test Function

The second example considers the bounded oscillatory function $g(z) = \cos(\pi z) / (1 + z^2)$ on $[0, 2]$. Figure 3 compares this function with its Durrmeyer–Beta approximations for five values of n . For smaller n , the approximations smooth the oscillations and reduce

the magnitudes of the positive and negative extrema. As n increases, these features are captured more closely.

Univariate Durrmeyer–Beta approximation: $g(z) = \cos(\pi z)/(1 + z^2)$

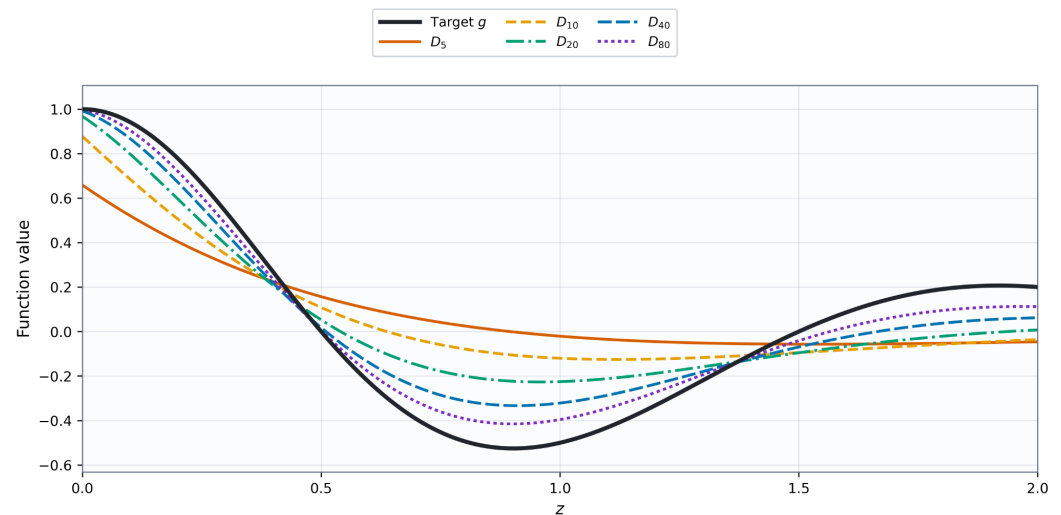


Figure 3. The target function $g(z) = \cos(\pi z)/(1 + z^2)$ and its Durrmeyer–Beta approximations on $[0, 2]$ for $n \in \{5, 10, 20, 40, 80\}$.

Figure 4 shows the corresponding pointwise absolute errors. The grid-based maximum error decreases from 0.5287 at $n = 5$ to 0.1103 at $n = 80$. At the larger tested indices, the maximum errors remain higher than those obtained for the first test function.

Absolute errors: $g(z) = \cos(\pi z)/(1 + z^2)$

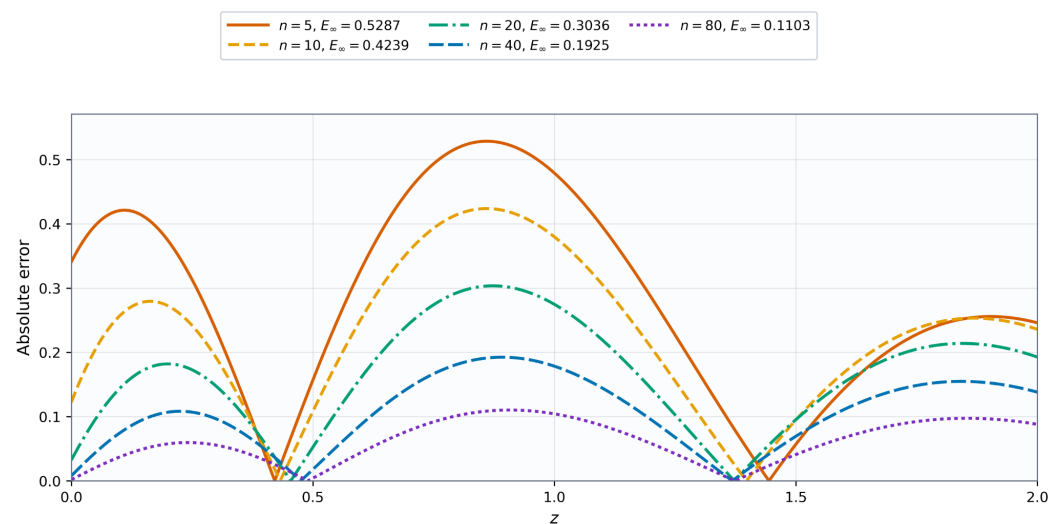


Figure 4. Pointwise absolute errors $|D_n(g; z) - g(z)|$ on $[0, 2]$ for $n \in \{5, 10, 20, 40, 80\}$. The legend reports the grid-based maximum error E_∞ for each n .

To quantify the accuracy for both test functions, let

$$y_q = h(z_q), \quad \hat{y}_q^{(n)} = D_n(h; z_q), \quad e_q^{(n)} = y_q - \hat{y}_q^{(n)}, \quad h \in \{f, g\},$$

where $z_q, q = 1, \dots, N_g$, are the evaluation points. The grid-based maximum error, root mean square error, mean absolute error, and coefficient of determination are defined by

$$E_{\infty}^{(n)} = \max_{1 \leq q \leq N_g} |e_q^{(n)}|, \quad (61)$$

$$\text{RMSE}^{(n)} = \left(\frac{1}{N_g} \sum_{q=1}^{N_g} |e_q^{(n)}|^2 \right)^{1/2}, \quad (62)$$

$$\text{MAE}^{(n)} = \frac{1}{N_g} \sum_{q=1}^{N_g} |e_q^{(n)}|, \quad (63)$$

$$R_n^2 = 1 - \frac{\sum_{q=1}^{N_g} |e_q^{(n)}|^2}{\sum_{q=1}^{N_g} |y_q - \bar{y}|^2}, \quad \bar{y} = \frac{1}{N_g} \sum_{q=1}^{N_g} y_q. \quad (64)$$

Table 2 summarizes these metrics for both functions. Over the tested index sequence, the three error measures decrease and R^2 increases. At $n = 80$, the grid-based maximum errors are 0.040030 for f and 0.110264 for g .

Table 2. Grid-based accuracy measures for the univariate approximations on $K_1 = [0, 2]$.

Test Function	n	E_{∞}	RMSE	MAE	R^2
f	5	0.580940	0.287828	0.267913	0.438913
	10	0.322264	0.174470	0.161457	0.793838
	20	0.162331	0.097922	0.090071	0.935058
	40	0.080287	0.052364	0.047912	0.981429
	80	0.040030	0.027171	0.024771	0.995000
g	5	0.528709	0.319608	0.281796	0.463148
	10	0.423892	0.252426	0.225516	0.665122
	20	0.303581	0.183479	0.164254	0.823074
	40	0.192496	0.119398	0.106498	0.925077
	80	0.110264	0.070036	0.062121	0.974221

Although the grid-based maximum error for g is smaller than that for f at $n = 5$, it is larger for every reported $n \geq 10$.

7.1.3. A Bounded Test Function on a Wider Interval

To examine approximation on a wider interval and the influence of reduced second-order regularity, consider

$$h_w(z) = \frac{1}{2} \left(1 + \frac{(z-10)|z-10|}{1+(z-10)^2} \right), \quad z \geq 0. \quad (65)$$

This function satisfies $0 < h_w(z) < 1, z \geq 0$. Its first derivative is

$$h'_w(z) = \frac{|z-10|}{(1+(z-10)^2)^2},$$

which is continuous on $[0, \infty)$. At the transition point, $h_w(10) = \frac{1}{2}, h'_w(10) = 0$.

The one-sided second derivatives satisfy $h''_w(10-) = -1, h''_w(10+) = 1$. Thus, $h_w \in C^1[0, \infty) \cap C_B[0, \infty), h_w \notin C^2[0, \infty)$. Moreover, $\|h'_w\|_{\infty} = \frac{3\sqrt{3}}{16}$. Consequently, h_w is globally Lipschitz continuous and belongs to $C_{BU}[0, \infty)$. Its operator integral is finite for every $n > 0$.

The bounded-function convergence result and the Lipschitz estimate established earlier therefore apply to this example. In particular, for $n > 2$,

$$|D_n(h_w; z) - h_w(z)| \leq \frac{3\sqrt{3}}{16} \sqrt{\mu_{n,2}(z)}.$$

Since the coefficients of the numerator of $\mu_{n,2}(z)$ are nonnegative, this gives

$$\sup_{0 \leq z \leq 20} |D_n(h_w; z) - h_w(z)| \leq \frac{3\sqrt{3}}{16} \sqrt{\frac{460n + 882}{(n-1)(n-2)}}. \quad (66)$$

The right-hand side tends to zero as $n \rightarrow \infty$, providing an explicit compact-uniform error bound on the wider interval.

The numerical experiment uses a uniform 1001-point grid on $[0, 20]$ and $n \in \{5, 10, 20, 40, 80\}$. The outer series is truncated so that its omitted weight mass is below 10^{-14} . The inner integrals are evaluated after the transformation $x = t/(1+t)$, splitting the integration interval at $x = 10/11$ and at additional points determined by the location and scale of the beta density. The requested absolute and relative quadrature tolerances are both 2×10^{-11} .

Selected inner integrals are also evaluated independently in the original beta-prime variable, with the integral split at $t = 10$. The largest observed difference in these checks is below 8×10^{-14} .

Table 3 shows that all three error measures decrease over the tested sequence of indices. The grid-based maximum error falls from 0.306212 at $n = 5$ to 0.083763 at $n = 80$.

Figure 5 shows how the approximations recover the transition around $z = 10$ as n increases. The larger errors occur in the transition region, while the approximations follow the nearly constant portions more closely. This example extends the numerical study to $[0, 20]$ within the bounded, uniformly continuous function setting used in the approximation results.

Table 3. Grid-based approximation errors for h_w on a uniform 1001-point grid in $[0, 20]$.

n	E_∞	RMSE	MAE
5	0.306212	0.158194	0.132682
10	0.242904	0.114535	0.088752
20	0.182553	0.078445	0.055376
40	0.128093	0.050638	0.032685
80	0.083763	0.031012	0.018526

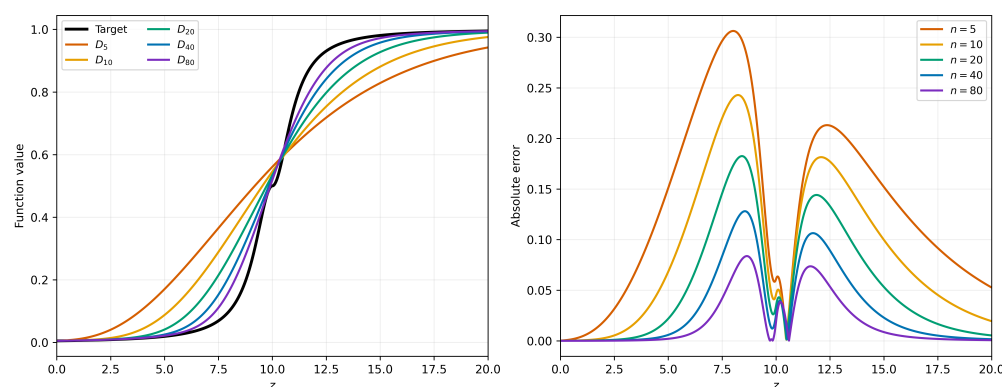


Figure 5. Approximation of the bounded function h_w on $[0, 20]$. Left: the target function and its approximations for $n = 5, 10, 20, 40, 80$. Right: the corresponding absolute errors. The target has a continuous first derivative and unequal one-sided second derivatives at $z = 10$.

7.2. Bivariate Numerical Examples

The bivariate experiments are performed on $K = [0, 2]^2$. We consider one smooth and one nonsmooth bounded test function. Both belong to $C_B(I)$ and therefore satisfy the hypothesis of Theorem 6.

7.2.1. Smooth Bounded Test Function

Define

$$\phi(z) = \frac{4z}{(1+z)^2}, \quad z \geq 0, \quad (67)$$

and

$$f_s(z_1, z_2) = \phi(z_1)\phi(z_2) = \frac{16z_1z_2}{(1+z_1)^2(1+z_2)^2}. \quad (68)$$

Since $0 \leq \phi(z) \leq 1$, the function f_s is smooth, bounded, and belongs to $C_B(I)$. The tensor-product structure gives

$$D_{n,n}(f_s; z_1, z_2) = D_n(\phi; z_1)D_n(\phi; z_2). \quad (69)$$

7.2.2. Nonsmooth Lipschitz Test Function

Define

$$\psi(z) = \frac{|z-1|}{1+z}, \quad z \geq 0, \quad (70)$$

and

$$f_{ns}(z_1, z_2) = \frac{1}{2}(\psi(z_1) + \psi(z_2)). \quad (71)$$

The function ψ is bounded and continuous but is not differentiable at $z = 1$. It is globally Lipschitz continuous and satisfies $|\psi(t) - \psi(z)| \leq 2|t - z|$, $t, z \geq 0$. Consequently,

$$|f_{ns}(t_1, t_2) - f_{ns}(z_1, z_2)| \leq |t_1 - z_1| + |t_2 - z_2|,$$

so $f_{ns} \in \text{Lip}_1^{(1,1)}$ and Theorem 8 applies. Moreover,

$$D_{n,n}(f_{ns}; z_1, z_2) = \frac{1}{2}(D_n(\psi; z_1) + D_n(\psi; z_2)). \quad (72)$$

For the numerical evaluation, put $x = t/(1+t)$. Then

$$\frac{t^k}{B(k+1, n)(1+t)^{k+n+1}} dt = \frac{x^k(1-x)^{n-1}}{B(k+1, n)} dx. \quad (73)$$

For the smooth factor, $\phi(t) = 4x(1-x)$, and hence

$$D_n(\phi; z) = \sum_{k=0}^{\infty} p_{n,k}(z) \frac{4n(k+1)}{(k+n+1)(k+n+2)}. \quad (74)$$

For the nonsmooth factor, $\psi(t) = |2x - 1|$. If $a = k+1$, $b = n$, and $I_x(a, b)$ denotes the regularized incomplete beta function, splitting the integral at $x = 1/2$ gives

$$A_{n,k}^{(\psi)} = 2I_{1/2}(a, b) - \frac{4a}{a+b}I_{1/2}(a+1, b) + \frac{2a}{a+b} - 1, \quad (75)$$

and

$$D_n(\psi; z) = \sum_{k=0}^{\infty} p_{n,k}(z) A_{n,k}^{(\psi)}. \quad (76)$$

The computations were performed for $n_1 = n_2 = n$ with $n \in \{5, 10, 20, 40, 80\}$ on a uniform 201×201 grid over K . The omitted mass of the outer series was kept below 10^{-14} at every grid point.

In Figures 6 and 7, the same color scale is used in every panel. Dark purple represents values near zero, blue and green represent intermediate values, and yellow represents values near one. The common scale permits a direct visual comparison between each target surface and its approximations.

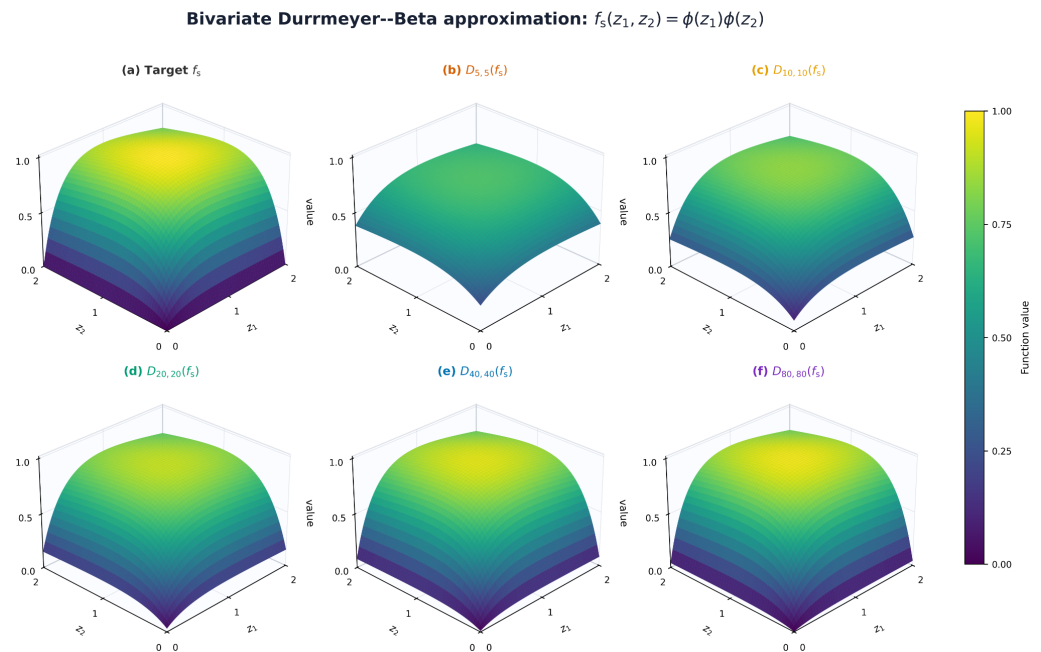


Figure 6. Approximation of the smooth test function f_s on $K = [0, 2]^2$ for $n_1 = n_2 = n \in \{5, 10, 20, 40, 80\}$. Panel (a) shows the target surface, while panels (b–f) show the corresponding operator approximations.

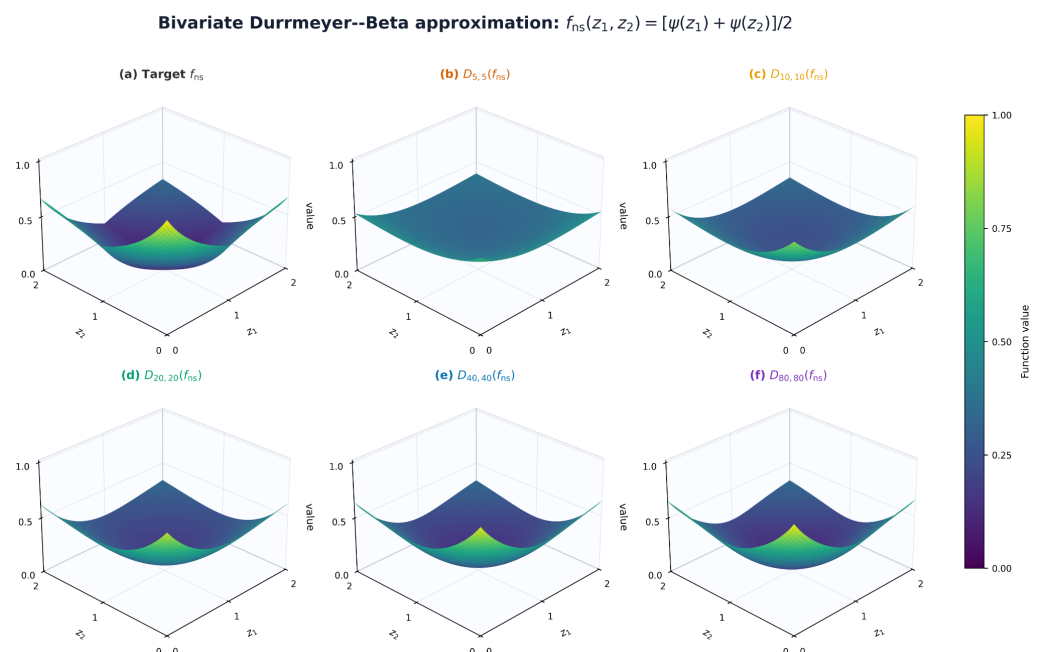


Figure 7. Approximation of the nonsmooth test function f_{ns} on $K = [0, 2]^2$ for $n_1 = n_2 = n \in \{5, 10, 20, 40, 80\}$. Panel (a) shows the target surface, while panels (b–f) show the corresponding operator approximations.

For either $F \in \{f_s, f_{ns}\}$, define

$$y_{ij} = F(z_{1,i}, z_{2,j}), \quad \hat{y}_{ij}^{(n)} = D_{n,n}(F; z_{1,i}, z_{2,j}), \quad e_{ij}^{(n)} = y_{ij} - \hat{y}_{ij}^{(n)}.$$

The bivariate grid contains $N_g = 201^2 = 40401$ points. The definitions of E_∞ , RMSE, MAE, and R^2 are the natural double-sum counterparts of Equations (61)–(64).

Table 4 summarizes the accuracy metrics for both bivariate test functions. All three error measures decrease over the tested index sequence. From $n = 5$ to $n = 80$, the grid-based maximum error decreases from 0.404545 to 0.047588 for f_s and from 0.323815 to 0.088747 for f_{ns} . At the larger tested indices, this maximum error decreases more slowly for f_{ns} , consistent with its reduced regularity along $z_1 = 1$ and $z_2 = 1$.

Table 4. Grid-based accuracy metrics for the bivariate approximation of the smooth and nonsmooth test functions on $K = [0, 2]^2$.

Test Function	$n_1 = n_2 = n$	E_∞	RMSE	MAE	R^2
f_s	5	0.404545	0.210954	0.196233	0.360811
	10	0.276917	0.128846	0.119667	0.761552
	20	0.165137	0.072044	0.066770	0.925448
	40	0.090680	0.038209	0.035357	0.979030
	80	0.047588	0.019690	0.018204	0.994431
f_{ns}	5	0.323815	0.145451	0.125607	0.221645
	10	0.241130	0.092464	0.077274	0.685448
	20	0.174605	0.055485	0.044070	0.886733
	40	0.124841	0.032496	0.024092	0.961150
	80	0.088747	0.018936	0.012918	0.986807

Figure 8 shows that the error contracts as n increases. For the nonsmooth example, the remaining error is concentrated near the lines $z_1 = 1$ and $z_2 = 1$.

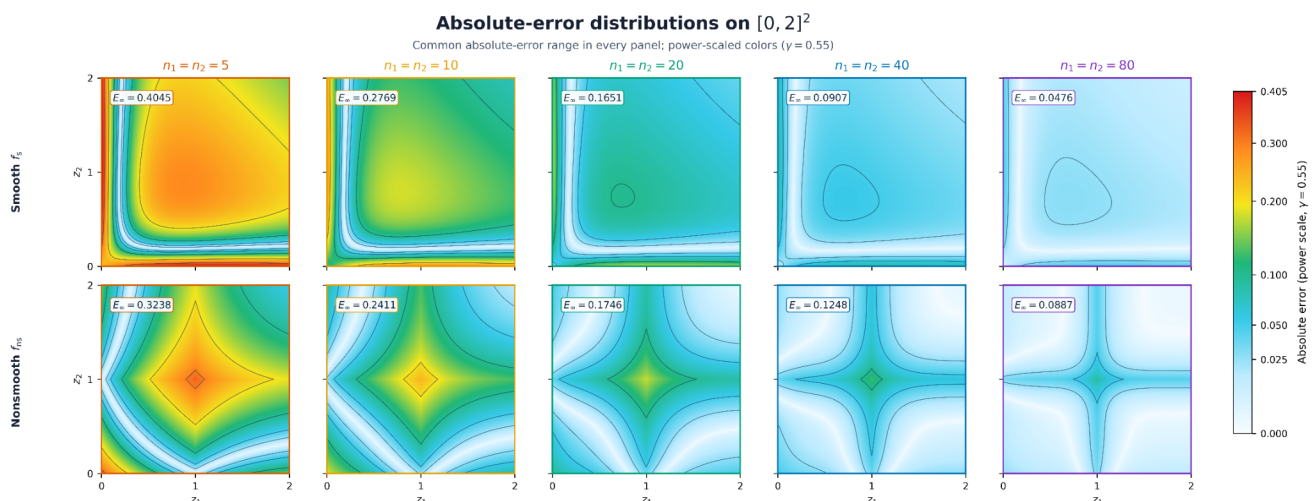


Figure 8. Absolute-error distributions for f_s (top row) and f_{ns} (bottom row) on $K = [0, 2]^2$. All panels use the same power-scaled color normalization with $\gamma = 0.55$ and common absolute-error contour levels.

7.3. Numerical Comparison with the Dunkl–Beta Operator

We complement the moment comparison in Section Quantitative Comparison with a Dunkl–Beta Operator with the bounded smooth test function $h(z) = \sin(\pi z)e^{-z/5}$, $z \geq 0$.

The operators are evaluated on the same uniform 201-point grid in $[0, 2]$, with $m + p = n$, $v = 0.3$, and $n \in \{5, 10, 20, 30, 40, 80\}$. The accuracy metrics are defined as in the preceding univariate experiments.

Both outer series use an omitted-weight threshold of 10^{-14} . For D_n and $z > 0$, the tail is computed from the negative-binomial distribution with shape nz and success probability $1/2$. For $S_n^{0.3}$, the Dunkl-weight recurrence gives a geometric tail bound, which is checked before the retained weights are normalized. Since $|h| \leq 1$, the error due to series truncation and this normalization is at most 2×10^{-14} in exact arithmetic. At $z = 0$, only the $k = 0$ term is needed for either operator.

The inner beta-prime integrals are evaluated on $[0, 200]$ by adaptive sine-weighted quadrature, with requested absolute and relative tolerances of 2×10^{-12} . The bound $|h(t)| \leq e^{-t/5}$ limits each omitted conditional integral beyond $t = 200$ to e^{-40} . Four representative inner integrals are also evaluated after the transformation $x = t/(1+t)$; the largest observed difference between the two procedures is below 2×10^{-15} .

The calculations use Python 3.13, NumPy 2.3.5, and SciPy 1.17.0. The accompanying script records quadrature error estimates and series cutoffs. Across the tested indices, the maximum retained series lengths over the grid range from 77 to 327 terms for D_n and from 46 to 276 terms for $S_n^{0.3}$. All omitted-weight bounds are below 10^{-14} .

Table 5 summarizes the errors, while Figure 9 shows the corresponding approximations. All three error measures decrease over the tested index sequence, with the Dunkl operator giving smaller values for this example. At $n = 80$, the maximum errors are 0.254587 for D_n and 0.213177 for $S_n^{0.3}$; the corresponding RMSE values are 0.138810 and 0.118412. These results complement the moment comparison for the specified function and parameter setting on $[0, 2]$.

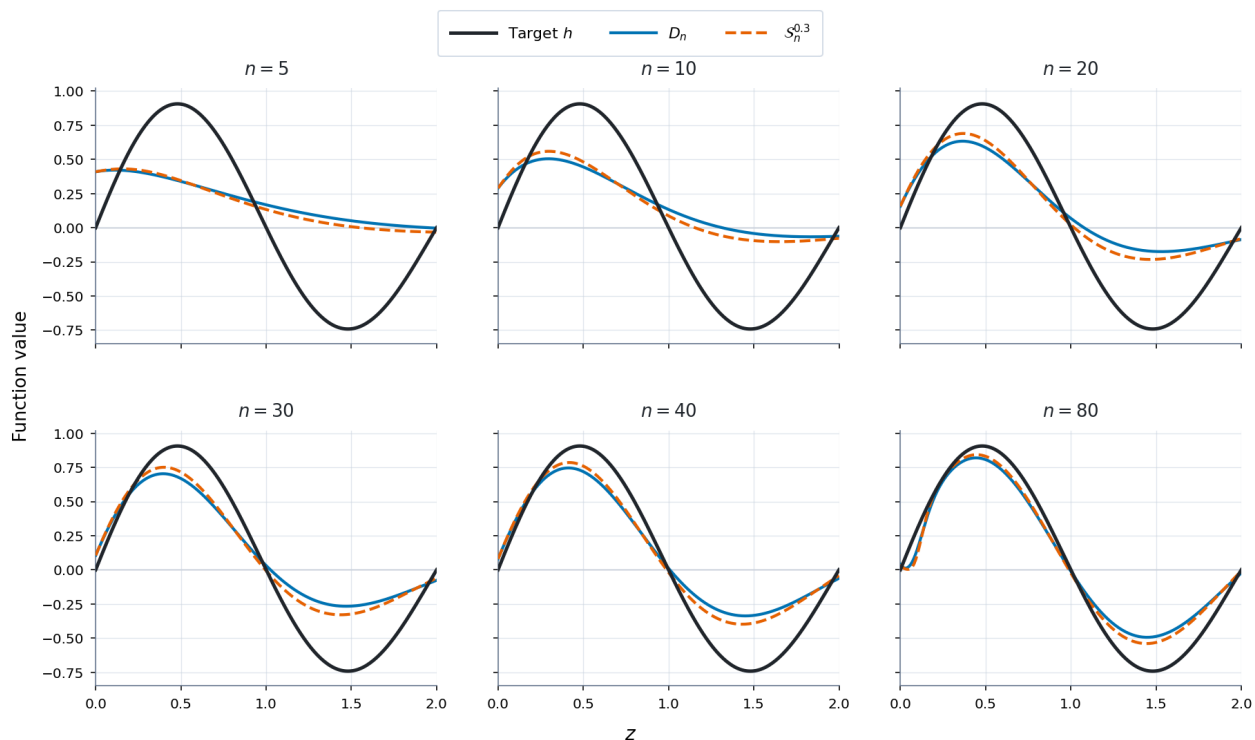


Figure 9. Approximations of $h(z) = \sin(\pi z)e^{-z/5}$ on $[0, 2]$ for $n = 5, 10, 20, 30, 40, 80$. The target function is shown in black, $D_n(h; z)$ in solid blue, and $S_n^{0.3}(h; z)$ in dashed orange, with matched indices $m + p = n$.

Table 5. Matched-index comparison for $h(z) = \sin(\pi z)e^{-z/5}$ on a uniform 201-point grid in $[0, 2]$. The Dunkl parameter is $\nu = 0.3$, with $m + p = n$.

n	Operator	E_∞	RMSE	MAE
5	D_n	0.798049	0.485856	0.426767
5	$\mathcal{S}_n^{0.3}$	0.753949	0.463041	0.406975
10	D_n	0.705478	0.414652	0.358350
10	$\mathcal{S}_n^{0.3}$	0.650657	0.381447	0.329671
20	D_n	0.569232	0.320171	0.270798
20	$\mathcal{S}_n^{0.3}$	0.509785	0.284285	0.240370
30	D_n	0.475461	0.261072	0.217878
30	$\mathcal{S}_n^{0.3}$	0.417216	0.226924	0.189235
40	D_n	0.406902	0.220244	0.182165
40	$\mathcal{S}_n^{0.3}$	0.351680	0.188670	0.155975
80	D_n	0.254587	0.138810	0.117407
80	$\mathcal{S}_n^{0.3}$	0.213177	0.118412	0.098616

8. Exploratory Comparison on Synthetic Noisy Signals

Positive linear operators have an established connection with smooth nonparametric estimation. Bernstein-based schemes have been used for nonparametric curve estimation [45], and a general framework linking random positive linear operators with nonparametric statistics has recently been developed [46]. Motivated by this connection, the following experiment examines the empirical smoothing behavior of D_n for a synthetic noisy signal.

The experiment examines signal reconstruction from noisy observations on a fixed sampling grid. The Durrmeyer–Beta smoother D_{80} is compared with three reference procedures: a Nadaraya–Watson Gaussian kernel smoother with bandwidth $b = 0.05$, an ARIMA(2, 0, 2) model, and an LSTM reconstruction network. These procedures represent nonparametric smoothing, classical linear time-series modeling [47], and nonlinear sequence reconstruction, respectively. Related applications have also explored LSTM autoencoder architectures [48]; the network used here is described in Section 8.2.

All methods receive the same noisy observations and are evaluated against the clean signal at the observed sampling points. The ARIMA order, Gaussian bandwidth, operator index, and LSTM training settings are fixed throughout the experiment. The comparison therefore examines reconstruction under a common experimental design rather than forecasting performance or performance after method-specific parameter optimization.

The results are summarized by the mean and empirical standard deviation of NRMSE, NMAE, and R^2 , describing average reconstruction accuracy and its variation across realizations. General considerations concerning the design and reporting of simulation studies are discussed by Morris et al. [49].

8.1. Signal Generation and Experimental Design

The base signal is

$$f_{\text{base}}(z) = M \frac{\pi z}{1 + z^2}, \quad z \in [0, 2], \quad M \sim \mathcal{U}(10, 50). \quad (77)$$

It is sampled at 100 equally spaced points, $z_i = \frac{2i}{99}$, $i = 0, \dots, 99$, and contaminated with additive Gaussian noise:

$$f_{\text{noisy}}(z_i) = f_{\text{base}}(z_i) + \varepsilon(z_i), \quad \varepsilon(z_i) \sim \mathcal{N}(0, \sigma^2). \quad (78)$$

The tested noise levels are $\sigma \in \{2, 4, 8, 12, 16\}$.

For each noise level, 30 signal–noise pairs are generated. For realization $r = 1, \dots, 30$, write

$$s_r(z_i) = M_r \frac{\pi z_i}{1 + z_i^2}, \quad y_{r,\sigma,i} = s_r(z_i) + \sigma \xi_{r,i},$$

where $M_r \sim \mathcal{U}(10, 50)$ and the $\xi_{r,i}$ are independent standard Gaussian variables, independent of the amplitudes.

The NumPy 1.21.0 random seed is reset to 42 when generating the data for each noise level. Consequently, the same amplitudes and standardized noise vectors are reused across noise levels. This provides a paired design for examining the effect of increasing σ . Within each noise level, all methods receive the same 30 noisy signals. The reported variation reflects both the random noise and the variation in signal amplitude.

8.2. Reconstruction Procedures

Durrmeyer–Beta smoother:

The operator index is fixed at $n = 80$ for every realization and noise level. A cubic interpolant is constructed from the observed samples using `scipy.interpolate.interp1d` in SciPy 1.7.0 with `kind='cubic'`. Its extension to the half-line is

$$F_{r,\sigma}(t) = \begin{cases} c_{r,\sigma}(t), & 0 \leq t \leq 2, \\ y_{r,\sigma,99}, & t > 2. \end{cases} \quad (79)$$

The reconstructed values are $\hat{s}_{r,\sigma,i}^D = D_{80}(F_{r,\sigma}; z_i)$. Thus, the operator acts on a bounded continuous extension of the observations, retaining the half-line integration domain in its definition.

The inner integrals are evaluated after the transformation $x = t/(1+t)$ using adaptive quadrature with requested absolute and relative tolerances of 10^{-12} and 10^{-11} , respectively. The beta-density mode is included as an integration breakpoint, with a subdivision limit of 100. For numerical endpoint handling, the transformed integrand is set to zero when $x \leq 10^{-10}$ or $x \geq 1 - 10^{-10}$.

The outer weights are generated recursively. A common cutoff across the evaluation grid is selected using a target cumulative-weight deficit below 10^{-14} , subject to an index cap of 2000. For fixed $n = 80$, no model coefficients are estimated from the observations; n remains the smoothing parameter.

Gaussian kernel smoother:

A Nadaraya–Watson estimator with a Gaussian kernel provides a direct nonparametric smoothing reference [50,51]:

$$\hat{s}_{r,\sigma,i}^K = \frac{\sum_{j=0}^{99} \exp\left(-\frac{(z_i - z_j)^2}{2b^2}\right) y_{r,\sigma,j}}{\sum_{j=0}^{99} \exp\left(-\frac{(z_i - z_j)^2}{2b^2}\right)}, \quad b = 0.05. \quad (80)$$

The bandwidth is fixed across all realizations and noise levels. Neither the bandwidth nor the Durrmeyer–Beta index is selected through a data-driven search.

ARIMA reconstruction:

An ARIMA(2,0,2) model is fitted separately to each noisy signal using `statsmodels`. The order is fixed in advance; AIC- or BIC-based order selection is not performed. Reconstruction uses the in-sample fitted values returned by the model.

The implementation makes one fitting attempt with the default settings. If fitting raises an exception, the noisy observations are returned as the baseline output. Warnings are suppressed, and the execution logs do not record optimizer convergence flags or exception counts. The ARIMA results reported below therefore summarize the outputs of this implemented procedure.

LSTM reconstruction network:

The network contains one unidirectional LSTM layer with input dimension 1 and hidden-state dimension 16, followed by a linear layer mapping the 16 features to one output at each time step. Each noisy signal is independently scaled to $[0, 1]$ and supplied as a sequence of length 100 with batch size 1.

A separate network is fitted to each noisy signal, using that same signal as the reconstruction target. Training uses mean squared error, the Adam optimizer, a learning rate of 0.01, and 100 epochs. The initialization seed in PyTorch 2.0.0 is fixed at 42 for each fit.

The implementation uses CPU tensors, without dropout, early stopping, or a separate validation set. The outputs are transformed back to the original signal scale before evaluation.

The architecture contains an LSTM layer and a timewise linear output, without a separate encoder–decoder bottleneck. It is therefore described here as an LSTM reconstruction network.

8.3. Accuracy Metrics

For realization r , let

$$A_r = \max_{0 \leq i \leq 99} s_r(z_i) - \min_{0 \leq i \leq 99} s_r(z_i) \quad (81)$$

denote the clean-signal range. For a reconstruction $\hat{s}_{r,\sigma,i}$, define

$$\begin{aligned} \text{NRMSE}_{r,\sigma} &= \frac{1}{A_r} \left[\frac{1}{100} \sum_{i=0}^{99} (\hat{s}_{r,\sigma,i} - s_r(z_i))^2 \right]^{1/2}, \\ \text{NMAE}_{r,\sigma} &= \frac{1}{100 A_r} \sum_{i=0}^{99} |\hat{s}_{r,\sigma,i} - s_r(z_i)|, \\ R_{r,\sigma}^2 &= 1 - \frac{\sum_{i=0}^{99} (\hat{s}_{r,\sigma,i} - s_r(z_i))^2}{\sum_{i=0}^{99} (s_r(z_i) - \bar{s}_r)^2}, \end{aligned} \quad (82)$$

where

$$\bar{s}_r = \frac{1}{100} \sum_{i=0}^{99} s_r(z_i).$$

These definitions agree with the implementation, which maps each clean signal to $[0, 1]$ and applies the same affine transformation to its noisy observations and reconstructed values before computing the metrics. The clean signal is used for evaluation and normalization, not as a fitting target.

For each metric $q_{r,\sigma}$, the reported mean and empirical standard deviation are

$$\bar{q}_\sigma = \frac{1}{30} \sum_{r=1}^{30} q_{r,\sigma}, \quad s_{q,\sigma} = \left[\frac{1}{30} \sum_{r=1}^{30} (q_{r,\sigma} - \bar{q}_\sigma)^2 \right]^{1/2}. \quad (83)$$

The denominator 30 agrees with the default `numpy.std` convention used in the code. These standard deviations describe across-realization variation rather than confidence intervals.

8.4. Results and Discussion

Table 6 reports the reconstruction metrics for the five noise levels. Raw noisy observations are included as a reference for assessing the effect of each reconstruction procedure.

Table 6. Reconstruction accuracy across 30 signal–noise pairs at each Gaussian noise level. Entries are means $\pm$ empirical standard deviations, computed with denominator 30. The settings are ARIMA(2,0,2), an LSTM reconstruction network with 16 hidden units, Gaussian kernel bandwidth $b = 0.05$, and Durrmeyer–Beta index $n = 80$.

σ	Metric	Raw Noisy Data	ARIMA(2,0,2)	LSTM Reconstruction	Gaussian Kernel	D_{80}
2	NRMSE	0.0495 $\pm$ 0.0241	0.0685 $\pm$ 0.0472	0.0365 $\pm$ 0.0060	0.0196 $\pm$ 0.0088	0.0228 $\pm$ 0.0059
	NMAE	0.0398 $\pm$ 0.0195	0.0354 $\pm$ 0.0127	0.0286 $\pm$ 0.0048	0.0147 $\pm$ 0.0073	0.0192 $\pm$ 0.0050
	R^2	0.9513 $\pm$ 0.0504	0.8890 $\pm$ 0.2524	0.9780 $\pm$ 0.0088	0.9926 $\pm$ 0.0079	0.9911 $\pm$ 0.0050
4	NRMSE	0.0991 $\pm$ 0.0482	0.0795 $\pm$ 0.0235	0.0492 $\pm$ 0.0226	0.0351 $\pm$ 0.0184	0.0321 $\pm$ 0.0136
	NMAE	0.0796 $\pm$ 0.0391	0.0522 $\pm$ 0.0195	0.0390 $\pm$ 0.0179	0.0278 $\pm$ 0.0149	0.0256 $\pm$ 0.0103
	R^2	0.8051 $\pm$ 0.2015	0.8896 $\pm$ 0.0717	0.9529 $\pm$ 0.0606	0.9748 $\pm$ 0.0305	0.9805 $\pm$ 0.0181
8	NRMSE	0.1982 $\pm$ 0.0963	0.1207 $\pm$ 0.0440	0.1088 $\pm$ 0.0886	0.0675 $\pm$ 0.0373	0.0545 $\pm$ 0.0295
	NMAE	0.1592 $\pm$ 0.0781	0.0819 $\pm$ 0.0253	0.0881 $\pm$ 0.0726	0.0544 $\pm$ 0.0302	0.0416 $\pm$ 0.0212
	R^2	0.2202 $\pm$ 0.8061	0.7350 $\pm$ 0.2479	0.6840 $\pm$ 0.6134	0.9045 $\pm$ 0.1202	0.9383 $\pm$ 0.0709
12	NRMSE	0.2973 $\pm$ 0.1445	0.1491 $\pm$ 0.0489	0.2040 $\pm$ 0.1573	0.1004 $\pm$ 0.0561	0.0785 $\pm$ 0.0452
	NMAE	0.2388 $\pm$ 0.1172	0.1114 $\pm$ 0.0408	0.1649 $\pm$ 0.1286	0.0811 $\pm$ 0.0454	0.0590 $\pm$ 0.0322
	R^2	−0.7545 $\pm$ 1.8137	0.6047 $\pm$ 0.2898	−0.0656 $\pm$ 1.6442	0.7876 $\pm$ 0.2693	0.8682 $\pm$ 0.1590
16	NRMSE	0.3964 $\pm$ 0.1926	0.1803 $\pm$ 0.0621	0.3090 $\pm$ 0.2086	0.1334 $\pm$ 0.0749	0.1030 $\pm$ 0.0607
	NMAE	0.3184 $\pm$ 0.1562	0.1367 $\pm$ 0.0514	0.2492 $\pm$ 0.1707	0.1079 $\pm$ 0.0606	0.0768 $\pm$ 0.0429
	R^2	−2.1192 $\pm$ 3.2244	0.4160 $\pm$ 0.4328	−1.2327 $\pm$ 2.8649	0.6243 $\pm$ 0.4776	0.7703 $\pm$ 0.2824

Table 6 shows that the relative performance depends on the noise level. At $\sigma = 2$, the Gaussian kernel smoother gives the lowest mean NRMSE and NMAE and the highest mean R^2 . For $\sigma \in \{4, 8, 12, 16\}$, D_{80} gives the best mean values of all three metrics among the tested procedures. The differences between its mean metrics and those of the fixed-bandwidth Gaussian smoother increase over these four noise levels.

At $\sigma = 16$, D_{80} yields an NRMSE of 0.1030 ± 0.0607 , an NMAE of 0.0768 ± 0.0429 , and an R^2 of 0.7703 ± 0.2824 . The corresponding Gaussian kernel results are 0.1334 ± 0.0749 , 0.1079 ± 0.0606 , and 0.6243 ± 0.4776 , respectively. The lower mean NMAE complements the NRMSE result, showing that the observed advantage is reflected in both absolute and squared-error measures.

The ARIMA baseline retains a positive mean R^2 at every tested noise level, reaching 0.4160 ± 0.4328 at $\sigma = 16$. The LSTM reconstruction network exhibits a larger decline in reconstruction accuracy at the higher noise levels, with a mean R^2 of $−1.2327 \pm 2.8649$ at $\sigma = 16$. These findings concern the implemented ARIMA procedure and the specified LSTM architecture and training settings.

For an individual realization, a negative R^2 indicates a larger sum of squared reconstruction errors than that obtained using the clean-signal mean as a constant estimate. A negative average does not imply that every realization has a negative value and does not establish a loss of structure in the underlying clean signal.

Both the operator index $n = 80$ and the Gaussian bandwidth $b = 0.05$ are held fixed, without data-driven optimization. Within these settings, the results support the use of D_{80} for reconstructing this signal family at the four higher noise levels, while the Gaussian smoother performs better at the lowest level. The empirical standard deviations

describe across-realization variation; the ordering of the means alone does not establish statistical significance.

Figures 10–12 show the mean NRMSE, NMAE, and R^2 , respectively, over 30 realizations at each noise level. The corresponding empirical standard deviations are reported in Table 6.

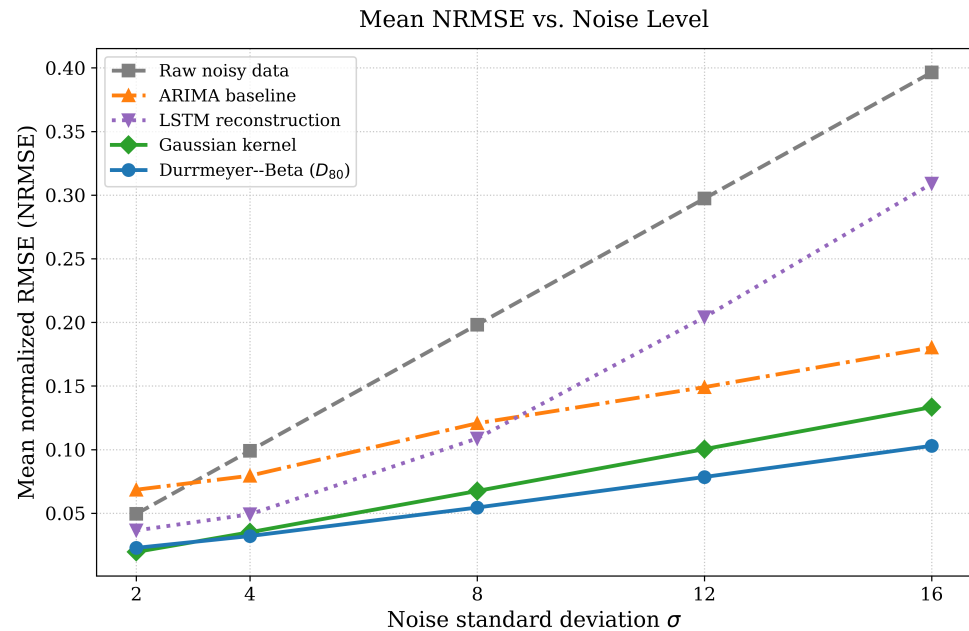


Figure 10. Mean NRMSE over 30 realizations at each Gaussian noise level. The Gaussian kernel smoother gives the lowest mean error at $\sigma = 2$, while D_{80} gives the lowest mean errors at $\sigma \in \{4, 8, 12, 16\}$.

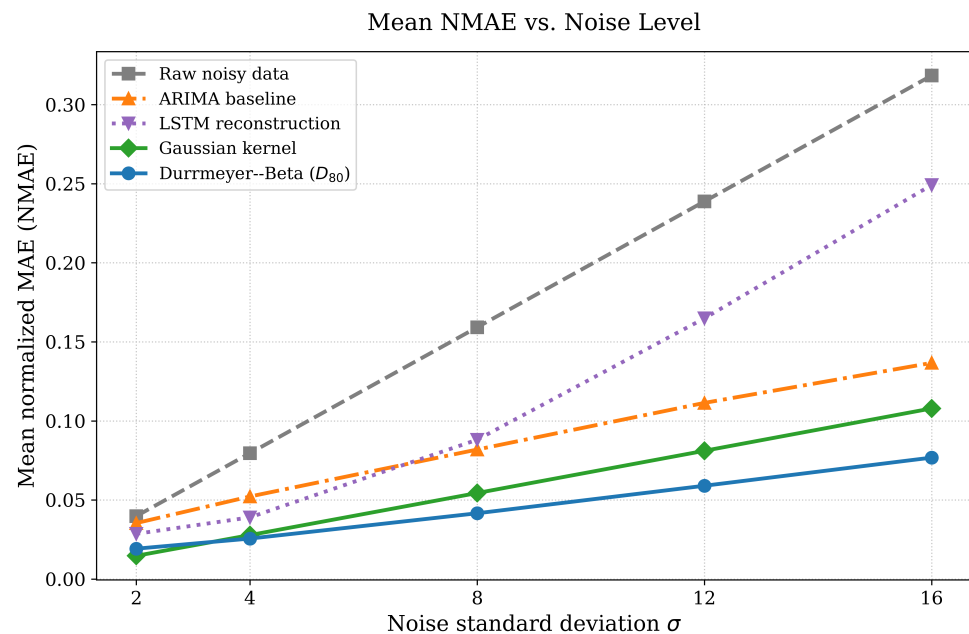


Figure 11. Mean NMAE over 30 realizations at each Gaussian noise level. The absolute-error measure shows the same ordering between the Gaussian kernel smoother and D_{80} as the NRMSE results.

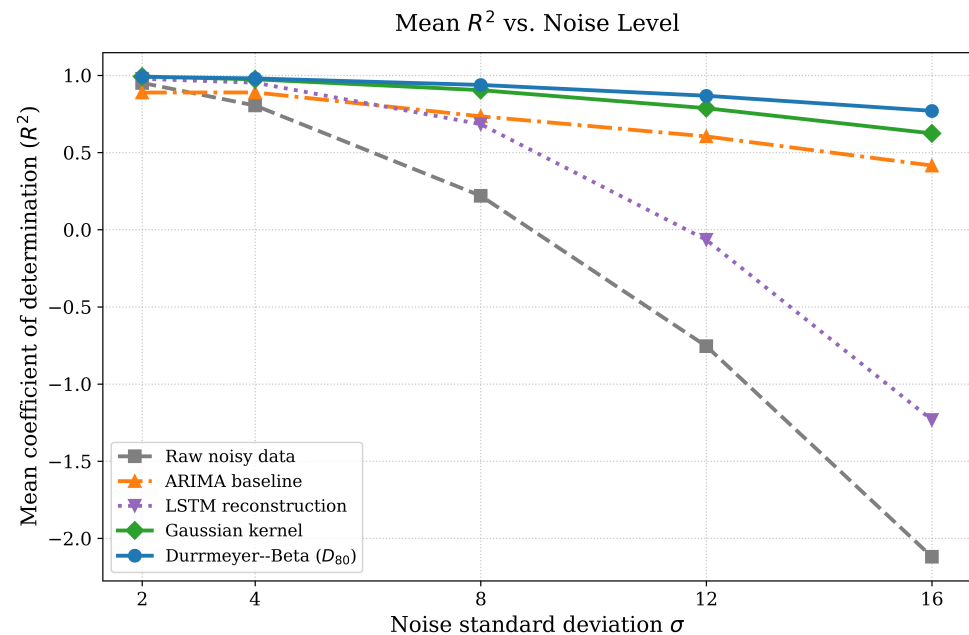


Figure 12. Mean coefficient of determination over 30 realizations at each Gaussian noise level. Negative mean R^2 values are retained. The corresponding empirical standard deviations are reported in Table 6.

9. Conclusions

This study connects the explicit moment structure of a Durrmeyer–Beta-type operator with its approximation properties and numerical behavior. Exact raw moments through degree four and the corresponding central moments were obtained under the conditions required for their existence. In particular, the fourth central moment is $O(n^{-2})$ uniformly on compact intervals. The moment identities yield local modulus estimates, Hölder–Lipschitz bounds, weighted Korovkin convergence, and a weighted $O(n^{-1})$ rate when the first and second derivatives are bounded.

The nonzero first central moment plays an essential role in these estimates, since the operator preserves constants but does not reproduce linear functions. The tensor-product extension carries the same moment-based approach to the bivariate setting, where it provides compact-uniform convergence, partial-modulus estimates, and coordinatewise Lipschitz bounds.

The comparisons with related Durrmeyer constructions place the chosen weight–kernel pairing within the existing literature. For the oscillatory test function used in the matched-index experiment, the Dunkl–Beta operator gives smaller grid-based errors, while both operators improve over the tested index sequence. The remaining numerical examples illustrate approximation of smooth and nonsmooth bounded functions. The experiment on $[0, 20]$ also examines a function whose second derivative does not exist at an interior point, extending the assessment to a wider interval and a different regularity class.

The repeated-realization experiment examines a further application: reconstruction from noisy observations on a fixed grid. Across 30 signal–noise pairs at each noise level, the Gaussian kernel smoother gives the best mean accuracy at $\sigma = 2$. For $\sigma = 4, 8, 12, 16$, the fixed choice $n = 80$ gives the lowest mean NRMSE and NMAE and the highest mean R^2 among the tested procedures. At $\sigma = 16$, its mean NRMSE is 0.1030, compared with 0.1334 for the Gaussian smoother. The empirical standard deviations reported alongside these means describe the variation across realizations. These findings demonstrate the smoothing potential of the operator within the specified reconstruction setting.

The present experiments use fixed method settings and a single signal family for the noisy-data comparison. Future work may examine additional signal families, different sampling densities, and data-driven selection of the operator index and kernel bandwidth. Unequal indices in the bivariate construction and Voronovskaya-type asymptotic formulas provide further directions for extending the analysis.

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Abbreviations and Notation

The following abbreviations and notation are used in this manuscript:

ARIMA	Autoregressive integrated moving average
LSTM	Long short-term memory
RMSE	Root mean square error
MAE	Mean absolute error
NRMSE	Normalized root mean square error
NMAE	Normalized mean absolute error
Notation	
$B(\cdot, \cdot)$	Beta function
$(a)_k$	Rising factorial
$p_{n,k}$	Discrete weights of D_n
$q_{n,k}$	Normalized beta-prime density
D_n	Univariate Durrmeyer–Beta operator
D_{n_1, n_2}	Bivariate tensor-product operator
L_n^M	Modified Lupaş–Agratini operator
$\mu_{n,r}$	Central moment of order r about z
E_∞	Maximum absolute error on the evaluation grid
R^2	Coefficient of determination

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