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Homogenization of a Monotone Robin Problem in a Domain with a Thick Oscillating Boundary

Patrizia Donato [1,*] and Klas Pettersson [2]

[1] Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS, Univ Rouen Normandie, F-76000 Rouen, France
[2] Independent Researcher, SE-435 39 Mölnlycke, Sweden; klas.pettersson@gmail.com
* Correspondence: patrizia.donato@univ-rouen.fr

Abstract

In this paper, the asymptotics of an elliptic boundary value problem in two dimensions with a thick oscillating boundary Γ^ε are characterized, as the period of oscillations ε tends to zero. On Γ^ε, a nonlinear Robin boundary condition is prescribed, where the nonlinear function is monotone, regular, and of polynomial growth. The asymptotics depend on both the geometric oscillations and the rapidly oscillating coefficients of the differential operator. The methods of Minty for monotone operators, together with periodic unfolding adapted to the problem domain, introduced in a previous paper by the authors, are applied. The homogenized problem is described, as well as the convergence of the energies.

Keywords: homogenization; nonlinear Robin conditions; periodic unfolding for thick oscillating boundaries

MSC: 35J25

1. Introduction

In this paper, we study the homogenization, as $\varepsilon \to 0$, of an elliptic problem depending on a positive parameter ε, in a two-dimensional domain Q^ε consisting of a fixed part Q_0 and a part B^ε containing a thick oscillating boundary, that is such that the height of the oscillations remains fixed (see Figures 1 and 2a). We write the boundary of the domain as $\partial Q^\varepsilon = \Gamma^\varepsilon \cup \Gamma_D \cup \Gamma_N^\varepsilon$, where Γ_D is the lower Dirichlet part of the boundary and Γ_N^ε the lateral Neumann boundaries.

More precisely, we consider the following boundary value problem with a nonlinear Robin condition on the oscillating boundary Γ^ε, a Dirichlet condition on Γ_D and a homogeneous Neumann condition on Γ_N^ε:

$$\begin{cases} -\mathrm{div}\left(A^\varepsilon \nabla u\right) = f & \text{in } Q^\varepsilon, \\ A^\varepsilon \nabla u \cdot \nu + \varepsilon \rho^\varepsilon h(u) = \varepsilon \tau^\varepsilon & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u \cdot \nu = 0 & \text{on } \Gamma_N^\varepsilon, \\ u = 0 & \text{on } \Gamma_D, \end{cases}$$

where the outward unit normal to the boundary of Q^ε is denoted by ν. In the problem, we assume that $f \in L^2(Q)$, the elliptic and bounded coefficient matrix $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ is periodically oscillating, and the nonlinear function h is monotone, regular, and of polynomial growth.

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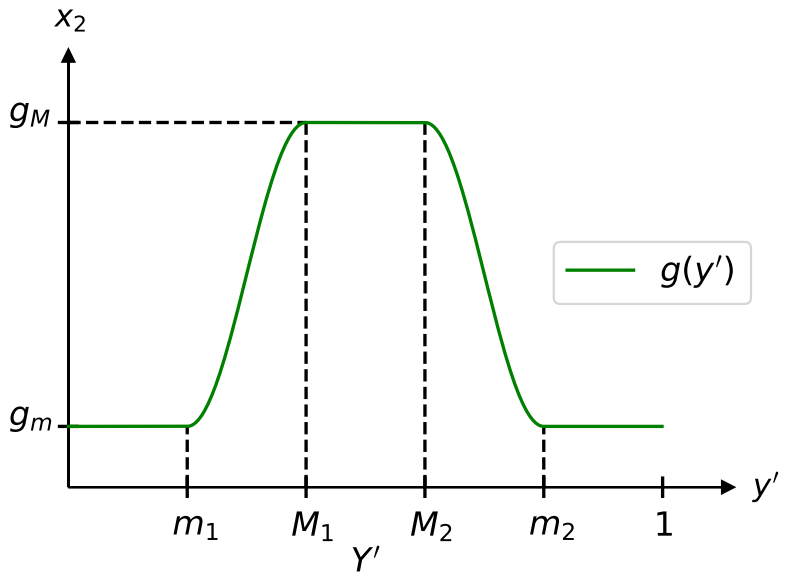


Figure 1. A period of the graph of g with the properties (4)–(9).

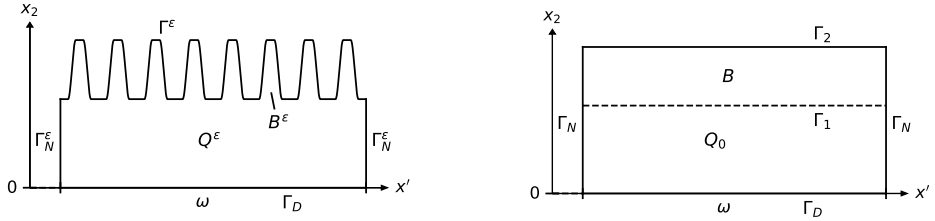


Figure 2. The domain Q^ε with cross-section ω and the oscillating boundary Γ^ε with $\varepsilon = 1/8$ (**a**) and the limiting domain Q with parts Q_0 and B (**b**).

The main difficulties in characterizing the asymptotic behavior of the solutions arise from the nonlinearity of the Robin boundary condition and the oscillations of the coefficient matrix in all spatial directions.

Indeed, the nature of the coefficients requires an adaptation of the unfolding method introduced in [1] to our domain. Moreover, the nonlinear condition, which is the main novelty of this paper, requires several specific arguments, including Minty's method, as explained below.

Nonlinear Robin conditions naturally appear in science (see e.g., [2]) as for instance in chemical reactions and models such as the Langmuir kinetics model (see e.g., [3]). For the homogenization in perforated domains with a nonlinear Robin condition, we refer for instance to [4].

The periodic unfolding method was introduced for homogenization in [5], subsequently developed in detail in [6], and is closely related to the method of two-scale convergence [7,8]. It is based on unfolding operators that map $x \in \mathbb{R}^N$ to higher dimensional $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ suitable for characterization of the asymptotics of the oscillations, where y belongs to a reference periodicity cell.

Asymptotics of the solutions to elliptic boundary value problems with oscillations only in the direction of the domain oscillations have been studied in [9] (see also [6]). Earlier studies can be found in [10,11]. The case of L^1 data was considered in [12]. Many contributions on thick oscillating interfaces may be found in [13,14] and the works citing these papers. In particular, a problem with nonlinear boundary conditions was studied in [15] for the Laplacian.

Here, in order to deal with the matrix field A^ε, we use the unfolding operators introduced in [1] by the authors for the homogenization of a domain with two components, separated by an oscillating interface, which has the same graph as the oscillating boundary considered in the present paper. We adapt here to our "one-sided" domain that some properties proved therein, in particular, a compactness result, which is essential.

Nevertheless, in this paper, we need several specific arguments, due to the difficulties related to the Robin nonlinear boundary condition. For that, we use a Minty's argument for monotone operators, which implies that we only need to pass to the limit in a boundary term containing a nonlinear factor of the form $h(\Phi^\varepsilon)$, where Φ^ε is a strongly convergent sequence of test functions. This approach avoids the need to pass directly to the limit in the boundary integral containing $h(u^\varepsilon)$, where u^ε is only weakly converging.

As usual in the unfolding method, we first obtain (see Theorem 4) an unfolded homogenized problem, which is posed in both variables x and y, then we derive in Theorem 3 the homogenized problem, posed in the variable $x \in Q$ only.

The homogenized problem, posed in the whole cylinder Q, is stated in Theorem 3. Its formal interpretation is given by

$$\begin{cases} -\mathrm{div}(A^0 \nabla u) = f \quad \text{in } Q_0 \\ -\dfrac{\partial}{\partial x_2}\left(a^0 \dfrac{\partial u}{\partial x_2}\right) + \rho^0 h(u) = \mathrm{meas}(Y'_{x_2}) f + \tau^0 \quad \text{in } B \\ A^0 \nabla u \cdot \nu - a^0 \dfrac{\partial u}{\partial x_2} = 0 \quad \text{on } \Gamma_1 \\ [u] = 0 \quad \text{on } \Gamma_1 \\ A^0 \nabla u \cdot \nu = 0 \quad \text{on } \Gamma_N \cap \overline{Q_0} \\ a^0 \dfrac{\partial u}{\partial x_2} = 0 \quad \text{on } \Gamma_2 \\ u = 0 \quad \text{on } \Gamma_D \end{cases} \tag{1}$$

where B is the fixed domain containing all the oscillations, Γ_1 is the interface between Q_0 and B, $[\cdot]$ denotes the jump of a function through Γ_1, and ν is the unit outward normal to the boundary of Q_0. Throughout the paper, $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure. Observe that, in B, we also have a partial differential equation, but where the derivatives are only with respect to x_2. Then, in B, the homogenized matrix reduces to a matrix with only one nonzero entry a^0. On Γ_1, there are the usual transmission conditions.

In Section 2, the setting of the problem is made precise. In Section 3, the tools from unfolding are included. In Section 4, the natural functional spaces for the problem are described and the weak compactness, and in Section 5, a preliminary result (Lemma 1) is proved, which enables passage to the limit in the boundary terms in the proof of Theorem 4. The main homogenization result is stated in Section 6 and proved in the subsequent Section 7. Finally, in Section 8, we establish the convergence of the energies.

2. Problem Statement

The construction of the oscillating domain and the precise hypotheses in the boundary value problem are presented in this section, as well as the well posedness of the problem.

Let us mention that this geometry correspond to a one-sided part of the two-component domain considered by the authors in [1], where the homogenization of a problem with a jump condition on the interface was studied.

We denote

$$Q = \omega \times (0, g_M) \subset \mathbb{R}^2,$$

with $\omega \subset \mathbb{R}$ nonempty, open, and bounded. In Q, for each ε, we will fix a subdomain Q^ε that has a thick periodically oscillating boundary Γ^ε with characteristic period ε, as illustrated in Figure 2a.

The domain Q may be viewed as a cylinder with cross-section ω and longitudinal direction pointed out by $(0, g_m)$, allowing one to view the problem as higher dimensional by considering $\omega \subset \mathbb{R}^{N-1}$. For this reason, we write $x \in \overline{Q}$ as $x = (x', x_2)$, with $x' \in \overline{\omega}$ and $x_2 \in [0, g_M]$.

To specify the boundary value problem, we partition the boundary of the domain as follows:

$$\partial Q^\varepsilon = \Gamma^\varepsilon \cup \Gamma_D \cup \Gamma_N^\varepsilon, \tag{2}$$

where Γ_D is the base of the cylinder Q and Γ_N^ε is the lateral boundary. That is, we set

$$\Gamma_D = \omega \times \{0\}, \qquad \Gamma_N^\varepsilon = \partial Q^\varepsilon \setminus (\Gamma_D \cup \Gamma^\varepsilon). \tag{3}$$

For the purpose of periodic unfolding, we denote periodicity cells by

$$Y = (0,1)^2, \qquad Y' = (0,1),$$

where analogously to $x \in \mathbb{R}^2$, we write $y \in \overline{Y}$ as $y = (y', y_2)$, where $y' \in \overline{Y'}$, $y_2 \in [0,1]$.

We prepare for constructing the boundary part Γ^ε. We assume $g : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$g \text{ is 1-periodic and Lipschitz continuous.} \tag{4}$$

The graph of g over one period, is denoted by

$$\Gamma = \{ (y', g(y')) : y' \in Y' \}, \tag{5}$$

and the minimum and maximum levels by

$$g_m = \inf \{ g(y') : y' \in Y' \}, \qquad g_M = \sup \{ g(y') : y' \in Y' \}. \tag{6}$$

Furthermore, we restrict g as follows:

$$\begin{cases} \text{There are } m_1, m_2, M_1, M_2 \in Y', \quad m_1 < M_1 < M_2 < m_2 \text{ such that} \\ g \text{ is strictly monotone on } (m_1, M_1) \text{ and on } (M_2, m_2). \end{cases} \tag{7}$$

and

$$g(y') = g_m \quad \text{if } y' \in (0, m_1) \cup (m_2, 1), \qquad g(y') = g_M \quad \text{if } y' \in (M_1, M_2). \tag{8}$$

The levels are assumed to be separated:

$$0 < g_m < g_M. \tag{9}$$

An illustration is shown in Figure 1.

Properties (4)–(9) are important for the asymptotics of the problem and serve to simplify the characterization while providing sufficient structure in the limit. The positive measure of the flat parts at g_m, g_M is one example and that the levels are separated is another.

We define Γ^ε and the problem domain Q^ε. Given $\varepsilon > 0$, we set

$$\Gamma^\varepsilon = \left\{ \left(x', g\left(\frac{x'}{\varepsilon}\right) \right) : x' \in \omega \right\}. \tag{10}$$

and

$$Q^\varepsilon \doteq \left\{ (x', x_2) \in Q : 0 < x_2 < g\left(\frac{x'}{\varepsilon}\right) \right\}. \tag{11}$$

The domain Q^ε has, as the upper boundary, the oscillating curve Γ^ε and is denoted by

$$B^\varepsilon = \left\{(x', x_2) \in Q^\varepsilon : x_2 > g_m\right\}$$

the subset of Q^ε containing the oscillating boundary.

We note that the one-dimensional measure of Γ^ε scales as $O(1/\varepsilon)$ and that B^ε is contained in the fixed size subdomain B on Q given by

$$B = Q \cap \{(x', x_2) : g_m < x_2 < g_M\}.$$

Then, we have a partition of Q into the two (limit) regions Q_0, and B, setting

$$Q_0 = Q \setminus \overline{B} = Q \cap \{(x', x_2) : x_2 < g_m\}, \qquad \Gamma_1 = Q \cap \{(x', x_2) : x_2 = g_m\},$$

where Γ_1 denotes the interfaces between them. Hence we have the following decompositions:

$$Q^\varepsilon = Q_0 \cup \Gamma_1 \cup B^\varepsilon, \qquad Q = Q_0 \cup \Gamma_1 \cup B.$$

The domain Q^ε together with its decomposition is illustrated in Figure 2a, while the limit partition of the domain Q is illustrated in Figure 2b.

We specify the boundary value problem in the domain Q^ε with the nonlinear Robin condition on Γ^ε, homogeneous Dirichlet condition on Γ_D, and homogeneous Neumann condition on Γ_N^ε:

$$\begin{cases} -\mathrm{div}\left(A^\varepsilon \nabla u\right) = f & \text{in } Q^\varepsilon, \\ A^\varepsilon \nabla u \cdot \nu + \varepsilon \rho^\varepsilon h(u) = \varepsilon \tau^\varepsilon & \text{on } \Gamma^\varepsilon, \\ A^\varepsilon \nabla u \cdot \nu = 0 & \text{on } \Gamma_N^\varepsilon, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{12}$$

where $f \in L^2(Q)$ and ν denotes the unit outward normal to the boundary of the domain Q^ε.

We make the following hypotheses:

Hypothesis 1 (H_1)**.** *A is a matrix field such that*

$$\begin{cases} i. \quad A \in L^\infty(\mathbb{R}^2), \text{ and is } Y\text{-periodic}, \\ ii. \quad A(x)\lambda \cdot \lambda \geq \alpha |\lambda|^2, \quad \lambda \in \mathbb{R}^2, \text{ a.e. } x \in \mathbb{R}^2, \end{cases} \tag{13}$$

for some constant $\alpha > 0$, and

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \quad \text{a.e. } x \in Q. \tag{14}$$

Hypothesis 2 (H_2)**.** *The function $\rho : \Gamma \to \mathbb{R}$ is such that*

i. $\rho \geq 0$,

ii. $y' \mapsto \rho(y', g(y'))$ *is a Y'-periodic function in $L^\infty(Y')$.*

We set

$$\rho^\varepsilon(x) = \rho\left(\frac{x'}{\varepsilon}, g\left(\frac{x'}{\varepsilon}\right)\right), \quad \text{a.e. } x \in \Gamma^\varepsilon. \tag{15}$$

Hypothesis 3 (H_3)**.** *The function $h : \mathbb{R} \to \mathbb{R}$ is an increasing C^1 continuous function, that is*

$$(h(t) - h(s))(t - s) \geq 0 \quad \text{for } t, s \in \mathbb{R}, \tag{16}$$

verifying

$$h(0) = 0. \tag{17}$$

We also suppose that h has a polynomial growth, that is for some positive constant c_h and an exponent $q \geq 1$, one has

$$|h(t)| \leq c_h(1 + |t|^q), \quad \text{for } t \in \mathbb{R}. \tag{18}$$

Hypothesis 4 (H_4)**.** *The function* $\tau : \Gamma \to \mathbb{R}$ *is such that*

$$T : y' \mapsto \tau(y', g(y')) \text{ is a } Y'\text{-periodic function in } L^2(Y').$$

We set

$$\tau_\varepsilon(x) = \tau\Big(\frac{x'}{\varepsilon}, g\Big(\frac{x'}{\varepsilon}\Big)\Big), \quad \text{a.e. } x \in \Gamma^\varepsilon. \tag{19}$$

Remark 1. *It is easy to check that the assumption* $H_2(ii)$ *implies that* ρ^ε *belongs to* $L^\infty(\Gamma^\varepsilon)$*. In addition, assumption* H_4 *implies that* τ^ε *is in* $L^2(\Gamma^\varepsilon)$*.*

Our goal is to characterize the asymptotics of the solutions u^ε to the problem (12) in the limit as ε tends to zero, under the hypotheses listed above.

Let us denote the Sobolev space V^ε of the $H^1(Q^\varepsilon)$ functions with zero trace on the base Γ_D, that is

$$V^\varepsilon = \{v \in H^1(Q^\varepsilon) : v = 0 \text{ on } \Gamma_D\},$$

equipped with the norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(Q^\varepsilon)}, \tag{20}$$

justified by the following Poincaré inequality.

Proposition 1 (Remark 2.1 in [1])**.** *The following estimate holds uniformly in* ε*:*

$$\|v\|_{L^2(Q^\varepsilon)} \le C\Big\|\frac{\partial v}{\partial x_2}\Big\|_{L^2(Q^\varepsilon)}, \quad v \in V^\varepsilon. \tag{21}$$

Remark 2. *By Lemma 2.2 in [1], there exists a constant* C *that is independent of* ε *such that*

$$\sqrt{\varepsilon}\|v\|_{L^2(\Gamma^\varepsilon)} \le C\,\|\nabla v\|_{L^2(Q^\varepsilon)}, \quad v \in V^\varepsilon, \tag{22}$$

for all small enough ε*. This tells us* ε*, in the boundary condition of problem* (12)*, is a natural scaling.*

Remark 3. *Following some arguments from [4], which make use of the Sobolev inclusions in dimension 2 and classical results ([16], see also ([17], Proposition 26.6)), thanks to assumption* (18) *on* h*, we deduce that, for fixed* ε*,*

$$u \in V^\varepsilon \to h(u) \in L^p(\Gamma^\varepsilon), \text{ is continuous for every } p \in [1, \infty). \tag{23}$$

In particular,

$$u, v \in V^\varepsilon \implies h(u)v \in L^1(\Gamma^\varepsilon). \tag{24}$$

This gives a sense to the variational formulation (25) *below.*

The variational formulation of problem (12) is the following:

$$\begin{cases} \text{Find } u^\varepsilon \in V^\varepsilon \text{ such that} \\ \displaystyle\int_{Q^\varepsilon} A^\varepsilon \nabla u^\varepsilon \cdot \nabla v\,dx + \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(u^\varepsilon) v\,ds = \int_{Q^\varepsilon} f v\,dx + \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon(s) v\,ds, \\ \text{for every } v \in V^\varepsilon. \end{cases} \tag{25}$$

Theorem 1. *Under the assumptions* H_1*–*H_4*, there exists a unique solution* $u^\varepsilon \in V^\varepsilon$ *to* (25)*.*

Proof. Let us prove first the existence of a solution. For any $\varepsilon > 0$, we define the operator

$$\mathcal{A}^\varepsilon : u \in V^\varepsilon \mapsto \mathcal{A}^\varepsilon(u) \in (V^\varepsilon)',$$

where

$$\langle \mathcal{A}^\varepsilon(u), v\rangle_{(V^\varepsilon)',V^\varepsilon} = \int_{Q^\varepsilon} A^\varepsilon \nabla u \cdot \nabla v \, dx + \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(u) v \, ds, \qquad v \in V^\varepsilon. \tag{26}$$

Remark that the right-hand side in (26) is well defined, and $\mathcal{A}^\varepsilon(u) \in (V^\varepsilon)'$ due to assumptions H_1–H_3 and from (23) from Remark 3. Moreover, from H_4, the Cauchy-Schwarz inequality, and the trace theorem, the linear map

$$L^\varepsilon : v \in V^\varepsilon \mapsto \int_{Q^\varepsilon} f v \, dx + \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon v \, ds \tag{27}$$

is bounded, and so an element of $(V^\varepsilon)'$.

We prove the existence of u such that $\mathcal{A}^\varepsilon(u) = L^\varepsilon$. To do that, we show that $\mathcal{A}^\varepsilon$ is surjective, verifying that the nonlinear operator $\mathcal{A}^\varepsilon : V^\varepsilon \to (V^\varepsilon)'$ satisfies the assumptions of the Minty–Browder Theorem, namely, it is monotone, coercive, and continuous.

Let $u, u_1 \in V^\varepsilon$. Then from the assumptions,

$$\begin{aligned}&\langle \mathcal{A}^\varepsilon(u) - \mathcal{A}^\varepsilon(u_1), u - u_1\rangle_{(V^\varepsilon)',V^\varepsilon}\\ &\quad = \int_{Q^\varepsilon} A^\varepsilon(\nabla u - \nabla u_1)\cdot(\nabla u - \nabla u_1)\, dx + \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon (h(u) - h(u_1))(u - u_1)\, ds \geq 0,\end{aligned}$$

so that $\mathcal{A}^\varepsilon$ is monotone. Actually, $\mathcal{A}^\varepsilon$ is strictly monotone, since $\langle \mathcal{A}^\varepsilon(u) - \mathcal{A}^\varepsilon(u_1), u - u_1\rangle_{(V^\varepsilon)',V^\varepsilon} = 0$ implies $\nabla u - \nabla u_1 = 0$. Then by (20), one gets $u = u_1$.

Let us prove that $\mathcal{A}^\varepsilon$ is coercive, that is

$$\lim_{\|u\|_{V^\varepsilon}\to+\infty} \frac{\langle \mathcal{A}^\varepsilon(u), u\rangle_{(V^\varepsilon)',V^\varepsilon}}{\|u\|_{V^\varepsilon}} = +\infty.$$

This follows from the assumption

$$\langle \mathcal{A}^\varepsilon(u), u\rangle_{(V^\varepsilon)',V^\varepsilon} \geq \alpha \|u\|^2_{V^\varepsilon},$$

with $\alpha > 0$. Finally, since A is linear, to prove the continuity of $\mathcal{A}^\varepsilon$, it is sufficient to prove the continuity of the map $F : u \in V^\varepsilon \mapsto F(u) \in (V^\varepsilon)'$, where

$$F(u) : v \in V^\varepsilon \mapsto \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(u) v \, ds,$$

which follows from (23) and ends the proof of existence.

The uniqueness follows from the strict monotonicity proved above. □

3. Periodic Unfolding Adapted to the Domain Q^ε

In [1], we introduced suitable operators for a domain with two components that were denoted Q_1^ε and Q_2^ε, separated by an interface (the same Γ^ε as here), and proved some related properties, including a weak compactness result.

In this section, we adapt these unfolding operators to the the domain of our problem; we also state (without proof), in this section, some results previously obtained. The main difference here is that we consider only one component Q^ε, which corresponds to the component Q_1^ε considered in [1].

A general exposition of the periodic unfolding method is available in [6].

To keep track of the domain density in B^ε, we will use the auxiliary sets

$$Y'_{x_2} = \{y' \in Y' : x_2 < g(y')\}, \qquad x_2 \in [g_m, g_M].$$

Note that Y'_{x_2} may be empty. Similarly, the boundary periodicity cell we denote by

$$\Gamma_{x_2} = \{y' \in Y' : x_2 = g(y')\} = g^{-1}\{x_2\} \cap Y', \tag{28}$$

that is the level set of x_2, which by assumption (7), for every $x_2 \in (g_m, g_M)$ consists of two points, as illustrated in Figure 3.

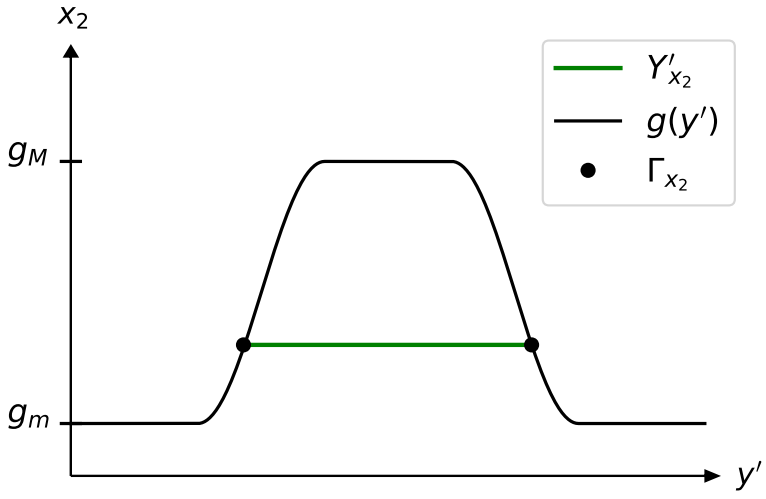


Figure 3. A period of the up-scaled boundary with indicated cross-section cells Y'_{x_2}.

In our analysis, we will use Γ_{x_2} in the oscillating band (g_m, g_M). We comment that, by construction, Γ_{x_2} consists of one interval at level g_M and two intervals at level g_m. It is empty for all $x_2 < g_m$ by assumption (9) and empty for all $x_2 > g_M$.

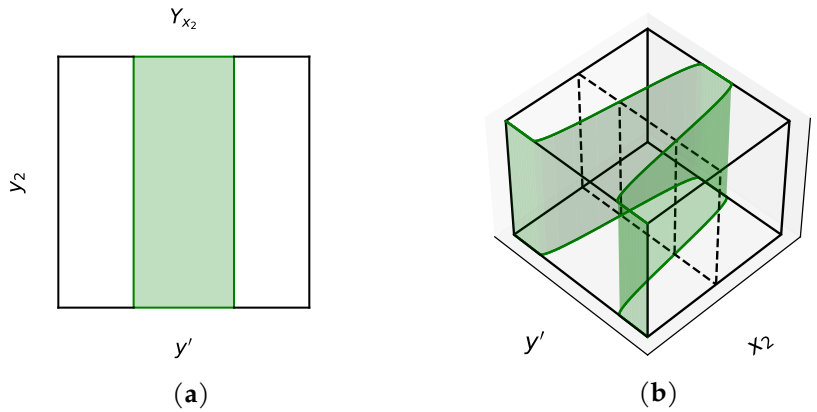
Observe that by construction, Y'_{x_2} is connected, and for some constant C,

$$0 < C \le \mathrm{meas}(Y'_{x_2}), \qquad x_2 \in (g_m, g_M). \tag{29}$$

To cover both dimensions, we set

$$Y_{x_2} = Y'_{x_2} \times (0,1) = \{(y', y_2) \in Y : x_2 < g(y')\}, \qquad x_2 \in [g_m, g_M], \tag{30}$$

and an illustration is shown in Figure 4.



(**a**) (**b**)

Figure 4. The periodicity cell Y_{x_2} (**a**) and an illustration of its dependence on the parameter x_2 in (**b**) where it is viewed as the section at x_2 (dashed) under the graph of g (cf. Figure 3).

By definition, for any $v \in L^1(Y)$, we have

$$\int_{Y_{x_2}} v(y)\, dy = \int_0^1 \int_{Y'_{x_2}} v(y', y_2)\, dy'\, dy_2. \tag{31}$$

We turn to periodic unfolding of $x \in \mathbb{R}^2$. The unique integer part $[x] \in \mathbb{Z}^2$ is defined for a.e. $x \in \mathbb{R}^2$ by

$$x - [x] \in Y = Y' \times (0,1) = (0,1)^2.$$

The corresponding fractional part $\{x\} \in [0,1)^2$ of x is then given by

$$\{x\} = x - [x] \in Y, \quad \text{a.e. } x \in \mathbb{R}^2,$$

and for every $\varepsilon > 0$,

$$x = \varepsilon\Big(\Big[\frac{x}{\varepsilon}\Big] + \Big\{\frac{x}{\varepsilon}\Big\}\Big), \quad \text{a.e. } x \in \mathbb{R}^2.$$

We denote

- $\Xi^\varepsilon_{Q_0} = \{\zeta \in \mathbb{Z}^2 : \varepsilon(\zeta + Y) \subset Q_0\}$, $\quad \Xi^\varepsilon_B = \{\zeta \in \mathbb{Z}^2 : \varepsilon(\zeta + Y) \subset B\}$;
- $\hat{Q}^\varepsilon_0 = \text{interior}\Big(\bigcup_{\zeta \in \Xi^\varepsilon_{Q_0}} \varepsilon(\zeta + \overline{Y})\Big)$, $\quad \hat{B}^\varepsilon = \text{interior}\Big(\bigcup_{\zeta \in \Xi^\varepsilon_B} \varepsilon(\zeta + \overline{Y})\Big)$;
- $\Lambda^\varepsilon_{Q_0} = Q_0 \setminus \hat{Q}^\varepsilon_0$, $\quad \Lambda^\varepsilon_B = B \setminus \hat{B}^\varepsilon$, $\quad \hat{B}^\varepsilon = B^\varepsilon \setminus \Lambda^\varepsilon_B$; (32)
- $D^{*\varepsilon} = \Big\{(x,y) \in \hat{B}^\varepsilon \times Y : g_m < \varepsilon\Big[\frac{x_2}{\varepsilon}\Big] + \varepsilon y_2 < g(y')\Big\}$;
- $D^* = \{(x,y) \in B \times Y : y \in Y_{x_2}\}$ (the limit set).

By definition, for every $v \in L^1(D^*)$, we have

$$\int_{D^*} v(x,y)\,dxdy = \int_B \int_{Y_{x_2}} v(x,y)\,dy\,dx. \tag{33}$$

We now adapt the definition of the unfolding operators introduced in [1], as well as some properties proved therein.

Definition 1. *The unfolding $T^\varepsilon v$ of a measurable function v on Q^ε is the function on $Q \times Y$ defined by*

$$T^\varepsilon v(x,y) = \begin{cases} v\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon y\Big) & \text{a.e. } (x,y) \in \big(\hat{Q}^\varepsilon_0 \times Y\big) \cup D^{*\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Gradients transform according to

$$\nabla_y T^\varepsilon v = \varepsilon T^\varepsilon \nabla v. \tag{34}$$

Since on a fixed domain T^ε reduces to the usual unfolding operator, then if $v \in L^1(Q_0)$,

$$\int_{Q_0} v\,dx - \int_{\Lambda^\varepsilon_{Q_0}} v\,dx = \int_{Q_0 \times Y} T^\varepsilon v\,dxdy. \tag{35}$$

Proposition 2. *The following properties hold:*

(i) For $v \in L^1(B^\varepsilon)$, one has

$$\int_{B^\varepsilon} v\,dx - \int_{\Lambda^\varepsilon_B \cap B^\varepsilon} v\,dx = \int_{\hat{B}^\varepsilon} v\,dx$$
$$= \int_{D^{*\varepsilon}} T^\varepsilon v\,dxdy = \int_{D^*} T^\varepsilon v\,dxdy + \int_{D^{*\varepsilon} \setminus D^*} T^\varepsilon v\,dxdy.$$

(ii) Let $\{v^\varepsilon\}$ *be a sequence of functions such that* $v^\varepsilon \in L^1(B^\varepsilon)$ *satisfying*

$$\lim_{\varepsilon\to 0}\int_{\Lambda_B^\varepsilon \cap B^\varepsilon} |v^\varepsilon|\,dx = 0, \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_{D^{*\varepsilon}\setminus D^*} T^\varepsilon|v^\varepsilon|\,dxdy = 0. \tag{36}$$

Then, as ε *tends to zero,*

$$\int_{B^\varepsilon} v^\varepsilon\,dx = \int_{D^*} T^\varepsilon v^\varepsilon\,dxdy + o(1) = \int_B \int_{Y_{x_2}} T^\varepsilon v^\varepsilon\,dx\,dy + o(1). \tag{37}$$

If (37) holds, we sometimes write

$$\int_{B^\varepsilon} v^\varepsilon\,dx \simeq \int_{D^*} T^\varepsilon v^\varepsilon\,dxdy. \tag{38}$$

We will use the following variant of Proposition 3.8 in [1].

Proposition 3. *Suppose that* $\{v^\varepsilon\}$ *and* $\{w^\varepsilon\}$ *are sequences in* $L^p(B^\varepsilon)$ *and* $L^q(B^\varepsilon)$*, respectively, such that* $\|v^\varepsilon\|_{L^p(B^\varepsilon)} + \|w^\varepsilon\|_{L^q(B^\varepsilon)} \le C$*, for some* $0 < 1/p + 1/q \le 1$*. Then*

$$\int_{D^*} T^\varepsilon(v^\varepsilon w^\varepsilon)\,dxdy = \int_{D^*} T^\varepsilon v^\varepsilon\, T^\varepsilon w^\varepsilon\,dxdy.$$

If moreover $1/p + 1/q < 1$*, then*

$$\int_{B^\varepsilon} v^\varepsilon w^\varepsilon\,dx \simeq \int_{D^*} T^\varepsilon v^\varepsilon\, T^\varepsilon w^\varepsilon\,dxdy.$$

For fixed $w \in L^q(B)$ *such that* $1/p + 1/q \le 1$*,*

$$\int_{B^\varepsilon} v^\varepsilon w\,dx \simeq \int_{D^*} T^\varepsilon v^\varepsilon\, T^\varepsilon w\,dxdy. \tag{39}$$

We define the partial unfolding in Q^ε, motivated by the fact that its ε-oscillations are not in the x_2 direction.

Let

$$\Xi_\omega^\varepsilon = \{\zeta \in \mathbb{Z}^1 : \varepsilon(\zeta + Y') \subset \omega\}, \qquad \hat{\omega}^\varepsilon = \text{interior}\Big(\bigcup_{\zeta\in\Xi_\omega^\varepsilon} \varepsilon(\zeta + \overline{Y'})\Big), \qquad \Lambda_\omega^\varepsilon = \omega \setminus \hat{\omega}^\varepsilon, \tag{40}$$

$$D'^*_\varepsilon = \{(x,y') \in \hat{\omega}^\varepsilon \times (g_m, g_M) \times Y' : y' \in Y'_{x_2}\},$$

$$D'^* = \{(x,y') \in B \times Y' : y' \in Y'_{x_2}\}, \quad \text{(the limit set of } D'^*_\varepsilon\text{)} \tag{41}$$

$$\Lambda_{\Gamma^\varepsilon}^\varepsilon = \Gamma^\varepsilon \cap (\Lambda_\omega^\varepsilon \times (g_m, g_M)), \tag{42}$$

where $\text{meas}(\Lambda_{\Gamma^\varepsilon}^\varepsilon) = O(1)$.

By definition, for every $v \in L^1(D'^*)$, we have

$$\int_{D'^*} v(x,y')\,dxdy' = \int_B \int_{Y'_{x_2}} v(x,y')\,dy'\,dx. \tag{43}$$

Definition 2. *The partial unfolding* $T'^\varepsilon v$ *of a measurable function* v *on* Q^ε *is the function on* $Q \times Y'$ *defined by*

$$T'^\varepsilon v(x,y') = \begin{cases} v\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big] + \varepsilon y', x_2\Big) & \text{a.e. } (x,y') \in D'^*_\varepsilon, \\ 0 & \text{elsewhere.} \end{cases}$$

Remark 4. *We note that*

$$T^\varepsilon f^\varepsilon(x,y) = f(y), \quad \text{a.e. } (x,y) \in (\hat{Q}_0^\varepsilon \times Y) \cup D^{*\varepsilon}.$$

with

$$f^\varepsilon(x) = f\Big(\frac{x}{\varepsilon}\Big) = f\Big(\Big\{\frac{x}{\varepsilon}\Big\}\Big), \quad \text{a.e. } x \in \mathbb{R}^2, \tag{44}$$

for any periodic function f obtained by periodically extending a measurable function on Y. Because Q_0 and B are bounded in $\mathbb{R}^2$, we have for any $f \in L^p(Q)$, $1 \le p < \infty$ that

$$T^\varepsilon f \to f \quad \text{strongly in } L^p((Q_0 \times Y) \cup D^*), \qquad \text{as } \varepsilon \to 0.$$

Similarly, if $\zeta \in L^2(\Gamma)$ is such that $y' \mapsto \zeta(y', g(y'))$ is Y'-periodic, we set

$$\zeta^\varepsilon(x) = \zeta\Big(\frac{x'}{\varepsilon}, g\Big(\frac{x'}{\varepsilon}\Big)\Big), \quad \text{a.e. } x \in \Gamma^\varepsilon,$$

then

$$T'^\varepsilon \zeta^\varepsilon(x, y') = T'^\varepsilon \zeta^\varepsilon(x', y') = \zeta(y', g(y')), \quad \text{a.e. } (x', y') \in \hat{\omega}^\varepsilon \times Y'.$$

Proposition 4. *The following properties hold:*

(i) For $v \in L^1(B^\varepsilon)$, one has

$$\int_{B^\varepsilon} v\,dx - \int_{\Lambda^\varepsilon_\omega \times (g_m, g_M)} v\,dx = \int_{B^\varepsilon \setminus (\Lambda^\varepsilon_\omega \times (g_m, g_M))} v\,dx = \int_{D'^*} T'^\varepsilon v\,dxdy'.$$

(ii) Let $\{v^\varepsilon\}$ be a sequence of functions such that $v^\varepsilon \in L^1(B^\varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} \int_{\Lambda^\varepsilon_\omega \times (g_m, g_M)} |v^\varepsilon|\,dx = 0. \tag{45}$$

Then, as ε tends to zero,

$$\int_{B^\varepsilon} v^\varepsilon\,dx - \int_{D'^*} T'^\varepsilon v^\varepsilon\,dxdy' \to 0. \tag{46}$$

4. Function Spaces and Weak Compactness

The compactness principle presented in this section (Theorem 2) is the main tool in our proof of the homogenized problem (Theorem 4). We first describe the main functional spaces that will be used (cf. [1]).

Definition 3. *Denote by $C^\infty_{\text{per}}(Y)$ (resp. $C^\infty_{\text{per},2}(Y)$) the subspace of $C^\infty(\mathbb{R}^2)$ of functions that are Y-periodic (resp. 1-periodic in y_2). For any open set F in $\overline{Y}$, we introduce the space $H^1_{\text{per}}(F)$ (resp. $H^1_{\text{per},2}(F)$) as the completion of $C^\infty_{\text{per}}(Y)$ (resp. $C^\infty_{\text{per},2}(Y)$) with respect to $H^1(F)$-norm*

$$\|v\|^2_{H^1_{\text{per}}(F)} = \int_F (|v|^2 + |\nabla v|^2)\,dy.$$

Definition 4.

1. *Let D^* and D'^* be defined by (32) and (41), respectively. Set*

$$Z = \{v = v(x, y) \in L^2(D^*) : \nabla_y v \in L^2(D^*)\}, \tag{47}$$

equipped with the norm

$$\|v\|^2_Z = \int_{D^*} \left(|v|^2 + |\nabla_y v|^2\right) dxdy,$$

and

$$Z' = \Big\{v = v(x, y') \in L^2(D'^*) : \frac{\partial v}{\partial x_2} \in L^2(D'^*),\ \frac{\partial v}{\partial y'} \in L^2(D'^*)\Big\}, \tag{48}$$

equipped with the norm

$$\|v\|^2_{Z'} = \int_{D'^*} \Big(v^2 + \Big(\frac{\partial v}{\partial x_2}\Big)^2 + \Big(\frac{\partial v}{\partial y'}\Big)^2\Big)\,dxdy'.$$

We also introduce the following subspace of Z of functions periodic in x_2:

$$Z_{\text{per},2} = \{v = v(x, y) \in Z : v(x, \cdot) \in H^1_{\text{per},2}(Y_{x_2}) \text{ a.e. } x \in B\}. \tag{49}$$

2. *Also set*

$$W = \Big\{ v = v(x,y) \in L^2(Q \times Y) : v_{|Q_0 \times Y} \in L^2(Q_0; H^1_{\text{per}}(Y)),\ v_{|D^*} \in Z_{\text{per},2},$$
$$\int_Y v(x,y)\,dy = 0 \ \text{a.e. } x \in Q_0,\ \int_{Y_{x_2}} v(x,y)\,dy = 0 \ \text{a.e. } x \in B \Big\}, \tag{50}$$

where the Y_{x_2} *integral is given by* (30) *and* (31)*, equipped with the norm*

$$\|v\|^2_W = \|v\|^2_{L^2(Q_0;H^1(Y))} + \|v\|^2_Z.$$

3. *Set*

$$V = \Big\{ v = v(x) \in L^2(Q) : \nabla v \in L^2(Q_0),\ \frac{\partial v}{\partial x_2} \in L^2(Q) \Big\}, \tag{51}$$

equipped with the norm

$$\|v\|^2_V = \|v\|^2_{L^2(Q)} + \|\nabla v\|^2_{L^2(Q_0)} + \Big\|\frac{\partial v}{\partial x_2}\Big\|^2_{L^2(B)},$$

and

$$V_0 = \{ v \in V :\ v = 0 \ on\ \Gamma_D \} \tag{52}$$

The weak compactness result below is a direct consequence of that proved in ([1], [Theorem 4.5]) and is an essential tool when studying the homogenization. It describes separately the limit in the fixed part Q_0 and in the part B^ε containing Γ^ε. Observe that (29)(i) means that the limit of the unfolded sequence is independent of y'.

Theorem 2 (**Weak compactness, [1]**). *Under assumptions* (4)–(9)*, let* $v^\varepsilon \in H^1(Q^\varepsilon)$ *for every* ε*, with*

$$\|v^\varepsilon\|_{H^1(Q^\varepsilon)} \le C. \tag{53}$$

Then there exists $v \in V$*, such that for a subsequence (still denoted* ε*),*

$$T^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(Q \times Y). \tag{54}$$

We also have, as ε *tends to zero,*

$$v^\varepsilon \rightharpoonup v \quad \text{weakly in } H^1(Q_0), \tag{55}$$
$$T^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(Q_0; H^1(Y)), \tag{56}$$
$$T^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } Z, \tag{57}$$
$$T'^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } Z'. \tag{58}$$

Furthermore, there exist $q \in W$ *such that*

$$T^\varepsilon \nabla v^\varepsilon \rightharpoonup \nabla v + \nabla_y q \quad \text{weakly in } L^2(Q_0 \times Y), \tag{59}$$
$$T^\varepsilon \nabla v^\varepsilon \rightharpoonup \Big(0, \frac{\partial v}{\partial x_2}\Big) + \nabla_y q \quad \text{weakly in } L^2(D^*), \tag{60}$$

as ε *tends to zero.*

Choosing $v_2^\varepsilon = 0$ in ([1], Corollary 4.8), we have the following result:

Corollary 1 ([1]). *Under the assumptions of Theorem 2, there exists $v \in L^2(\omega; H^1(g_m, g_M))$ such that, along some subsequence,*

$$\widetilde{v^\varepsilon} \rightharpoonup \begin{cases} v & \text{in } Q_0, \\ \operatorname{meas}(Y'_{x_2}) v & \text{in } B, \end{cases}$$

and

$$\widetilde{\nabla v^\varepsilon} \rightharpoonup \begin{cases} \nabla v & \text{in } Q_0, \\ \Big(\displaystyle\int_{Y_{x_2}} \frac{\partial q}{\partial y'} \, dy, \operatorname{meas}(Y'_{x_2}) \frac{\partial v}{\partial x_2} \Big) & \text{in } B, \end{cases}$$

weakly in $L^2(Q)$ as ε tends to zero, where $\sim$ denotes the zero extension to the whole of Q.

5. Limit of the Boundary Integral

In this section, we state a variation of a result from ([1], Theorem 5.2) that identifies the limit of the boundary integral over Γ^ε for our problem.

Lemma 1. *Suppose that $\zeta : \Gamma \to \mathbb{R}$ is such that*

(i) $y' \mapsto \zeta(y', g(y'))$ is Y'-periodic,

(ii) $\zeta \in L^2(\Gamma)$, and set

$$\zeta^\varepsilon(x) = \zeta\Big(\frac{x'}{\varepsilon}, g\Big(\frac{x'}{\varepsilon}\Big)\Big), \quad \text{a.e. } x \in \Gamma^\varepsilon,$$

and let ζ^0 be defined by

$$\zeta^0(x) = \zeta^0(x_2) = \int_{\Gamma_{x_2}} \zeta(y', x_2) \, dH_0(y'), \quad \text{a.e. } x \in B, \tag{61}$$

where H_0 is the zero-dimensional Hausdorff measure, and where Γ^ε is defined in (28). Let also $\{\sigma^\varepsilon\}$ and $\{\Psi^\varepsilon\}$ be sequences such that $\sigma^\varepsilon, \Psi^\varepsilon \in H^1(Q^\varepsilon)$, verifying

$$\begin{aligned} &(i) \quad T'^\varepsilon \sigma^\varepsilon \rightharpoonup \sigma \text{ weakly in } Z', \\ &\text{and} \\ &(ii) \quad \Psi^\varepsilon \equiv 1, \quad \text{or} \\ &\zeta \in L^\infty(\Gamma) \quad \text{and} \quad T'^\varepsilon \Psi^\varepsilon \to \Psi \text{ strongly in } Z', \\ &\operatorname{supp} \Psi^\varepsilon \subset K, \text{ for some compact } K \subset Q, \end{aligned} \tag{62}$$

with both σ and Ψ independent of y', the space Z' being defined in (48). Then, as ε tends to zero,

$$\varepsilon \int_{\Gamma^\varepsilon} \zeta^\varepsilon \, \sigma^\varepsilon \Psi^\varepsilon \, ds \to \int_B \zeta^0 \, \sigma \, \Psi \, dx, \tag{63}$$

where ζ^0 is defined by (61).

Proof. We first unfold the integral as follows, using (40), (42), Definition 2, and Remark 4:

$$\begin{aligned} \varepsilon \int_{\Gamma^\varepsilon} \zeta^\varepsilon \sigma^\varepsilon \Psi^\varepsilon \, ds = &\int_{\hat{\omega}^\varepsilon} \int_{Y'} \zeta(y', g(y'))(T'^\varepsilon \sigma^\varepsilon \, T'^\varepsilon \Psi^\varepsilon)(x', g(y'), y') \Big| \frac{\partial g}{\partial y'}(y') \Big| dy' \, dx' \\ &+ \varepsilon \int_{\Lambda^\varepsilon_{\Gamma^\varepsilon}} \zeta^\varepsilon \sigma^\varepsilon \Psi^\varepsilon \, ds + o(1), \end{aligned} \tag{64}$$

as ε tends to zero. Using assumptions (i) and (ii), where the limits are independent of y', arguing as in ([1], Theorem 5.2) the first integral on the right-hand side converges as follows:

$$\lim_{\varepsilon\to 0}\int_{\hat{\omega}^\varepsilon}\int_{Y'}\zeta(y',g(y'))(T'^\varepsilon\sigma^\varepsilon\, T'^\varepsilon\Psi^\varepsilon)(x',g(y'),y')\Big|\frac{\partial g}{\partial y'}(y')\Big|dy'\,dx' = \int_B \zeta^0\sigma\Psi\,dx,$$

where ζ^0 is defined by (61).

We turn to the integral over $\Lambda^\varepsilon_{\Gamma^\varepsilon}$ in (64). Suppose first that $\Psi^\varepsilon \equiv 1$.

Then by the Hölder inequality,

$$\varepsilon\Big|\int_{\Lambda^\varepsilon_{\Gamma^\varepsilon}}\zeta^\varepsilon\sigma^\varepsilon\,ds\Big| \le \sqrt{\varepsilon}\|\zeta^\varepsilon\|_{L^2(\Lambda^\varepsilon_{\Gamma^\varepsilon})}\sqrt{\varepsilon}\|\sigma^\varepsilon\|_{L^2(\Lambda^\varepsilon_{\Gamma^\varepsilon})}. \tag{65}$$

On the other hand, by a change of variable and the periodicity, we obtain

$$\begin{aligned}\int_{\Lambda^\varepsilon_{\Gamma^\varepsilon}}(\zeta^\varepsilon)^2\,ds &\le \int_{\omega\setminus\hat{\omega}^\varepsilon}\zeta\Big(\frac{x'}{\varepsilon},g\Big(\frac{x'}{\varepsilon}\Big)\Big)^2\Big(1+\frac{1}{\varepsilon^2}\Big|\frac{\partial g}{\partial y'}\Big(\frac{x'}{\varepsilon}\Big)\Big|^2\Big)^{1/2}dx'\\ &\le C\varepsilon\int_0^1\zeta(y',g(y'))^2\Big(1+\frac{1}{\varepsilon^2}\Big|\frac{\partial g}{\partial y'}(y')\Big|^2\Big)^{1/2}dy' \le C_1,\end{aligned}$$

while the boundedness of $\sqrt{\varepsilon}\|\sigma^\varepsilon\|_{L^2(\Lambda^\varepsilon_{\Gamma^\varepsilon})}$ follows from the assumption $\sigma^\varepsilon \in H^1(Q^\varepsilon)$ using (22). This together with (65) allows us to conclude that

$$\lim_{\varepsilon\to 0}\ \varepsilon\int_{\Lambda^\varepsilon_{\Gamma^\varepsilon}}\zeta^\varepsilon\sigma^\varepsilon\,ds = 0$$

for this case.

For the case of a compactly supported Ψ^ε in (ii), the integral is zero for all sufficiently small ε, and thus (63) holds. Then (63) holds for all cases listed in (ii). □

6. Homogenization Result

To characterize the asymptotics of the solutions u^ε to (12), we first give some preliminary definitions and then state our main result Theorem 3. We follow the standard procedure of first proving the unfolded homogenization (Theorem 4). The convergence of the energies is postponed to the last subsection (Proposition 6).

To define the effective matrix, for $\lambda \in \mathbb{R}^2$, let $\chi^{Q_0}_\lambda \in H^1_{\mathrm{per}}(Y)$ be the unique solution of average zero to the problem

$$-\mathrm{div}_y(A\nabla_y\chi^{Q_0}_\lambda) = \mathrm{div}_y(A\lambda) \quad \text{in } Y, \tag{66}$$

that is

$$\int_Y A\nabla_y\chi^{Q_0}_\lambda\cdot\nabla\varphi\,dy = -\int_Y A\lambda\cdot\nabla_y\varphi\,dy,$$

for any $\varphi \in H^1_{\mathrm{per}}(Y)$.

Similarly, for $x_2 \in (g_m, g_M)$ and $\lambda \in \mathbb{R}^2$, let $\chi^B_{\lambda,x_2} = \chi^B_{\lambda,x_2}(y) \in H^1_{\mathrm{per}}(Y_{x_2})$ be solutions to the following equation,

$$\begin{cases} -\mathrm{div}_y(A\nabla_y\chi^B_\lambda) = \mathrm{div}_y(A\lambda) & \text{in } Y_{x_2},\\ A_i\nabla_y\chi^B_\lambda\cdot\nu = -A\lambda\cdot\nu & \text{on } \partial Y_{x_2}\setminus\partial Y,\end{cases} \tag{67}$$

where the x_2 parameter is suppressed in χ^B_λ. The variational formulation of (67) is

$$\int_{Y_{x_2}} A\nabla_y\chi^B_\lambda\cdot\nabla_y\varphi\,dy = -\int_{Y_{x_2}} A\lambda\cdot\nabla_y\varphi\,dy, \quad \varphi\in H^1_{\mathrm{per}}(Y_{x_2}).$$

We will use the following notation for averages over Y_{x_2}:

$$M_{Y_{x_2}}(v) = \frac{1}{\text{meas}(Y_{x_2})} \int_{Y_{x_2}} v(y)\, dy, \quad x_2 \in (g_m, g_M), \quad v \in L^1(Y_{x_2}), \tag{68}$$

where Y_{x_2} is defined by (30). As in (66), we select unique solutions by adding the constraint

$$M_{Y_{x_2}}(\chi_\lambda^B) = 0. \tag{69}$$

The effective matrice is defined in Q by

$$A^0\lambda = \begin{cases} \int_Y A(\lambda + \nabla_y \chi_\lambda^{Q_0})\, dy & \text{in } Q_0, \\ \int_{Y_{x_2}} A(\lambda + \nabla_y \chi_\lambda^B)\, dy & \text{in } B. \end{cases} \tag{70}$$

From the ellipticity of A (13), it follows that the the matrix A^0 is constant in Q_0 and satisfies the ellipticity condition (13)(ii) in Q_0.

On the other hand, by ([1], Lemma 6.1), used from one component only, we derive that in the band B, the homogenized matrix depends on x_2, is positive semidefinite, and has the form

$$A^0 = \begin{pmatrix} 0 & 0 \\ 0 & (A^0)_{22} \end{pmatrix} \quad \text{in } B. \tag{71}$$

We can write $(A^0)_{22}$ explicitly in B by

$$(A^0)_{22} = (A^0)_{22}(x_2) = \int_{Y_{x_2}} \Big(A_{21}(y) \frac{\partial \chi_2^B}{\partial y_1} + A_{22}(y) \Big(1 + \frac{\partial \chi_2^B}{\partial y_2} \Big) \Big)\, dy,$$

where χ_2^B is given by χ_λ^B for $\lambda = (0, 1)$ and

$$a^0 \doteq (A^0)_{22} \geq C > 0 \quad \text{in } B. \tag{72}$$

From (71), it is indicated that the homogenized problem in B contains only a single term:

$$-\text{div}(A^0 \nabla u) = -\frac{\partial}{\partial x_2} \Big(a^0 \frac{\partial u}{\partial x_2} \Big), \quad \text{a.e. in } B. \tag{73}$$

Recall that V_0 is defined by (52) and that $v \in V_0$ is a function on $Q_0 \cup \Gamma_1 \cup B$. In Theorem 3 below, the limit problem in Q is solved by a function $u \in V_0$.

Theorem 3 (**The homogenized problem in Q**). *Under assumptions (4)–(9) and H_1–H_4, let $\{u^\varepsilon\}$ be the sequence of solutions $u^\varepsilon \in V^\varepsilon$ to problem (12). Let also A^0 and a^0 be given by (70)–(72).*
Then there exists $u \in V_0$ (cf. (52)) such that in the fixed region Q_0, as ε tends to zero,

$$u^\varepsilon \rightharpoonup u \quad \textit{weakly in } H^1(Q_0), \tag{74}$$

$$\nabla u^\varepsilon \rightharpoonup \nabla u \quad \textit{weakly in } L^2(Q_0), \tag{75}$$

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u \quad \textit{weakly in } L^2(Q_0), \tag{76}$$

and in the Γ^ε region B,

$$\widetilde{u^\varepsilon} \rightharpoonup \text{meas}(Y'_{x_2}) u \quad \textit{weakly in } L^2(B), \tag{77}$$

$$\widetilde{\nabla u^\varepsilon} \rightharpoonup \text{meas}(Y'_{x_2}) \Big(M_{Y_{x_2}} \Big(\frac{\partial \chi_2^B}{\partial y'} \Big) \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_2} \Big) \quad \textit{weakly in } L^2(B), \tag{78}$$

$$A^\varepsilon \widetilde{\nabla u^\varepsilon} \rightharpoonup A^0 \nabla u = \Big(0, a^0 \frac{\partial u}{\partial x_2} \Big) \quad \textit{weakly in } L^2(B), \tag{79}$$

where χ_2^B are defined by (67) for $\lambda = e_2$, and $\sim$ denotes the zero extension to Q.
The limit function $u \in V_0$ is the unique solution to the following homogenized problem:

$$\begin{aligned}&\int_{Q_0} A^0 \nabla u \cdot \nabla \varphi \, dx + \int_B a^0 \frac{\partial u}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \, dx + \int_B \rho^0 h(u) \varphi \, dx \\ &= \int_{Q_0} f \varphi \, dx + \int_B \mathrm{meas}(Y'_{x_2}) f \varphi \, dx + \int_B \tau^0 \varphi \, dx,\end{aligned} \tag{80}$$

for all $\varphi \in V_0$, which is the variational form of problem (1). Here ρ^0 and τ^0 are defined by (61), written for ζ equal to ρ and τ, respectively.

Remark 5. *Under the assumption of Theorem 3, problem (80) is exactly the variational form of problem (1), already mentioned in the introduction.*

We will prove Theorem 3 in the next section, as a consequence of Theorem 4 below, where we identify the unfolded homogenized problem, and of Corollary 2. The convergence of energies is then considered in Proposition 6 of Section 8.

Recall that the spaces Z and Z' are defined in (47)–(48).

Theorem 4 (The unfolded homogenized problem). *Under assumptions (4)–(9) and H_1–H_4, let $\{u^\varepsilon\}$ be the sequence of solution $u^\varepsilon \in V^\varepsilon$ to problem (12). Then there exists $u \in V_0$ such that as ε tends to zero,*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(Q_0), \tag{81}$$

$$T^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(Q_0; H^1(Y)), \tag{82}$$

$$T^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } Z, \tag{83}$$

$$T'^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } Z'. \tag{84}$$

Also, u^ε and u satisfy the convergences stated in Corollary 1, written for $v^\varepsilon = u^\varepsilon$ and $v = u$, respectively.
Furthermore, there exists $q \in W$ such that

$$T^\varepsilon \nabla u^\varepsilon \rightharpoonup \nabla u + \nabla_y q \quad \text{weakly in } L^2(Q_0 \times Y), \tag{85}$$

$$T^\varepsilon \nabla u^\varepsilon \rightharpoonup \left(0, \frac{\partial u}{\partial x_2}\right) + \nabla_y q \quad \text{weakly in } L^2(D^*), \tag{86}$$

as ε tends to zero, where W is defined by (50).
Let ρ^0 and τ^0 be defined by (61) for ζ equal to ρ and τ, respectively. The function $(u, q) \in V_0 \times W$ is the unique solution to the following unfolded homogenized problem:

$$\begin{aligned}&\int_{Q_0 \times Y} A(y)\left(\nabla u + \nabla_y q\right) \cdot \left(\nabla \varphi + \nabla_y \Psi\right) dxdy \\ &\quad + \int_B \int_{Y_{x_2}} A(y)\left(\left(0, \frac{\partial u}{\partial x_2}\right) + \nabla_y q\right) \cdot \left(\left(0, \frac{\partial \varphi}{\partial x_2}\right) + \nabla_y \Psi\right) dy \, dx + \int_B \rho^0 h(u) \varphi \, dx \\ &\qquad = \int_{Q_0} f \varphi \, dx + \int_B \mathrm{meas}(Y'_{x_2}) f \varphi \, dx + \int_B \tau^0 \varphi \, dx,\end{aligned} \tag{87}$$

for every $\varphi \in V_0$ and $\Psi \in W$.

We state the factorization of q, which is a special case of Corollary 6.5 in [1].

Corollary 2. *Under the hypotheses of Theorem 4, with u and q given as therein. Let* $\chi_k^{Q_0}$ *and* χ_k^B *be the solutions to the problems (66) and (67), for* $\lambda = e_k$ *with* $\{e_k\}_{k=1}^2$*, the standard basis in* $\mathbb{R}^2$*. Then*

$$q = \begin{cases} \displaystyle\sum_{k=1}^{2} \chi_k^{Q_0} \frac{\partial u}{\partial x_k} & \text{a.e. in } Q_0 \times Y, \\ \chi_2^B \dfrac{\partial u}{\partial x_2} & \text{a.e. in } D^*. \end{cases}$$

7. Proof of the Homogenization Result

We first prove the following a priori estimate:

Proposition 5. *Under the assumption of Theorem 3, let* $f \in L^2(Q)$ *and let* u^ε *be the solutions to (25). Then there is a constant C that is independent of* ε *such that for all small enough* $\varepsilon > 0$*,*

$$\|u^\varepsilon\|_{H^1(Q^\varepsilon)} \leq C. \tag{88}$$

Proof. Choosing $v = u^\varepsilon$ in the variational formulation (25), we have

$$\int_{Q^\varepsilon} A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx + \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon h(u^\varepsilon) u^\varepsilon \, ds = \int_{Q^\varepsilon} f u^\varepsilon \, dx + \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon u^\varepsilon \, ds.$$

Because g is Lipschitz-continuous, from the periodicity assumption H_4 on τ and classical results on oscillating functions, we have for small enough ε,

$$\begin{aligned} \varepsilon \int_{\Gamma^\varepsilon} \tau_\varepsilon^2 \, ds &= \varepsilon \int_\omega \tau\Big(\frac{x'}{\varepsilon}, g\Big(\frac{x'}{\varepsilon}\Big)\Big)^2 \Big(1 + \frac{1}{\varepsilon^2}\Big|\frac{\partial g}{\partial y'}\Big(\frac{x'}{\varepsilon}\Big)\Big|^2\Big)^{1/2} dx' \\ &\leq c \int_\omega \tau\Big(\frac{x'}{\varepsilon}, g\Big(\frac{x'}{\varepsilon}\Big)\Big)^2 dx' \leq c_2, \end{aligned}$$

where c_2 is independent of ε.
Consequently, from assumptions H_1–H_4, and (22), we have

$$\begin{aligned} \alpha \|\nabla u^\varepsilon\|_{L^2(Q^\varepsilon)}^2 &\leq \|f\|_{L^2(Q)} \|u^\varepsilon\|_{L^2(Q^\varepsilon)} + \varepsilon \|\tau^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \|u^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \\ &\leq c \|u^\varepsilon\|_{L^2(Q^\varepsilon)} + c_1 \|\nabla u^\varepsilon\|_{L^2(Q^\varepsilon)}, \end{aligned} \tag{89}$$

for all small enough ε. Hence, the Poincaré inequality from Proposition 1 gives (88) for ε sufficiently small. □

Proof of Theorem 4. Observe that Proposition 5 together with the weak compactness stated in Theorem 2 implies that for a subsequence (still denoted ε), there exist $u \in V_0$ and $q \in W$ such that (81)–(86) hold. The function $u \in V_0$ as u^ε has zero trace on Γ_D.
We now pass to the limit in the variational form (25) of the equation for u^ε. A priori, u^ε is weakly converging along some subsequence, and because of the nonlinearity of the operator $\mathcal{A}^\varepsilon$ defined in (26), we cannot pass directly to the limit as ε tends to zero. Since from the proof of Theorem 1, $\mathcal{A}^\varepsilon$ is monotone, we use here the Minty's argument for monotone operators [18].
Using that $u^\varepsilon \in V^\varepsilon$ solves (25), we have for any Φ^ε such that $\Phi^\varepsilon \in V^\varepsilon$,

$$\begin{aligned} 0 &\leq \langle \mathcal{A}^\varepsilon(u^\varepsilon) - \mathcal{A}^\varepsilon(\Phi^\varepsilon), u^\varepsilon - \Phi^\varepsilon \rangle \\ &= \int_{Q^\varepsilon} f(x)(u^\varepsilon - \Phi^\varepsilon) \, dx - \int_{Q^\varepsilon} A^\varepsilon \nabla \Phi^\varepsilon \cdot \nabla (u^\varepsilon - \Phi^\varepsilon) \, dx \\ &\quad + \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon(s)(u^\varepsilon - \Phi^\varepsilon) \, ds - \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(\Phi^\varepsilon)(u^\varepsilon - \Phi^\varepsilon) \, ds, \end{aligned} \tag{90}$$

where $\mathcal{A}^\varepsilon$ is given by (26).

From the weak compactness (Theorem 2), the characterization of the weak limit of the gradient in D^* (60) is given by (86). This motivates the choice of the function Φ^ε done below in (90).
Let $\varphi, \psi \in C_0^\infty(Q)$ and $\xi \in C^\infty_{\mathrm{per}}(Y)$ and set

$$\widetilde{\Phi}^\varepsilon(x,y) = \varphi(x) + \varepsilon\psi(x)\xi(y) - \varepsilon\theta^\varepsilon(x)\frac{\partial\varphi}{\partial x'}(x)y', \quad (x,y) \in Q \times Y,$$

where $\theta^\varepsilon = \theta^\varepsilon(x', x_2)$ is a smooth cutoff for the purpose of continuity over the interface Γ_1 between Q_0 and B; to do that, we choose θ^ε to be the piecewise linear and continuous function such that

$$\theta^\varepsilon = \begin{cases} 1 & \text{in } B \setminus R_\varepsilon, \\ 0 & \text{on } Q_0, \end{cases} \tag{91}$$

where $R_\varepsilon = \omega \times (g_m, g_m + \varepsilon)$. By construction, $\varepsilon|\nabla\sigma^\varepsilon|$ is bounded with respect to ε, and its support is contained in $\overline{R_\varepsilon}$, cf. Figure 2b.
Let us now Y-periodize the function y' and set

$$\Phi^\varepsilon(x) = \widetilde{\Phi}^\varepsilon\Big(x, \frac{x}{\varepsilon}\Big), \quad x \in Q.$$

To pass to the limit in (90), using (81)–(86), we first calculate the limits of the gradient of the selected function Φ^ε, as well as of its unfolded gradient.
We have (see Remark 4)

$$\begin{aligned}
\nabla\Phi^\varepsilon(x) &= \nabla_x\widetilde{\Phi}^\varepsilon\Big(x, \frac{x}{\varepsilon}\Big) + \frac{1}{\varepsilon}\nabla_y\widetilde{\Phi}^\varepsilon\Big(x, \frac{x}{\varepsilon}\Big) \\
&= \Big(0, \frac{\partial\varphi}{\partial x_2}(x)\Big) + (1 - \theta^\varepsilon(x))\Big(\frac{\partial\varphi}{\partial x'}(x), 0\Big) + \psi(x)\nabla_y\xi\Big(\frac{x}{\varepsilon}\Big) \\
&\quad + \varepsilon\xi\Big(\frac{x}{\varepsilon}\Big)\nabla\psi(x) - \varepsilon\frac{\partial\varphi}{\partial x'}(x)\Big\{\frac{x'}{\varepsilon}\Big\}\nabla\theta^\varepsilon(x) - \varepsilon\theta^\varepsilon(x)\Big\{\frac{x'}{\varepsilon}\Big\}\nabla\frac{\partial\varphi}{\partial x'}(x), \quad \text{a.e. } x \in Q,
\end{aligned}$$

where we arranged suitably the terms. For its unfolded gradient, using Definition 1, we obtain

$$\begin{aligned}
T^\varepsilon\nabla\Phi^\varepsilon &= \Big(0, T^\varepsilon\frac{\partial\varphi}{\partial x_2}\Big) + T^\varepsilon(1 - \theta^\varepsilon)\Big(T^\varepsilon\frac{\partial\varphi}{\partial x'}, 0\Big) + (T^\varepsilon\psi)\nabla_y\xi \\
&\quad + \varepsilon\xi T^\varepsilon\nabla\psi - \varepsilon(T^\varepsilon\frac{\partial\varphi}{\partial x'})y'T^\varepsilon\nabla\theta^\varepsilon - \varepsilon(T^\varepsilon\theta^\varepsilon)y'T^\varepsilon\nabla\frac{\partial\varphi}{\partial x'},
\end{aligned} \tag{92}$$

a.e. in $\big(\hat{Q}_0^\varepsilon \times Y\big) \cup D^{*\varepsilon}$, and 0 elsewhere.
By (33), (35), Proposition 3, the a priori estimate (88), and using Proposition 2(ii), it follows from (90) that

$$\begin{aligned}
0 \le{}& \int_{Q_0} f(u^\varepsilon - \Phi^\varepsilon)\,dx + \int_B\int_{Y_{x_2}} T^\varepsilon f\,(T^\varepsilon u^\varepsilon - T^\varepsilon\Phi^\varepsilon)\,dy\,dx \\
&- \int_{Q_0\times Y} A(y)\,T^\varepsilon\nabla\Phi^\varepsilon \cdot T^\varepsilon\nabla(u^\varepsilon - \Phi^\varepsilon)\,dxdy \\
&- \int_B\int_{Y_{x_2}} A(y)\,T^\varepsilon\nabla\Phi^\varepsilon \cdot T^\varepsilon\nabla(u^\varepsilon - \Phi^\varepsilon)\,dy\,dx \\
&+ \varepsilon\int_{\Gamma^\varepsilon}\tau^\varepsilon(s)(u^\varepsilon - \Phi^\varepsilon)\,ds - \varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon(s)h(\Phi^\varepsilon)(u^\varepsilon - \Phi^\varepsilon)\,ds.
\end{aligned} \tag{93}$$

Observe that from (91) and (92), we have

$$T^\varepsilon\Phi^\varepsilon \to \varphi \quad \text{strongly in } L^2((Q_0 \times Y) \cup D^*), \tag{94}$$

$$T^\varepsilon\nabla\Phi^\varepsilon \to \nabla\varphi + \psi\nabla_y\xi \quad \text{strongly in } L^2(Q_0 \times Y), \tag{95}$$

as ε tends to zero. Moreover, as $\varepsilon|\nabla\theta^\varepsilon| \le C$ in B, uniformly in ε, and $\theta^\varepsilon = 1$ in B at a distance greater than ε from Γ_1, we have

$$T^\varepsilon\nabla\Phi^\varepsilon \to \Big(0, \frac{\partial\varphi}{\partial x_2}\Big) + \psi\nabla_y\xi \quad \text{strongly in } L^2(D^*), \tag{96}$$

as ε tends to zero.
We first compute the limits of the two boundary terms in (93), using Lemma 1.
For the first one, since τ^0, u, and φ do not depend on y', we can apply the lemma for $\zeta = \tau$, $\sigma^\varepsilon = u^\varepsilon - \Phi^\varepsilon$, and $\Psi^\varepsilon = 1$. Indeed, τ^ε is defined by (19) with $\tau \in L^2(\Gamma)$; moreover, applying Theorem 2 to Φ^ε and using (86), we deduce that

$$T'^\varepsilon \sigma^\varepsilon \rightharpoonup u(x) - \varphi(x) \quad \text{weakly in } Z',$$

as ε tends to zero. Then, by Lemma 1, we obtain

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon(s)(u^\varepsilon - \Phi^\varepsilon)\, ds = \int_B \tau^0(x)(u - \varphi)\, dx. \tag{97}$$

For the second boundary integral, let us show that

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(\Phi^\varepsilon)(u^\varepsilon - \Phi^\varepsilon)\, ds = \int_B \rho^0(x) h(\varphi)(u - \varphi)\, dx. \tag{98}$$

For this purpose, we use again Lemma 1, where now we choose $\zeta = \rho$, noting that $\rho \in L^\infty(\Gamma)$, $\sigma^\varepsilon = u^\varepsilon - \Psi^\varepsilon$ as above, and $\Psi^\varepsilon = h(\Phi^\varepsilon)$, which is compactly supported, due to the construction of Φ^ε and the assumption (17) in H_3.
It remains to verify that

$$T'^\varepsilon \Psi^\varepsilon = h(T'^\varepsilon \Phi^\varepsilon) \to h(\varphi) \quad \text{strongly in } Z', \tag{99}$$

as ε tends to zero, where, thanks to assumption (17), we interchanged T'^ε and h.
The same argument used to derive convergences (94) and (96) from identity (92) can be used here for T'^ε and gives

$$\begin{aligned} T'^\varepsilon \Phi^\varepsilon &\to \varphi \quad \text{strongly in } L^2(D'^*), \\ \frac{\partial}{\partial x_2}(T'^\varepsilon \Phi^\varepsilon) &\to \frac{\partial \varphi}{\partial x_2} \quad \text{strongly in } L^2(D'^*), \\ \frac{\partial}{\partial y'}(T'^\varepsilon \Phi^\varepsilon) &\to 0 \quad \text{strongly in } L^2(D'^*), \end{aligned}$$

as ε tends to zero, and therefore by definition (48),

$$T'^\varepsilon \Phi^\varepsilon \to \varphi \quad \text{strongly in } Z', \tag{100}$$

as ε tends to zero.
On the other hand, by the chain rule, we have

$$\frac{\partial}{\partial x_2}(T'^\varepsilon \Psi^\varepsilon) = h'(T'^\varepsilon \Phi^\varepsilon) \frac{\partial}{\partial x_2}(T'^\varepsilon \Phi^\varepsilon) \tag{101}$$

and

$$\frac{\partial}{\partial y'}(T'^\varepsilon \Psi^\varepsilon) = h'(T'^\varepsilon \Phi^\varepsilon) \frac{\partial}{\partial y'}(T'^\varepsilon \Phi^\varepsilon). \tag{102}$$

This, together with the regularity properties of h, and (100)–(102) imply convergence (99). Hence, by Lemma 1, convergence (98) holds, as the limit functions are independent of y'. Consequently, by passing to the limit in (93), using (94)–(98), as well as convergences (81)–(86), we obtain the inequality

$$\begin{aligned} 0 \le & \int_{Q_0} f(u - \varphi)\, dx + \int_B \operatorname{meas}(Y_{x_2}) f(u - \varphi)\, dx \\ & + \int_B \tau^0(x)(u - \varphi)\, dx - \int_B \rho^0(x) h(\varphi)(u - \varphi)\, dx \\ & - \int_{Q_0} \int_Y A(y)(\nabla \varphi + \psi \nabla_y \xi) \cdot (\nabla u + \nabla_y q - \nabla \varphi - \psi \nabla_y \xi)\, dy\, dx \\ & - \int_B \int_{Y_{x_2}} A(y)\Big((0, \frac{\partial \varphi}{\partial x_2}) + \psi \nabla_y \xi\Big) \cdot \Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q - \Big(0, \frac{\partial \varphi}{\partial x_2}\Big) - \psi \nabla_y \xi\Big)\, dy\, dx. \end{aligned} \tag{103}$$

Let now $\delta \in \mathbb{R}$. By the density of the tensor product $C_0^\infty(Q) \otimes C_{\text{per}}^\infty(Y)$ in W, we can replace in (103) the product $\psi(x)\xi(y)$ by $q + \delta\Psi$, with $\Psi = \Psi(x,y) \in W$. Also, using the density of smooth functions in V_0, we can choose $\varphi = u + \delta\phi$, where ϕ is an arbitrary function in V_0. This leads to

$$\begin{aligned} 0 \le & -\delta \int_{Q_0} f\phi\, dx - \delta \int_B \text{meas}(Y_{x_2}) f\phi\, dx - \delta \int_B \tau^0 \phi\, dx + \delta \int_B \rho^0 h(u)\phi\, dx \\ & + \delta \int_{Q_0} \int_Y A(y)\big(\nabla(u+\delta\phi) + \nabla_y q + \delta\nabla_y\Psi\big)\cdot\big(\nabla\phi + \nabla_y\Psi\big)\, dy\, dx \\ & + \delta \int_B \int_{Y_{x_2}} A(y)\Big((0, \frac{\partial(u+\delta\phi)}{\partial x_2}) + \nabla_y q + \delta\nabla_y\Psi\Big)\cdot\Big((0, \frac{\partial\phi}{\partial x_2}) + \nabla_y\Psi\Big)\, dy\, dx. \end{aligned} \tag{104}$$

Dividing by $|\delta| > 0$, and passing to the limit as δ tends to zero, we obtain the unfolded Equation (87).

Because $\rho^0 \ge 0$, and A^0 has the structure given by (70)–(72), the associated operator is strictly monotone, so the solution to (87) is unique in the space $V_0 \times W$. The uniqueness implies that (81)–(86) hold for the full sequences. □

Proof of Theorem 3. We first show that the function u is the unique solution in V_0 to problem (80), (1).

Applying Theorem 4 to the sequence u^ε, choosing $\Psi = 0$ in (87), gives

$$\begin{aligned} & \int_{Q_0\times Y} A(y)(\nabla u + \nabla_y q)\cdot\nabla\varphi\, dxdy \\ & \quad + \int_B \int_{Y_{x_2}} A(y)\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\cdot\Big(0, \frac{\partial\varphi}{\partial x_2}\Big)\, dy\, dx + \int_B \rho^0 h(u)\varphi\, dx \\ & = \int_{Q_0} f\varphi\, dx + \int_B \text{meas}(Y'_{x_2}) f\varphi\, dx, + \int_B \tau^0\varphi\, dx, \end{aligned} \tag{105}$$

for every $\varphi \in V_0$.

For almost every $x \in B$, using Corollary 2,

$$\int_{Y_{x_2}} A\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\, dy = \frac{\partial u}{\partial x_2}\int_{Y_{x_2}} A(e_2 + \nabla_y\chi_2^B)\, dy = \Big(0, a^0\frac{\partial u}{\partial x_2}\Big) = A^0\nabla u, \tag{106}$$

where we have used (71), (70), and (72).

Replacing in Q_0 the expression of q given in Corollary 2, from (70), (105), and (106), we also have that

$$\int_Y A(y)(\nabla u + \nabla_y q)\, dy = A^0\nabla u, \tag{107}$$

for almost every $x \in Q_0$, and the function u satisfies the variational Equation (80). Hence, u is the unique solution in V_0 to the limit problem (1).

We turn to the weak convergence of u^ε, ∇u^ε, and $A^\varepsilon\nabla u^\varepsilon$. Convergences (74), (75), (77), and (78) follow from Corollaries 1 and 2.

We turn to prove the convergences (76) and (79). With $\Phi \in C_0^\infty(B)^2$, we have by (86) and Proposition 2(ii), as ε tends to zero,

$$\begin{aligned} \int_B A^\varepsilon\nabla u^\varepsilon\cdot\Phi\, dx &= \int_{B^\varepsilon} A^\varepsilon\nabla u^\varepsilon\cdot\Phi\, dx = \int_B\int_{Y_{x_2}} A(y)T^\varepsilon\nabla u^\varepsilon\cdot T^\varepsilon\Phi\, dy\, dx + o(1) \\ &\to \int_B \Phi\cdot\int_{Y_{x_2}} A(y)\Big((0, \frac{\partial u}{\partial x_2}) + \nabla_y q\Big)\, dy\, dx = \int_B A^0\nabla u\cdot\Phi\, dx, \end{aligned}$$

where (106) was used in the last step. We have proved (79).

Then (76) is obtained using (85) and (107):

$$\begin{aligned} \int_{Q_0} A^\varepsilon\nabla u^\varepsilon\cdot\Phi\, dx &= \int_{Q_0}\int_Y A(y)T^\varepsilon\nabla u^\varepsilon\cdot T^\varepsilon\Phi\, dy\, dx + o(1) \\ &\to \int_{Q_0}\Phi\cdot\int_Y A(y)(\nabla u + \nabla_y q)\, dy\, dx = \int_{Q_0} A^0\nabla u\cdot\Phi\, dx, \end{aligned}$$

as ε tends to zero, for any $\Phi \in C_0^\infty(Q_0)^2$. □

8. Energy Convergence

In this section, a proof of convergence of the energies of the solutions u^ε to (12) is presented. The limit energy is the energy of the solution u to (80). Moreover, when the boundary condition is linear (that is $h(s) \equiv s$), we also prove the convergence of the unfolded energies to that of the solution (u, q) to (87).

In the proof, we will make use of the following lemma for real sequences:

Lemma 2. *Let x_n^k be a finite number of bounded sequences of real numbers, such that $\liminf_n x_n^k \geq x^k$ for some x^k, and $\lim_n \sum_k x_n^k = \sum_k x^k$. Then $\lim_n x_n^k = x^k$ for each k.*

Proof. By standard properties of the limit inferior and limit superior for bounded sequences of real numbers:

$$\begin{aligned} x^m &\leq \liminf_n x_n^m \leq \limsup_n x_n^m = \limsup_n \Big(\sum_k x_n^k + \sum_{k\neq m} (-x_n^k) \Big) \\ &\leq \limsup_n \sum_k x_n^k + \limsup_n \sum_{k\neq m} (-x_n^k) \leq \limsup_n \sum_k x_n^k + \sum_{k\neq m} \limsup_n (-x_n^k) \\ &= \sum_k x^k + \sum_{k\neq m} (-\liminf_n x_n^k) \leq \sum_k x^k - \sum_{k\neq m} x^k = x^m. \end{aligned}$$

This shows that

$$\liminf_n x_n^m = \limsup_n x_n^m,$$

and thus $\lim_n x_n^m = x^m$. □

We set

$$E^\varepsilon = \int_{Q^\varepsilon} A^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx + \varepsilon \int_{\Gamma^\varepsilon} \rho^\varepsilon(s) h(u^\varepsilon) u^\varepsilon \, ds, \tag{108}$$

$$E^0 = \int_{Q_0} A^0 \nabla u \cdot \nabla u \, dx + \int_B a^0 \Big(\frac{\partial u}{\partial x_2} \Big)^2 dx + \int_B \rho^0 h(u) u \, dx. \tag{109}$$

Remark 6. *Observe that the energy is well defined by (108) even if A is not symmetric, as by the ellipticity assumption (13)(ii), we have $K(x)\lambda \cdot \lambda = 0$ for $\lambda \in \mathbb{R}^2$, a.e. $x \in \mathbb{R}^2$, where K is the non-symmetric part of A, that is $K = (A - A^T)/2$.*

In the sequel, for simplicity, we use the following notation:

$$\Theta(y') = \rho(y', g(y')) \Big| \frac{\partial g}{\partial y'}(y') \Big|. \tag{110}$$

Then we define the corresponding unfolded energies by

$$\begin{aligned} E_T^\varepsilon = &\int_{Q_0 \times Y} A(y) \, T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla u^\varepsilon \, dxdy + \int_B \int_{Y_{x_2}} A(y) \, T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla u^\varepsilon \, dxdy \\ &+ \int_\omega \int_{Y'} \Theta(y') \; T'^\varepsilon h(u^\varepsilon) \; T'^\varepsilon(u^\varepsilon)(x', g(y'), y') \, dy' \, dx', \end{aligned} \tag{111}$$

$$\begin{aligned} E_T^0 = &\int_{Q_0 \times Y} A(y) (\nabla_x u + \nabla_y q) \cdot (\nabla_x u + \nabla_y q) \, dxdy \\ &+ \int_{D^*} A(y) \Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q \Big) \cdot \Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q \Big) \, dxdy + \int_B \rho^0 h(u) u \, dx. \end{aligned} \tag{112}$$

Proposition 6. *Under assumptions (4)–(9) and H_1–H_4, let $\{u^\varepsilon\}$ be the sequence of solution $u^\varepsilon \in V^\varepsilon$ to problem (12) and let the respective energies be given by (108)–(112).*

- *Then the energies of the problem converge, namely,*

$$\lim_{\varepsilon\to 0} E^\varepsilon = E^0 = E^0_T. \tag{113}$$

- *Suppose now that the boundary condition is linear, which is a particular case of H_3, so that we assume*

$$h(s) \equiv s, \qquad \text{for every } s \in \mathbb{R}. \tag{114}$$

Then in addition to (113), we also have the following convergence of the unfolded energies:

$$\lim_{\varepsilon\to 0} E^\varepsilon_T = E^0_T\,, \tag{115}$$

as well as the separate convergences

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{Q_0\times Y} A(y)\, T^\varepsilon\nabla u^\varepsilon \cdot T^\varepsilon\nabla u^\varepsilon \, dxdy \\ = \int_{Q_0\times Y} A(y)(\nabla_x u + \nabla_y q)\cdot(\nabla_x u + \nabla_y q)\, dxdy.\end{aligned} \tag{116}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_B \int_{Y_{x_2}} A(y)\, T^\varepsilon\nabla u^\varepsilon \cdot T^\varepsilon\nabla u^\varepsilon \, dx\, dy \\ = \int_{D^*} A(y)\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\cdot\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\, dxdy.\end{aligned} \tag{117}$$

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\omega \int_{Y'} \Theta(y')\; T'^\varepsilon h(u^\varepsilon)\; T'^\varepsilon(u^\varepsilon)(x', g(y'), y')\, dy'\, dx' \\ = \int_B \rho^0 u^2\, dx = \int_\omega \int_{Y'} \Theta(y') u^2(x', g(y'))\, dy'\, dx'.\end{aligned} \tag{118}$$

Proof. Let $u^\varepsilon \in V^\varepsilon$ be the solution to problem (12), $u \in V_0$ the solution to problem (80), and $q \in W$ such that u, q is the unique solution to (87). By (25), we may pass to the limit in E^ε, using convergences (81)–(86), so that

$$\begin{aligned}E^\varepsilon &= \int_{Q^\varepsilon} f u^\varepsilon\, dx + \varepsilon \int_{\Gamma^\varepsilon} \tau^\varepsilon(s) u^\varepsilon\, ds \\ &\to \int_{Q_0} f u\, dx + \int_B \mathrm{meas}(Y'_{x_2}) f u\, dx + \int_B \tau^0 u\, dx = E^0,\end{aligned} \tag{119}$$

as ε tends to zero, which gives the first limit in (115). Choosing u, q as test functions in (80) and (87), respectively, it is clear that

$$E^0 = E^0_T. \tag{120}$$

It remains the case to prove the convergence of the unfolded energy, assuming that the problem is linear.
We observe first that since ρ^0 is defined by (61) written for ζ equal to ρ, recalling formula (5.9) from [1], which was proved using the coarea formula, we can write

$$\int_B \rho^0 u^2\, dx = \int_\omega \int_{Y'} \Theta(y') u^2(x', g(y'))\, dy'\, dx'. \tag{121}$$

From the ellipticity of A and convergences (85)–(86), using Propositions 2 and 4, and (121), we have

$$\begin{aligned}E^0_T &= \int_{Q_0\times Y} A(y)(\nabla u + \nabla_y q)\cdot(\nabla u + \nabla_y q)\, dxdy \\ &\quad + \int_{D^*} A(y)\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\cdot\Big(\Big(0, \frac{\partial u}{\partial x_2}\Big) + \nabla_y q\Big)\, dxdy + \int_B \rho^0 u^2\, dx \\ &\le \liminf_{\varepsilon\to 0} \int_{Q_0\times Y} A(y) T^\varepsilon\nabla u^\varepsilon \cdot T^\varepsilon\nabla u^\varepsilon\, dxdy \\ &\quad + \liminf_{\varepsilon\to 0} \int_{D^*} A(y) T^\varepsilon\nabla u^\varepsilon \cdot T^\varepsilon\nabla u^\varepsilon\, dxdy + \int_\omega \int_{Y'} \Theta(y') u^2(x', g(y'))\, dy'\, dx'.\end{aligned} \tag{122}$$

Let us prove that

$$\liminf_{\varepsilon\to 0}\int_{\omega}\int_{Y'}\Theta(y')\;T'^{\varepsilon}(u^{\varepsilon})^2(x',g(y'),y')\,dy'\,dx' \geq \int_{\omega}\int_{Y'}\Theta(y')u^2(x',g(y'))\,dy'\,dx'. \tag{123}$$

From (84), and the weak sequential continuity of the trace from the set D'^{*} defined by (41) to the set

$$B_g \doteq \{(x',x_2,y')\in B\times Y' : x_2 = g(y')\},$$

it results that

$$\gamma_{B_g}(T'^{\varepsilon}u^{\varepsilon}) \rightharpoonup \gamma_{B_g}(u) \qquad \text{in } L^2(B_g), \tag{124}$$

where γ_{B_g} denotes the trace operator on B_g, that is

$$\gamma_{B_g}(T'^{\varepsilon}u^{\varepsilon})(x',y') = T'^{\varepsilon}u^{\varepsilon}(x',g(y'),y'), \quad \text{a.e. } (x',x_2,y')\in B_g,$$

so that

$$U^{\varepsilon} \rightharpoonup U \qquad \text{in } L^2(\omega\times Y'), \tag{125}$$

where

$$U^{\varepsilon}(x',y') = \gamma_{B_g}(T'^{\varepsilon}u^{\varepsilon})(x',y') = T'^{\varepsilon}u^{\varepsilon}(x',g(y'),y'), \quad (x',y')\in\omega\times Y',$$

and

$$U(x',y') = \gamma_{B_g}(u)(x',y') = u(x',g(y')), \quad (x',y')\in\omega\times Y'.$$

This implies (123), which used in (122) gives

$$\begin{aligned}
E_T^0 &\leq \liminf_{\varepsilon\to 0}\int_{Q_0\times Y} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy + \liminf_{\varepsilon\to 0}\int_{D^*} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy\\
&\quad + \liminf_{\varepsilon\to 0}\int_{\omega}\int_{Y'}\Theta(y')\;T'^{\varepsilon}(u^{\varepsilon})^2(x',g(y'),y')\,dy'\,dx'\\
&\leq \liminf_{\varepsilon\to 0}\Big(\int_{Q_0\times Y} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy + \int_{D^*} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy\\
&\qquad + \int_{\omega}\int_{Y'}\Theta(y')\;T'^{\varepsilon}(u^{\varepsilon})^2(x',g(y'),y')\,dy'\,dx'\Big)\\
&\leq \limsup_{\varepsilon\to 0}\Big(\int_{Q_0\times Y} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy + \int_{D^*} A(y)T^{\varepsilon}\nabla u^{\varepsilon}\cdot T^{\varepsilon}\nabla u^{\varepsilon}\,dxdy\\
&\qquad + \int_{\omega}\int_{Y'}\Theta(y')\;T'^{\varepsilon}(u^{\varepsilon})^2(x',g(y'),y')\,dy'\,dx'\Big)\\
&\leq \lim_{\varepsilon\to 0}\Big(\int_{Q^{\varepsilon}} A^{\varepsilon}\nabla u^{\varepsilon}\cdot\nabla u^{\varepsilon}\,dx + \varepsilon\int_{\Gamma^{\varepsilon}}\rho^{\varepsilon}(s)(u^{\varepsilon})^2\,ds\Big) = E^0 = E_T^0,
\end{aligned}$$

where in the last inequality we applied Proposition 4. Hence, the above inequalities are actually equalities and they are all limits. Then, using Lemma 2 together with (121), we obtain convergences (116)–(118), which completes the proof. □

Corollary 3. *Under the hypotheses of Theorem 4, and assuming the linearity of the boundary condition given by (114), convergences (85) and (86) are strong and read*

$$T^{\varepsilon}\nabla u^{\varepsilon} \to \nabla u + \sum_{k=1}^{2}\frac{\partial u}{\partial x_k}\nabla_y\chi_k^{Q_0} \quad \textit{strongly in } L^2(Q_0\times Y), \tag{126}$$

$$T^{\varepsilon}\nabla u^{\varepsilon} \to \Big(0,\frac{\partial u}{\partial x_2}\Big) + \frac{\partial u}{\partial x_2}\nabla_y\chi_2^{B} \quad \textit{strongly in } L^2(D^*), \tag{127}$$

as ε tends to zero. Furthermore,

$$\lim_{\varepsilon\to 0}\;\varepsilon\int_{\Gamma^{\varepsilon}}\rho^{\varepsilon}(u^{\varepsilon}-u)^2\,ds = 0. \tag{128}$$

Proof. We write

$$\begin{aligned}
&\int_{Q_0\times Y} A(y)\Big(T^\varepsilon\nabla u^\varepsilon - \nabla u - \nabla_y q\Big)\cdot\Big(T^\varepsilon\nabla u^\varepsilon - \nabla u - \nabla_y q\Big)\,dxdy\\
&\quad+\int_{D^*} A(y)\Big(T^\varepsilon\nabla u^\varepsilon - \Big(0,\frac{\partial u}{\partial x_2}\Big) - \nabla_y q\Big)\cdot\Big(T^\varepsilon\nabla u^\varepsilon - \Big(0,\frac{\partial u}{\partial x_2}\Big) - \nabla_y q\Big)\,dxdy\\
&\quad=\int_{Q_0\times Y} A(y)T^\varepsilon\nabla u^\varepsilon\cdot T^\varepsilon\nabla u^\varepsilon\,dxdy+\int_{D^*} A(y)T^\varepsilon\nabla u^\varepsilon\cdot T^\varepsilon\nabla u^\varepsilon\,dxdy\\
&\qquad-\int_{Q_0\times Y} A(y)T^\varepsilon\nabla u^\varepsilon\cdot(\nabla u+\nabla_y q)\,dxdy-\int_{D^*} A(y)T^\varepsilon\nabla u^\varepsilon\cdot\Big(\Big(0,\frac{\partial u}{\partial x_2}\Big)+\nabla_y q\Big)\,dxdy\\
&\qquad-\int_{Q_0\times Y} A(y)(\nabla u+\nabla_y q)\cdot T^\varepsilon\nabla u^\varepsilon\,dxdy-\int_{D^*} A(y)\Big(\Big(0,\frac{\partial u}{\partial x_2}\Big)+\nabla_y q\Big)\cdot T^\varepsilon\nabla u^\varepsilon\,dxdy\\
&\qquad+\int_{Q_0\times Y} A(y)(\nabla u+\nabla_y q)\cdot\Big(\Big(0,\frac{\partial u}{\partial x_2}\Big)+\nabla_y q\Big)\,dxdy\\
&\qquad+\int_{D^*} A(y)\Big(\Big(0,\frac{\partial u}{\partial x_2}\Big)+\nabla_y q\Big)\cdot\Big(\Big(0,\frac{\partial u}{\partial x_2}\Big)+\nabla_y q\Big)\,dxdy,
\end{aligned}$$

where q is given by Corollary 2.
From (116)–(117), and convergences (85)–(86), the right-hand side of this equality goes to 0, as ε tends to zero. By (13)(ii), replacing q by its expression given by Corollary 2, we obtain

$$\begin{aligned}
&\alpha\int_{Q_0\times Y}\Big|T^\varepsilon\nabla u^\varepsilon-\nabla u-\sum_{k=1}^{2}\frac{\partial u}{\partial x_k}\nabla_y\chi_k^{Q_0}\Big|^2\,dxdy\\
&\quad+\alpha\int_{D^*}\Big|T^\varepsilon\nabla u^\varepsilon-\Big(0,\frac{\partial u}{\partial x_2}\Big)-\frac{\partial u}{\partial x_2}\nabla_y\chi_2^{B}\Big|^2\,dxdy\to 0,
\end{aligned}$$

which implies convergences (126)–(127).
We turn now to (128). We write

$$\varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon(x)(u^\varepsilon-u)^2\,ds=\varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon(u^\varepsilon)^2\,ds-\varepsilon\int_{\Gamma^\varepsilon}2\rho^\varepsilon u^\varepsilon u\,ds+\varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon u^2\,ds. \tag{129}$$

By the same arguments used to prove (6.65) and (6.67) in [1], we obtain

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon(u^\varepsilon)^2\,ds=\int_B\rho^0u^2\,dx. \tag{130}$$

and

$$\varepsilon\int_{\Gamma^\varepsilon}\rho^\varepsilon u^\varepsilon u\,ds\to\int_B\rho^0u^2\,dx, \tag{131}$$

as ε tends to zero.
Passing to the limit in (129), using (130)–(131), gives (128). □

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