mathematics

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Insight on Generalizing Forced Proportional Delay Equations via a Developed Analytical Solution

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Abstract

This paper develops a unified analytical approach for a broad class of delay differential equations with proportional delay and an arbitrary forcing term. The models under consideration incorporate a proportional delay characterized by a contraction of the independent time variable, a structure that arises naturally in various applied settings. The proposed approach is based on power series expansions, which yield explicit recursive relations for the solution coefficients and provide a systematic procedure for constructing the corresponding solutions. The resulting solution is naturally decomposed into two complementary components: one associated with the underlying homogeneous delay equation and the other representing the contribution of the external forcing function. In addition, rigorous convergence analysis is presented, establishing the absolute convergence of the resulting series representations. Building upon the general formulation, several exact closed-form solutions are derived for important classes of forcing functions, including polynomial, periodic, and exponential functions. A range of distinguished and limiting cases is also investigated to illustrate the versatility, consistency, and unifying character of the proposed methodology. The results extend and encompass several existing findings in the literature while providing a systematic and effective analytical tool for a broad class of forced delay models.

Keywords: Ambartsumian equation; delay; forcing term; series

MSC: 34K06; 34K07; 65L03

1. Introduction

Delay differential equations (DDEs) constitute an essential class in which the evolution of a system depends not only on its present state but also on its past behavior [1–3]. A wide range of applications for DDEs arise in modeling population dynamics [4], biomedical/biological sciences and engineering [5–7], and applied sciences [8–10], where memory effects or aftereffects cannot be neglected [11]. A particular kind of DDE is formed by equations with proportional delays, where a scaled version of the independent variable is involved [12,13]. Notable models of such a category are often referred to as pantograph-type equations [14–16] and its different forms [17,18]. Additional examples include the Ambartsumian-type equations which are of practical interest in astrophysical studies of radiative transfer [19–21]. Some of its generalizations have been considered in [22–24] and also in a variable form [25]. Applications of pantograph-type equations arise in electrified railway systems [26–28], the current collection system for an electric locomotive [29,30],

Check for updates

Academic Editor: Baccouch Mahboub

Received: 2 September 2026
Revised: 4 October 2026
Accepted: 20 October 2026
Published: 25 October 2026

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analytic number theory [31], and nonlinear dynamical systems [32]. Over time, the Ambartsumian equation and its generalizations have been analyzed utilizing various analytical and numerical techniques. Despite these developments, the major ratio of these studies have focused primarily on the homogeneous forms or those with specific types of forcing terms. So, considering that a general forcing function represents an additional complexity, it remains less explored from an analytical perspective, which may significantly broaden the scope of potential applications. The present work aims to fill this gap by considering the Ambartsumian equation with a general forcing term in the form

$$y'(t) = -y(t) + \frac{\xi}{q}\, y\left(\frac{t}{q}\right) + f(t), \quad y(0) = \lambda, \quad q > 1, \tag{1}$$

where ξ and q are real constants while λ is the initial condition. In the absence of the forcing terms $f(t)$, and if $\xi = 1$, the forced Ambartsumian Equation (FAE) (1) becomes the standard Ambartsumian equation (SAE) [20,21]. The literature is rich with numerous numerical and analytical approaches to treat different kinds of differential equations. For numerical approaches, there are the spectral methods [33], the tau method [34], and the finite difference method [35]. However, numerical approaches are either restricted to particular forms of DDEs with/without a forcing term, require auxiliary transformations or additional assumptions, or provide numerical approximations rather than explicit analytical representations. The analytical approaches include the Laplace transform (LT) [36–38], the Adomian decomposition method (ADM) [39–41], the homotopy perturbation method (HPM) [42–44], the differential transform method (DTM) [45,46], the homotopy analysis method (HAM) [47–49], the power series method (PSM) [50], and the ansatz method [51].

Although there is availability in the varieties of analytical approaches, the PSM is preferred in this work, not only for its simplicity but also for its capability of obtaining closed-form solutions. These observations help identify the methodological gap addressed by the present work. Thus, the main objective is to introduce a unified analytical treatment capable of obtaining closed-form solutions under a variety of analytic forcing functions. To achieve this target, a power series approach is developed to determine the series solution in terms of recursively defined coefficients. This leads to explicit series solutions depending on the system parameters and the derivatives of the forcing term. Beyond, this paper presents a rigorous convergence proof, ensuring the validity of the obtained series representation. Our approach also identifies the conditions under which the series is truncated, leading to exact solutions in different forms. Different types of forcing functions are to be investigated, including polynomial, periodic, and exponential forms, to illustrate the versatility of the proposed approach. The rest of the paper is structured as follows. Section 2 applies the series approach on the problem (1) and also introduces the closed-series solution. A convergence analysis of the resulting solution is addressed in Section 3. Section 4 reformulates the series solution in a compact form. Subsequent Sections 4–7 focus on deriving exact solutions for specific forms of the forcing function and presenting equivalent representations that facilitate further analysis. Additional examples are provided to demonstrate the applicability of the method. This paper concludes with a summary of the main findings and potential directions for future research.

2. Series Approach

The present approach is based on assuming the following series forms for $y(t)$ and $f(t)$:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n, \quad f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n. \tag{2}$$

We have

$$y\left(\frac{t}{q}\right) = \sum_{n=0}^{\infty} \xi q^{-n} a_n t^n, \;\; y'(t) = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n. \tag{3}$$

Substituting into the given equation gives

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n = -\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} \xi q^{-(n+1)} a_n t^n + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n. \tag{4}$$

Hence,

$$\sum_{n=0}^{\infty} t^n \Big[(n+1) a_{n+1} - \Big(\xi q^{-(n+1)} - 1 \Big) a_n - \frac{f^{(n)}(0)}{n!} \Big] = 0, \tag{5}$$

which yields the recurrence relation

$$a_{n+1} = \frac{\xi q^{-(n+1)} - 1}{n+1} a_n + \frac{f^{(n)}(0)}{(n+1)!}, \qquad n \geq 0. \tag{6}$$

For $n = 0, 1, 2$, we obtain

$$\begin{cases} a_1 = (\xi q^{-1} - 1) a_0 + f(0), \\ a_2 = \frac{1}{2}(\xi q^{-2} - 1) a_1 + \frac{1}{2!} f^{(1)}(0) \\ \quad = \frac{1}{2!}(\xi q^{-2} - 1)(\xi q^{-1} - 1) a_0 + \frac{1}{2!} \left[(\xi q^{-2} - 1) f(0) + f^{(1)}(0) \right], \\ a_3 = \frac{1}{3}(\xi q^{-3} - 1) a_2 + \frac{1}{3!} f^{(2)}(0) \\ \quad = \frac{1}{3!}(\xi q^{-1} - 1)(\xi q^{-2} - 1)(\xi q^{-3} - 1) a_0 + \\ \quad \frac{1}{3!} \Big[(\xi q^{-2} - 1)(\xi q^{-3} - 1) f(0) + (\xi q^{-3} - 1) f^{(1)}(0) + f^{(2)}(0) \Big]. \end{cases} \tag{7}$$

The structure of the above first few terms leads to

$$a_n = \frac{a_0}{n!} \prod_{k=1}^{n} (\xi q^{-k} - 1) + \frac{1}{n!} \sum_{j=0}^{n-1} f^{(j)}(0) \prod_{k=j+2}^{n} (\xi q^{-k} - 1), \qquad n \geq 1. \tag{8}$$

Note that $\prod_{k=j+2}^{n} (\xi q^{-k} - 1) = 1$ when $j = n - 1$. Introducing

$$\mu_k = \xi q^{-k} - 1, \tag{9}$$

and noting that $a_0 = \lambda$ (from applying the initial condition), then the coefficients can be rewritten as

$$a_n = \frac{\lambda}{n!} \prod_{k=1}^{n} \mu_k + \frac{1}{n!} \sum_{j=0}^{n-1} f^{(j)}(0) \prod_{k=j+2}^{n} \mu_k. \tag{10}$$

Hence, the solution admits the series representation

$$y(t) = \lambda \sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{k=1}^{n} \mu_k + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n-1} f^{(j)}(0) \prod_{k=j+2}^{n} \mu_k, \tag{11}$$

or simply

$$y(t) = \lambda \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!} + \sum_{n=1}^{\infty} \sigma_n \frac{t^n}{n!}, \tag{12}$$

where

$$\alpha_0 = 1, \;\; \alpha_n = \prod_{k=1}^{n} \mu_k, \;\; \sigma_n = \sum_{j=0}^{n-1} \beta_{j,n} f^{(j)}(0), \;\; \beta_{j,n} = \prod_{k=j+2}^{n} \mu_k, \;\; \mu_k = \xi q^{-k} - 1, \;\; n \geq 1. \tag{13}$$

Remark 1. *The coefficients α_n and σ_n have specific asymptotic behavior as described below. For $|q| > 1$, we have $\mu_k = \xi q^{-k} - 1 \longrightarrow -1$ as $k \to \infty$. Hence, there exists a positive constant C such that $|\mu_k| \leq C, \forall k \geq 1$. Therefore, the product defining α_n satisfies $|\alpha_n| = \prod_{k=1}^{n} |\mu_k| \leq C^n$. Similarly, for $\beta_{j,n}$ we obtain $|\beta_{j,n}| = \prod_{k=j+2}^{n} |\mu_k| \leq C^{n-j-1}$. Thus, the coefficients σ_n satisfy $|\sigma_n| \leq \sum_{j=0}^{n-1} C^{n-j-1} |f^{(j)}(0)|$.*

3. Convergence Analysis

This section analyzes the convergence of the obtained solution consisting of the two series $\sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}$ and $\sum_{n=1}^{\infty} \sigma_n \frac{t^n}{n!}$. For the first series, we assume that $b_n(t) = \alpha_n \frac{t^n}{n!}$. Using the relation $\alpha_{n+1} = \alpha_n \mu_{n+1}$, we observe by the ratio test that

$$\left| \frac{b_{n+1}(t)}{b_n(t)} \right| = |\mu_{n+1}| \cdot \frac{|t|}{n+1}. \tag{14}$$

Since $\mu_{n+1} = \xi q^{-(n+1)} - 1 \longrightarrow -1$ as $n \to \infty$ ($|q| > 1$), it follows that $|\mu_{n+1}|$ is bounded. Accordingly,

$$\lim_{n \to \infty} \left| \frac{b_{n+1}(t)}{b_n(t)} \right| = 0. \tag{15}$$

Moreover, suppose for the second series that $c_n(t) = \sigma_n \frac{t^n}{n!}$, then

$$\left| \frac{c_{n+1}(t)}{c_n(t)} \right| = \frac{|\sigma_{n+1}|}{|\sigma_n|} \cdot \frac{|t|}{n+1}. \tag{16}$$

From the formulas $\sigma_n = \sum_{j=0}^{n-1} \beta_{j,n} f^{(j)}(0)$ and $\beta_{j,n} = \prod_{k=j+2}^{n} \mu_k$, we have $\beta_{j,n+1} = \mu_{n+1} \beta_{j,n}$ and

$$\sigma_{n+1} = \mu_{n+1} \sum_{j=0}^{n-1} \beta_{j,n} f^{(j)}(0) + f^{(n)}(0) = \mu_{n+1} \sigma_n + f^{(n)}(0). \tag{17}$$

Therefore,

$$\frac{\sigma_{n+1}}{\sigma_n} = \mu_{n+1} + \frac{f^{(n)}(0)}{\sigma_n}. \tag{18}$$

Assume that the function $f(t)$ is analytic in a neighborhood of $t = 0$, i.e., there exist constants $M, R > 0$ such that $|f^{(n)}(0)| \leq M \frac{n!}{R^n}$.

From Remark 1, we have $|\sigma_n| \leq C^{n-1} \sum_{j=0}^{n-1} C^{-j} |f^{(j)}(0)|$, where $C > 0$. Consequently, $\frac{f^{(n)}(0)}{\sigma_n} \longrightarrow 0$. Thus,

$$\lim_{n \to \infty} \frac{\sigma_{n+1}}{\sigma_n} = \lim_{n \to \infty} \mu_{n+1} = -1, \tag{19}$$

and therefore,

$$\lim_{n \to \infty} \left| \frac{c_{n+1}(t)}{c_n(t)} \right| = \lim_{n \to \infty} |\mu_{n+1}| \cdot \frac{|t|}{n+1} = 0. \tag{20}$$

By the ratio test, the series $\sum_{n=1}^{\infty} c_n(t)$ converges absolutely for all $t \in \mathbb{R}$.

4. Equivalent Expansion Form

Expanding the series (12), one can write

$$y(t) = \lambda \left(\alpha_0 + \alpha_1 t + \alpha_2 \frac{t^2}{2!} + \alpha_3 \frac{t^3}{3!} + \cdots \right) + \sigma_1 t + \sigma_2 \frac{t^2}{2!} + \sigma_3 \frac{t^3}{3!} + \cdots, \tag{21}$$

From the recurrence relation $\sigma_{n+1} = \mu_{n+1}\sigma_n + f^{(n)}(0)$, it then follows

$$\begin{aligned} \sigma_1 &= f(0), \\ \sigma_2 &= \mu_2\sigma_1 + f'(0), \\ \sigma_3 &= \mu_3\sigma_2 + f''(0), \\ &\vdots \end{aligned} \tag{22}$$

So

$$y(t) = \lambda\left(\alpha_0 + \alpha_1 t + \alpha_2\frac{t^2}{2!} + \alpha_3\frac{t^3}{3!} + \cdots\right) + f(0)t + (\mu_2\sigma_1 + f'(0))\frac{t^2}{2!} + (\mu_3\sigma_2 + f''(0))\frac{t^3}{3!} + \cdots, \tag{23}$$

which can be written as

$$y(t) = \lambda(\alpha_0 + \alpha_1 t) + (\lambda\alpha_2 + \mu_2\sigma_1)\frac{t^2}{2!} + (\lambda\alpha_3 + \mu_3\sigma_2)\frac{t^3}{3!} + \cdots + \left(f(0)t + f'(0)\frac{t^2}{2!} + f''(0)\frac{t^3}{3!} + \cdots\right), \tag{24}$$

i.e.,

$$y(t) = \lambda(\alpha_0 + \alpha_1 t) + \int_0^t f(s)\,ds + \sum_{n=2}^{\infty}(\lambda\alpha_n + \mu_n\sigma_{n-1})\frac{t^n}{n!}. \tag{25}$$

Inserting $\alpha_0 = 1$ and $\alpha_1 = \mu_1$ into Equation (25) and using the recursive relation $\alpha_{n+1} = \mu_{n+1}\alpha_n$, we obtain

$$y(t) = \lambda(1 + \mu_1 t) + \int_0^t f(s)\,ds + \sum_{n=1}^{\infty}(\lambda\alpha_n + \sigma_n)\mu_{n+1}\frac{t^{n+1}}{(n+1)!}. \tag{26}$$

The solution in this form facilitates the derivation of various solutions for the problem (1) once $f(t)$ is assigned. Although Formulas (26) provide an explicit series solution representation in terms of convergent infinite series, its direct implementation in numerical computations necessarily requires truncation to a finite number of terms.

Furthermore, it will be used in the next section to establish several exact solutions when $f(t)$ takes the form of polynomials of different degrees. In this context, we start our next section with the simplest case in which $f(t)$ represents a constant function, just to make our analysis as clear as possible. Then, in subsequent sections, the results are to be generalized to include polynomial functions of higher degrees. Additionally, a unified solution form is to be provided when $f(t)$ is a polynomial of an arbitrary positive degree.

5. Special Cases vs. Exact Solutions

5.1. $f(t) = d_0$

Let $f(t) = d_0$ (constant), then $f(0) = d_0$ and $f^{(j)}(0) = 0\ \forall j \geq 1$. Thus,

$$\sigma_n = d_0\prod_{k=2}^{n}\mu_k, \quad \alpha_n = \prod_{k=1}^{n}\mu_k. \tag{27}$$

Substituting into Equation (26) gives

$$y(t) = \lambda(1 + \mu_1 t) + d_0 t + \sum_{n=1}^{\infty}\left(\lambda\prod_{k=1}^{n}\mu_k + d_0\prod_{k=2}^{n}\mu_k\right)\mu_{n+1}\frac{t^{n+1}}{(n+1)!}, \tag{28}$$

or

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + (\lambda\mu_1 + d_0)\sum_{n=1}^{\infty}\prod_{k=2}^{n+1}\mu_k\frac{t^{n+1}}{(n+1)!}. \tag{29}$$

If $\lambda\mu_1 + d_0 = 0$, i.e., $\mu_1 = -\frac{d_0}{\lambda}$ or $\xi q^{-1} = 1 - \frac{d_0}{\lambda}$. This yields $y(t) = \lambda$ as the trivial solution to $y'(t) = -y(t) + \left(1 - \frac{d_0}{\lambda}\right) y\left(\frac{t}{q}\right) + d_0,\ y(0) = \lambda$. Moreover, if $\mu_2 = 0$, i.e., $\xi = q^2$, then another exact form arises as

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t,\ \mu_1 = q - 1, \tag{30}$$

for

$$y'(t) = -y(t) + q\, y\left(\frac{t}{q}\right) + d_0, \quad y(0) = \lambda. \tag{31}$$

Similarly, if $\mu_3 = 0$, i.e., $\xi = q^3$, we get the exact solution

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + (\lambda\mu_1 + d_0)\mu_2 \frac{t^2}{2!},\ \mu_1 = q^2 - 1,\ \mu_2 = q - 1, \tag{32}$$

for

$$y'(t) = -y(t) + q^2\, y\left(\frac{t}{q}\right) + d_0, \quad y(0) = \lambda. \tag{33}$$

Generally, if $\mu_r = 0$ or $\xi = q^r$ $(r \geq 2)$, we obtain

$$y(t) = \lambda + (\lambda\mu_1 + d_0) \sum_{n=0}^{r-2} \prod_{k=2}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}, \quad \mu_k = q^{r-k} - 1, \tag{34}$$

as the exact solution for

$$y'(t) = -y(t) + q^{r-1}\, y\left(\frac{t}{q}\right) + d_0, \quad y(0) = \lambda. \tag{35}$$

5.2. $f(t) = d_0 + d_1 t$

If $f(t) = d_0 + d_1 t$, then $f(0) = d_0$, $f^{(1)}(0) = d_1$, and $f^{(j)}(0) = 0\ \forall j \geq 2$. From the definition of σ_n, we have

$$\sigma_1 = f(0) = d_0,\ \sigma_n = d_0 \prod_{k=2}^{n} \mu_k + d_1 \prod_{k=3}^{n} \mu_k,\ n \geq 2. \tag{36}$$

Hence,

$$(\lambda\alpha_n + \sigma_n)\mu_{n+1} = (\lambda\mu_1\mu_2 + d_0\mu_2 + d_1) \prod_{k=3}^{n+1} \mu_k,\ n \geq 2. \tag{37}$$

Therefore, Equation (26) becomes

$$y(t) = \lambda(1 + \mu_1 t) + d_0 t + d_1 \frac{t^2}{2!} + (\lambda\mu_1 + d_0)\mu_2 \frac{t^2}{2!} + [(\lambda\mu_1 + d_0)\mu_2 + d_1] \sum_{n=2}^{\infty} \prod_{k=3}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}, \tag{38}$$

or

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + [(\lambda\mu_1 + d_0)\mu_2 + d_1] \sum_{n=1}^{\infty} \prod_{k=3}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}. \tag{39}$$

It is clear that the selection $d_1 = -(\lambda\mu_1 + d_0)\mu_2$ leads to the exact solution $y(t) = \lambda + (\lambda\mu_1 + d_0)t$ for $y'(t) = -y(t) + \frac{\xi}{q} y\left(\frac{t}{q}\right) + d_0 - (\lambda\mu_1 + d_0)\mu_2 t,\ y(0) = \lambda$, where $\mu_1 = \xi/q - 1$ and $\mu_2 = \xi/q^2 - 1$.

Additionally, if $\mu_3 = 0$, i.e., $\xi = q^3$, we obtain

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + [(\lambda\mu_1 + d_0)\mu_2 + d_1]\frac{t^2}{2!},\ \mu_1 = q^2 - 1,\ \mu_2 = q - 1, \tag{40}$$

for

$$y'(t) = -y(t) + q^2\, y\left(\frac{t}{q}\right) + d_0 + d_1 t, \quad y(0) = \lambda. \tag{41}$$

Also, when $\mu_4 = 0$ or $\xi = q^4$, the exact solution reads

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + [(\lambda\mu_1 + d_0)\mu_2 + d_1]\left(\frac{t^2}{2!} + \mu_3\frac{t^3}{3!}\right), \tag{42}$$

with

$$\begin{cases} \mu_1 = q^3 - 1, \\ \mu_2 = q^2 - 1, \\ \mu_3 = q - 1, \end{cases} \tag{43}$$

for

$$y'(t) = -y(t) + q^3\, y\left(\frac{t}{q}\right) + d_0 + d_1 t, \quad y(0) = \lambda. \tag{44}$$

For $m \geq 3$ and $\mu_m = 0$ or $\xi = q^m$, one can derive

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\sum_{n=0}^{m-3}\prod_{k=3}^{n+2}\mu_k\frac{t^{n+2}}{(n+2)!}, \quad \mu_k = q^{m-k} - 1, \tag{45}$$

as the exact solution of the problem

$$y'(t) = -y(t) + q^{m-1}\, y\left(\frac{t}{q}\right) + d_0 + d_1 t, \quad y(0) = \lambda. \tag{46}$$

5.3. $f(t) = d_0 + d_1 t + d_2 t^2$

This section considers $f(t) = d_0 + d_1 t + d_2 t^2$ which implies $f(0) = d_0$, $f^{(1)}(0) = d_1$, $f^{(2)}(0) = 2d_2$, and $f^{(j)}(0) = 0\ \forall j \geq 3$. This gives

$$\begin{cases} \sigma_1 = f(0) = d_0, \\ \sigma_2 = \mu_2\sigma_1 + f^{(1)}(0) = \mu_2 d_0 + d_1, \\ \sigma_n = d_0\prod_{k=2}^{n}\mu_k + d_1\prod_{k=3}^{n}\mu_k + 2d_2\prod_{k=4}^{n}\mu_k,\ n \geq 3. \end{cases} \tag{47}$$

Accordingly, we find

$$(\lambda\alpha_n + \sigma_n)\mu_{n+1} = \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right]\prod_{k=4}^{n+1}\mu_k,\ n \geq 3. \tag{48}$$

Thus, the solution takes the form

$$y(t) = \lambda(1 + \mu_1 t) + d_0 t + d_1\frac{t^2}{2!} + 2d_2\frac{t^3}{3!} + (\lambda\mu_1 + d_0)\mu_2\frac{t^2}{2!} + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\mu_3\frac{t^3}{3!} + \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right]\sum_{n=3}^{\infty}\prod_{k=4}^{n+1}\mu_k\frac{t^{n+1}}{(n+1)!}. \tag{49}$$

i.e.,

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\frac{t^2}{2!} + \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right] \times \sum_{n=2}^{\infty}\prod_{k=4}^{n+1}\mu_k \frac{t^{n+1}}{(n+1)!}. \tag{50}$$

From this equation, one can find the following exact form when $\mu_4 = 0$ ($\xi = q^4$):

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\frac{t^2}{2!} + \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right]\frac{t^3}{3!}. \tag{51}$$

for the problem

$$y'(t) = -y(t) + q^3 y\left(\frac{t}{q}\right) + d_0 + d_1 t + d_2 t^2, \quad y(0) = \lambda. \tag{52}$$

Moreover, if $\mu_5 = 0$ ($\xi = q^5$), we get

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\frac{t^2}{2!} + \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right]\left(\frac{t^3}{3!} + \mu_4\frac{t^4}{4!}\right), \tag{53}$$

with

$$\begin{cases} \mu_1 = q^4 - 1, \\ \mu_2 = q^3 - 1, \\ \mu_3 = q^2 - 1, \\ \mu_4 = q - 1, \end{cases} \tag{54}$$

for

$$y'(t) = -y(t) + q^4 y\left(\frac{t}{q}\right) + d_0 + d_1 t + d_2 t^2, \quad y(0) = \lambda. \tag{55}$$

Repeating this analysis l-times ($l \geq 4$) under the condition $\mu_l = 0$ ($\xi = q^l$), we obtain

$$y(t) = \lambda + (\lambda\mu_1 + d_0)t + \left[(\lambda\mu_1 + d_0)\mu_2 + d_1\right]\frac{t^2}{2!} + \left[(\lambda\mu_1 + d_0)\mu_2\mu_3 + \mu_3 d_1 + 2d_2\right] \times \sum_{n=0}^{l-4}\prod_{k=4}^{n+3}\mu_k \frac{t^{n+3}}{(n+3)!}, \quad \mu_k = q^{l-k} - 1, \tag{56}$$

as the exact solution of the problem

$$y'(t) = -y(t) + q^{l-1} y\left(\frac{t}{q}\right) + d_0 + d_1 t + d_2 t^2, \quad y(0) = \lambda. \tag{57}$$

5.4. $f(t) = \sum_{i=0}^{\rho} d_i t^i$

This section generalizes the obtained results in the previous sections by considering the general polynomial function $f(t) = \sum_{i=0}^{\rho} d_i t^i = d_0 + d_1 t + d_2 t^2 + \cdots + d_\rho t^\rho$, where ρ is positive and finite. In this case, we have $f(0) = d_0$, $f'(0) = d_1$, $f''(0) = 2!d_2$, $\cdots$, $f^{(\rho)}(0) = \rho! d_\rho$, and $f^{(j)}(0) = 0\ \forall j \geq \rho + 1$. Recall that $\sigma_n = \sum_{j=0}^{n-1}\beta_{j,n} f^{(j)}(0)$ where $\beta_{j,n} = \prod_{k=j+2}^{n}\mu_k$, one can decompose σ_n in the form

$$\sigma_n = \sum_{j=0}^{\rho} \beta_{j,n} f^{(j)}(0) + \sum_{j=\rho+1}^{n-1} \beta_{j,n} f^{(j)}(0), \quad n \geq 2, \tag{58}$$

where $\sigma_1 = f(0)$ by definition. Since $f^{(j)}(0) = 0 \ \forall j \geq \rho + 1$, then the summation truncates to

$$\sigma_n = \sum_{j=0}^{\rho} \beta_{j,n} f^{(j)}(0). \tag{59}$$

Thus,

$$\sigma_n = \beta_{0,n} d_0 + \beta_{1,n} d_1 + \beta_{2,n}(2! d_2) + \cdots + \beta_{\rho,n}(\rho! d_\rho) \tag{60}$$

$$= d_0 \prod_{k=2}^{n} \mu_k + d_1 \prod_{k=3}^{n} \mu_k + 2! d_2 \prod_{k=4}^{n} \mu_k + \cdots + \rho! d_\rho \prod_{k=p+2}^{n} \mu_k. \tag{61}$$

Factorizing leads to

$$\sigma_n = \left[d_0 \prod_{k=2}^{\rho+1} \mu_k + d_1 \prod_{k=3}^{\rho+1} \mu_k + \cdots + \rho! d_\rho \right] \prod_{k=\rho+2}^{n} \mu_k, \tag{62}$$

or equivalently

$$\sigma_n = \sum_{j=0}^{\rho} j! d_j \left(\prod_{k=j+2}^{\rho+1} \mu_k \right) \left(\prod_{k=\rho+2}^{n} \mu_k \right). \tag{63}$$

On the other hand, we express α_n as

$$\alpha_n = \prod_{k=1}^{n} \mu_k = \left(\prod_{k=1}^{\rho+1} \mu_k \right) \left(\prod_{k=\rho+2}^{n} \mu_k \right). \tag{64}$$

Therefore,

$$(\lambda \alpha_n + \sigma_n) \mu_{n+1} = \left[\lambda \prod_{k=1}^{\rho+1} \mu_k + \sum_{j=0}^{\rho} j! d_j \prod_{k=j+2}^{\rho+1} \mu_k \right] \prod_{k=\rho+2}^{n+1} \mu_k, \quad n \geq 2. \tag{65}$$

Hence, the solution is finally obtained as

$$y(t) = \lambda(1 + \mu_1 t) + (\lambda \mu_1 + d_0) \mu_2 \frac{t^2}{2!} + \sum_{i=0}^{\rho} d_i \frac{t^{i+1}}{i+1} + \left[\lambda \prod_{k=1}^{\rho+1} \mu_k + \sum_{j=0}^{\rho} j! d_j \prod_{k=j+2}^{\rho+1} \mu_k \right] \times \sum_{n=2}^{\infty} \prod_{k=\rho+2}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}. \tag{66}$$

The solution can take the compact form:

$$y(t) = \lambda(1 + \mu_1 t) + (\lambda \mu_1 + d_0) \mu_2 \frac{t^2}{2!} + \sum_{i=0}^{\rho} d_i \frac{t^{i+1}}{i+1} + \Omega \sum_{n=2}^{\infty} \prod_{k=\rho+2}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}, \tag{67}$$

where

$$\Omega = \lambda \prod_{k=1}^{\rho+1} \mu_k + \sum_{j=0}^{\rho} j! d_j \prod_{k=j+2}^{\rho+1} \mu_k. \tag{68}$$

It is quite obvious that the obtained solutions in the previous sections can be recovered as special cases from the compact form (67)–(68) by just setting $\rho = 0$, $\rho = 1$, and $\rho = 2$, respectively.

Remark 2. *The implemented assumptions in this section arise naturally from the algebraic structure of the transformed differential equation and are required to obtain the corresponding explicit closed-form representations. These assumptions are used to separate different parameter regimes and they do not invalidate the general analytical formulation developed in the preceding sections. The cases excluded by a particular non-degeneracy assumption correspond to special or limiting parameter configurations and can be considered separately when necessary.*

6. Verification

In Ref. [52], the problem

$$y'(t) = ay(t) + by(ct) + \sum_{i=0}^{\rho} d_i t^i, \quad y(0) = \lambda, \quad |c| < 1, \tag{69}$$

has been solved in the form

$$y(t) = \sum_{i=0}^{\rho} \chi_i t^i + (\lambda - \chi_0) \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \left(a + bc^{k-1} \right) \right) \frac{t^n}{n!}. \tag{70}$$

where the coefficients χ_i were recurrently given by

$$\begin{cases} \chi_i = -\sum_{j=0}^{\rho-i-1} \frac{\prod_{k=j}^{\rho-i-1} (\rho-k)}{\prod_{k=i}^{\rho-j} \left(a+bc^k \right)} d_{\rho-j} - \frac{d_i}{a+bc^i}, & \text{if } \rho \geq 2, \\ \chi_0 = -\frac{d_1}{(a+b)(a+bc)} - \frac{d_0}{a+b}, \quad \chi_1 = -\frac{d_1}{a+bc}, & \text{if } \rho = 1, \\ \chi_0 = -\frac{d_0}{a+b}, & \text{if } \rho = 0, \end{cases} \tag{71}$$

assuming that $a + bc^i$ is not zero at $i = 0, 1, \ldots, \rho$. Setting $a = -1$, $b = \xi/q$, and $c = 1/q$ in Equations (70) and (71), noting that $a + bc^{k-1} = \xi q^{-k} - 1 = \mu_k$, yields

$$y(t) = \sum_{i=0}^{\rho} \chi_i t^i + (\lambda - \chi_0) \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!}, \tag{72}$$

with

$$\begin{cases} \chi_i = -\sum_{j=0}^{\rho-i-1} \frac{\prod_{k=j}^{\rho-i-1} (\rho-k)}{\prod_{k=i}^{\rho-j} \mu_{k+1}} d_{\rho-j} - \frac{d_i}{\mu_{i+1}}, & \text{if } \rho \geq 2, \\ \chi_0 = -\frac{d_1}{\mu_1 \mu_2} - \frac{d_0}{\mu_1}, \quad \chi_1 = -\frac{d_1}{\mu_2}, & \text{if } \rho = 1, \\ \chi_0 = -\frac{d_0}{\mu_1}, & \text{if } \rho = 0, \end{cases} \tag{73}$$

where $\xi q^{-(k+1)} - 1 = \mu_{k+1}$ and $a + bc^i = \xi q^{-(i+1)} - 1 = \mu_{i+1}$. In order to compare between our solution and the above one reported in [52], it may be sufficient to examine some cases such as the constant function ($\rho = 0$), the first-degree polynomial ($\rho = 1$), and the second-degree polynomial ($\rho = 2$). Of course, other cases for higher values of ρ can also be checked as desired.

For $\rho = 0$, Equations (72) and (73) give

$$\begin{cases} y(t) = \chi_0 + (\lambda - \chi_0) \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!}, \\ \chi_0 = -\frac{d_0}{\mu_1}. \end{cases} \tag{74}$$

Hence, the solution becomes

$$y(t) = \chi_0 + (\lambda - \chi_0) + (\lambda - \chi_0) \sum_{n=1}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!} = \lambda + \left(\lambda + \frac{d_0}{\mu_1}\right) \mu_1 \sum_{n=1}^{\infty} \prod_{k=2}^{n} \mu_k \frac{t^n}{n!}$$
$$= \lambda + (\lambda\mu_1 + d_0)t + (\lambda\mu_1 + d_0) \sum_{n=1}^{\infty} \prod_{k=2}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}, \tag{75}$$

which agrees with our result in Equation (29).

For $\rho = 1$, the solution is

$$\begin{cases} y(t) = \chi_0 + \chi_1 t + (\lambda - \chi_0) \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!}, \\ \chi_0 = -\frac{d_1}{\mu_1\mu_2} - \frac{d_0}{\mu_1}, \quad \chi_1 = -\frac{d_1}{\mu_2}, \end{cases} \tag{76}$$

or

$$y(t) = \lambda + \chi_1 t + (\lambda - \chi_0) \sum_{n=1}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!} = \lambda + \chi_1 t + (\lambda - \chi_0)\mu_1 t + (\lambda - \chi_0)\mu_1 \sum_{n=2}^{\infty} \prod_{k=2}^{n} \mu_k \frac{t^n}{n!},$$
$$= \lambda + (\lambda\mu_1 + \chi_1 - \chi_0\mu_1)t + (\lambda - \chi_0)\mu_1\mu_2 \sum_{n=2}^{\infty} \prod_{k=3}^{n} \mu_k \frac{t^n}{n!},$$
$$= \lambda + (\lambda\mu_1 + d_0)t + [(\lambda\mu_1 + d_0)\mu_2 + d_1] \sum_{n=1}^{\infty} \prod_{k=3}^{n+1} \mu_k \frac{t^{n+1}}{(n+1)!}. \tag{77}$$

This result corresponds to our solution in Equation (39).

For $\rho = 2$, the solution is

$$\begin{cases} y(t) = \chi_0 + \chi_1 t + \chi_2 t^2 + (\lambda - \chi_0) \sum_{n=0}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!}, \\ \chi_i = -\sum_{j=0}^{1-i} \frac{\prod_{k=j}^{1-i}(2-k)}{\prod_{k=i}^{2-j} \mu_{k+1}} d_{2-j} - \frac{d_i}{\mu_{i+1}}, \quad i = 0, 1, 2. \end{cases} \tag{78}$$

or

$$y(t) = \lambda + \chi_1 t + \chi_2 t^2 + (\lambda - \chi_0) \sum_{n=1}^{\infty} \prod_{k=1}^{n} \mu_k \frac{t^n}{n!}$$
$$= \lambda + (\lambda\mu_1 + \chi_1 - \chi_0\mu_1)t + \left[2\chi_2 + (\lambda - \chi_0)\mu_1\mu_2\right] \frac{t^2}{2!} +$$
$$(\lambda - \chi_0)\mu_1\mu_2\mu_3 \sum_{n=3}^{\infty} \prod_{k=4}^{n} \mu_k \frac{t^n}{n!}. \tag{79}$$

From the second equation in (78), the expressions for χ_0, χ_1, and χ_2 can be determined as

$$\chi_0 = -\frac{1}{\mu_1\mu_2\mu_3}(\mu_3 d_1 + 2d_2) - \frac{d_0}{\mu_1},$$
$$\chi_1 = -\frac{2d_2}{\mu_2\mu_3} - \frac{d_1}{\mu_2}, \qquad \chi_2 = -\frac{d_2}{\mu_3}. \tag{80}$$

By inserting these expressions into Equation (79) and simplifying, we obtain the same result given in Equation (50).

Remark 3. *This section confirms the agreement of our results with the corresponding ones in Ref. [52] when $f(t)$ is of polynomial type. It was seen that the results in the literature were recovered as special cases of the present ones. However, this work offers advantages over Ref. [52], including*

its direct applicability to deduce compact series solutions for other types of analytic functions $f(t)$. For declaration, the next sections address this point for the two other types of $f(t)$ in periodic and exponential forms.

7. Solutions for Other Types of $f(t)$

7.1. Periodic Forced Term: $f(t) = \Omega \sin(\omega t)$

Suppose that $f(t) = \Omega \sin(\omega t)$; then the derivatives at $t = 0$ are

$$\begin{aligned} f^{(2j)}(0) &= 0, \\ f^{(2j+1)}(0) &= \Omega(-1)^j \omega^{2j+1}, \quad j \geq 0. \end{aligned} \tag{81}$$

Since only odd derivatives are nonzero, then

$$\sigma_n = \sum_{j=0}^{n-1} \beta_{j,n} f^{(j)}(0) = \sum_{j=0}^{\left[\frac{n}{2}\right]-1} \beta_{2j+1,n} f^{(2j+1)}(0), \tag{82}$$

where $\left[\frac{n}{2}\right]$ equals the greatest integer less than or equal to $\frac{n}{2}$. Thus,

$$\sigma_n = \Omega \sum_{j=0}^{\left[\frac{n}{2}\right]-1} (-1)^j \omega^{2j+1} \prod_{k=2j+3}^{n} \mu_k. \tag{83}$$

The solution becomes

$$y(t) = \lambda(1 + \mu_1 t) + \int_0^t \Omega \sin(\omega s)\, ds + \sum_{n=1}^{\infty} (\lambda \alpha_n + \sigma_n) \mu_{n+1} \frac{t^{n+1}}{(n+1)!}. \tag{84}$$

Hence,

$$y(t) = \lambda(1 + \mu_1 t) + \frac{\Omega}{\omega}(1 - \cos(\omega t)) + \sum_{n=1}^{\infty} \left[\lambda \prod_{k=1}^{n+1} \mu_k + \Omega \sum_{j=0}^{\left[\frac{n}{2}\right]-1} (-1)^j \omega^{2j+1} \prod_{k=2j+3}^{n+1} \mu_k \right] \frac{t^{n+1}}{(n+1)!}. \tag{85}$$

7.2. Exponential Form: $f(t) = \eta e^{\nu t}$

Let $f(t) = \eta e^{\nu t}$, then

$$f^{(j)}(0) = \eta \nu^j, \quad j \geq 0, \tag{86}$$

and

$$\sigma_n = \eta \sum_{j=0}^{n-1} \nu^j \prod_{k=j+2}^{n} \mu_k. \tag{87}$$

This leads to the compact form solution

$$y(t) = \lambda(1 + \mu_1 t) + \frac{\eta}{\nu}(e^{\nu t} - 1) + \sum_{n=1}^{\infty} \left[\lambda \prod_{k=1}^{n+1} \mu_k + \eta \sum_{j=0}^{n-1} \nu^j \prod_{k=j+2}^{n+1} \mu_k \right] \frac{t^{n+1}}{(n+1)!}. \tag{88}$$

Based on the obtained results, the present approach offers a unified treatment to handle a wide class of the FAE with different types of forcing functions including polynomial of arbitrary degree, periodic term, and an exponential term. The main advantage lies in expressing σ_n in terms of the derivatives of any analytic function $f(t)$ at the initial time, i.e., $f^{(j)}(0)$, $j \geq 0$, which significantly simplifies the analysis.

8. Conclusions

This paper introduced a comprehensive study on solving an Ambartsumian-type delay equation with a general forcing function. A power series approach was adopted to derive an explicit representation of the series solution with prescribed coefficients in terms of the model's parameters and the derivatives of the forcing function. A rigorous convergence analysis was presented, showing that the obtained series solution is convergent under broad conditions, ensuring its applicability over the problem's domain. The developed analysis was capable of deriving exact solutions in different forms when the forcing function satisfies specific structural conditions. The results demonstrate the effectiveness and efficiency of the proposed unified approach, making it possible to analyze a wide class of DDEs with proportional delays. In addition to recovering the known results in the literature as special cases, the analysis extends the applicability to more general forcing functions, offering new insights and solution forms. This suggests exploring extensions of the developed approach to include nonlinear models in addition to systems of delay equations as future works. In this context, the symmetry reduction approach is useful for obtaining exact analytical solutions of invariant DDEs, as pointed out by Dorodnitsyn et al. [53].

Funding: The author extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2026-504).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

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