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Global Strong Solutions to the One-Dimensional Planar Compressible Magneto-Micropolar Equations with Vacuum

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Abstract

We consider the one-dimensional planar isentropic compressible magneto-micropolar equations on a bounded interval, with homogeneous Dirichlet boundary conditions and initial data that may contain vacuum, the initial density being allowed to vanish on an open subset of the interval. For large initial data satisfying four natural compatibility conditions, we prove the global existence and uniqueness of strong solutions. The proof rests on a priori estimates in which the magnetic and micro-rotation couplings are controlled by the dissipation, through a coercive functional built from material-derivative testing and a density bound that requires no positive lower bound. To the best of our knowledge, this is the first global well-posedness result, with vacuum and large initial data, for the compressible magneto-micropolar equations in one space dimension.

Keywords: compressible magneto-micropolar equations; vacuum; large initial data; global strong solutions; uniqueness

MSC: 35Q35; 76W05; 76N10; 35D35; 35A02

1. Introduction

The dynamics of viscous compressible fluids have been studied intensively, and the compressible magnetohydrodynamic (MHD) equations, which couple the compressible Navier–Stokes equations with the induction equation for the magnetic field, are among their most important extensions. The magneto-micropolar equations enlarge this system to fluids with microstructure: in addition to the density, the velocity and the magnetic field, the angular velocity of the microscopic particles, the micro-rotation field, is taken as an unknown and obeys its own evolution equation. Analytically, the micro-rotation is coupled to the velocity through skew-symmetric first-order terms and is governed by a parabolic equation carrying, besides these terms, a zeroth-order damping; it interacts with the magnetic field only indirectly, through the velocity. Such equations were proposed by Ahmadi and Shahinpoor [1], within the framework of the micropolar fluids of Eringen [2], to describe electrically conducting fluids whose suspended particles undergo rotation. Models for fluids with suspended particles also arise in enhanced oil recovery, where nanofluid flooding through porous media has been studied numerically [3].

We consider the one-dimensional planar motion of an isentropic compressible magneto-micropolar fluid on the interval $\Omega = (0, 1)$, with $0 < T < \infty$. The unknowns are the density $\rho \geq 0$, the longitudinal velocity $u \in \mathbb{R}$, the transverse velocity $w \in \mathbb{R}^2$, the longitudinal and transverse micro-rotations $\omega_1 \in \mathbb{R}$ and $\tilde{\omega} \in \mathbb{R}^2$, and the transverse magnetic field $b \in \mathbb{R}^2$, governed by

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$$\begin{cases} \rho_t + (\rho u)_x = 0, & (1)\\ \rho(u_t + uu_x) + \left(P + \frac{1}{2}|b|^2\right)_x = (2\mu + \lambda)\, u_{xx}, & (2)\\ \rho(w_t + uw_x) - b_x = (\mu + \zeta)\, w_{xx} + 2\zeta\, J\tilde{\omega}_x, & (3)\\ \rho(\omega_{1,t} + u\omega_{1,x}) + 4\zeta\, \omega_1 = (2\mu' + \lambda')\, \omega_{1,xx}, & (4)\\ \rho(\tilde{\omega}_t + u\tilde{\omega}_x) + 4\zeta\, \tilde{\omega} = \mu'\, \tilde{\omega}_{xx} + 2\zeta\, Jw_x, & (5)\\ b_t + (ub)_x - w_x = \nu\, b_{xx}. & (6) \end{cases}$$

where $P = P(\rho) = a\rho^\gamma$ with $a > 0$, $\gamma > 1$, and $J(a_1, a_2) = (-a_2, a_1)$ is the 90° rotation on $\mathbb{R}^2$; it is skew-symmetric, $J^\top = -J$, and $|Jv| = |v|$. The viscosity constants satisfy

$$\mu,\ \nu,\ \zeta,\ \mu' > 0, \qquad 2\mu' + 3\lambda' \geq 0, \qquad 2\mu + 3\lambda - 4\zeta \geq 0, \tag{7}$$

which entail the non-degeneracy of all diffusion coefficients,

$$2\mu + \lambda \ \geq\ \frac{4}{3}(\mu + \zeta)\ > 0, \qquad 2\mu' + \lambda' \ \geq\ \frac{4}{3}\mu'\ > 0. \tag{8}$$

System (1)–(6) is the one-dimensional planar reduction of the three-dimensional compressible magneto-micropolar equations; following the planar ansatz of Chen and Wang [4], this reduction is carried out in Appendix A. Appendix A computes both the magnetic force and induction terms and the longitudinal and transverse micro-rotation equations.

The system is supplemented with homogeneous Dirichlet boundary conditions and initial data

$$(u, w, \omega_1, \tilde{\omega}, b)\big|_{\partial\Omega} = 0, \qquad (\rho, u, w, \omega_1, \tilde{\omega}, b)\big|_{t=0} = (\rho_0, u_0, w_0, \omega_{1,0}, \tilde{\omega}_0, b_0). \tag{9}$$

Throughout, we assume the initial total mass is positive and finite,

$$0 < m := \int_\Omega \rho_0\, dx < \infty. \tag{10}$$

The initial density is non-negative and may vanish on an open subset of Ω so that the system degenerates at the vacuum; this, together with the coupling among the velocity, the micro-rotation and the magnetic field, is the main source of difficulty.

The mathematical theory of compressible viscous flows with vacuum is by now well developed. For the one-dimensional Navier–Stokes equations, the global existence of solutions with large data, when the density has a positive lower bound, goes back to Kazhikhov and Shelukhin [5]. When the density may vanish, the local existence and uniqueness of strong solutions were established by Choe and Kim and by Cho and Kim [6,7]. For the three-dimensional Cauchy problem, Huang, Li and Xin [8] proved global existence of classical solutions with small initial energy, allowing large oscillations and vacuum.

For the compressible MHD equations, the situation is analogous, and it is convenient to recall the planar results according to the way the vacuum is allowed to appear. When the density has a positive lower bound, the global existence, uniqueness and regularity of solutions with large initial data were obtained by Chen and Wang [4], who also showed that no vacuum forms in finite time. When the density may vanish initially, Zhang proved the global existence and uniqueness of strong solutions to the isentropic planar MHD equations on a bounded interval [9], and Fan, Huang and Li established global strong solutions to the full, non-isentropic, planar MHD equations with large data and vacuum [10]; the local existence and uniqueness of strong solutions with vacuum, in several space dimensions, are due to Fan and Yu [11]. For vacuum free boundary problems of the planar and axisymmetric MHD equations, the reader may consult [12–16] and the references therein.

Micropolar fluids were introduced by Eringen [2], and a systematic treatment is given in the monograph of Łukaszewicz [17]. For the one-dimensional compressible micropolar equations, the local and global existence and the regularity of solutions were established by Mujaković [18–20]; when the initial density may vanish, the global existence of strong solutions in one dimension was obtained by Chen [21], and global classical solutions of the heat-conducting system with vacuum were obtained by Feng and Zhu [22].

The magneto-micropolar equations were proposed by Ahmadi and Shahinpoor [1] and have so far been studied mainly in two and three dimensions, where global finite-energy weak solutions and global strong or classical solutions for data close to an equilibrium have been obtained by several authors, including Wu and Zhang [23], Wei, Guo and Li [24], Xu [25] and, for the non-resistive system without spin viscosity, Feng, Hong and Zhu [26]; for the full system without heat conductivity, see Pan, Wang and Zhang [27]. To the best of our knowledge, however, the one-dimensional, or planar, compressible magneto-micropolar equations with large initial data and vacuum have not been treated: the available results on the compressible magneto-micropolar system are confined to two or three dimensions, while the one-dimensional results that allow vacuum concern the micropolar equations without a magnetic field.

In this paper, we study the one-dimensional planar isentropic compressible magneto-micropolar system (1)–(9) with large initial data, allowing the initial density to vanish on an open set. We establish the global existence and uniqueness of strong solutions; see Theorem 1. To the best of our knowledge, this is the first result on the one-dimensional, or planar, compressible magneto-micropolar equations with vacuum. The problem lies at the intersection of the isentropic planar MHD equations with vacuum, treated by Zhang [9], and the one-dimensional compressible micropolar equations with vacuum, treated by Chen [21]; here, the two couplings are handled simultaneously, in the strong-solution class with density in $L^\infty(0,T;H^2(\Omega))$. The initial density and the other fields belong to $H^2(\Omega)$, the velocity, micro-rotation and magnetic fields satisfy the homogeneous Dirichlet condition, and the total mass is positive. Four weighted compatibility conditions, stated in Theorem 1, control the initial material accelerations; no smallness of the data is imposed.

Our proof rests on global a priori estimates of the solution, designed so that all constants are independent of any positive lower bound of the density; we now describe the main difficulties and how each is treated. The first is the vacuum itself: since the density may vanish, the natural energy of the system is weighted by the density and degenerates where the density does, so the velocity cannot be controlled directly in the vacuum region; following Choe and Kim, Cho and Kim, and Fan and Yu [6,7,11], we therefore work with $\sqrt{\rho}$-weighted quantities and with the material derivatives of the unknowns. The second is the coupling of the velocity, the micro-rotation and the magnetic field, whose cross terms obstruct the closure of the energy estimate: the magnetic coupling is of Alfvén type and its leading terms cancel, whereas the micro-rotation coupling is governed by the skew-symmetric operator J and is absorbed by the dissipation together with the zeroth-order damping of the micro-rotation, for arbitrary positive coefficients and without any smallness condition. The third concerns pressure regularity at vacuum: the density H^2 bound does not in general control P_{xx} over the full range $\gamma > 1$ considered here. We therefore estimate the velocity up to H^2, whereas the magnetic field, which carries no density weight, is shown to be more regular by means of an estimate on its time derivative. Finally, for the uniqueness, we use a difference estimate; the only term that is delicate in the presence of vacuum is the one carrying the difference of the densities, and it is controlled by the fact that, in one space dimension, the material acceleration belongs to $L^2(0,T;L^\infty(\Omega))$ so that this term becomes a Gronwall term. The treatment of the vacuum and of the magneto-hydrodynamic part follows the strong-solution theory of Choe and Kim, Cho and Kim,

Zhang, and Fan and Yu [6,7,9,11], while the additional estimates concern the simultaneous magnetic and micro-rotation couplings, both in the material-derivative estimate and in the local construction. In the latter, the frozen transport equation is first solved at the H^1 level; the H^2 density bound is recovered from the continuity and momentum equations after the nonlinear limit. The one-dimensional embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ controls the products and the density-difference terms. This embedding and the scalar effective-flux argument do not give the corresponding bounds in higher dimensions.

The rest of this paper is organized as follows. The main result is stated precisely in the next section; the a priori estimates are derived in Section 3; and the local existence, the global existence and the uniqueness are established in Sections 4–6, respectively. Concluding remarks are given in Section 7, and the planar system is derived from the three-dimensional equations in Appendix A.

2. Main Result

Theorem 1 (Global strong solution, H^2 level)**.** *Assume* (7) *and* (10)*. Let the initial data satisfy*

$$\rho_0 \geq 0, \quad \rho_0 \in H^2(\Omega), \qquad u_0,\, w_0,\, \omega_{1,0},\, \tilde{\omega}_0,\, b_0 \in H_0^1(\Omega) \cap H^2(\Omega),$$

together with the compatibility conditions: there exist $g_1, g_3 \in L^2(\Omega)$ and $g_2, g_4 \in L^2(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned}
(2\mu+\lambda)(u_0)_{xx} - \Big(P(\rho_0) + \frac{1}{2}|b_0|^2\Big)_x &= \sqrt{\rho_0}\, g_1,\\
(\mu+\zeta)(w_0)_{xx} + 2\zeta\, J(\tilde{\omega}_0)_x + (b_0)_x &= \sqrt{\rho_0}\, g_2,\\
(2\mu'+\lambda')(\omega_{1,0})_{xx} - 4\zeta\, \omega_{1,0} &= \sqrt{\rho_0}\, g_3,\\
\mu'(\tilde{\omega}_0)_{xx} + 2\zeta\, J(w_0)_x - 4\zeta\, \tilde{\omega}_0 &= \sqrt{\rho_0}\, g_4.
\end{aligned}$$

Then for any $0 < T < \infty$, there exists a unique global strong solution $(\rho, u, w, \omega_1, \tilde{\omega}, b)$ to (1)–(9) *on $\Omega \times (0,T)$ such that*

$$\begin{aligned}
&\rho \in L^\infty(0,T;H^2(\Omega)), \quad \rho \geq 0, \quad \rho_t \in L^\infty(0,T;H^1(\Omega)),\\
&u, w, \omega_1, \tilde{\omega} \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)),\\
&b \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)) \cap L^2(0,T;H^3(\Omega)),\\
&\sqrt{\rho}\, u_t,\ \sqrt{\rho}\, w_t,\ \sqrt{\rho}\, \omega_{1,t},\ \sqrt{\rho}\, \tilde{\omega}_t,\ b_t \in L^\infty(0,T;L^2(\Omega)),\\
&u_t, w_t, \omega_{1,t}, \tilde{\omega}_t, b_t \in L^2(0,T;H^1(\Omega)).
\end{aligned}$$

Remark 1 (Compatibility conditions)**.** *The functions g_1, g_2, g_3, g_4 describe the initial density-weighted material derivatives of $u, w, \omega_1, \tilde{\omega}$, respectively; they are not additional freely prescribed initial fields. For a smooth solution with positive density, evaluation of the four equations at $t = 0$ gives*

$$\begin{aligned}
\sqrt{\rho_0}\, u_t(0) &= g_1 - \sqrt{\rho_0}\, u_0 u_{0,x}, \qquad \sqrt{\rho_0}\, w_t(0) = g_2 - \sqrt{\rho_0}\, u_0 w_{0,x},\\
\sqrt{\rho_0}\, \omega_{1,t}(0) &= g_3 - \sqrt{\rho_0}\, u_0 \omega_{1,0,x}, \quad \sqrt{\rho_0}\, \tilde{\omega}_t(0) = g_4 - \sqrt{\rho_0}\, u_0 \tilde{\omega}_{0,x}.
\end{aligned}$$

Thus the compatibility conditions control the initial weighted time derivatives in $L^2(\Omega)$, since the convective terms on the right are controlled by the stated initial norms. When ρ_0 vanishes, the spatial identities in Theorem 1 are the definition of compatibility: their left-hand sides must vanish on $\{\rho_0 = 0\}$, and the values of g_i on that set are immaterial and may be chosen to be zero. No division by the density is required.

The magnetic field has the prescribed initial value b_0, and its equation contains no density weight. In particular,

$$b_t(0) = \nu b_{0,xx} - (u_0 b_0)_x + w_{0,x} \in L^2(\Omega)$$

is already controlled by the H^2 initial norms; no analogous weighted compatibility condition is needed for b_0. All fields attain their prescribed initial values strongly in $H^1(\Omega)$ by their stated time regularity.

3. A Priori Estimates

Throughout, we use the notation $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $Q_T := \Omega \times [0,T]$, $\bar{\mu} := 2\mu + \lambda$ and $\bar{\mu}' := 2\mu' + \lambda'$. The operator $D_t := \partial_t + u\partial_x$ denotes the material derivative. Constants $C, C_0, \ldots$ depend only on the data and the fixed parameters and may change from line to line.

The calculations are first carried out for smooth positive-density solutions and then passed to the strong solutions by regularization. The compatible approximation with $\rho_0^\delta = \rho_0 + \delta$, $0 < \delta \le 1$ is constructed in Section 4; all bounds below are independent of δ.

The effective viscous flux

$$F := \bar{\mu} u_x - P - \frac{1}{2}|b|^2$$

will be used repeatedly; by (2), it satisfies $F_x = \rho D_t u$.

We first establish the energy, density upper bound and magnetic H^1 estimate. The first-order energy then controls the velocity and micro-rotation gradients and the density gradient. The material-derivative estimate precedes the H^2 density bound, and the magnetic time-derivative estimate finally yields the additional H^3 regularity of b.

3.1. Lower-Order Estimates

We first record the basic energy inequality.

Lemma 1. *Let $(\rho, u, w, \omega_1, \tilde{\omega}, b)$ be a smooth solution of (1)–(9) on $[0,T]$. Set*

$$\mathcal{E}_L(t) = \int_\Omega \Big[\frac{1}{2}\rho\big(u^2 + |w|^2 + \omega_1^2 + |\tilde{\omega}|^2\big) + \frac{a}{\gamma - 1}\rho^\gamma + \frac{1}{2}|b|^2 \Big] dx.$$

Then for all $t \in [0,T]$,

$$\begin{aligned} \mathcal{E}_L(t) + \int_0^t \int_\Omega \Big(& \bar{\mu} u_x^2 + \mu |w_x|^2 + (2\mu' + \lambda')\omega_{1,x}^2 \\ & + \mu' |\tilde{\omega}_x|^2 + \nu |b_x|^2 + 4\zeta \omega_1^2 \Big) dx\, ds \le \mathcal{E}_L(0). \end{aligned} \tag{11}$$

Proof. The estimate is obtained by testing the equations against the unknowns themselves and summing. We shall use two elementary identities. First, for any field f vanishing on $\partial\Omega$, integration by parts together with the continuity equation (1) gives

$$\int_\Omega \rho u \frac{1}{2}(f^2)_x \, dx = -\frac{1}{2}\int_\Omega (\rho u)_x f^2 \, dx = \frac{1}{2}\int_\Omega \rho_t f^2 \, dx,$$

and therefore

$$\int_\Omega \rho \, (f_t + u f_x) f \, dx = \frac{d}{dt}\int_\Omega \frac{1}{2}\rho f^2 \, dx.$$

Second, by (1) and $u|_{\partial\Omega} = 0$,

$$-\int_\Omega P \, u_x \, dx = \frac{a}{\gamma - 1}\frac{d}{dt}\int_\Omega \rho^\gamma \, dx.$$

Testing (2)–(6) by u, w, ω_1, $\tilde{\omega}$ and b, respectively, integrating over Ω and discarding the boundary terms by (9), we are led to the five balance laws

$$\frac{d}{dt}\int_\Omega \Big(\frac{1}{2}\rho u^2 + \frac{a}{\gamma-1}\rho^\gamma\Big)dx + \bar{\mu}\int_\Omega u_x^2\,dx = \frac{1}{2}\int_\Omega |b|^2 u_x\,dx, \tag{12}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|w|^2\,dx + (\mu+\zeta)\int_\Omega |w_x|^2\,dx = \int_\Omega b_x\cdot w\,dx - 2\zeta\int_\Omega (J\tilde{\omega})\cdot w_x\,dx, \tag{13}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho\,\omega_1^2\,dx + (2\mu'+\lambda')\int_\Omega \omega_{1,x}^2\,dx + 4\zeta\int_\Omega \omega_1^2\,dx = 0, \tag{14}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\tilde{\omega}|^2\,dx + \mu'\int_\Omega |\tilde{\omega}_x|^2\,dx + 4\zeta\int_\Omega |\tilde{\omega}|^2\,dx = 2\zeta\int_\Omega (Jw_x)\cdot\tilde{\omega}\,dx, \tag{15}$$

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|b|^2\,dx + \nu\int_\Omega |b_x|^2\,dx = -\frac{1}{2}\int_\Omega u_x|b|^2\,dx + \int_\Omega w_x\cdot b\,dx, \tag{16}$$

where the magnetic term in (16) has been simplified through

$$\int_\Omega (ub)_x\cdot b\,dx = \int_\Omega u_x|b|^2\,dx + \frac{1}{2}\int_\Omega u(|b|^2)_x\,dx = \frac{1}{2}\int_\Omega u_x|b|^2\,dx.$$

We now add (12)–(16) and examine the coupling terms on the right-hand side, which fall into three groups. The magnetic pressure terms cancel,

$$\frac{1}{2}\int_\Omega |b|^2 u_x\,dx - \frac{1}{2}\int_\Omega u_x|b|^2\,dx = 0,$$

and, after an integration by parts, so do the two Alfvén coupling terms,

$$\int_\Omega b_x\cdot w\,dx + \int_\Omega w_x\cdot b\,dx = -\int_\Omega b\cdot w_x\,dx + \int_\Omega w_x\cdot b\,dx = 0.$$

It remains to treat the micro-rotation coupling. By the skew-symmetry $J^\top = -J$,

$$\int_\Omega (Jw_x)\cdot\tilde{\omega}\,dx = -\int_\Omega (J\tilde{\omega})\cdot w_x\,dx,$$

so that the two cross terms combine into the single term $-4\zeta\int_\Omega (J\tilde{\omega})\cdot w_x\,dx$. Since $|J\tilde{\omega}| = |\tilde{\omega}|$, Young's inequality yields

$$-4\zeta\int_\Omega (J\tilde{\omega})\cdot w_x\,dx \le \zeta\int_\Omega |w_x|^2\,dx + 4\zeta\int_\Omega |\tilde{\omega}|^2\,dx.$$

The term $\zeta\int_\Omega |w_x|^2\,dx$ is absorbed by the dissipation $(\mu+\zeta)\int_\Omega |w_x|^2\,dx$, which leaves $\mu\int_\Omega |w_x|^2\,dx$, while $4\zeta\int_\Omega |\tilde{\omega}|^2\,dx$ is absorbed by the damping term $4\zeta\int_\Omega |\tilde{\omega}|^2\,dx$ already present on the left-hand side of (15). Integrating the resulting differential inequality over $[0,t]$ gives (11). □

We next derive a uniform upper bound for the density, via the effective viscous flux.

Lemma 2. *Under the hypotheses of Theorem 1, any smooth solution on $[0,T]$ satisfies*

$$0 \le \rho(x,t) \le \bar{\rho} := \|\rho_0\|_{L^\infty}\, e^{2C_\psi}, \qquad (x,t)\in\overline{Q_T}, \tag{17}$$

where C_ψ depends only on $\mathcal{E}_L(0), m, T$ and the fixed parameters $\mu,\lambda,\zeta,a,\gamma$. In particular, $\bar{\rho}$ is independent of any positive lower bound of ρ_0, hence uniform in δ for the approximate problem.

Proof. By means of the continuity equation (1), the momentum equation (2) can be cast in the conservative form $(\rho u)_t = \Phi_x$, where

$$\Phi := F - \rho u^2 = \bar{\mu}\,u_x - \rho u^2 - P - \frac{1}{2}|b|^2. \tag{18}$$

Notice that the micro-rotation field $(\omega_1, \tilde{\omega})$ does not enter the flux Φ. We introduce the associated potential

$$\psi(x,t) := \frac{1}{\bar{\mu}}\Big[\int_0^t \Phi(x,\tau)\,d\tau + \int_0^x \rho_0(\xi)\,u_0(\xi)\,d\xi\Big].$$

so that, in view of $(\rho u)_t = \Phi_x$,

$$\psi_x = \frac{1}{\bar{\mu}}\Big[\int_0^t \Phi_x\,d\tau + \rho_0 u_0\Big] = \frac{\rho u}{\bar{\mu}}, \qquad \psi_t = \frac{1}{\bar{\mu}}\Phi = u_x - \frac{1}{\bar{\mu}}\Big(\rho u^2 + P + \frac{1}{2}|b|^2\Big).$$

Hence the two contributions ρu^2 cancel, and the material derivative of ψ reduces to

$$D_t\psi := \psi_t + u\psi_x = u_x - \frac{1}{\bar{\mu}}\Big(P + \frac{1}{2}|b|^2\Big).$$

Since $D_t\rho = -\rho u_x$ by (1), it follows that

$$\begin{aligned} D_t\big(\rho e^{\psi}\big) &= e^{\psi} D_t\rho + \rho e^{\psi} D_t\psi = \rho e^{\psi}\Big[-u_x + u_x - \frac{1}{\bar{\mu}}\Big(P + \frac{1}{2}|b|^2\Big)\Big] \\ &= -\frac{1}{\bar{\mu}}\Big(P + \frac{1}{2}|b|^2\Big)\rho e^{\psi} \le 0, \end{aligned}$$

since $P = a\rho^{\gamma} \ge 0$, $|b|^2 \ge 0$ and $\bar{\mu} > 0$. Therefore, ρe^{ψ} is non-increasing along the particle trajectories $\dot{X} = u(X,t)$.

It remains to bound ψ. For any $g \in W^{1,1}(\Omega)$, the elementary inequality

$$\|g\|_{L^\infty} \le \|g_x\|_{L^1} + \Big|\int_\Omega g\,dx\Big|$$

holds, as is seen from

$$g(x) = \int_\Omega g\,dy + \int_\Omega \int_y^x g_x(s)\,ds\,dy.$$

We apply it to ψ. On the one hand, by the mass conservation $\int_\Omega \rho\,dx = m$ and Lemma 1,

$$\begin{aligned} \|\psi_x\|_{L^1} = \frac{1}{\bar{\mu}}\int_\Omega \rho|u|\,dx &\le \frac{1}{\bar{\mu}}\Big(\int_\Omega \rho\,dx\Big)^{1/2}\Big(\int_\Omega \rho u^2\,dx\Big)^{1/2} \\ &\le \frac{\sqrt{m}}{\bar{\mu}}\sqrt{2\mathcal{E}_L(0)}. \end{aligned}$$

On the other hand, integrating (18) over Ω gives

$$\int_\Omega \Phi\,dx = \bar{\mu}\,[u]_0^1 - \int_\Omega \rho u^2\,dx - \int_\Omega P\,dx - \frac{1}{2}\int_\Omega |b|^2\,dx,$$

where $[u]_0^1 = 0$ by (9) and the remaining terms are controlled by $C\mathcal{E}_L(0)$; thus, $\big|\int_\Omega \Phi\,dx\big| \le C$ uniformly in t, and consequently

$$\Big|\int_\Omega \psi\,dx\Big| \le \frac{1}{\bar{\mu}}\Big[\Big|\int_0^t\int_\Omega \Phi\,dx\,d\tau\Big| + \|\rho_0 u_0\|_{L^1}\Big] \le \frac{1}{\bar{\mu}}(CT + C_0).$$

Combining these two bounds, we conclude that

$$\|\psi\|_{L^\infty(Q_T)} \le C_\psi = C_\psi(\mathcal{E}_L(0), m, T, \bar{\mu}, \gamma, a),$$

that is, $e^{-C_\psi} \le e^{\psi} \le e^{C_\psi}$.

Finally, for a smooth velocity u vanishing on $\partial\Omega$, the flow map $x_0 \mapsto X(t;x_0)$ is a diffeomorphism of $\overline{\Omega}$. Given $(x,t) \in \overline{Q_T}$, let x_0 be such that $X(t;x_0) = x$. The monotonicity of ρe^{ψ} then gives

$$\rho(x,t)\,e^{\psi(x,t)} \le \rho_0(x_0)\,e^{\psi(x_0,0)} \le \|\rho_0\|_{L^\infty} e^{C_\psi},$$

and dividing by $e^{\psi(x,t)} \ge e^{-C_\psi}$ gives the upper bound (17). □

We complete the lower-order estimates with a bound on the magnetic field.

Lemma 3. *Under the hypotheses of Theorem 1, any smooth solution on* $[0,T]$ *satisfies*

$$\sup_{0\le t\le T}\int_\Omega |b_x(t)|^2\,dx+\int_0^T\int_\Omega |b_{xx}|^2\,dx\,dt\le C, \tag{19}$$

so that

$$b\in L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega))$$

and $b_t\in L^2(0,T;L^2(\Omega))$*, where* C *depends only on* $\mathcal{E}_L(0)$*, the initial magnetic energy* $\int_\Omega |(b_0)_x|^2\,dx$*, and the fixed parameters; in particular,* C *is independent of* δ*.*

Proof. Since the induction equation (6) carries no factor ρ, this estimate is self-contained, relying only on Lemma 1. We multiply (6) by $-b_{xx}$ and integrate over Ω; all boundary terms vanish because $b|_{\partial\Omega}=0$, and hence $b_t|_{\partial\Omega}=0$. Using

$$\int_\Omega b_t(-b_{xx})\,dx=\frac{d}{dt}\frac{1}{2}\int_\Omega |b_x|^2\,dx,\qquad \int_\Omega \nu b_{xx}(-b_{xx})\,dx=-\nu\int_\Omega |b_{xx}|^2\,dx,$$

we obtain

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |b_x|^2\,dx+\nu\int_\Omega |b_{xx}|^2\,dx=\int_\Omega (u_xb+ub_x)\,b_{xx}\,dx-\int_\Omega w_x\,b_{xx}\,dx.$$

Since $u|_{\partial\Omega}=b|_{\partial\Omega}=0$, the one-dimensional embedding gives

$$\|u\|_{L^\infty}^2\le\int_\Omega u_x^2\,dx,\qquad \|b\|_{L^\infty}^2\le\int_\Omega |b_x|^2\,dx,$$

so that, by the Cauchy–Schwarz inequality, the three terms on the right-hand side are estimated by

$$\begin{aligned}
\int_\Omega u_x b\,b_{xx}\,dx &\le \Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2},\\
\int_\Omega ub_x\,b_{xx}\,dx &\le \Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2},\\
-\int_\Omega w_x\,b_{xx}\,dx &\le \Big(\int_\Omega |w_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2}.
\end{aligned}$$

Young's inequality then gives

$$\frac{d}{dt}\int_\Omega |b_x|^2\,dx+\nu\int_\Omega |b_{xx}|^2\,dx\le\frac{8}{\nu}\Big(\int_\Omega u_x^2\,dx\Big)\Big(\int_\Omega |b_x|^2\,dx\Big)+\frac{2}{\nu}\int_\Omega |w_x|^2\,dx. \tag{20}$$

Since u_x and w_x are bounded in $L^2(0,T;L^2(\Omega))$ by Lemma 1 and $b_0\in H^1(\Omega)$, Gronwall's inequality applied to (20) yields

$$\sup_{0\le t\le T}\int_\Omega |b_x|^2\,dx\le C;$$

integrating (20) over $[0,T]$ then bounds the dissipation term, which is (19). Finally, the induction equation gives $b_t=\nu b_{xx}-u_xb-ub_x+w_x$, and the bounds $\|b\|_{L^\infty}\le C$ and $\int_\Omega |b_x|^2\,dx\le C$ imply

$$\int_\Omega |b_t|^2\,dx\le C\Big(\int_\Omega |b_{xx}|^2\,dx+\int_\Omega u_x^2\,dx+\int_\Omega |w_x|^2\,dx\Big),$$

whence $b_t\in L^2(0,T;L^2(\Omega))$. □

3.2. Higher-Order Estimates

We now control the first spatial derivatives of the velocity fields, together with the weighted material derivatives, and then the second-order and higher bounds. The structural set-up comes first: an L^∞ bound on u_x, the differential identity produced by material-derivative testing the resolution of the micro-rotation coupling, and the coercivity of the resulting functional. The closure follows, through an estimate of the right-hand side and a Gronwall argument.

Throughout, C denotes a constant depending only on the data through $\mathcal{E}_L(0)$, m, $\bar{\rho}$ (Lemma 2), the bound of Lemma 3, the coefficients and T; its value may change from line to line. We write $\dot{f} := f_t + uf_x$ for the material derivative.

The first ingredient is a pointwise bound on u_x, obtained from the effective viscous flux F; recall that $F_x = \rho\dot{u}$ by (2).

Lemma 4. *There is a constant C as above such that, for a.e. $t \in [0,T]$,*

$$\|u_x\|_{L^\infty} \le C\Big(1 + \Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + \Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2}\Big). \tag{21}$$

Proof. Since $|\Omega| = 1$, every $g \in W^{1,1}(\Omega)$ satisfies $\|g\|_{L^\infty} \le \|g\|_{L^1} + \|g_x\|_{L^1}$. Applied to F,

$$\|F\|_{L^1} \le \bar{\mu}\|u_x\|_{L^1} + \|P\|_{L^1} + \frac{1}{2}\||b|^2\|_{L^1} \le \bar{\mu}\Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + C,$$

where $\|P\|_{L^1}$ and $\||b|^2\|_{L^1}$ are bounded by Lemma 1. For F_x,

$$\|F_x\|_{L^1} = \int_\Omega \rho|\dot{u}|\,dx \le \Big(\int_\Omega \rho\,dx\Big)^{1/2}\Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2} = \sqrt{m}\Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2}.$$

Hence

$$\|F\|_{L^\infty} \le \bar{\mu}\Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + \sqrt{m}\Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2} + C.$$

Since $\|P\|_{L^\infty} = a\bar{\rho}^\gamma \le C$ by Lemma 2 and $\|b\|_{L^\infty} \le C$ by Lemma 3, the identity $\bar{\mu}u_x = F + P + |b|^2/2$ gives

$$\bar{\mu}\|u_x\|_{L^\infty} \le \|F\|_{L^\infty} + C,$$

which is (21). □

We next test the equations against the material derivatives. Multiply (2)–(5) by $\dot{u}, \dot{w}, \dot{\omega}_1, \dot{\tilde{\omega}}$, respectively, and integrate over Ω. Since $f|_{\partial\Omega} = 0$ forces $\dot{f}|_{\partial\Omega} = 0$, as $f_t = 0$ and $u = 0$ there,

$$\int_\Omega f_{xx}\dot{f}\,dx = \int_\Omega f_{xx}f_t\,dx + \int_\Omega f_{xx}uf_x\,dx = -\frac{d}{dt}\frac{1}{2}\int_\Omega |f_x|^2\,dx - \frac{1}{2}\int_\Omega u_x|f_x|^2\,dx.$$

For the pressure, we avoid ρ_x entirely: with $\dot{u} = u_t + uu_x$, $P_t = -uP_x - \gamma Pu_x$, and one integration by parts,

$$\int_\Omega P_x\,\dot{u}\,dx = \int_\Omega P_x u_t\,dx + \int_\Omega uP_x u_x\,dx = -\frac{d}{dt}\int_\Omega Pu_x\,dx - \gamma\int_\Omega Pu_x^2\,dx, \tag{22}$$

where the term $\int_\Omega uP_x u_x\,dx$ cancels. We treat $|b|^2/2$ in the same way and substitute b_t from (6); the term $\int_\Omega u(b\cdot b_x)u_x\,dx$ cancels in the same manner, and the four identities read

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\bar\mu}{2}\int_\Omega u_x^2\,dx-\int_\Omega Pu_x\,dx-\frac{1}{2}\int_\Omega |b|^2u_x\,dx\right]+\int_\Omega \rho\dot u^2\,dx\\
&\quad=-\frac{\bar\mu}{2}\int_\Omega u_x^3\,dx+\gamma\int_\Omega Pu_x^2\,dx-\nu\int_\Omega (b\cdot b_{xx})u_x\,dx\\
&\qquad+\int_\Omega |b|^2u_x^2\,dx-\int_\Omega (b\cdot w_x)u_x\,dx,
\end{aligned}\tag{23}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\mu+\zeta}{2}\int_\Omega |w_x|^2\,dx+\int_\Omega b\cdot w_x\,dx\right]+\int_\Omega \rho|\dot w|^2\,dx\\
&\quad=-\frac{\mu+\zeta}{2}\int_\Omega u_x|w_x|^2\,dx+2\zeta\int_\Omega (J\tilde\omega_x)\cdot\dot w\,dx+\nu\int_\Omega b_{xx}\cdot w_x\,dx\\
&\qquad-\int_\Omega u_x\,b\cdot w_x\,dx+\int_\Omega |w_x|^2\,dx,
\end{aligned}\tag{24}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\bar\mu'}{2}\int_\Omega \omega_{1,x}^2\,dx+2\zeta\int_\Omega \omega_1^2\,dx\right]+\int_\Omega \rho\dot\omega_1^2\,dx\\
&\quad=-\frac{\bar\mu'}{2}\int_\Omega u_x\omega_{1,x}^2\,dx+2\zeta\int_\Omega u_x\omega_1^2\,dx,
\end{aligned}\tag{25}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\mu'}{2}\int_\Omega |\tilde\omega_x|^2\,dx+2\zeta\int_\Omega |\tilde\omega|^2\,dx\right]+\int_\Omega \rho|\dot{\tilde\omega}|^2\,dx\\
&\quad=-\frac{\mu'}{2}\int_\Omega u_x|\tilde\omega_x|^2\,dx+2\zeta\int_\Omega u_x|\tilde\omega|^2\,dx+2\zeta\int_\Omega (Jw_x)\cdot\dot{\tilde\omega}\,dx,
\end{aligned}\tag{26}$$

The cross terms in (24)–(26) combine into an exact time derivative. Splitting the material derivative into its temporal and convective parts, we find that the convective contributions cancel pointwise by skew-symmetry,

$$(J\tilde\omega_x)\cdot w_x+(Jw_x)\cdot\tilde\omega_x=-(Jw_x)\cdot\tilde\omega_x+(Jw_x)\cdot\tilde\omega_x=0,$$

so that the convective contribution vanishes,

$$2\zeta\int_\Omega u[(J\tilde\omega_x)\cdot w_x+(Jw_x)\cdot\tilde\omega_x]dx=0.$$

Furthermore, integrating $\int_\Omega (Jw_{xt})\cdot\tilde\omega\,dx$ by parts and using $J^\top=-J$, one gets

$$\int_\Omega (Jw_x)\cdot\tilde\omega_t\,dx=\frac{d}{dt}\int_\Omega (Jw_x)\cdot\tilde\omega\,dx-\int_\Omega (J\tilde\omega_x)\cdot w_t\,dx,$$

so the temporal parts give

$$2\zeta\int_\Omega (J\tilde\omega_x)\cdot\dot w\,dx+2\zeta\int_\Omega (Jw_x)\cdot\dot{\tilde\omega}\,dx=2\zeta\frac{d}{dt}\int_\Omega (Jw_x)\cdot\tilde\omega\,dx.\tag{27}$$

Thus the skew micro-rotation coupling balances at the level of material derivatives exactly as it does in the energy identity of Lemma 1, one order higher.

Adding (23)–(26) and moving the time derivative (27) to the left,

$$\begin{aligned}
&\frac{d}{dt}\Psi+\int_\Omega \rho\dot u^2\,dx+\int_\Omega \rho|\dot w|^2\,dx+\int_\Omega \rho\dot\omega_1^2\,dx+\int_\Omega \rho|\dot{\tilde\omega}|^2\,dx\\
&\quad=-\frac{\bar\mu}{2}\int_\Omega u_x^3\,dx-\frac{\mu+\zeta}{2}\int_\Omega u_x|w_x|^2\,dx-\frac{\bar\mu'}{2}\int_\Omega u_x\omega_{1,x}^2\,dx-\frac{\mu'}{2}\int_\Omega u_x|\tilde\omega_x|^2\,dx\\
&\qquad+2\zeta\int_\Omega u_x\omega_1^2\,dx+2\zeta\int_\Omega u_x|\tilde\omega|^2\,dx+\gamma\int_\Omega Pu_x^2\,dx+\int_\Omega |b|^2u_x^2\,dx+\int_\Omega |w_x|^2\,dx\\
&\qquad-\nu\int_\Omega (b\cdot b_{xx})u_x\,dx+\nu\int_\Omega b_{xx}\cdot w_x\,dx-2\int_\Omega (b\cdot w_x)u_x\,dx,
\end{aligned}\tag{28}$$

where

$$\begin{aligned}\Psi := \frac{\bar{\mu}}{2}\int_\Omega u_x^2\,dx + \frac{\mu+\zeta}{2}\int_\Omega |w_x|^2\,dx + \frac{\bar{\mu}'}{2}\int_\Omega \omega_{1,x}^2\,dx + \frac{\mu'}{2}\int_\Omega |\tilde{\omega}_x|^2\,dx \\ + 2\zeta\int_\Omega \omega_1^2\,dx + 2\zeta\int_\Omega |\tilde{\omega}|^2\,dx - \int_\Omega P u_x\,dx \\ - \frac{1}{2}\int_\Omega |b|^2 u_x\,dx + \int_\Omega b\cdot w_x\,dx - 2\zeta\int_\Omega (Jw_x)\cdot\tilde{\omega}\,dx.\end{aligned}$$

The functional is coercive. Indeed, by Young's inequality and Lemmas 1–3,

$$\Big|\int_\Omega P u_x\,dx\Big| + \Big|\frac{1}{2}\int_\Omega |b|^2 u_x\,dx\Big| \le \frac{\bar{\mu}}{4}\int_\Omega u_x^2\,dx + C, \qquad \Big|\int_\Omega b\cdot w_x\,dx\Big| \le \varepsilon\int_\Omega |w_x|^2\,dx + C,$$

while the $(w_x, \tilde{\omega})$ quadratic form satisfies

$$\begin{aligned}&\frac{\mu+\zeta}{2}\int_\Omega |w_x|^2\,dx + 2\zeta\int_\Omega |\tilde{\omega}|^2\,dx - 2\zeta\int_\Omega (Jw_x)\cdot\tilde{\omega}\,dx \\ &\ge \frac{\mu+\zeta}{2}\int_\Omega |w_x|^2\,dx - 2\zeta\Big(\int_\Omega |w_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\tilde{\omega}|^2\,dx\Big)^{1/2} + 2\zeta\int_\Omega |\tilde{\omega}|^2\,dx,\end{aligned}$$

whose discriminant

$$(2\zeta)^2 - 4\cdot\frac{\mu+\zeta}{2}\cdot 2\zeta = -4\mu\zeta$$

is negative; since the leading coefficient is positive, it is positive definite for *every* admissible $\mu, \zeta > 0$; deducting the $\varepsilon\int_\Omega |w_x|^2\,dx$ above leaves it positive definite provided $\varepsilon < \mu/2$. Hence, for some $c_0 > 0$,

$$\begin{aligned}c_0\Big(\int_\Omega u_x^2\,dx + \int_\Omega |w_x|^2\,dx + \int_\Omega \omega_{1,x}^2\,dx + \int_\Omega |\tilde{\omega}_x|^2\,dx \\ + \int_\Omega \omega_1^2\,dx + \int_\Omega |\tilde{\omega}|^2\,dx\Big) - C \le \Psi \le C(\text{same sum}) + C.\end{aligned} \tag{29}$$

It remains to estimate the right-hand side of (28). The two-sided bound (29) is algebraic and uses only the preceding lower-order estimates; it does not yet bound Ψ in time. The time-integrated dissipation in Lemma 1 supplies the integrable coefficients in the following Gronwall estimate.

Fix $C_* \ge C + 1$, where C is the constant subtracted in (29), so that $\Psi + C_* \ge 1$. Note that (29) gives

$$\int_\Omega |\phi|^2\,dx \le c_0^{-1}(\Psi + C_*) \qquad \text{for each } \phi \in \{u_x, w_x, \omega_{1,x}, \tilde{\omega}_x, \omega_1, \tilde{\omega}\}.$$

We first treat the cubic terms with a gradient factor, $\int_\Omega u_x^3\,dx$, $\int_\Omega u_x|w_x|^2\,dx$, $\int_\Omega u_x\omega_{1,x}^2\,dx$ and $\int_\Omega u_x|\tilde{\omega}_x|^2\,dx$. Each is of the form $c\int_\Omega u_x\phi^2\,dx$ with ϕ a gradient; by Lemma 4,

$$\begin{aligned}\Big|c\int_\Omega u_x\phi^2\,dx\Big| &\le C\Big(1 + \Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + \Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2}\Big)\int_\Omega |\phi|^2\,dx \\ &\le C\int_\Omega |\phi|^2\,dx + C\Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\int_\Omega |\phi|^2\,dx \\ &\quad + \varepsilon\int_\Omega \rho\dot{u}^2\,dx + C_\varepsilon\Big(\int_\Omega |\phi|^2\,dx\Big)^2.\end{aligned}$$

Here the first term is bounded by $C(\Psi + C_*)$ in view of (29). The two remaining terms other than $\varepsilon\int_\Omega \rho\dot{u}^2\,dx$ are bounded by

$$C\Big(\Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + \int_\Omega |\phi|^2\,dx\Big)(\Psi + C_*),$$

whose coefficient belongs to $L^1(0,T)$ by Lemma 1.

For the cubic terms with a zeroth-order factor, $2\zeta \int_\Omega u_x \omega_1^2\,dx$ and $2\zeta \int_\Omega u_x|\tilde{\omega}|^2\,dx$, we use integration by parts to place a derivative on the micro-rotation factors. Since $u|_{\partial\Omega} = 0$ and

$$\|u\|_{L^\infty} \le \Big(\int_\Omega u_x^2\,dx\Big)^{1/2},$$

$$\begin{aligned}
2\zeta \int_\Omega u_x|\tilde{\omega}|^2\,dx &= -4\zeta \int_\Omega u\,(\tilde{\omega}\cdot\tilde{\omega}_x)\,dx \\
&\le 4\zeta\Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\Big(\int_\Omega |\tilde{\omega}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\tilde{\omega}_x|^2\,dx\Big)^{1/2} \\
&\le \varepsilon \int_\Omega |\tilde{\omega}_x|^2\,dx + \frac{4\zeta^2}{\varepsilon}\int_\Omega u_x^2\,dx \int_\Omega |\tilde{\omega}|^2\,dx,
\end{aligned}$$

where

$$\varepsilon \int_\Omega |\tilde{\omega}_x|^2\,dx \le \varepsilon c_0^{-1}(\Psi + C_*),$$

as $\int_\Omega |\tilde{\omega}_x|^2\,dx$ enters Ψ, and

$$\int_\Omega u_x^2\,dx \int_\Omega |\tilde{\omega}|^2\,dx \le C c_0^{-1}\Big(\int_\Omega u_x^2\,dx\Big)(\Psi + C_*),$$

whose coefficient lies in $L^1(0,T)$ by Lemma 1. The term $2\zeta \int_\Omega u_x\omega_1^2\,dx$ is treated identically. This avoids any need for the time integrability of $\int_\Omega |\tilde{\omega}|^2\,dx$.

The quadratic terms with bounded coefficient are direct: by Lemmas 2 and 3,

$$\gamma \int_\Omega P u_x^2\,dx \le \gamma a \bar{\rho}^\gamma \int_\Omega u_x^2\,dx, \qquad \int_\Omega |b|^2 u_x^2\,dx \le \int_\Omega |b_x|^2\,dx \int_\Omega u_x^2\,dx,$$

and these, together with the term $\int_\Omega |w_x|^2\,dx$ on the right-hand side of (28), are all bounded by $C(\Psi + C_*)$.

For the magnetic terms carrying b_{xx}, since $\|b\|_{L^\infty} \le C$ by Lemma 3,

$$\begin{aligned}
&-\nu \int_\Omega (b\cdot b_{xx})u_x\,dx + \nu \int_\Omega b_{xx}\cdot w_x\,dx \\
&\quad \le C\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2}\Big(\Big(\int_\Omega u_x^2\,dx\Big)^{1/2} + \Big(\int_\Omega |w_x|^2\,dx\Big)^{1/2}\Big) \\
&\quad \le C\int_\Omega |b_{xx}|^2\,dx + C(\Psi + C_*),
\end{aligned}$$

where $b_{xx} \in L^2(0,T;L^2(\Omega))$ by (19).

Likewise, the magnetic term carrying $b\cdot w_x$ obeys

$$-2\int_\Omega (b\cdot w_x)u_x\,dx \le C\Big(\int_\Omega |w_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega u_x^2\,dx\Big)^{1/2} \le C(\Psi + C_*).$$

Finally, we close the estimate. The gradient-factor bounds above produce four copies of $\varepsilon \int_\Omega \rho \dot{u}^2\,dx$; choosing $\varepsilon \le 1/8$, these are absorbed into the left-hand side of (28). Since $s^{1/2} \le 1 + s$ for $s \ge 0$, the above estimates give

$$\begin{aligned}
&\frac{d}{dt}(\Psi + C_*) + \frac{1}{2}\Big(\int_\Omega \rho\dot{u}^2\,dx + \int_\Omega \rho|\dot{w}|^2\,dx + \int_\Omega \rho\dot{\omega}_1^2\,dx + \int_\Omega \rho|\dot{\tilde{\omega}}|^2\,dx\Big) \\
&\quad \le C\Big(1 + \int_\Omega u_x^2\,dx + \int_\Omega |w_x|^2\,dx + \int_\Omega \omega_{1,x}^2\,dx + \int_\Omega |\tilde{\omega}_x|^2\,dx\Big)(\Psi + C_*) \\
&\qquad + C\Big(1 + \int_\Omega u_x^2\,dx + \int_\Omega |w_x|^2\,dx + \int_\Omega \omega_{1,x}^2\,dx + \int_\Omega |\tilde{\omega}_x|^2\,dx + \int_\Omega |b_{xx}|^2\,dx\Big).
\end{aligned} \tag{30}$$

Both coefficients belong to $L^1(0,T)$, by (11) and (19). Since $\rho_0, u_0, w_0, \omega_{1,0}, \tilde{\omega}_0 \in H^2(\Omega)$, $\Psi(0) < \infty$; no compatibility condition is used here. Gronwall's inequality yields $\sup_{0\le t\le T}\Psi(t) \le C$; with (29) and one integration of (30), we arrive at the following.

Lemma 5. *Under the assumptions of Theorem 1, the strong solution satisfies, with C as above and in particular independent of any regularization parameter δ,*

$$\begin{aligned}\sup_{0\le t\le T}\Big(\int_\Omega u_x^2\,dx+\int_\Omega |w_x|^2\,dx+\int_\Omega \omega_{1,x}^2\,dx+\int_\Omega |\tilde\omega_x|^2\,dx\Big)\\ +\int_0^T\Big(\int_\Omega \rho\dot u^2\,dx+\int_\Omega \rho|\dot w|^2\,dx+\int_\Omega \rho\dot\omega_1^2\,dx+\int_\Omega \rho|\dot{\tilde\omega}|^2\,dx\Big)\,dt\le C.\end{aligned}\tag{31}$$

Moreover, $\omega_1,\tilde\omega\in L^2(0,T;H^2(\Omega))$.

Proof. The bound (31) was established above. For the last assertion, (4) gives

$$\bar\mu'\omega_{1,xx}=\rho\dot\omega_1+4\zeta\omega_1,$$

whence

$$\Big(\int_\Omega \omega_{1,xx}^2\,dx\Big)^{1/2}\le\frac{1}{\bar\mu'}\Big(\sqrt{\bar\rho}\Big(\int_\Omega\rho\dot\omega_1^2\,dx\Big)^{1/2}+4\zeta\Big(\int_\Omega\omega_1^2\,dx\Big)^{1/2}\Big)\in L^2(0,T),$$

because, by (31),

$$\Big(\int_\Omega\rho\dot\omega_1^2\,dx\Big)^{1/2}\in L^2(0,T),\qquad\Big(\int_\Omega\omega_1^2\,dx\Big)^{1/2}\in L^\infty(0,T).$$

Similarly, (5) gives

$$\mu'\tilde\omega_{xx}=\rho\dot{\tilde\omega}+4\zeta\tilde\omega-2\zeta Jw_x,$$

so that

$$\begin{aligned}\Big(\int_\Omega|\tilde\omega_{xx}|^2\,dx\Big)^{1/2}\le\frac{1}{\mu'}\Big(\sqrt{\bar\rho}\Big(\int_\Omega\rho|\dot{\tilde\omega}|^2\,dx\Big)^{1/2}+4\zeta\Big(\int_\Omega|\tilde\omega|^2\,dx\Big)^{1/2}\\ +2\zeta\Big(\int_\Omega|w_x|^2\,dx\Big)^{1/2}\Big)\in L^2(0,T).\quad\square\end{aligned}$$

We now upgrade the density to $L^\infty(0,T;H^1(\Omega))$ and obtain $u,w\in L^2(0,T;H^2(\Omega))$. The two are coupled through the elliptic relation for u_{xx}. Constants C are as above.

Lemma 6. *Under the assumptions of Theorem 1,*

$$\sup_{0\le t\le T}\int_\Omega\rho_x^2\,dx\le C,\qquad i.e.,\qquad\rho\in L^\infty(0,T;H^1(\Omega)),\tag{32}$$

and, consequently, $u,w\in L^2(0,T;H^2(\Omega))$. All constants are independent of the regularization parameter δ.

Proof. Writing the continuity equation (1) as $D_t\rho=-\rho u_x$ with $D_t=\partial_t+u\partial_x$ and differentiating in x,

$$D_t(\rho_x)=\rho_{xt}+u\rho_{xx}=-2u_x\rho_x-\rho u_{xx}.\tag{33}$$

Testing (33) with ρ_x and using

$$\int_\Omega u\rho_{xx}\rho_x\,dx=\int_\Omega u\,\frac12(\rho_x^2)_x\,dx=-\frac12\int_\Omega u_x\rho_x^2\,dx$$

since $u|_{\partial\Omega}=0$, the boundary term vanishes and we obtain

$$\frac{d}{dt}\int_\Omega\rho_x^2\,dx=-3\int_\Omega u_x\rho_x^2\,dx-2\int_\Omega\rho\,u_{xx}\rho_x\,dx.\tag{34}$$

We bound u_{xx} elliptically: from (2),

$$\bar\mu u_{xx}=\rho\dot u+P_x+b\cdot b_x,$$

so by Lemmas 2 and 3, using

$$\Big(\int_\Omega P_x^2\,dx\Big)^{1/2} = a\gamma\Big(\int_\Omega \rho^{2(\gamma-1)}\rho_x^2\,dx\Big)^{1/2} \le a\gamma\bar\rho^{\gamma-1}\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2}$$

and

$$\Big(\int_\Omega (b\cdot b_x)^2\,dx\Big)^{1/2} \le \|b\|_{L^\infty}\Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2} \le C,$$
$$\Big(\int_\Omega u_{xx}^2\,dx\Big)^{1/2} \le C\Big(\Big(\int_\Omega \rho\dot u^2\,dx\Big)^{1/2} + \Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2} + 1\Big). \tag{35}$$

To close the estimate, for the first term of (34), Lemmas 4 and 5, which give $u_x \in L^\infty(0,T;L^2(\Omega))$ and $\sqrt{\rho}\,\dot u \in L^2(0,T;L^2(\Omega))$, yield $\|u_x\|_{L^\infty} \in L^1(0,T)$, hence

$$\Big|3\int_\Omega u_x\rho_x^2\,dx\Big| \le 3\|u_x\|_{L^\infty}\int_\Omega \rho_x^2\,dx.$$

For the second term, using $\rho \le \bar\rho$ and (35),

$$\begin{aligned}\Big|2\int_\Omega \rho u_{xx}\rho_x\,dx\Big| &\le 2\bar\rho\Big(\int_\Omega u_{xx}^2\,dx\Big)^{1/2}\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2}\\ &\le C\Big(\Big(\int_\Omega \rho\dot u^2\,dx\Big)^{1/2} + \Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2} + 1\Big)\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2}\\ &\le C\int_\Omega \rho\dot u^2\,dx + C\int_\Omega \rho_x^2\,dx + C,\end{aligned}$$

where the last term is integrable in time by Lemma 5; the ρ_x contribution inside the bound (35) re-enters only as the Gronwall term $C\int_\Omega \rho_x^2\,dx$. Thus

$$\frac{d}{dt}\int_\Omega \rho_x^2\,dx \le C\big(1+\|u_x\|_{L^\infty}\big)\int_\Omega \rho_x^2\,dx + C\Big(1+\int_\Omega \rho\dot u^2\,dx\Big),$$

where both coefficients lie in $L^1(0,T)$. Since $\rho_0 \in H^2(\Omega)$, Gronwall's inequality gives (32).

We finally bound the second-order velocities. With $\rho_x \in L^\infty(0,T;L^2(\Omega))$ now available, (35) gives

$$\int_0^T\int_\Omega u_{xx}^2\,dx\,dt \le C\int_0^T\Big(\int_\Omega \rho\dot u^2\,dx + \int_\Omega \rho_x^2\,dx + 1\Big)dt \le C,$$

so that $u \in L^2(0,T;H^2(\Omega))$. For w, Equation (3) reads

$$(\mu+\zeta)w_{xx} = \rho\dot w - b_x - 2\zeta J\tilde\omega_x$$

(no pressure), whence

$$\int_0^T\int_\Omega |w_{xx}|^2\,dx\,dt \le C\int_0^T\Big(\int_\Omega \rho|\dot w|^2\,dx + \int_\Omega |b_x|^2\,dx + \int_\Omega |\tilde\omega_x|^2\,dx\Big)dt \le C$$

by Lemmas 3 and 5, so $w \in L^2(0,T;H^2(\Omega))$. □

We next bound the (weighted) time derivatives. The order matters: the H^2 bound for the density couples ρ_{xx} to u_{xxx}, and u_{xxx} involves $\dot u_x$ through the momentum equation; hence the present estimate precedes the H^2 density bound. Constants C are as above; $\dot f = f_t + uf_x$, and Poincaré's inequality gives

$$\Big(\int_\Omega |\dot f|^2\,dx\Big)^{1/2} \le C\Big(\int_\Omega |\dot f_x|^2\,dx\Big)^{1/2}$$

since $\dot f|_{\partial\Omega} = 0$.

Lemma 7. *Under the assumptions of Theorem 1, in particular, the four compatibility conditions,*

$$\begin{aligned}\sup_{0\le t\le T}\int_\Omega \rho\big(\dot u^2+|\dot w|^2+\dot\omega_1^2+|\dot{\tilde\omega}|^2\big)dx\\ +\int_0^T\int_\Omega\big(\dot u_x^2+|\dot w_x|^2+\dot\omega_{1,x}^2+|\dot{\tilde\omega}_x|^2\big)dx\,dt\le C,\end{aligned}\tag{36}$$

and, consequently,

$$\begin{aligned}&\sqrt\rho\,u_t,\sqrt\rho\,w_t,\sqrt\rho\,\omega_{1,t},\sqrt\rho\,\tilde\omega_t\in L^\infty(0,T;L^2(\Omega)),\\ &u_t,w_t,\omega_{1,t},\tilde\omega_t\in L^2(0,T;H^1(\Omega)),\\ &u,w,\omega_1,\tilde\omega\in L^\infty(0,T;H^2(\Omega)).\end{aligned}$$

All constants are independent of δ.

Proof. We treat the scalar fields first, with u as template; we write (2) as $\rho\dot u=\bar\mu u_{xx}-(P+|b|^2/2)_x$, apply ∂_t and test with $\dot u$. Using

$$\int_\Omega\partial_t(\rho\dot u)\,\dot u\,dx=\frac{d}{dt}\frac12\int_\Omega\rho\dot u^2\,dx+\frac12\int_\Omega\rho_t\dot u^2\,dx,$$

the identity for the viscous term

$$\bar\mu\int_\Omega u_{xxt}\,\dot u\,dx=-\bar\mu\int_\Omega u_{xt}\,\dot u_x\,dx,\qquad u_{xt}=\dot u_x-u_x^2-uu_{xx},$$

and the identity

$$-\int_\Omega(P+|b|^2/2)_{xt}\,\dot u\,dx=\int_\Omega(P+|b|^2/2)_t\,\dot u_x\,dx,$$

we are led to

$$\begin{aligned}\frac{d}{dt}\frac12\int_\Omega\rho\dot u^2\,dx+\bar\mu\int_\Omega\dot u_x^2\,dx=-\int_\Omega\rho u\,\dot u\,\dot u_x\,dx+\bar\mu\int_\Omega(u_x^2+uu_{xx})\dot u_x\,dx\\ +\int_\Omega(P_t+b\cdot b_t)\dot u_x\,dx,\end{aligned}\tag{37}$$

where we have used

$$\frac12\int_\Omega\rho_t\dot u^2\,dx=\int_\Omega\rho u\,\dot u\,\dot u_x\,dx$$

together with $\rho_t=-(\rho u)_x$. We now estimate the three terms on the right-hand side of (37), using Lemmas 2–6 and Poincaré's inequality. Since $\|u\|_{L^\infty}\le C$ by Lemma 5, the first term satisfies

$$\begin{aligned}\Big|\int_\Omega\rho u\,\dot u\,\dot u_x\,dx\Big|&\le\sqrt{\bar\rho}\,\|u\|_{L^\infty}\Big(\int_\Omega\rho\dot u^2\,dx\Big)^{1/2}\Big(\int_\Omega\dot u_x^2\,dx\Big)^{1/2}\\ &\le\varepsilon\int_\Omega\dot u_x^2\,dx+C\int_\Omega u_x^2\,dx\int_\Omega\rho\dot u^2\,dx\le\varepsilon\int_\Omega\dot u_x^2\,dx+C\int_\Omega\rho\dot u^2\,dx.\end{aligned}$$

For the second term, the interpolation

$$\Big(\int_\Omega u_x^4\,dx\Big)^{1/2}=\|u_x\|_{L^4}^2\le\|u_x\|_{L^\infty}\Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\le C\Big(1+\Big(\int_\Omega\rho\dot u^2\,dx\Big)^{1/2}\Big)$$

gives

$$\bar\mu\int_\Omega u_x^2\,\dot u_x\,dx\le\varepsilon\int_\Omega\dot u_x^2\,dx+C\Big(1+\int_\Omega\rho\dot u^2\,dx\Big),$$

while, since $u_{xx}\in L^2(0,T;L^2(\Omega))$ by Lemma 6,

$$\begin{aligned}\bar\mu\int_\Omega uu_{xx}\,\dot u_x\,dx&\le C\Big(\int_\Omega u_x^2\,dx\Big)^{1/2}\Big(\int_\Omega u_{xx}^2\,dx\Big)^{1/2}\Big(\int_\Omega\dot u_x^2\,dx\Big)^{1/2}\\ &\le\varepsilon\int_\Omega\dot u_x^2\,dx+C\int_\Omega u_{xx}^2\,dx.\end{aligned}$$

For the third term,

$$\Big(\int_\Omega P_t^2\,dx\Big)^{1/2} \le \Big(\int_\Omega u^2 P_x^2\,dx\Big)^{1/2} + \gamma\Big(\int_\Omega P^2 u_x^2\,dx\Big)^{1/2} \le C\Big(\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2} + 1\Big)$$

and

$$\Big(\int_\Omega (b\cdot b_t)^2\,dx\Big)^{1/2} \le C\Big(\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2} + 1\Big)$$

(the latter bound from $b_t = \nu b_{xx} - u_x b - u b_x + w_x$) so that

$$\int_\Omega (P_t + b\cdot b_t)\dot u_x\,dx \le \varepsilon\int_\Omega \dot u_x^2\,dx + C\Big(\int_\Omega \rho_x^2\,dx + \int_\Omega |b_{xx}|^2\,dx + 1\Big),$$

the source term lying in $L^1(0,T)$. Choosing ε small, we obtain

$$\frac{d}{dt}\int_\Omega \rho\dot u^2\,dx + \bar\mu\int_\Omega \dot u_x^2\,dx \le C\int_\Omega \rho\dot u^2\,dx + f(t), \qquad f\in L^1(0,T).$$

The initial datum is controlled by compatibility: at $t=0$, (2) gives

$$\rho_0\dot u(0) = \bar\mu u_{0,xx} - (P_0 + \frac{1}{2}|b_0|^2)_x = \sqrt{\rho_0}\,g_1$$

by the first compatibility condition, with $g_1\in L^2(\Omega)$. Hence $\sqrt{\rho_0}\,\dot u(0) = g_1$ on $\{\rho_0>0\}$ and $=0$ on $\{\rho_0=0\}$ so that

$$\Big(\int_\Omega \rho\dot u^2\,dx\Big)^{1/2}\Big|_{t=0} \le \Big(\int_\Omega g_1^2\,dx\Big)^{1/2} < \infty.$$

Gronwall's inequality then yields

$$\sup_{0\le t\le T}\int_\Omega \rho\dot u^2\,dx + \int_0^T\int_\Omega \dot u_x^2\,dx\,dt \le C.$$

For ω_1, the same calculation has no pressure term. The damping contributes $4\zeta\int_\Omega \dot\omega_1^2\,dx$ on the left and the correction $4\zeta\int_\Omega u\omega_{1,x}\dot\omega_1\,dx$ on the right, where

$$\Big|4\zeta\int_\Omega u\omega_{1,x}\dot\omega_1\,dx\Big| \le 4\zeta\|u\|_{L^\infty}\|\omega_{1,x}\|_{L^2}\|\dot\omega_1\|_{L^2} \le \eta\|\dot\omega_{1,x}\|_{L^2}^2 + C_\eta.$$

Here Poincaré's inequality and Lemma 5 apply, and $\eta>0$ is chosen small enough to absorb the gradient term.

For the coupled block, testing the time derivatives of (3) and (5) by $\dot w$ and $\dot{\tilde\omega}$, respectively, gives

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}(\dot w,\dot{\tilde\omega})\|_{L^2}^2 + (\mu+\zeta)\|\dot w_x\|_{L^2}^2 + \mu'\|\dot{\tilde\omega}_x\|_{L^2}^2 + 4\zeta\|\dot{\tilde\omega}\|_{L^2}^2 = -4\zeta\int_\Omega (J\dot{\tilde\omega})\cdot\dot w_x\,dx + \mathcal{R}_c,$$

where, using $w_{xt} = \dot w_x - u_x w_x - u w_{xx}$ and its micro-rotation counterpart,

$$\begin{aligned}\mathcal{R}_c = &-\int_\Omega \rho u\big(\dot w\cdot\dot w_x + \dot{\tilde\omega}\cdot\dot{\tilde\omega}_x\big)\,dx\\ &+\int_\Omega\big[(\mu+\zeta)(u_x w_x + u w_{xx}) + 2\zeta u J\tilde\omega_x - b_t\big]\cdot\dot w_x\,dx\\ &+\int_\Omega\big[\mu'(u_x\tilde\omega_x + u\tilde\omega_{xx}) + 2\zeta u J w_x\big]\cdot\dot{\tilde\omega}_x\,dx + 4\zeta\int_\Omega u\tilde\omega_x\cdot\dot{\tilde\omega}\,dx.\end{aligned}$$

The leading coupling is absorbed by the gradient dissipation and micro-rotation damping: for $\varepsilon \in (\zeta, \mu + \zeta)$,

$$\left|4\zeta \int_\Omega (J\dot{\tilde{\omega}}) \cdot \dot{w}_x \, dx\right| \le \varepsilon \int_\Omega |\dot{w}_x|^2 \, dx + \frac{4\zeta^2}{\varepsilon} \int_\Omega |\dot{\tilde{\omega}}|^2 \, dx, \qquad \varepsilon < \mu + \zeta, \quad \frac{4\zeta^2}{\varepsilon} < 4\zeta, \tag{38}$$

so that both pieces are absorbed, for every admissible $\mu, \zeta > 0$.

Fix such an ε. By Hölder's and Poincaré's inequalities, followed by Young's inequality with a smaller parameter $\eta > 0$,

$$\begin{aligned} |\mathcal{R}_c| &\le C\|\sqrt{\rho}(\dot{w}, \dot{\tilde{\omega}})\|_{L^2}\big(\|\dot{w}_x\|_{L^2} + \|\dot{\tilde{\omega}}_x\|_{L^2}\big) \\ &\quad + C\big(1 + \|u_x\|_{L^\infty} + \|w_{xx}\|_{L^2} + \|\tilde{\omega}_{xx}\|_{L^2} + \|b_t\|_{L^2}\big)\big(\|\dot{w}_x\|_{L^2} + \|\dot{\tilde{\omega}}_x\|_{L^2}\big) \\ &\le \eta\big(\|\dot{w}_x\|_{L^2}^2 + \|\dot{\tilde{\omega}}_x\|_{L^2}^2\big) + C_\eta\|\sqrt{\rho}(\dot{w}, \dot{\tilde{\omega}})\|_{L^2}^2 \\ &\quad + C_\eta\big(1 + \|u_x\|_{L^\infty}^2 + \|w_{xx}\|_{L^2}^2 + \|\tilde{\omega}_{xx}\|_{L^2}^2 + \|b_t\|_{L^2}^2\big). \end{aligned}$$

Here $\|u\|_{L^\infty}$ and the L^2 norms of $w_x, \tilde{\omega}_x$ are bounded by Lemma 5. The last line is integrable in time: use Lemma 6 for $u, w \in L^2(0,T;H^2(\Omega))$, Lemma 5 for $\tilde{\omega} \in L^2(0,T;H^2(\Omega))$, and Lemma 3 for $b_t \in L^2(0,T;L^2(\Omega))$. Choose η below the remaining gradient dissipation in (38). The compatibility conditions control the initial weighted norm. Adding the scalar estimates and applying Gronwall's inequality gives (36).

As consequences, writing $\phi_t = \dot{\phi} - u\phi_x$ for $\phi \in \{u, w, \omega_1, \tilde{\omega}\}$ and using the bounds already proved, we obtain $\sqrt{\rho}\,\phi_t \in L^\infty(0,T;L^2(\Omega))$ and $\phi_t \in L^2(0,T;H^1(\Omega))$. Finally, writing (2) as

$$\bar{\mu}u_{xx} = \rho\dot{u} + P_x + b \cdot b_x,$$

where

$$\Big(\int_\Omega \rho^2\dot{u}^2\,dx\Big)^{1/2} \le \sqrt{\bar{\rho}}\Big(\int_\Omega \rho\dot{u}^2\,dx\Big)^{1/2}, \qquad \Big(\int_\Omega P_x^2\,dx\Big)^{1/2} \le C\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2},$$
$$\Big(\int_\Omega (b \cdot b_x)^2\,dx\Big)^{1/2} \le C$$

are bounded in $L^\infty(0,T)$, we obtain $u \in L^\infty(0,T;H^2(\Omega))$. Likewise, $w, \omega_1, \tilde{\omega} \in L^\infty(0,T;H^2(\Omega))$ follow from (3)–(5), in which $\rho\dot{w}, \rho\dot{\omega}_1, \rho\dot{\tilde{\omega}}$ are bounded in $L^\infty(0,T;L^2(\Omega))$ by (36) and the remaining terms by Lemmas 3 and 5. □

We next upgrade the density to $L^\infty(0,T;H^2(\Omega))$. Together with Lemma 7, this completes the a priori bounds for $\rho, u, w, \omega_1, \tilde{\omega}$ in the class of Theorem 1.

Lemma 8. *Under the assumptions of Theorem 1,*

$$\sup_{0\le t\le T}\int_\Omega \rho_{xx}^2\,dx \le C,$$

i.e., $\rho \in L^\infty(0,T;H^2(\Omega))$, and $\rho_t \in L^\infty(0,T;H^1(\Omega))$. The constants are independent of δ.

Proof. Differentiating (33) in x gives

$$D_t(\rho_{xx}) = -3u_x\rho_{xx} - 3u_{xx}\rho_x - \rho u_{xxx}.$$

Testing with ρ_{xx} and using

$$\int_\Omega u\rho_{xxx}\rho_{xx}\,dx = -\frac{1}{2}\int_\Omega u_x\rho_{xx}^2\,dx,$$

we obtain

$$\frac{d}{dt}\int_\Omega \rho_{xx}^2\,dx = -5\int_\Omega u_x\rho_{xx}^2\,dx - 6\int_\Omega u_{xx}\rho_x\rho_{xx}\,dx - 2\int_\Omega \rho\, u_{xxx}\rho_{xx}\,dx. \tag{39}$$

Differentiating (2) once more,

$$\bar{\mu}u_{xxx} = \rho_x\dot{u} + \rho\dot{u}_x + P''\rho_x^2 + P'\rho_{xx} + |b_x|^2 + b\cdot b_{xx}. \tag{40}$$

Substituting into the last term of (39), the contribution of $P'\rho_{xx}$ has a favorable sign,

$$-\frac{2}{\bar{\mu}}\int_\Omega \rho\,P'\,\rho_{xx}^2\,dx \le 0 \qquad (\rho\ge 0,\ P' = a\gamma\rho^{\gamma-1}\ge 0),$$

and is discarded. The remaining terms are estimated by Gronwall-type factors. We use Lemmas 2–7, in particular, $\|u_x\|_{L^\infty} \le C$ and $u_{xx}\in L^\infty(0,T;L^2(\Omega))$ from Lemma 7, together with

$$\rho P'' = a\gamma(\gamma-1)\rho^{\gamma-1}\le C, \qquad \|\rho_x\|_{L^\infty}\le \|\rho_x\|_{L^1} + \|\rho_{xx}\|_{L^1}\le C\Big(1+\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2}\Big),$$

the latter bound by (32). Thus

$$\begin{aligned}
&-5\int_\Omega u_x\rho_{xx}^2\,dx \le C\int_\Omega \rho_{xx}^2\,dx,\\
&-6\int_\Omega u_{xx}\rho_x\rho_{xx}\,dx \le C\|\rho_x\|_{L^\infty}\Big(\int_\Omega u_{xx}^2\,dx\Big)^{1/2}\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2} \le C + C\int_\Omega \rho_{xx}^2\,dx,\\
&-\frac{2}{\bar{\mu}}\int_\Omega \rho(\rho_x\dot{u}+\rho\dot{u}_x)\rho_{xx}\,dx \le C\Big(\|\dot{u}\|_{L^\infty}\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2} + \Big(\int_\Omega \dot{u}_x^2\,dx\Big)^{1/2}\Big)\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2}\\
&\qquad \le C\int_\Omega \dot{u}_x^2\,dx + C\int_\Omega \rho_{xx}^2\,dx,\\
&-\frac{2}{\bar{\mu}}\int_\Omega \rho P''\rho_x^2\rho_{xx}\,dx \le C\|\rho_x\|_{L^4}^2\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2} \le C + C\int_\Omega \rho_{xx}^2\,dx,\\
&-\frac{2}{\bar{\mu}}\int_\Omega \rho(|b_x|^2 + b\cdot b_{xx})\rho_{xx}\,dx \le C\Big(1+\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2}\Big)\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2}\\
&\qquad \le C + C\int_\Omega |b_{xx}|^2\,dx + C\int_\Omega \rho_{xx}^2\,dx,
\end{aligned}$$

where, by Poincaré's inequality and (32),

$$\begin{aligned}
&\|\dot{u}\|_{L^\infty}\le C\Big(\int_\Omega \dot{u}_x^2\,dx\Big)^{1/2},\\
&\|\rho_x\|_{L^4}^2 \le \|\rho_x\|_{L^\infty}\Big(\int_\Omega \rho_x^2\,dx\Big)^{1/2} \le C\Big(1+\Big(\int_\Omega \rho_{xx}^2\,dx\Big)^{1/2}\Big).
\end{aligned}$$

Summing yields

$$\frac{d}{dt}\int_\Omega \rho_{xx}^2\,dx \le C\int_\Omega \rho_{xx}^2\,dx + C\Big(1+\int_\Omega \dot{u}_x^2\,dx + \int_\Omega |b_{xx}|^2\,dx\Big),$$

where the last bracket belongs to $L^1(0,T)$ because $\dot{u}_x$ and b_{xx} belong to $L^2(0,T;L^2(\Omega))$ by Lemmas 7 and 3. As $\rho_0\in H^2(\Omega)$, Gronwall's inequality gives the first assertion. Finally,

$$\partial_x\rho_t = -\rho_{xx}u - 2\rho_x u_x - \rho u_{xx}$$

has all factors in $L^\infty(0,T;L^2(\Omega))$, so $\rho_t\in L^\infty(0,T;H^1(\Omega))$. □

Remark 2. *The velocity fields are* not *asserted in* $L^2(0,T;H^3(\Omega))$. *For the isentropic pressure* $P = a\rho^\gamma$, *the bound* $u\in L^2(0,T;H^3(\Omega))$ *would require*

$$P_{xx} = P''\rho_x^2 + P'\rho_{xx}\in L^2(\Omega),$$

but the density H^2 bound alone does not ensure this pressure estimate for all the exponents $\gamma > 1$ allowed here.

The regularity stated in Theorem 1 is the one of Choe–Kim [6] for isentropic flows with vacuum, in which u_{xx} is obtained from

$$\bar{\mu} u_{xx} = \rho \dot{u} + P_x + b \cdot b_x,$$

where only $P_x = P' \rho_x$ enters, with P' bounded. The magnetic field, governed by a non-degenerate parabolic equation, does retain $b \in L^2(0,T;H^3(\Omega))$.

Finally, we upgrade the magnetic field. Together with Lemmas 6–8, this completes the entire set of δ-uniform a priori estimates in the class of Theorem 1.

Lemma 9. *Under the assumptions of Theorem 1,*

$$\sup_{0 \le t \le T} \int_\Omega |b_t|^2 \, dx + \int_0^T \int_\Omega |b_{tx}|^2 \, dx \, dt \le C, \tag{41}$$

and consequently

$$b \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega)), \qquad b_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

The constants are independent of δ, and no compatibility condition on b_0 is required.

Proof. We argue through the time derivative b_t, which vanishes on $\partial\Omega$, and then lift by the equation. As in Lemma 3, the estimate carries no ρ-weight.

We first explain why the second spatial derivative is avoided. Differentiating (6) twice in x and testing with b_{xx} would produce the viscous term

$$\nu \int_\Omega b_{xxxx} b_{xx} \, dx = \nu [b_{xxx} b_{xx}]_{\partial\Omega} - \nu \int_\Omega |b_{xxx}|^2 \, dx,$$

whose boundary contribution does *not* vanish under the fixed boundary: evaluating (6) on $\partial\Omega$, where $b = u = w = 0$ and hence $b_t = 0$ and $(ub)_x = u_x b + u b_x = 0$, gives

$$b_{xx}\big|_{\partial\Omega} = -\frac{1}{\nu} w_x\big|_{\partial\Omega}, \tag{42}$$

which is generically nonzero. We therefore work with b_t instead.

We estimate b_t first. Since $b|_{\partial\Omega} = 0$ for all t, also $b_t|_{\partial\Omega} = 0$. Differentiating (6) in t,

$$b_{tt} = \nu \, b_{xxt} - (ub)_{xt} + w_{xt},$$

and testing with b_t over Ω, the viscous term integrates by parts with no boundary contribution,

$$\nu \int_\Omega b_{xxt} b_t \, dx = -\nu \int_\Omega |b_{tx}|^2 \, dx,$$

so that

$$\frac{d}{dt} \frac{1}{2} \int_\Omega |b_t|^2 \, dx + \nu \int_\Omega |b_{tx}|^2 \, dx = -\int_\Omega (ub)_{xt} \, b_t \, dx + \int_\Omega w_{xt} \, b_t \, dx. \tag{43}$$

For the convective term, $(ub)_t = u_t b + u b_t$ gives

$$(ub)_{xt} = u_{tx} b + u_t b_x + u_x b_t + u b_{tx},$$

and the last summand integrates by parts as

$$-\int_\Omega u b_{tx} b_t \, dx = -\int_\Omega u \, \frac{1}{2} (|b_t|^2)_x \, dx = \frac{1}{2} \int_\Omega u_x |b_t|^2 \, dx$$

by $u|_{\partial\Omega} = 0$; combining it with the term $-\int_\Omega u_x|b_t|^2\,dx$,

$$-\int_\Omega (ub)_{xt}\, b_t\,dx = -\int_\Omega u_{tx}\, b\, b_t\,dx - \int_\Omega u_t\, b_x\, b_t\,dx - \frac{1}{2}\int_\Omega u_x|b_t|^2\,dx.$$

Using $\|u_x\|_{L^\infty} \le C$ from Lemma 7 and

$$\|b\|_{L^\infty} + \Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2} \le C$$

from Lemma 3, we have

$$\begin{aligned}
&\Big|\frac{1}{2}\int_\Omega u_x|b_t|^2\,dx\Big| \le C\int_\Omega |b_t|^2\,dx,\\
&\Big|\int_\Omega u_{tx}\, b\, b_t\,dx\Big| \le \Big(\int_\Omega u_{tx}^2\,dx\Big)^{1/2}\|b\|_{L^\infty}\Big(\int_\Omega |b_t|^2\,dx\Big)^{1/2} \le \int_\Omega u_{tx}^2\,dx + C\int_\Omega |b_t|^2\,dx,\\
&\Big|\int_\Omega u_t\, b_x\, b_t\,dx\Big| \le \|u_t\|_{L^\infty}\Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2}\Big(\int_\Omega |b_t|^2\,dx\Big)^{1/2}\\
&\qquad \le \|u_t\|_{L^\infty}^2\int_\Omega |b_x|^2\,dx + C\int_\Omega |b_t|^2\,dx,
\end{aligned}$$

where

$$\|u_t\|_{L^\infty} \le \Big(\int_\Omega u_{tx}^2\,dx\Big)^{1/2}$$

by Poincaré's inequality, since $u_t|_{\partial\Omega} = 0$. For the induction term,

$$\int_\Omega w_{xt} b_t\,dx = \int_\Omega w_{tx} b_t\,dx \le \int_\Omega |w_{tx}|^2\,dx + C\int_\Omega |b_t|^2\,dx.$$

Hence

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |b_t|^2\,dx + 2\nu\int_\Omega |b_{tx}|^2\,dx\\
&\quad \le C\int_\Omega |b_t|^2\,dx + C\Big(\int_\Omega u_{tx}^2\,dx + \int_\Omega |w_{tx}|^2\,dx + \|u_t\|_{L^\infty}^2\int_\Omega |b_x|^2\,dx\Big).
\end{aligned}$$

The last bracket is integrable in time: since $\|u_t\|_{L^\infty}^2 \le \int_\Omega u_{tx}^2\,dx$ and $\int_\Omega |b_x|^2\,dx \le C$ by Lemma 3, its integral over $(0,T)$ is bounded by $C\int_0^T\int_\Omega (u_{tx}^2 + |w_{tx}|^2)\,dx\,dt \le C$ by Lemma 7, which gives $u_t, w_t \in L^2(0,T;H^1(\Omega))$. The initial datum is finite without any compatibility condition: evaluating (6) at $t = 0$,

$$b_t(0) = \nu\, b_{0,xx} - (u_0 b_0)_x + w_{0,x} \in L^2(\Omega),$$

since $b_0 \in H^2(\Omega)$ and $u_0, w_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ so that $(u_0b_0)_x = u_{0,x}b_0 + u_0 b_{0,x} \in L^2(\Omega)$. Gronwall's inequality yields (41), that is, $b_t \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$.

We then lift to H^2 and H^3. Rewrite (6) as the elliptic identity

$$\nu\, b_{xx} = b_t + (ub)_x - w_x. \tag{44}$$

Its right-hand side lies in $L^\infty(0,T;L^2(\Omega))$: $b_t \in L^\infty(0,T;L^2(\Omega))$ by (41);

$$\begin{aligned}
\Big(\int_\Omega |(ub)_x|^2\,dx\Big)^{1/2} &\le \|u_x\|_{L^\infty}\Big(\int_\Omega |b|^2\,dx\Big)^{1/2}\\
&\quad + \|u\|_{L^\infty}\Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2} \in L^\infty(0,T)
\end{aligned}$$

by Lemmas 3 and 7; and $w_x \in L^\infty(0,T;L^2(\Omega))$ since $w \in L^\infty(0,T;H^2(\Omega))$. Therefore, $b \in L^\infty(0,T;H^2(\Omega))$. Differentiating (44) once in x,

$$\nu\, b_{xxx} = b_{tx} + (ub)_{xx} - w_{xx},$$

where $b_{tx} \in L^2(0,T;L^2(\Omega))$ by (41), $w_{xx} \in L^\infty(0,T;L^2(\Omega))$ by Lemma 7, and, by $b, u \in L^\infty(0,T;H^2(\Omega))$,

$$\Big(\int_\Omega |(ub)_{xx}|^2\,dx\Big)^{1/2} \le \Big(\int_\Omega u_{xx}^2\,dx\Big)^{1/2}\|b\|_{L^\infty} + 2\|u_x\|_{L^\infty}\Big(\int_\Omega |b_x|^2\,dx\Big)^{1/2} + \|u\|_{L^\infty}\Big(\int_\Omega |b_{xx}|^2\,dx\Big)^{1/2} \in L^\infty(0,T).$$

Hence $b \in L^2(0,T;H^3(\Omega))$. □

4. Local Existence

The a priori estimates of Section 3 are complemented by the following local existence and uniqueness result. For fixed physical coefficients and domain, the local time is controlled by the H^2 norms of the initial fields and the L^2 norms of $g_1, \dots, g_4$, independently of a positive density lower bound. These are the quantities needed in the continuation argument.

Proposition 1. *Under the hypotheses of Theorem 1, there is a time $\tau_* > 0$ depending only on the fixed physical coefficients, Ω, $\|\rho_0\|_{H^2}$, $\|(u_0, w_0, \omega_{1,0}, \tilde\omega_0, b_0)\|_{H^2}$ and $\|(g_1,g_2,g_3,g_4)\|_{L^2}$, but* not *on $\inf_\Omega \rho_0$, such that, with $T_* = \min\{T, \tau_*\}$, there is a unique strong solution $(\rho, u, w, \omega_1, \tilde\omega, b)$ of (1)–(9) on $\Omega \times (0, T_*)$ in the regularity class of Theorem 1 with T replaced by T_*.*

Proof. We use compatible positive-density data and successive approximation, following the linearized construction of Cho–Kim [7]. Write $\Phi = (u, w, \omega_1, \tilde\omega)$ to collect the four equations carrying the density weight, and let $K \ge 1$ bound the initial norms in the proposition. In this proof, $\|\cdot\|_p$ denotes the $L^p(\Omega)$ norm. Constants C_K may also depend on the fixed coefficients and Ω, but not on the density lower bound.

For fields vanishing at the endpoints, the viscous and micro-rotation terms satisfy the identity

$$\begin{aligned}&\int_\Omega [\bar\mu u_x^2 + (\mu+\zeta)|w_x|^2 + \bar\mu'\omega_{1,x}^2 + \mu'|\tilde\omega_x|^2 + 4\zeta(\omega_1^2 + |\tilde\omega|^2) + 4\zeta J\tilde\omega\cdot w_x]\,dx\\ &\quad = \int_\Omega [\bar\mu u_x^2 + \mu|w_x|^2 + \bar\mu'\omega_{1,x}^2 + \mu'|\tilde\omega_x|^2 + 4\zeta\omega_1^2 + \zeta|w_x + 2J\tilde\omega|^2]\,dx \ge c\|\Phi_x\|_2^2,\end{aligned}$$

where $c > 0$ depends only on the fixed physical coefficients.

Set $\rho_0^\delta = \rho_0 + \delta$ and $b_0^\delta = b_0$ for $0 < \delta \le 1$. Define Φ_0^δ by the four elliptic compatibility equations in Theorem 1, with ρ_0, b_0 replaced by $\rho_0^\delta, b_0^\delta$ and the same g_i. Coercivity, Poincaré's inequality and the spatial equations give $\|\Phi_0^\delta\|_{H^2} \le C_K$ and

$$\begin{aligned}\|\Phi_0^\delta - \Phi_0\|_{H^2} &\le C\big(\|\sqrt{\rho_0+\delta} - \sqrt{\rho_0}\|_\infty \|(g_1,g_2,g_3,g_4)\|_2 \\ &\qquad + \|(P(\rho_0+\delta) - P(\rho_0))_x\|_2\big)\\ &\le C_K\big(\sqrt{\delta} + \delta^{\min\{1,\gamma-1\}}\big) \longrightarrow 0.\end{aligned}$$

Here $s^{\gamma-1}$ is Hölder continuous with exponent $\min\{1, \gamma-1\}$ on bounded non-negative intervals.

For fixed δ, we suppress the superscript on the initial data.

Step 1 (linearized cascade).

Let $v(0) = u_0$, $v \in L^\infty(0,T_*;H^2(\Omega)\cap H_0^1(\Omega))$ and $v_t \in L^2(0,T_*;H_0^1(\Omega))$, with

$$\operatorname*{ess\,sup}_{0<t<T_*} \|v(t)\|_{H^2}^2 + \int_0^{T_*} \|v_t\|_{H^1}^2\,dt \le M.$$

We use the continuous H^1 representative of v. The constants $M = M(K)$ and $T_* = T_*(K)$ will be chosen below. Define the new fields by solving, in this order:

(i) The linear transport equation for the density,

$$\rho_t + (\rho v)_x = 0, \qquad \rho|_{t=0} = \rho_0 \geq 0. \tag{45}$$

Along the characteristics $\dot{X}(t,y) = v(t, X(t,y))$, $X(0,y) = y$, one has

$$\rho(t, X) = \rho_0(y) \exp\Big(- \int_0^t v_x \, ds \Big),$$

so that $\rho \in C([0,T_*]; H^1(\Omega))$, $\rho_t \in C([0,T_*]; L^2(\Omega))$, and

$$\delta \exp\Big(- \int_0^{T_*} \|v_x\|_\infty \, dt \Big) \leq \rho \leq \|\rho_0\|_\infty \exp\Big(\int_0^{T_*} \|v_x\|_\infty \, dt \Big).$$

This is the density regularity used in the frozen cascade.

(ii) The decoupled linear parabolic equation for ω_1,

$$\rho(\omega_{1,t} + v\omega_{1,x}) + 4\zeta\omega_1 = \bar{\mu}' \, \omega_{1,xx}, \qquad \omega_1|_{\partial\Omega} = 0, \ \omega_1|_{t=0} = \omega_{1,0}. \tag{46}$$

(iii) The *coupled* linear parabolic block for $(w, \tilde{\omega}, b)$,

$$\rho(w_t + vw_x) - b_x = (\mu + \zeta) w_{xx} + 2\zeta J \tilde{\omega}_x, \tag{47}$$

$$\rho(\tilde{\omega}_t + v\tilde{\omega}_x) + 4\zeta\tilde{\omega} = \mu' \tilde{\omega}_{xx} + 2\zeta J w_x, \tag{48}$$

$$b_t + (vb)_x - w_x = \nu b_{xx}, \tag{49}$$

with

$$w|_{\partial\Omega} = \tilde{\omega}|_{\partial\Omega} = b|_{\partial\Omega} = 0$$

and the prescribed initial data. The principal part

$$\mathrm{diag}\big((\mu + \zeta)\partial_{xx}, \, \mu'\partial_{xx}, \, \nu\partial_{xx} \big)$$

is non-degenerate by (7); the couplings $2\zeta J(\cdot)_x$, b_x, w_x are first order, and the damping 4ζ is of order zero.

(iv) The decoupled linear parabolic equation for u, with the ρ, b just produced,

$$\rho(u_t + vu_x) + \Big(P(\rho) + \frac{1}{2}|b|^2 \Big)_x = \bar{\mu} \, u_{xx}, \qquad u|_{\partial\Omega} = 0, \ u|_{t=0} = u_0. \tag{50}$$

For fixed $\delta > 0$, the characteristic formula gives a positive mass coefficient $\rho \in C([0,T_*]; H^1(\Omega))$ with $\rho_t \in C([0,T_*]; L^2(\Omega))$. The remaining equations have constant positive spatial diffusion coefficients and lower-order couplings. Use the Dirichlet Laplacian eigenfunctions for the Galerkin construction and project the $H^2 \cap H_0^1$ initial data; the initial spatial residuals converge in L^2. The positive mass matrices, time-difference-quotient energy estimates and spatial elliptic estimates give, after passing to the Galerkin limit with δ fixed, the unique linear solution with $\Phi, b \in L^\infty(0,T_*; H^2(\Omega))$ and $\Phi_t, b_t \in L^2(0,T_*; H_0^1(\Omega))$. This construction may use the fixed positive lower bound. Step 2 then applies the compatibility conditions to this linear solution to obtain bounds independent of δ.

Step 2 (bounds independent of $\inf \rho_0$*).*

Restrict $T_* \leq 1$ and $T_* M \leq 1$. The time integral of v_t gives

$$\|v\|_{L^\infty(0,T_*;H^1(\Omega))} \leq \|u_0\|_{H^1} + \sqrt{T_*}\|v_t\|_{L^2(0,T_*;H^1(\Omega))} \leq C_K, \qquad \int_0^{T_*} \|v_x\|_\infty \, dt \leq C T_* \sqrt{M}.$$

Differentiating (45) once and testing by ρ_x yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho_x\|_2^2 &= -\frac{3}{2}\int_\Omega v_x \rho_x^2 \, dx - \int_\Omega \rho v_{xx} \rho_x \, dx \\ &\leq \frac{3}{2}\|v_x\|_\infty \|\rho_x\|_2^2 + \|\rho\|_\infty \|v_{xx}\|_2 \|\rho_x\|_2. \end{aligned}$$

Since $\int_0^{T_*} \|v_{xx}\|_2\,dt \le T_*\sqrt{M}$, this gives

$$\|\rho_x(t)\|_2 \le \exp\!\left(\frac{3}{2}\int_0^t \|v_x\|_\infty\,ds\right)\left(\|\rho_{0,x}\|_2 + \|\rho\|_{L^\infty(\Omega\times(0,T_*))}\int_0^t \|v_{xx}\|_2\,ds\right) \le C_K.$$

Consequently, $\|\rho\|_{L^\infty(0,T_*;H^1(\Omega))}+\|\rho_t\|_{L^\infty(0,T_*;L^2(\Omega))} \le C_K$, by $\rho_t = -v\rho_x - \rho v_x$. Testing (47)–(49) by $w, \tilde{\omega}, b$ and using $\rho_t + (\rho v)_x = 0$ gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{1}{2}\rho\big(|w|^2+|\tilde{\omega}|^2\big)+\frac{1}{2}|b|^2\Big]dx + (\mu+\zeta)\int_\Omega |w_x|^2\,dx \\
&\quad + \mu'\int_\Omega |\tilde{\omega}_x|^2\,dx + \nu\int_\Omega |b_x|^2\,dx + 4\zeta\int_\Omega |\tilde{\omega}|^2\,dx \\
&\quad = \int_\Omega b_x\cdot w\,dx + \int_\Omega w_x\cdot b\,dx - 4\zeta\int_\Omega (J\tilde{\omega})\cdot w_x\,dx - \frac{1}{2}\int_\Omega v_x|b|^2\,dx.
\end{aligned} \tag{51}$$

By the two computations of Lemma 1, the Alfvén pair, that is, the first two terms on the right, cancels after an integration by parts, while, by $J^\top = -J$, $|J\tilde{\omega}| = |\tilde{\omega}|$ and Young's inequality,

$$-4\zeta\int_\Omega (J\tilde{\omega})\cdot w_x\,dx \le \zeta\int_\Omega |w_x|^2\,dx + 4\zeta\int_\Omega |\tilde{\omega}|^2\,dx;$$

the first term on the right is absorbed by $(\mu+\zeta)\int_\Omega |w_x|^2\,dx$ and the second term by the damping. The remaining term obeys

$$-\frac{1}{2}\int_\Omega v_x|b|^2\,dx \le \frac{1}{2}\|v_x\|_{L^\infty}\int_\Omega |b|^2\,dx$$

and is a Gronwall term.

Adding the scalar micro-rotation energy, and testing (49) by $-b_{xx}$, gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|b_x\|_2^2 + \nu\|b_{xx}\|_2^2 &= \int_\Omega \big((vb)_x - w_x\big)\cdot b_{xx}\,dx \\
&\le \big(2\|v_x\|_2\|b_x\|_2 + \|w_x\|_2\big)\|b_{xx}\|_2 \\
&\le \frac{\nu}{2}\|b_{xx}\|_2^2 + \frac{4}{\nu}\|v_x\|_2^2\|b_x\|_2^2 + \frac{1}{\nu}\|w_x\|_2^2.
\end{aligned}$$

Here $\|v\|_\infty \le \|v_x\|_2$ and $\|b\|_\infty \le \|b_x\|_2$. Testing (50) by u also gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u^2\,dx + \bar{\mu}\|u_x\|_2^2 &= \int_\Omega (P+|b|^2/2)u_x\,dx \\
&\le \|P+|b|^2/2\|_2\|u_x\|_2 \le \frac{\bar{\mu}}{2}\|u_x\|_2^2 + C_K.
\end{aligned}$$

Gronwall's inequality and the coupled energy therefore yield

$$\sup_{t\le T_*}\big(\|\sqrt{\rho}\,\Phi\|_2^2 + \|b\|_{H^1}^2\big) + \int_0^{T_*}\big(\|\Phi_x\|_2^2 + \|b_{xx}\|_2^2\big)\,dt \le C_K.$$

Testing the four spatial momentum equations by Φ, and using the coercivity above, gives

$$\begin{aligned}
c\|\Phi_x\|_2^2 &\le \|\sqrt{\rho}\,\Phi_t\|_2\|\sqrt{\rho}\,\Phi\|_2 + \|v\|_\infty\|\sqrt{\rho}\,\Phi_x\|_2\|\sqrt{\rho}\,\Phi\|_2 \\
&\quad + \|P+|b|^2/2\|_2\|u_x\|_2 + \|b\|_2\|w_x\|_2 \\
&\le C_K\|\sqrt{\rho}\,\Phi_t\|_2 + C_K\|\Phi_x\|_2 \\
&\le \frac{c}{2}\|\Phi_x\|_2^2 + C_K(1+\|\sqrt{\rho}\,\Phi_t\|_2).
\end{aligned}$$

Thus, $\|\Phi_x\|_2^2 \le C_K(1+\|\sqrt{\rho}\,\Phi_t\|_2)$. The spatial equations and the induction equation then give

$$\begin{aligned}
\|\Phi_{xx}\|_2 + \|b_{xx}\|_2 &\le C_K\big(1+\|\sqrt{\rho}\,\Phi_t\|_2 + \|b_t\|_2 + \|\Phi_x\|_2 + \|(vb)_x\|_2\big) \\
&\le C_K\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2\big), \\
\|(\Phi,b)\|_{H^2}^2 &\le C_K\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2\big), \\
\|\Phi_x\|_\infty &\le C_K\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2\big).
\end{aligned}$$

Differentiate the equations in time and test by Φ_t, b_t. The magnetic cross terms cancel, and coercivity gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2 + c\|(\Phi_{tx}, b_{tx})\|_2^2 \\ \le -\int_\Omega \rho_t|\Phi_t|^2\,dx - \int_\Omega \rho_t v\Phi_x\cdot\Phi_t\,dx - \int_\Omega \rho v_t\Phi_x\cdot\Phi_t\,dx \\ + \int_\Omega (P_t + b\cdot b_t)u_{tx}\,dx - \frac{1}{2}\int_\Omega v_x|b_t|^2\,dx - \int_\Omega (v_t b)_x\cdot b_t\,dx.\end{aligned}$$

For the density-time terms, integrate by parts using $\rho_t = -(\rho v)_x$; Hölder's and Young's inequalities give

$$\begin{aligned}\Big|\int_\Omega \rho_t\big(|\Phi_t|^2 + v\Phi_x\cdot\Phi_t\big)\,dx\Big| \\ \le C_K(\|\sqrt{\rho}\,\Phi_t\|_2 + \|\Phi_x\|_2)\|\Phi_{tx}\|_2 \\ + C_K\|v_x\|_\infty\|\Phi_x\|_2\|\sqrt{\rho}\,\Phi_t\|_2 + C_K\|\Phi_{xx}\|_2\|\sqrt{\rho}\,\Phi_t\|_2 \\ \le \eta\|\Phi_{tx}\|_2^2 + C_{K,\eta}(1+\|v_x\|_\infty)\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2\big).\end{aligned}$$

The terms containing v_t are estimated without squaring its norm:

$$\begin{aligned}\Big|\int_\Omega \rho v_t\Phi_x\cdot\Phi_t\,dx\Big| + \Big|\int_\Omega (v_t b)_x\cdot b_t\,dx\Big| \\ \le C_K\|v_t\|_{H^1}\big(\|\Phi_x\|_\infty\|\sqrt{\rho}\,\Phi_t\|_2 + \|b_t\|_2\big) \\ \le C_K\|v_t\|_{H^1}\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2\big).\end{aligned}$$

Finally, $\|P_t\|_2 \le C_K\|\rho_t\|_2 \le C_K$ gives

$$\begin{aligned}\Big|\int_\Omega (P_t + b\cdot b_t)u_{tx}\,dx\Big| + \frac{1}{2}\Big|\int_\Omega v_x|b_t|^2\,dx\Big| \\ \le C_K(1+\|b_t\|_2)\|u_{tx}\|_2 + \frac{1}{2}\|v_x\|_\infty\|b_t\|_2^2 \\ \le \eta\|\Phi_{tx}\|_2^2 + C_{K,\eta}(1+\|v_x\|_\infty)(1+\|b_t\|_2^2).\end{aligned}$$

Fix $0 < \eta < c/4$. Absorbing these terms yields

$$\frac{d}{dt}\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2 + c\|(\Phi_{tx}, b_{tx})\|_2^2 \le C_K(1+\|v_x\|_\infty + \|v_t\|_{H^1})\big(1+\|(\sqrt{\rho}\,\Phi_t, b_t)\|_2^2\big).$$

The initial norm is bounded by C_K, by compatibility and Remark 1. The integral of the Gronwall coefficient is at most $C_K(T_* + T_*\sqrt{M} + \sqrt{T_* M})$. Choose $M = M(K)$ larger than twice the resulting bound when this integral is at most one, also including the Dirichlet heat-solution bound for u_0. Then choose $T_* = T_*(K) \le \min\{1, T\}$ so that $T_* M \le 1$ and this coefficient integral is at most one. We obtain

$$\operatorname*{ess\,sup}_{0<t<T_*}\big(\|(\sqrt{\rho}\,\Phi_t, b_t)(t)\|_2^2 + \|(\Phi, b)(t)\|_{H^2}^2\big) + \int_0^{T_*}\|(\Phi_t, b_t)\|_{H^1}^2\,dt \le C_K.$$

In particular, u satisfies the bounds imposed on v in Step 1. All constants and the common interval are independent of δ.

Step 3 (iteration and uniqueness).

Convergence of the iteration. Let u^0 solve the Dirichlet heat equation with initial value u_0. Inductively, let $(\rho^{k+1}, \Phi^{k+1}, b^{k+1})$ solve (45)–(50) with $v = u^k$. The heat-solution estimate and Step 2 give the same bounds for every iterate on $[0, T_*]$. We compare consecutive iterates, as in Cho–Kim [7], writing

$$\bar\rho^{k+1} = \rho^{k+1} - \rho^k, \qquad \bar\Phi^{k+1} = \Phi^{k+1} - \Phi^k, \qquad \bar b^{k+1} = b^{k+1} - b^k.$$

Here $\bar{u}^k = u^k - u^{k-1}$ is the longitudinal component of the preceding difference. For $k \geq 1$, the density difference satisfies

$$\bar{\rho}_t^{k+1} + (u^k \bar{\rho}^{k+1})_x = -(\rho^k \bar{u}^k)_x, \qquad \bar{\rho}^{k+1}(0) = 0.$$

Since

$$\|(\rho^k \bar{u}^k)_x\|_2 \leq \|\rho^k\|_\infty \|\bar{u}_x^k\|_2 + \|\rho_x^k\|_2 \|\bar{u}^k\|_\infty \leq C_K \|\bar{u}_x^k\|_2,$$

the transport estimate and Hölder's inequality in time give

$$\sup_{t \leq T_*} \|\bar{\rho}^{k+1}(t)\|_2^2 \leq C_K \Big(\int_0^{T_*} \|\bar{u}_x^k\|_2 \, dt \Big)^2 \leq C_K T_* \int_0^{T_*} \|\bar{u}_x^k\|_2^2 \, dt.$$

Subtract the four momentum equations and the magnetic equation, and test by $\bar{\Phi}^{k+1}, \bar{b}^{k+1}$. The viscous and micro-rotation terms are coercive as above, and the magnetic cross terms cancel. The two terms caused by the frozen convective velocity and the density difference satisfy

$$\begin{aligned}
&\Big| \int_\Omega \bar{\rho}^{k+1} (\Phi_t^k + u^{k-1} \Phi_x^k) \cdot \bar{\Phi}^{k+1} \, dx \Big| \\
&\quad \leq \|\bar{\rho}^{k+1}\|_2 \|\Phi_t^k + u^{k-1} \Phi_x^k\|_\infty \|\bar{\Phi}^{k+1}\|_2 \\
&\quad \leq \varepsilon \|\bar{\Phi}_x^{k+1}\|_2^2 + C_\varepsilon \|\Phi_t^k + u^{k-1} \Phi_x^k\|_\infty^2 \|\bar{\rho}^{k+1}\|_2^2, \\
&\Big| \int_\Omega \rho^{k+1} \bar{u}^k \Phi_x^k \cdot \bar{\Phi}^{k+1} \, dx \Big| + \Big| \int_\Omega (\bar{u}^k b^k)_x \cdot \bar{b}^{k+1} \, dx \Big| \\
&\quad \leq C_K \|\bar{u}_x^k\|_2 \|(\sqrt{\rho^{k+1}} \bar{\Phi}^{k+1}, \bar{b}^{k+1})\|_2 \\
&\quad \leq \eta \|\bar{u}_x^k\|_2^2 + C_{K,\eta} \|(\sqrt{\rho^{k+1}} \bar{\Phi}^{k+1}, \bar{b}^{k+1})\|_2^2.
\end{aligned}$$

The pressure and magnetic-pressure differences obey

$$\|P(\rho^{k+1}) - P(\rho^k)\|_2 \leq C_K \|\bar{\rho}^{k+1}\|_2, \qquad \||b^{k+1}|^2 - |b^k|^2\|_2 \leq C_K \|\bar{b}^{k+1}\|_2.$$

Apply the preceding estimates with $\eta/2$ and fix ε by the dissipation coefficients. Multiplying the energy inequality by two gives

$$\begin{aligned}
&\frac{d}{dt} \|(\sqrt{\rho^{k+1}} \bar{\Phi}^{k+1}, \bar{b}^{k+1})\|_2^2 + c \|(\bar{\Phi}_x^{k+1}, \bar{b}_x^{k+1})\|_2^2 \\
&\quad \leq C_{K,\eta} \|(\sqrt{\rho^{k+1}} \bar{\Phi}^{k+1}, \bar{b}^{k+1})\|_2^2 + \eta \|\bar{u}_x^k\|_2^2 \\
&\qquad + C_K \big(1 + \|\Phi_t^k + u^{k-1} \Phi_x^k\|_\infty^2 \big) \|\bar{\rho}^{k+1}\|_2^2.
\end{aligned}$$

By Step 2 and $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, $\int_0^{T_*} \|\Phi_t^k + u^{k-1} \Phi_x^k\|_\infty^2 \, dt \leq C_K$. The density estimate and Gronwall's inequality therefore yield

$$\begin{aligned}
&\sup_{t \leq T_*} \|(\sqrt{\rho^{k+1}} \bar{\Phi}^{k+1}, \bar{b}^{k+1})(t)\|_2^2 + \int_0^{T_*} \|(\bar{\Phi}_x^{k+1}, \bar{b}_x^{k+1})\|_2^2 \, dt \\
&\quad \leq C_K e^{C_{K,\eta} T_*} (\eta + T_*) \int_0^{T_*} \|\bar{u}_x^k\|_2^2 \, dt.
\end{aligned}$$

Choose $\eta > 0$ first, and then reduce T_*, so that the coefficient on the right is less than $1/4$. Hence

$$\int_0^{T_*} \|\bar{u}_x^{k+1}\|_2^2 \, dt \leq \frac{1}{4} \int_0^{T_*} \|\bar{u}_x^k\|_2^2 \, dt,$$

and the successive differences are summable in the displayed norms.

For fixed $\delta > 0$, the positive lower bound also gives convergence of Φ^k, b^k in $C([0, T_*]; L^2(\Omega))$; interpolation with their uniform H^2 bounds gives convergence in $C([0, T_*]; H^1(\Omega))$. The densities converge in $C([0, T_*]; L^2(\Omega))$ and, by their uniform H^1 bounds, in $C([0, T_*]; H^s(\Omega))$ for $1/2 < s < 1$. These convergences and weak convergence

of the time derivatives pass the linearized equations to the nonlinear system. The limit satisfies the bounds of Step 2, and its transport equation gives $\rho \in C([0,T_*];H^1(\Omega))$.

Recovery of the density regularity. The continuity and longitudinal momentum equations imply

$$\partial_t \rho_x + u\partial_x \rho_x + \left(2u_x + \frac{a\gamma}{\bar{\mu}}\rho^\gamma\right)\rho_x = -\frac{1}{\bar{\mu}}\left(\rho^2 \dot{u} + \rho b \cdot b_x\right).$$

Since $H^1(\Omega)$ is an algebra and $\dot{u} = u_t + uu_x \in L^2(0,T_*;H^1(\Omega))$, its coefficients and source satisfy

$$\begin{aligned}\left\|2u_x + \frac{a\gamma}{\bar{\mu}}\rho^\gamma\right\|_{H^1} &\le C_K,\\ \|\rho^2\dot{u} + \rho b \cdot b_x\|_{H^1} &\le C\|\rho\|_{H^1}^2\|\dot{u}\|_{H^1} + C\|\rho\|_{H^1}\|b\|_{H^1}\|b_x\|_{H^1}\\ &\le C_K(1 + \|u_t\|_{H^1}).\end{aligned}$$

The H^1 transport estimate, with these coefficients fixed and initial value $\rho_{0,x} \in H^1$, reads

$$\begin{aligned}\frac{d}{dt}\|\rho_x\|_{H^1}^2 &\le C\left(1 + \|u_x\|_\infty + \left\|2u_x + \frac{a\gamma}{\bar{\mu}}\rho^\gamma\right\|_{H^1}\right)\|\rho_x\|_{H^1}^2\\ &\quad + \frac{2}{\bar{\mu}}\|\rho^2\dot{u} + \rho b \cdot b_x\|_{H^1}\|\rho_x\|_{H^1}\\ &\le C_K\|\rho_x\|_{H^1}^2 + C_K(1 + \|u_t\|_{H^1}^2).\end{aligned}$$

This estimate first constructs an H^1 solution of the linear equation by regularization; uniqueness in L^2 identifies it with the already constructed derivative ρ_x. No boundary value for ρ_x is needed, since $u = 0$ at the endpoints. Gronwall's inequality gives $\rho \in L^\infty(0,T_*;H^2(\Omega))$ and hence $\rho_t = -(\rho u)_x \in L^\infty(0,T_*;H^1(\Omega))$, uniformly in δ. The magnetic equation also gives $\nu b_{xxx} = b_{tx} + (ub)_{xx} - w_{xx} \in L^2(0,T_*;L^2(\Omega))$, so $b \in L^2(0,T_*;H^3(\Omega))$.

Passage to vacuum. Restore the superscripts δ. Compactness of $H^2(\Omega)$ in $H^1(\Omega)$ and the uniform $L^2(0,T_*;H^1(\Omega))$ time-derivative bounds give, along a subsequence,

$$(\rho^\delta, \Phi^\delta, b^\delta) \longrightarrow (\rho, \Phi, b) \quad \text{strongly in } C([0,T_*];H^1(\Omega)).$$

The time derivatives converge weakly in $L^2(0,T_*;H^1(\Omega))$, and the $L^\infty(0,T_*;H^2(\Omega))$ bounds pass weakly star. Moreover, $\sqrt{\rho^\delta} \to \sqrt{\rho}$ uniformly, which identifies the weak-star limit of $\sqrt{\rho^\delta}\Phi_t^\delta$ with $\sqrt{\rho}\,\Phi_t$ in $L^\infty(0,T_*;L^2(\Omega))$. Continuity of P' on the common bounded density range gives $P(\rho^\delta) \to P(\rho)$ in $C([0,T_*];H^1(\Omega))$, so the nonlinear terms pass to the limit. The strong $C([0,T_*];H^1(\Omega))$ convergence preserves the prescribed initial values and Dirichlet traces. The limit is non-negative and belongs to the class of Theorem 1. The difference estimate of Section 6 applies on $[0,T_*]$ and gives uniqueness. □

5. Global Existence

We prove the existence part of Theorem 1 by continuing the local solution of Proposition 1 with the a priori bounds of Section 3. This is the standard continuation argument of Cho–Kim [7] and Fan–Yu [11]; what makes it work here is that the local time T_* of Proposition 1 depends on the data only through norms controlled by the a priori estimates and that those estimates are finite on every bounded time interval.

Proof of the existence part of Theorem 1. Fix $0 < T < \infty$. Let $T_{\max}$ be the supremum of all $T' > 0$ for which (1)–(9) admits a strong solution in the class of Theorem 1 on $[0,T']$. By Proposition 1, $T_{\max} \ge T_* > 0$, and by the uniqueness in Proposition 1, the solutions for different $T' < T_{\max}$ are restrictions of a single solution on $[0,T_{\max})$. We show $T_{\max} > T$.

Step 1 (a priori bounds are finite up to $T_{\max}$).

The bounds of Lemmas 1–9 hold for this solution on each $[0,T']$ with $T' < T_{\max}$, with constants depending on the data and on T' only through an increasing finite function of T'; each Gronwall step contributes a factor $e^{CT'}$, which is finite for finite T'. Hence, if $T_{\max} \le T$, then

$$\operatorname*{ess\,sup}_{0<t<T_{\max}} \Big(\|\rho(t)\|_{H^2}^2 + \|(u,w,\omega_1,\tilde{\omega},b)(t)\|_{H^2}^2 + \big\|\big(\sqrt{\rho}\,\dot{u},\sqrt{\rho}\,\dot{w},\sqrt{\rho}\,\dot{\omega}_1,\sqrt{\rho}\,\dot{\tilde{\omega}}\big)(t)\big\|_{L^2}^2 + \|b_t(t)\|_{L^2}^2 \Big) \le K < \infty, \tag{52}$$

with K depending only on the data and on T.

Step 2 (compatibility at regular times).

There is a full-measure set $\mathcal{G} \subset (0,T_{\max})$ on which the spatial equations, the H^2 bounds and the weighted acceleration bounds all hold. Choose $t_0 \in \mathcal{G}$. By (52), $\rho(t_0) \in H^2(\Omega)$ and $(u,w,\omega_1,\tilde{\omega},b)(t_0) \in H_0^1(\Omega) \cap H^2(\Omega)$; evaluating (2), (3), (4) and (5) at t_0 gives

$$\begin{aligned}
&(2\mu+\lambda)u_{xx} - \Big(P(\rho) + \frac{1}{2}|b|^2\Big)_x = \rho\dot{u} = \sqrt{\rho}\,\tilde{g}_1,\\
&(\mu+\zeta)w_{xx} + 2\zeta J\tilde{\omega}_x + b_x = \rho\dot{w} = \sqrt{\rho}\,\tilde{g}_2,\\
&(2\mu'+\lambda')\omega_{1,xx} - 4\zeta\omega_1 = \rho\dot{\omega}_1 = \sqrt{\rho}\,\tilde{g}_3,\\
&\mu'\tilde{\omega}_{xx} + 2\zeta J w_x - 4\zeta\tilde{\omega} = \rho\dot{\tilde{\omega}} = \sqrt{\rho}\,\tilde{g}_4,
\end{aligned}$$

where $\tilde{g}_1,\dots,\tilde{g}_4$ denote $\sqrt{\rho}\,\dot{u}(t_0)$, $\sqrt{\rho}\,\dot{w}(t_0)$, $\sqrt{\rho}\,\dot{\omega}_1(t_0)$ and $\sqrt{\rho}\,\dot{\tilde{\omega}}(t_0)$, all of which belong to $L^2(\Omega)$ by (52). Thus the state at t_0 satisfies the hypotheses of Theorem 1, hence those of Proposition 1, with $\|(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3,\tilde{g}_4)\|_{L^2}^2 \le K$. The field b needs no compatibility condition.

Step 3 (restart and contradiction).

All the norms on which the local time of Proposition 1 depends are bounded by K, uniformly in t_0; hence there is a $\tau = \tau(K) > 0$, independent of $t_0 \in \mathcal{G}$, such that for every such t_0, the problem with initial time t_0 has a strong solution on $[t_0, t_0+\tau]$ in the class of Theorem 1. By uniqueness this coincides with the solution already constructed on their common interval, so the latter solution extends to $[0, t_0+\tau]$.

Choose $t_0 \in \mathcal{G}$ with $t_0 > T_{\max} - \tau/2$, which is possible since $\mathcal{G}$ has full measure. This yields a solution on $[0,t_0+\tau] \supset [0, T_{\max}+\tau/2]$, contradicting the definition of $T_{\max}$. Hence $T_{\max} > T$; since T was arbitrary, $T_{\max} = +\infty$, and in particular the solution exists on $[0,T]$ in the class of Theorem 1. Global uniqueness is established in Section 6. □

6. Uniqueness

We complete the proof of Theorem 1 by showing that two strong solutions with the same data coincide. This is the standard difference estimate of Cho–Kim [7] and Fan–Yu [11]; the feature that renders the vacuum harmless here is one-dimensional. Each solution has $u_t, w_t, \omega_{1,t}, \tilde{\omega}_t \in L^2(0,T;H^1(\Omega))$, so the material accelerations $\dot{u}, \dot{w}, \dot{\omega}_1, \dot{\tilde{\omega}}$ lie in $L^2(0,T;L^\infty(\Omega))$, since $H^1 \hookrightarrow L^\infty$ in one dimension, and the terms carrying the density difference become Gronwall terms.

Proof of the uniqueness part of Theorem 1. Let $(\rho_i,u_i,w_i,\omega_{1,i},\tilde{\omega}_i,b_i)$, $i=1,2$, be two solutions in the class of Theorem 1 on $[0,T]$ with the same initial data, and write $\bar{\rho} = \rho_1 - \rho_2$, $\bar{u} = u_1 - u_2$, and likewise $\bar{w}, \bar{\omega}_1, \bar{\tilde{\omega}}, \bar{b}$. By the regularity, all of $\rho_i, (\rho_i)_x, u_i, (u_i)_x, \dots, b_i, (b_i)_x$ belong to $L^\infty(\Omega\times(0,T))$ and $0 \le \rho_i \le \bar{R}$.

6.1. Density Difference

Subtracting the mass equations,

$$\partial_t \bar{\rho} + \partial_x(\rho_1 \bar{u}) + \partial_x(\bar{\rho}\, u_2) = 0. \tag{53}$$

Testing by $\bar{\rho}$ and using $u_2 = \bar{u} = 0$ at the endpoints gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \bar{\rho}^2\,dx = -\frac{1}{2}\int_\Omega (u_2)_x \bar{\rho}^2\,dx - \int_\Omega \big((\rho_1)_x \bar{u} + \rho_1 \bar{u}_x\big)\bar{\rho}\,dx.$$

By Hölder's, Poincaré's and Young's inequalities, the right-hand side is bounded by

$$\begin{aligned}&\frac{1}{2}\|(u_2)_x\|_{L^\infty}\|\bar{\rho}\|_{L^2}^2 + \big(\|(\rho_1)_x\|_{L^\infty}\|\bar{u}\|_{L^2} + \|\rho_1\|_{L^\infty}\|\bar{u}_x\|_{L^2}\big)\|\bar{\rho}\|_{L^2}\\ &\qquad \le C_\varepsilon \|\bar{\rho}\|_{L^2}^2 + \varepsilon \|\bar{u}_x\|_{L^2}^2.\end{aligned}$$

Thus

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \bar{\rho}^2\,dx \le C\int_\Omega \bar{\rho}^2\,dx + \varepsilon \int_\Omega \bar{u}_x^2\,dx, \tag{54}$$

with $\varepsilon > 0$ at our disposal.

6.2. Longitudinal Velocity

Subtracting (2) gives

$$\rho_1(\partial_t \bar{u} + u_1 \partial_x \bar{u}) + \bar{\rho}\,\dot{u}_2 + \rho_1 \bar{u}\,(u_2)_x + \big(P(\rho_1) - P(\rho_2)\big)_x + \frac{1}{2}\big(|b_1|^2 - |b_2|^2\big)_x = \bar{\mu}\,\bar{u}_{xx},$$

where $\dot{u}_2 = \partial_t u_2 + u_2(u_2)_x$. Testing by $\bar{u}$ and using

$$\begin{aligned}&\partial_t \rho_1 + \partial_x(\rho_1 u_1) = 0,\\ &\frac{d}{dt}\frac{1}{2}\int_\Omega \rho_1 \bar{u}^2\,dx + \bar{\mu}\int_\Omega \bar{u}_x^2\,dx\\ &= -\int_\Omega \bar{\rho}\,\dot{u}_2\,\bar{u}\,dx - \int_\Omega \rho_1 \bar{u}^2 (u_2)_x\,dx\\ &\qquad + \int_\Omega \big(P(\rho_1) - P(\rho_2)\big)\bar{u}_x\,dx + \frac{1}{2}\int_\Omega \big(|b_1|^2 - |b_2|^2\big)\bar{u}_x\,dx.\end{aligned}$$

The second term is bounded by $\|(u_2)_x\|_{L^\infty}\int_\Omega \rho_1 \bar{u}^2\,dx$; since $|P(\rho_1) - P(\rho_2)| \le C|\bar{\rho}|$ and $\big||b_1|^2 - |b_2|^2\big| \le C|\bar{b}|$ on the bounded ranges, the last two terms are bounded by

$$\varepsilon \int_\Omega \bar{u}_x^2\,dx + C_\varepsilon\Big(\int_\Omega \bar{\rho}^2\,dx + \int_\Omega |\bar{b}|^2\,dx\Big);$$

and, crucially,

$$\begin{aligned}\Big|\int_\Omega \bar{\rho}\,\dot{u}_2\,\bar{u}\,dx\Big| &\le \|\dot{u}_2\|_{L^\infty}\Big(\int_\Omega \bar{\rho}^2\,dx\Big)^{1/2}\Big(\int_\Omega \bar{u}^2\,dx\Big)^{1/2}\\ &\le \varepsilon \int_\Omega \bar{u}_x^2\,dx + C_\varepsilon \|\dot{u}_2\|_{L^\infty}^2 \int_\Omega \bar{\rho}^2\,dx,\end{aligned} \tag{55}$$

using Poincaré's inequality (as $\bar{u}|_{\partial\Omega} = 0$) and $\dot{u}_2 \in L^2(0,T;L^\infty(\Omega))$, so $\|\dot{u}_2\|_{L^\infty}^2 \in L^1(0,T)$. The micro-rotation differences $\bar{\omega}_1, \bar{\tilde{\omega}}$ are handled in the same way, upon testing the differences of (4) and (5) by $\bar{\omega}_1, \bar{\tilde{\omega}}$: the terms $\int_\Omega \bar{\rho}\,\dot{\omega}_{1,2}\,\bar{\omega}_1\,dx$ and $\int_\Omega \bar{\rho}\,\dot{\tilde{\omega}}_2 \cdot \bar{\tilde{\omega}}\,dx$ are Gronwall terms by the same bound, and the damping 4ζ only helps.

6.3. Magnetic Field and Couplings

Subtracting (6),

$$\partial_t \bar{b} + \partial_x(u_1 \bar{b}) + \partial_x(\bar{u}\, b_2) - \bar{w}_x = \nu\, \bar{b}_{xx};$$

testing by $\bar{b}$,

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega |\bar{b}|^2\,dx + \nu\int_\Omega |\bar{b}_x|^2\,dx\\ &\quad = -\frac{1}{2}\int_\Omega (u_1)_x|\bar{b}|^2\,dx - \int_\Omega (\bar{u}_x b_2 + \bar{u}\, b_{2,x})\cdot \bar{b}\,dx + \int_\Omega \bar{w}_x\cdot\bar{b}\,dx,\end{aligned}$$

where the first term is bounded by $C\int_\Omega |\bar{b}|^2\,dx$. Since $\|b_2\|_{L^\infty} + \|b_{2,x}\|_{L^\infty} \le C$, Poincaré's inequality bounds the second term by

$$\varepsilon\int_\Omega \bar{u}_x^2\,dx + C_\varepsilon\int_\Omega |\bar{b}|^2\,dx.$$

When the $\bar{w}, \bar{\tilde{\omega}}, \bar{b}$ equations are added, the inter-field couplings are exactly those of Lemma 1: the Alfvén pair cancels,

$$\int_\Omega \bar{b}_x\cdot\bar{w}\,dx + \int_\Omega \bar{w}_x\cdot\bar{b}\,dx = 0,$$

and the micro-rotation pair contributes $-4\zeta\int_\Omega (J\bar{\tilde{\omega}})\cdot\bar{w}_x\,dx$, absorbed by $(\mu+\zeta)\int_\Omega |\bar{w}_x|^2\,dx$ and the damping $4\zeta\int_\Omega |\bar{\tilde{\omega}}|^2\,dx$ through $|J\bar{\tilde{\omega}}| = |\bar{\tilde{\omega}}|$ and Young's inequality.

6.4. Closure

Consider the difference functional

$$\mathcal{D}(t) := \frac{1}{2}\int_\Omega \rho_1\big(\bar{u}^2 + |\bar{w}|^2 + \bar{\omega}_1^2 + |\bar{\tilde{\omega}}|^2\big)dx + \frac{1}{2}\int_\Omega |\bar{b}|^2\,dx + \frac{1}{2}\int_\Omega \bar{\rho}^2\,dx.$$

Adding (54) and the field estimates, and choosing ε small enough that all terms carrying ε are absorbed by the corresponding dissipation, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{D} + c_1\Big(&\int_\Omega \bar{u}_x^2\,dx + \int_\Omega |\bar{w}_x|^2\,dx + \int_\Omega \bar{\omega}_{1,x}^2\,dx\\ &+ \int_\Omega |\bar{\tilde{\omega}}_x|^2\,dx + \int_\Omega |\bar{b}_x|^2\,dx\Big)\\ &\le C\Big(1 + \|\dot{u}_2\|_{L^\infty}^2 + \|\dot{w}_2\|_{L^\infty}^2 + \|\dot{\omega}_{1,2}\|_{L^\infty}^2 + \|\dot{\tilde{\omega}}_2\|_{L^\infty}^2\Big)\,\mathcal{D},\end{aligned} \tag{56}$$

with $c_1 > 0$; the coefficient belongs to $L^1(0,T)$, as noted after (55). As $\mathcal{D}(0) = 0$, Gronwall's inequality forces $\mathcal{D} \equiv 0$ on $[0,T]$; then $\bar{\rho} \equiv 0$, and the dissipation in (56) together with the homogeneous boundary conditions gives

$$\bar{u} = \bar{w} = \bar{\omega}_1 = \bar{\tilde{\omega}} = \bar{b} \equiv 0.$$

The two solutions coincide.

Together with the existence proved in Sections 4 and 5, this completes the proof of Theorem 1. □

7. Conclusions

In this paper, we have established the global existence and uniqueness of strong solutions to the one-dimensional planar isentropic compressible magneto-micropolar equations with large initial data and vacuum; the density is allowed to vanish on an open subset of the interval. The results couple, in a single setting, the planar magnetohydrodynamic equations with vacuum and the one-dimensional compressible micropolar equations with vacuum; they also show that the magnetic and the micro-rotation couplings can be controlled simultaneously by the dissipation, for arbitrary positive viscosities and without any smallness assumption.

Several questions remain open. Because the pressure is isentropic, the velocity has been estimated only in H^2; whether higher regularity, or a classical solution, can be propagated in the presence of vacuum is not addressed here. The non-isentropic, heat-conducting system and the multidimensional problem are natural directions for further study.

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Appendix A. Derivation of the Planar System

System (1)–(6) is the one-dimensional planar reduction of the three-dimensional compressible magneto-micropolar equations. We recall the latter system in the form considered by Wu and Zhang [23]: for the density $\rho \geq 0$, the velocity $\mathbf{u}$, the micro-rotation $\mathbf{m}$, which is denoted w in that reference and renamed here to avoid a clash with the transverse velocity, and the magnetic field $\mathbf{H}$, all functions of $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}_+$,

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P \\ \qquad = (\mu + \zeta)\Delta \mathbf{u} + (\mu + \lambda - \zeta)\nabla(\nabla \cdot \mathbf{u}) + 2\zeta \nabla \times \mathbf{m} + (\nabla \times \mathbf{H}) \times \mathbf{H}, \\ (\rho \mathbf{m})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{m}) + 4\zeta \mathbf{m} \\ \qquad = \mu' \Delta \mathbf{m} + (\mu' + \lambda')\nabla(\nabla \cdot \mathbf{m}) + 2\zeta \nabla \times \mathbf{u}, \\ \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \qquad \nabla \cdot \mathbf{H} = 0, \end{cases} \tag{A1}$$

with $P = a\rho^\gamma$ and the coefficients as in (7).

Following the planar reduction introduced by Chen and Wang [4] for the magnetohydrodynamic equations, we look for solutions of (A1) that depend on the single space variable $x := x_1 \in \Omega$ and have the form

$$\mathbf{u} = (u,\, w_1,\, w_2), \qquad \mathbf{m} = (\omega_1,\, \tilde{\omega}_1,\, \tilde{\omega}_2), \qquad \mathbf{H} = (h,\, b_1,\, b_2), \tag{A2}$$

with u, ω_1, h scalar and $w = (w_1, w_2)$, $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2)$, $b = (b_1, b_2)$ all functions of (x, t). Since the fields are independent of x_2 and x_3, the constraint

$$\nabla \cdot \mathbf{H} = h_x = 0$$

forces h to be constant in x, and the first component of the induction equation gives $h_t = 0$; we consider the case of the constant longitudinal field $h \equiv 1$. Throughout, we use that, for a field $\mathbf{F} = (F_1, F_2, F_3)$ depending only on x,

$$\begin{aligned} \nabla \times \mathbf{F} &= (0,\, -\partial_x F_3,\, \partial_x F_2), \\ \nabla \cdot \mathbf{F} &= \partial_x F_1, \qquad \Delta \mathbf{F} = \partial_x^2 \mathbf{F}. \end{aligned} \tag{A3}$$

Appendix A.1. Magnetic Terms

By (A3) with $h \equiv 1$, we have

$$\nabla \times \mathbf{H} = (0, -b_{2,x}, b_{1,x}),$$

and hence the Lorentz force

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = \Big(-\frac{1}{2}\big(|b|^2\big)_x,\, b_{1,x},\, b_{2,x} \Big) = \Big(-\frac{1}{2}\big(|b|^2\big)_x,\, b_x \Big), \tag{A4}$$

whose longitudinal part supplies the magnetic pressure $(|b|^2)_x/2$ in (2) and whose transverse part supplies b_x in (3). For the induction equation,

$$\nabla \times (\mathbf{u} \times \mathbf{H}) = \big(0,\ -(ub_1)_x + w_{1,x},\ -(ub_2)_x + w_{2,x}\big),$$
$$-\nabla \times (\nu \nabla \times \mathbf{H}) = (0,\ \nu b_{1,xx},\ \nu b_{2,xx}),$$

so that its longitudinal component reduces to $h_t = 0$ and its transverse component to $b_t + (ub)_x - w_x = \nu b_{xx}$, which is (6).

Appendix A.2. Micropolar Couplings

Since $Jv = (-v_2, v_1)$, the transverse parts of the coupling terms are, by (A3),

$$\big(2\zeta\, \nabla \times \mathbf{m}\big)_{\mathrm{tr}} = 2\zeta(-\tilde{\omega}_{2,x}, \tilde{\omega}_{1,x}) = 2\zeta J\tilde{\omega}_x, \qquad \big(2\zeta\, \nabla \times \mathbf{u}\big)_{\mathrm{tr}} = 2\zeta(-w_{2,x}, w_{1,x}) = 2\zeta J w_x,$$

which are the coupling terms in (3) and (5); the longitudinal parts of $\nabla \times \mathbf{m}$ and $\nabla \times \mathbf{u}$ vanish, so (2) and (4) carry no such coupling.

Appendix A.3. Viscous Terms

As $\nabla \cdot \mathbf{u} = u_x$ and $\nabla \cdot \mathbf{m} = \omega_{1,x}$, we have

$$\nabla(\nabla \cdot \mathbf{u}) = (u_{xx}, 0, 0), \qquad \nabla(\nabla \cdot \mathbf{m}) = (\omega_{1,xx}, 0, 0);$$

together with $\Delta = \partial_x^2$, this yields $(2\mu + \lambda)u_{xx}$ and $(\mu + \zeta)w_{xx}$ for the longitudinal and transverse velocity, and $(2\mu' + \lambda')\omega_{1,xx}$ and $\mu'\tilde{\omega}_{xx}$ for the micro-rotation, as in (2)–(5).

Appendix A.4. The Planar System

Writing the convective terms in non-conservative form by means of the continuity equation and collecting the longitudinal and transverse components of (A1) under the ansatz (A2), one obtains exactly the planar system (1)–(6); the coefficient conditions (7) are inherited from the three-dimensional system.

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