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Complete Bernstein Functions from Gamma Ratios with Square Root Arguments

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Abstract

We study Gamma ratios evaluated at a shifted square root and finite products of their positive powers. For each ratio, we prove the complete Bernstein property and compute the representing measure explicitly. The symmetry of the Gamma parameters gives a simple formula for the boundary phase, which determines when a product remains a complete Bernstein function. For orders of, at most, one half, the exact condition is that the weighted sum of the orders does not exceed one. For mixed orders, we give an exact boundary-phase criterion and an explicit sufficient condition, and we explain the role of complementary orders. We also determine the killing and drift coefficients, prove absolute continuity of the representing measures, and derive the corresponding Lévy densities. These formulas yield resolvent representations and subordinate semigroups for non-negative self-adjoint operators.

Keywords: complete Bernstein function; Gamma ratio; Pick function; Stieltjes representation; Lévy measure; spectral multiplier; subordination

MSC: 26A48; 33B15; 30H10; 47A60; 60G51

1. Introduction

Complete Bernstein functions connect complex analysis, spectral theory and probability. Their holomorphic extensions preserve the upper half plane, their Stieltjes representations express them as positive superpositions of resolvent multipliers, and their Lévy and Khintchine representations generate subordinate semigroups. The model function (λ^s, with $0 < s < 1$) exhibits all three properties. For more general multipliers, the same structure may be less apparent, and an explicit representing measure contains information that a monotonicity statement alone does not provide.

The Gamma function offers a natural setting for this question. Inequalities and complete monotonicity for Gamma ratios were studied by Bustoz and Ismail [1] and Grinshpan and Ismail [2]. Berg and Pedersen developed methods based on Pick functions and complete monotonicity for functions related to the Gamma function [3,4]; Berg obtained further explicit Stieltjes representations [5]. More recently, Berg, Çetinkaya and Karp [6] investigated logarithmic complete monotonicity of products of ordinary and basic Gamma ratios with linear arguments. These works provide the analytic background for the present study. We use the general theory of Bernstein functions proposed in [7]. Berg [5] obtained Stieltjes integral representations for the function expressed as

$$\Phi(x) = \frac{\Gamma(x+1)^{1/x}(1+1/x)^x}{x}$$

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and its logarithm. Berg, Çetinkaya and Karp [6] studied products of Gamma ratios and basic Gamma ratios whose arguments are linear functions of the variable. They gave conditions for these products to be logarithmically completely monotone. These conditions depend on the sign of an explicit elementary function. These results are related to our work, but the problems are different. Previous papers studied Stieltjes representations and logarithmic complete monotonicity. In this paper, we prove that the functions in (2) and certain products of their positive powers are complete Bernstein functions. In our case, the expression

$$\sqrt{(1/2-s)^2+z/\kappa^2}$$

determines a spectral endpoint. By using the Gamma reflection formula, we calculate the boundary values and the continuous argument of the function on both sides of this endpoint. This allows us to obtain the explicit Stieltjes density (8). For products, we obtain the sharp condition (25), the exact phase condition (36), and the behavior of the drift when the total order is one. These results cannot be obtained simply by applying the formula for a single factor, since a product of complete Bernstein functions is not always a complete Bernstein function. The general results on Pick functions and Stieltjes inversion are already known. Our main contributions are the explicit formulas and the exact conditions for products.

Our concern is a specific square-root substitution and the product problem it creates. Related spectral formulas occur in the study of fractional GJMS operators on hyperbolic spaces [8]. Factorization questions also arise in conformal geometry; see, for example, Case and Malchiodi [9] for GJMS operators on special Einstein products. These geometric works motivate the consideration of products, but no identification with a particular geometric operator is needed for our results.

1.1. Notation and the Basic Multiplier

Fix $0 < s < 1$ and $\kappa > 0$. Throughout the paper, for a single multiplier, we use

$$a = \frac{1}{2}+s, \qquad b = \frac{1}{2}-s, \qquad \nu(z) = \sqrt{b^2+z/\kappa^2}, \qquad h = \kappa|b|. \tag{1}$$

Thus, $a+b=1$ and $a-b=2s$. The square root is the principal one and positive on the positive real axis. We set

$$m_{\kappa,s}(z) = \kappa^{2s}\frac{\Gamma(a+\nu(z))}{\Gamma(b+\nu(z))}, \qquad z \in \Omega_h := \mathbb{C}\setminus(-\infty,-h^2]. \tag{2}$$

At a pole of the denominator, the quotient is understood as $\Gamma(a+\nu(z))(1/\Gamma)(b+\nu(z))$—in particular, the zero at $z=0$ when $s>1/2$ is not a singularity. We write $m = m_{\kappa,s}$ when the parameters are fixed.

For a product, subscripts identify the factors.

$$\begin{gathered} a_j = \tfrac{1}{2}+s_j, \quad b_j = \tfrac{1}{2}-s_j, \quad \nu_j(z) = \sqrt{b_j^2+z/\kappa_j^2}, \quad h_j = \kappa_j|b_j|, \\ m_j = m_{\kappa_j,s_j}, \qquad F(z) = \prod_{j=1}^{N} m_j(z)^{\gamma_j}, \qquad \gamma_j > 0. \end{gathered} \tag{3}$$

Powers are defined on $\mathbb{C}\setminus(-\infty,0]$ using a holomorphic logarithm of each factor, chosen to be real on $(0,\infty)$. The existence of these logarithms is proven below. We reserve q, d for the killing and drift coefficients of a single Bernstein function and Q, D for those of a product. Thus, these coefficients never conflict with a, b in (1). The positive spectral

variable is λ, the Stieltjes integration variable is t, the Lévy integration variable is r, and the semigroup time variable is τ.

The scaling identity, i.e.,

$$m_{\kappa,s}(z) = \kappa^{2s} m_{1,s}(z/\kappa^2) \tag{4}$$

reduces the problem for a single factor to $\kappa = 1$. In a product, however, the relative scales determine different thresholds (h_j^2) and affect the sum of the boundary phases. In particular, if $\theta_{\kappa,s}(t) = \operatorname{Im} \operatorname{Log} m_{\kappa,s}(-t + \mathrm{i}0)$, then

$$\theta_{\kappa,s}(t) = \theta_{1,s}(t/\kappa^2). \tag{5}$$

The phase profile is universal after rescaling; its value at a fixed t generally depends on κ. In particular, different scales can lead to different positive critical values for factors of complementary orders. Thus, the endpoint powers alone cannot determine whether the product is a Pick function (see Example 2).

1.2. Main Results

Our first result gives the complete Bernstein representation, i.e.,

$$m_{\kappa,s}(\lambda) = q_{\kappa,s} + \int_{h^2}^{\infty} \frac{\lambda}{\lambda + t} \rho_{\kappa,s}(t)\, \mathrm{d}t, \tag{6}$$

where

$$q_{\kappa,s} = \begin{cases} \kappa^{2s}/\Gamma(1-2s), & 0 < s < 1/2, \\ 0, & 1/2 \le s < 1, \end{cases} \tag{7}$$

and, for $t > h^2$,

$$\rho_{\kappa,s}(t) = \frac{\kappa^{2s} \sin(\pi s)}{\pi^2 t} |\Gamma(a + \mathrm{i}y)|^2 \sinh(\pi y), \qquad y = \sqrt{t/\kappa^2 - b^2}. \tag{8}$$

There is no drift or singular part in this representation. At $s = 1/2$, it reduces to the usual representation of $\sqrt{\lambda}$, independently of κ.

The product problem is more delicate, since complete Bernstein functions are not closed under arbitrary multiplication. If all $0 < s_j \le 1/2$, we prove the sharp criterion (25): the product is a complete Bernstein function exactly when its total order $\sigma = \sum_{j=1}^{N} \gamma_j s_j$ does not exceed one.

Necessity follows from $F(\lambda) \sim \lambda^{\sigma}$ and the fact that Bernstein functions grow, at most, linearly. Sufficiency follows from the stronger boundary estimate of $0 \le \theta_{\kappa,s}(t) \le \pi s$ for $s \le 1/2$. The density of the representing measure is given by (33); it combines the boundary moduli with the sine of the continuous total phase.

When $\sigma = 1$, the drift coefficient is exactly one; this term must be separated from the boundary density. For mixed orders, we prove an exact criterion in terms of the continuous, unreduced sum of the boundary phases. An explicit sufficient condition is given in (41), with $c(s) = s$ for $0 < s \le 1/2$ and $c(s) = 1$ for $1/2 < s < 1$.

This condition is sharp when every order exceeds one half, but it is not necessary for arbitrary mixed products. Indeed, the complementary identity of $m_{\kappa,s} m_{\kappa,1-s} = \lambda$ gives exact phase cancellation. This distinction separates a useful bound applied to each factor from a classification of all mixed products.

Section 2 establishes the phase and inversion tools. Section 3 proves the representation for a single factor, including the uniform estimates needed at the two ends of the upper half plane. Sections 4 and 5 treat products. Section 6 derives Lévy densities and their leading

behavior. Section 7 gives the operator consequences, and Section 8 discusses the scope of the results.

2. Complete Bernstein Functions and Boundary Phases

2.1. Representations and Elementary Consequences

Definition 1. *A function $(f : (0,\infty) \to [0,\infty))$ is a Bernstein function if it is of class $C^\infty((0,\infty))$ and f' is completely monotone, that is,*

$$(-1)^n f^{(n+1)}(\lambda) \geq 0 \quad (n \geq 0,\ \lambda > 0).$$

It is a complete Bernstein function if its Lévy measure has a completely monotone density.

The Lévy and Khintchine representation is

$$f(\lambda) = q + d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda r})\, \mu(\,\mathrm{d}r), \qquad q, d \geq 0, \tag{9}$$

where the Lévy measure (μ) satisfies

$$\int_{(0,\infty)} (1 \wedge r)\, \mu(\,\mathrm{d}r) < \infty. \tag{10}$$

Here and throughout the paper, $p \wedge q := \min\{p, q\}$ for real numbers p and q. For a complete Bernstein function, $\mu(\,\mathrm{d}r) = k(r)\,\mathrm{d}r$ with $(-1)^n k^{(n)}(r) \geq 0$ for all $n \geq 0$. The constant $q = f(0+)$ is the killing coefficient, and $d = \lim_{\lambda\to\infty} f(\lambda)/\lambda$ is the drift coefficient.

We use the following standard equivalences [7] (Chapter 6).

Proposition 1. *For $f : (0,\infty) \to [0,\infty)$, the following statements are equivalent.*

(i) *f is a complete Bernstein function;*
(ii) *f extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$ and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$;*
(iii) *f has the unique representation of*

$$f(z) = q + dz + \int_{(0,\infty)} \frac{z}{z+t}\, \eta(\,\mathrm{d}t), \qquad \int_{(0,\infty)} \frac{\eta(\,\mathrm{d}t)}{1+t} < \infty, \tag{11}$$

where $q, d \geq 0$ and η is a positive measure.

We call η the Stieltjes representing measure of f. It is distinct from the Lévy measure (μ). Absolute continuity of η is not assumed in the proposition; it will be proven for the multipliers considered here.

For any positive measure (ξ) on $(0,\infty)$, a point $(t_0 > 0)$ is called an atom if $\xi(\{t_0\}) > 0$. Its mass is $\xi(\{t_0\})$. Thus, saying that the Stieltjes measure (η) has no atoms means $\eta(\{t\}) = 0$ for every $t > 0$.

Lemma 1. *Every Bernstein function satisfies $f(\lambda) = O(1+\lambda)$ as $\lambda \to \infty$. If $f(0+) = 0$, then $f(\lambda)/\lambda$ is nonincreasing on $(0,\infty)$.*

Proof. Both facts are standard consequences of the representation of a Bernstein function and its concavity (see [7] (Chapter 3)). Indeed, $1 - e^{-\lambda r} \leq \min\{1, \lambda r\}$ proves the linear bound directly from (9). Concavity and $f(0+) = 0$ give $f(x) \geq (x/y) f(y)$ for $0 < x < y$. □

2.2. A Boundary-Phase Principle with Angular End Behavior

The logarithm used below is an analytic logarithm, normalized to be real on the positive axis. Its imaginary part is a continuous argument; it is not reduced modulo 2π after products are formed.

Proposition 2. *Let g be holomorphic without zeros in $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ and positive on $(0, \infty)$. Suppose its normalized logarithm (L) has a continuous imaginary part ($\phi = \operatorname{Im} L$) on $\overline{\mathbb{H}} \setminus \{0\}$, with*

$$\phi(x) = 0 \ (x > 0), \qquad 0 \le \phi(-t) \le \pi \ (t > 0).$$

Suppose, for some $\alpha_0, \alpha_\infty \in [0, 1]$,

$$\begin{aligned} \sup_{0 \le \omega \le \pi} |\phi(re^{i\omega}) - \alpha_0 \omega| &\longrightarrow 0 \quad (r \downarrow 0), \\ \sup_{0 \le \omega \le \pi} |\phi(Re^{i\omega}) - \alpha_\infty \omega| &\longrightarrow 0 \quad (R \to \infty). \end{aligned} \tag{12}$$

Then, $0 \le \phi(z) \le \pi$ in $\mathbb{H}$, and $\operatorname{Im} g(z) \ge 0$ there. If g also extends holomorphically to the slit plane, it is a complete Bernstein function.

Proof. Fix $z_0 \in \mathbb{H}$ and $\varepsilon > 0$. According to (12), choose $r < |z_0| < R$ so that $-\varepsilon \le \phi \le \pi + \varepsilon$ on both semicircles of radii r and R. The same inequalities hold on the two real boundary segments, by hypothesis. On the bounded half annulus, i.e.,

$$\{z \in \mathbb{H} : r < |z| < R\},$$

ϕ is harmonic and continuous on its closure. The maximum and minimum principles therefore yield $-\varepsilon \le \phi(z_0) \le \pi + \varepsilon$. Letting $\varepsilon \downarrow 0$ proves the bound. Since $g = e^L$, we have $\operatorname{Im} g = e^{\operatorname{Re} L} \sin \phi \ge 0$. The last assertion follows from Proposition 1. □

Remark 1. *The limits in (12) depend on the angle (ω). For example, $g(z) = z^s$ has phase $s\omega$ on every semicircle. Requiring a single constant phase limit at infinity would exclude this basic example and would not justify the product arguments below.*

2.3. Recovering the Entire Representing Measure

Proposition 3. *Let $f \in$ CBF have representation (11). Suppose that on an open interval ($I \subset (0, \infty)$), $f(-t + i\varepsilon)$ converges locally uniformly as $\varepsilon \downarrow 0$ to a continuous finite function ($B(t)$). Then,*

$$\eta|_I = \rho(t)\, dt, \qquad \rho(t) = \frac{\operatorname{Im} B(t)}{\pi t}. \tag{13}$$

Moreover, if $f(-t_0 + i\varepsilon)$ remains bounded as $\varepsilon \downarrow 0$ for some $t_0 > 0$, then $\eta(\{t_0\}) = 0$.

Proof. Divide (11) by z and put $H(z) = f(z)/z$. Then,

$$H(z) = \frac{q}{z} + d + \int_{(0,\infty)} \frac{\eta(du)}{z + u}.$$

For $0 < \ell < v$ with $[\ell, v] \subset I$ and endpoints that are not atoms, Stieltjes inversion gives

$$\eta((\ell, v)) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_\ell^v \operatorname{Im} H(-t + i\varepsilon)\, dt.$$

The boundary convergence is uniform on $[\ell, v]$, since this interval is bounded away from zero. Passing the limit inside the integral yields

$$\eta((\ell, v)) = \int_\ell^v \frac{\operatorname{Im} B(t)}{\pi t}\, \mathrm{d}t.$$

Such intervals determine the measure on I, proving absolute continuity, not merely absence of atoms.

For the last assertion, take imaginary parts directly in (11) at $z = -t_0 + \mathrm{i}\varepsilon$.

$$\operatorname{Im} f(-t_0 + \mathrm{i}\varepsilon) = d\varepsilon + \int_{(0,\infty)} \frac{\varepsilon u}{(u - t_0)^2 + \varepsilon^2}\, \eta(\mathrm{d}u).$$

Every term is non-negative. An atom of mass $c > 0$ at t_0 would contribute ct_0/ε, contradicting boundedness. This argument also excludes cancellation by the rest of the measure. □

In applications, the boundary values are locally uniform away from finitely many positive thresholds. Proposition 3 gives absolute continuity on the complementary intervals; boundedness at the thresholds excludes the remaining possible atoms. The representing measure is defined on $(0, \infty)$, so the value at zero is accounted for separately by q.

3. A Single Gamma Ratio

We use a, b, ν, h, m as in (1) and (2).

3.1. Analyticity and Uniform Asymptotics

Lemma 2. *The m function is holomorphic on Ω_h, positive on $(0, \infty)$, and has no zeros on $\mathbb{C} \setminus (-\infty, 0]$. As $|z| \to \infty$ in the closed upper half plane, with upper boundary values on the negative axis,*

$$m(z) = z^s\big(1 + O(|z|^{-1})\big), \tag{14}$$

uniformly in $0 \le \operatorname{Arg} z \le \pi$.

Proof. For $z \in \Omega_h$, the principal root satisfies $\operatorname{Re} \nu(z) > 0$. Thus, $\operatorname{Re}(a + \nu) > a > 0$, and the numerator has no pole. The reciprocal Gamma function is entire, so the denominator cannot create a pole. This proves holomorphy on all of Ω_h.

A zero would require $b + \nu = -n$ for some integer ($n \ge 0$). Since $b > -1/2$ and $\operatorname{Re} \nu > 0$, values $n \ge 1$ are impossible. For $n = 0$, one must have $b < 0$ and $\nu = -b$, which gives $z = 0$. This point is excluded from the slit plane. For $\lambda > 0$, $\nu(\lambda) > |b|$, so both Gamma arguments are positive, and $m(\lambda) > 0$.

The Gamma ratio expansion [10] (§5.11(iii)) is uniform on each fixed sector ($|\operatorname{Arg} w| \le \pi - \delta$) and gives

$$\frac{\Gamma(w + a)}{\Gamma(w + b)} = w^{a-b}\left(1 + \frac{(a - b)(a + b - 1)}{2w} + O(|w|^{-2})\right).$$

Since $a + b = 1$, the term in w^{-1} vanishes. For z in the closed upper half plane with a sufficiently large modulus, $w = \nu(z)$ lies in the closed first quadrant, and $|w| \asymp |z|^{1/2}$. Thus, all these values of w lie in one fixed sector $|\operatorname{Arg} w| \le \pi - \delta$ with $0 < \delta < \pi/2$, including those on the positive imaginary axis. Consequently, the remainder is uniform up to $\operatorname{Arg} z = \pi$.

Furthermore, with consistent branches in the upper half plane,

$$\nu(z) = \kappa^{-1} z^{1/2}\left(1 + \frac{\kappa^2 b^2}{z}\right)^{1/2} = \kappa^{-1} z^{1/2}\big(1 + O(|z|^{-1})\big)$$

uniformly on the same closed half plane. Substitution into $m(z) = \kappa^{2s}\nu(z)^{2s}(1 + O(|\nu(z)|^{-2}))$ proves (14). Thus, uniformity on the boundary follows directly from the Gamma expansion; it is not inferred merely from separate limits on the negative axis. □

Lemma 3. *As $z \to 0$ through the closed upper half plane,*

$$m(z) = \begin{cases} q_{\kappa,s} + O(z), & 0 < s < 1/2, \\ z^{1/2}, & s = 1/2, \\ C_{\kappa,s} z + O(z^2), & 1/2 < s < 1, \end{cases} \qquad C_{\kappa,s} = \frac{\kappa^{2s-2}\Gamma(2s)}{2s-1} > 0. \tag{15}$$

The remainders are uniform in the angle.

Proof. If $b > 0$, then ν is analytic near zero and $\nu(0) = b$. Therefore $m(0) = \kappa^{2s}\Gamma(a + b)/\Gamma(2b) = q_{\kappa,s} > 0$, and a Taylor expansion gives the first line. If $b = 0$, the recurrence of $\Gamma(1 + w) = w\Gamma(w)$ gives $m(z) = \kappa\nu(z) = \sqrt{z}$ exactly.

If $b < 0$, put $\beta = -b = s - 1/2 > 0$. Taylor expansion gives

$$\nu(z) = \beta + \frac{z}{2\kappa^2\beta} + O(z^2), \qquad b + \nu(z) = \frac{z}{\kappa^2(2s-1)} + O(z^2).$$

Since $1/\Gamma(w) = w + O(w^2)$ as $w \to 0$ and $a + \beta = 2s$, multiplication yields the stated coefficient. The Taylor expansions hold in full disks around zero when $b \neq 0$, which proves the claimed uniformity. □

3.2. Boundary Values and the Continuous Argument

For $t > h^2$, put $y = \sqrt{t/\kappa^2 - b^2}$, so $\nu(-t + \mathrm{i}0) = \mathrm{i}y$.

Lemma 4. *For $t > h^2$,*

$$m(-t + \mathrm{i}0) = \frac{\kappa^{2s}}{\pi}|\Gamma(a + \mathrm{i}y)|^2\big[\cos(\pi s)\cosh(\pi y) + \mathrm{i}\sin(\pi s)\sinh(\pi y)\big]. \tag{16}$$

In particular, its imaginary part is strictly positive.

Proof. Euler's reflection formula [10] (§5.5(ii)), with $b = 1 - a$, yields

$$\Gamma(a - \mathrm{i}y)\Gamma(b + \mathrm{i}y) = \frac{\pi}{\sin\pi(a - \mathrm{i}y)}.$$

Solving this identity for the reciprocal of $\Gamma(b + \mathrm{i}y)$ and multiplying by $\Gamma(a + \mathrm{i}y)$ gives

$$\frac{\Gamma(a + \mathrm{i}y)}{\Gamma(b + \mathrm{i}y)} = \frac{|\Gamma(a + \mathrm{i}y)|^2}{\pi}\sin\pi(a - \mathrm{i}y).$$

Now,

$$\sin\pi(a - \mathrm{i}y) = \sin(\pi a)\cosh(\pi y) - \mathrm{i}\cos(\pi a)\sinh(\pi y),$$

and $\sin(\pi a) = \cos(\pi s)$, $\cos(\pi a) = -\sin(\pi s)$. This proves the formula and the sign because $\sin(\pi s) > 0$ and $y > 0$. □

Let $L = \operatorname{Log} m$ be the holomorphic logarithm on the slit plane normalized by $L(\lambda) = \log m(\lambda)$ for $\lambda > 0$. Define

$$\theta_{\kappa,s}(t) = \operatorname{Im} L(-t + \mathrm{i}0), \qquad t > 0.$$

Lemma 5. *The boundary phase is continuous on $(0, \infty)$ and has the following values.*

(i) *If* $0 < s < 1/2$, *it is zero for* $0 < t \le h^2$, *and for* $t > h^2$,

$$\theta_{\kappa,s}(t) = \arctan\big(\tan(\pi s)\tanh(\pi y)\big) \in (0, \pi s). \tag{17}$$

(ii) *If* $s = 1/2$, *it equals* $\pi/2$ *for all* $t > 0$.

(iii) *If* $1/2 < s < 1$, *it equals* π *for* $0 < t \le h^2$, *and for* $t > h^2$,

$$\theta_{\kappa,s}(t) = \pi - \arctan\big(\tan(\pi(1-s))\tanh(\pi y)\big) \in (\pi s, \pi). \tag{18}$$

In every case $\theta_{\kappa,s}(t) \to \pi s$ *as* $t \to \infty$.

Proof. For $0 < t < h^2$, write $x = \sqrt{b^2 - t/\kappa^2} \in (0, |b|)$. If $b > 0$, the two real Gamma arguments $(a + x, b + x)$ are positive, so $m(-t) > 0$. If $b < 0$, then $b + x \in (b, 0) \subset (-1, 0)$, where $\Gamma(b + x) = \Gamma(1 + b + x)/(b + x) < 0$, whereas $\Gamma(a + x) > 0$. Thus, $m(-t) < 0$.

The logarithmic phase is determined without ambiguity arising from multiples of 2π. Near zero, it is 0 on the gap when $b > 0$, according to the positive constant in (15). When $b < 0$, the expansion of $m(z) = C_{\kappa,s} z(1 + O(z))$ gives the upper boundary phase (π). The multiplier has no zeros on the gap, so continuity keeps these phases unchanged until the threshold.

Above the threshold, (16) lies in the first quadrant if $s < 1/2$ and in the second if $s > 1/2$. Taking the ratio of its imaginary and real parts gives (17) in the first case. In the second case, the angle of the reflected point in the first quadrant gives (18). Since $0 < \tanh(\pi y) < 1$, the asserted strict inequalities follow. When $s = 1/2$, $m = \sqrt{z}$ gives the phase directly.

As $t \downarrow h^2$, $y \downarrow 0$ and the two formulas approach 0 and π, respectively. The threshold value is finite and nonzero when $h > 0$, so the phase is continuous there. Finally, $\tanh(\pi y) \to 1$ as $t \to \infty$, yielding the stated limit. All these phases are the continuous boundary values of the same normalized logarithm. □

Corollary 1. *Put*

$$\delta(s) = \begin{cases} 0, & s < 1/2, \\ 1/2, & s = 1/2, \\ 1, & s > 1/2. \end{cases} \tag{19}$$

Uniformly for $0 \le \omega \le \pi$,

$$\begin{aligned} \operatorname{Im} L(re^{i\omega}) &= \delta(s)\omega + o(1) \quad (r \downarrow 0), \\ \operatorname{Im} L(Re^{i\omega}) &= s\omega + O(R^{-1}) \quad (R \to \infty). \end{aligned} \tag{20}$$

Proof. Each line of (15) has the form of $cz^{\delta(s)}(1 + o(1))$ with $c > 0$ and a uniform remainder. Taking its logarithm gives the first relation near zero. At infinity, (14) gives $L(z) = s \operatorname{Log} z + \operatorname{Log}(1 + O(|z|^{-1}))$. The possible additive integer multiple of $2\pi i$ is zero because the logarithms agree on the positive axis and continue across the connected large upper half annulus. The uniform remainder gives (20). □

3.3. The Representing Measure

Theorem 1. *For all* $\kappa > 0$ *and* $0 < s < 1$, $m_{\kappa,s}$ *is a complete Bernstein function. Its representation is* (6), *with* (7) *and* (8). *The drift is zero, and the representing measure is absolutely continuous, supported on* $[h^2, \infty)$ *when* $h > 0$ *and on* $(0, \infty)$ *when* $h = 0$.

Proof. Step 1. Lemma 2 gives holomorphy and a normalized logarithm. Lemma 5 places its boundary phase in $[0, \pi]$. Corollary 1 verifies the two angular hypotheses of Proposition 2,

with $\alpha_0 = \delta(s)$ and $\alpha_\infty = s$. Both belong to $[0,1]$. Thus, m preserves the upper half plane and Proposition 1 gives $m \in$ CBF.

Step 2. According to Lemma 3, $q = m(0+)$ is given by (7). According to Lemma 2,

$$d = \lim_{\lambda\to\infty} \frac{m(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \lambda^{s-1}(1 + O(\lambda^{-1})) = 0.$$

Step 3. On the gap of $0 < t < h^2$, the boundary value is real. Above the threshold, its imaginary part is

$$\operatorname{Im} m(-t + \mathrm{i}0) = \frac{\kappa^{2s}\sin(\pi s)}{\pi} |\Gamma(a + \mathrm{i}y)|^2 \sinh(\pi y).$$

The square root and Gamma factors converge to continuous upper boundary values, uniformly on each compact interval avoiding h^2 and zero. Proposition 3 therefore gives zero density on the gap and (8) above it. When $h > 0$, the value of the multiplier at the threshold is $\kappa^{2s}\Gamma(a)/\Gamma(b)$, which is finite. Hence, there is no atom at h^2. Since all other positive points belong to intervals of absolute continuity, no singular measure remains. The density is strictly positive above h^2, proving the support assertion. □

Proposition 4. *For every* $0 < s < 1$,

$$\rho_{\kappa,s}(t) \sim \frac{\sin(\pi s)}{\pi} t^{s-1} \qquad (t \to \infty). \tag{21}$$

If $s \neq 1/2$, *then, as* $t \downarrow h^2$,

$$\rho_{\kappa,s}(t) \sim \frac{\kappa^{2s-1}\sin(\pi s)\Gamma(a)^2}{\pi h^2} \sqrt{t - h^2}. \tag{22}$$

For $s = 1/2$, $m_{\kappa,1/2}(\lambda) = \sqrt{\lambda}$ *and* $\rho_{\kappa,1/2}(t) = 1/(\pi\sqrt{t})$.

Proof. The asymptotic formula for the Gamma function along vertical lines gives $|\Gamma(a + \mathrm{i}y)|^2 = 2\pi y^{2a-1}e^{-\pi y}(1 + o(1))$ as $y \to \infty$. Since $2a - 1 = 2s$ and $\sinh(\pi y) \sim e^{\pi y}/2$, substitution into (8) gives $\rho(t) \sim \kappa^{2s}\sin(\pi s)y^{2s}/(\pi t)$, proving (21). At the threshold, $|\Gamma(a + \mathrm{i}y)|^2 \to \Gamma(a)^2$, $\sinh(\pi y) \sim \pi y$, and $y = \sqrt{t - h^2}/\kappa$. This gives (22). Finally, for $s = 1/2$ use $|\Gamma(1 + \mathrm{i}y)|^2 = \pi y/\sinh(\pi y)$ in (8). □

The estimates also verify directly that $\int_0^\infty \rho(t)/(1 + t)\,\mathrm{d}t < \infty$. Indeed, for $s \neq 1/2$, the density vanishes like $\sqrt{t - h^2}$ at the positive threshold, whereas for $s = 1/2$, it behaves like $t^{-1/2}$ near the zero threshold. After division by $1 + t$, the tail is of order t^{s-2}.

4. Sharp Product Theorem for Orders of, at Most, One Half

Assume $N \geq 1$, $\kappa_j > 0$, $0 < s_j \leq 1/2$, and $\gamma_j > 0$. Use the notation of (3) and set

$$\sigma = \sum_{j=1}^{N} \gamma_j s_j, \qquad \beta = \sum_{j=1}^{N} \gamma_j \delta(s_j), \qquad \theta_j(t) = \theta_{\kappa_j,s_j}(t). \tag{23}$$

The logarithm defining F is $L_F(z) = \sum_j \gamma_j \operatorname{Log} m_j(z)$; in particular, the continuous boundary phase is

$$\Psi(t) = \sum_{j=1}^{N} \gamma_j \theta_j(t). \tag{24}$$

It is this sum, before reduction modulo 2π, that enters the argument.

Theorem 2. *Under these assumptions,*

$$F \in \mathrm{CBF} \quad \Longleftrightarrow \quad \sigma \le 1. \tag{25}$$

When $\sigma \le 1$, the drift coefficient is

$$D = \begin{cases} 0, & \sigma < 1, \\ 1, & \sigma = 1. \end{cases} \tag{26}$$

Proof. Sufficiency. Each m_j has no zeros on the slit plane. Therefore, $F = e^{L_F}$ is holomorphic and has no zeros there, with $F(\lambda) > 0$ for $\lambda > 0$. Lemma 5 gives

$$0 \le \Psi(t) \le \pi \sum_j \gamma_j s_j = \pi\sigma \le \pi.$$

For these orders, $0 \le \delta(s_j) \le s_j$, so $0 \le \beta \le \sigma \le 1$. Summing the finitely many uniform expansions in (20) gives

$$\begin{aligned} \operatorname{Im} L_F(re^{\mathrm{i}\omega}) &= \beta\omega + o(1) && (r \downarrow 0), \\ \operatorname{Im} L_F(Re^{\mathrm{i}\omega}) &= \sigma\omega + O(R^{-1}) && (R \to \infty), \end{aligned}$$

uniformly in $0 \le \omega \le \pi$. Thus, every hypothesis of Proposition 2 holds, and $F \in \mathrm{CBF}$.

Necessity. Multiplying the asymptotic expansions of the factors gives

$$F(\lambda) = \lambda^{\sigma}\big(1 + O(\lambda^{-1})\big). \tag{27}$$

If $\sigma > 1$, this contradicts Lemma 1. Hence, F is not even a Bernstein function in that case.

The drift. For $\sigma \le 1$, divide (27) by λ and pass to the limit. The limit is zero if $\sigma < 1$ and one if $\sigma = 1$, proving (26). □

Explicit Boundary Moduli and the Product Measure

For $t > 0$, define

$$J(t) = \{j : t > h_j^2\}, \qquad r_j(t) = |m_j(-t + \mathrm{i}0)|. \tag{28}$$

For an index with $t \le h_j^2$, necessarily $s_j < 1/2$, put

$$x_j(t) = \sqrt{b_j^2 - t/\kappa_j^2}, \qquad r_j(t) = \kappa_j^{2s_j} \frac{\Gamma(a_j + x_j(t))}{\Gamma(b_j + x_j(t))}. \tag{29}$$

For $j \in J(t)$, put $y_j(t) = \sqrt{t/\kappa_j^2 - b_j^2}$. Formula (16) gives

$$r_j(t) = \frac{\kappa_j^{2s_j}}{\pi} |\Gamma(a_j + \mathrm{i}y_j(t))|^2 \sqrt{\cos^2(\pi s_j) + \sinh^2(\pi y_j(t))}. \tag{30}$$

Indeed, the square of the modulus of the bracket in (16) is

$$\cos^2(\pi s_j)\cosh^2(\pi y_j) + \sin^2(\pi s_j)\sinh^2(\pi y_j) = \cos^2(\pi s_j) + \sinh^2(\pi y_j).$$

The θ_j phases are those in Lemma 5; in particular, inactive factors have phase zero in this section.

Theorem 3. *If* $0 < s_j \le 1/2$ *and* $\sigma \le 1$, *then*

$$F(\lambda) = Q + D\lambda + \int_0^\infty \frac{\lambda}{\lambda + t}\rho_F(t)\,\mathrm{d}t, \tag{31}$$

where D is given by (26),

$$Q = \begin{cases} \displaystyle\prod_{j=1}^{N}\left(\frac{\kappa_j^{2s_j}}{\Gamma(1-2s_j)}\right)^{\gamma_j}, & s_j < 1/2 \text{ for every } j, \\ 0, & \text{otherwise}, \end{cases} \tag{32}$$

and

$$\rho_F(t) = \frac{1}{\pi t}\left(\prod_{j=1}^{N} r_j(t)^{\gamma_j}\right)\sin\Psi(t). \tag{33}$$

The representing measure is absolutely continuous.

Proof. Theorem 2 gives a positive representing measure and the drift (D). Taking the limit of the factors at zero gives (32). Since the powers use their normalized logarithms,

$$F(-t + \mathrm{i}0) = \exp\left(\sum_j \gamma_j \log r_j(t) + \mathrm{i}\Psi(t)\right) = \prod_j r_j(t)^{\gamma_j} e^{\mathrm{i}\Psi(t)}.$$

The imaginary part is non-negative because $0 \le \Psi \le \pi$.

On each compact interval avoiding zero and the finitely many positive h_j^2, every factor and its normalized logarithm has a continuous upper boundary extension. Convergence of the product to the displayed boundary value is consequently uniform there. Proposition 3 gives (33) and absolute continuity on these intervals. At a positive threshold, each factor has a finite nonzero upper boundary value; the chosen positive powers therefore remain bounded. The same proposition rules out an atom at every such point. A singular measure supported on a finite set consists of atoms, so no singular part remains. This proves the full representation, not only its absolutely continuous component. □

Corollary 2. *Under Theorem 3, suppose at least one order satisfies* $s_j < 1/2$, *and set* $h_*^2 = \min_j h_j^2$. *Then,*

$$\rho_F(t) = 0 \quad (0 < t < h_*^2), \qquad \rho_F(t) > 0 \quad (t > h_*^2).$$

If all $s_j = 1/2$, *then* $F(\lambda) = \lambda^\sigma$. *Its density is* $\sin(\pi\sigma)t^{\sigma-1}/\pi$ *when* $\sigma < 1$; *when* $\sigma = 1$, $F(\lambda) = \lambda$, *and the representing measure is zero.*

Proof. Below h_*^2, all phases vanish. Above it, at least one phase is positive. For any finite t and any factor with $s_j < 1/2$, Lemma 5 gives the strict inequality of $\theta_j(t) < \pi s_j$, even if that factor is inactive. All weights are positive, so $0 < \Psi(t) < \pi\sigma \le \pi$. The product modulus is strictly positive; hence, so is the density. If all orders are one half, each factor is $\sqrt{\lambda}$, and the assertions follow directly from (33). □

Proposition 5. *If* $\sigma < 1$, *then*

$$\rho_F(t) \sim \frac{\sin(\pi\sigma)}{\pi} t^{\sigma-1} \qquad (t \to \infty). \tag{34}$$

If $\sigma = 1$ *and at least one* $s_j < 1/2$, *there is* $c > 0$ *such that* $\rho_F(t) = O(e^{-c\sqrt{t}})$ *as* $t \to \infty$.

Proof. The uniform asymptotic expansion (14) gives $\prod_j r_j(t)^{\gamma_j} = t^\sigma(1 + O(t^{-1}))$, while $\Psi(t) \to \pi\sigma$. If $\sigma < 1$, the limiting sine is positive and (34) follows.

Suppose $\sigma = 1$. For $s_j < 1/2$, let $A_j = \tan(\pi s_j) > 0$. For all sufficiently large t, the mean value theorem and $1 - \tanh u \le 2e^{-2u}$ imply

$$\begin{aligned} 0 < \pi s_j - \theta_j(t) &= \arctan A_j - \arctan(A_j \tanh(\pi y_j(t))) \\ &\le 2A_j e^{-2\pi y_j(t)}. \end{aligned}$$

Factors with $s_j = 1/2$ contribute zero to this difference. Since $y_j(t) \sim \sqrt{t}/\kappa_j$ and there are finitely many factors, $0 < \pi - \Psi(t) = O(e^{-c\sqrt{t}})$ for some $c > 0$. Use $\sin\Psi = \sin(\pi - \Psi) \le \pi - \Psi$ and $\prod_j r_j^{\gamma_j}/t = O(1)$ in (33). □

Example 1. *For $s_1 = 1/4$, $s_2 = 1/3$ and $\gamma_1 = \gamma_2 = 1$, the total order is $\sigma = 7/12$. Thus, $m_{\kappa_1,1/4} m_{\kappa_2,1/3}$ is a complete Bernstein function with zero drift. The support of its representing measure begins at $\min\{\kappa_1^2/16, \kappa_2^2/36\}$ and has strictly positive density above that point. Between unequal thresholds, only the factor with the smaller threshold contributes a nonzero phase, while both factors contribute to the modulus.*

5. Mixed Orders and Exact Phase Conditions

We now allow $0 < s_j < 1$. Retain $\sigma, \beta, \theta_j, \Psi$ from (23) and (24), with δ defined by (19). Lemmas 2 and 3 imply

$$F(z) = C_0 z^\beta (1 + o(1)) \quad (z \to 0), \qquad F(z) = z^\sigma (1 + O(|z|^{-1})) \quad (|z| \to \infty), \tag{35}$$

where $C_0 > 0$ and the expansions are uniform on closed upper semicircles. Here, $z^0 = 1$ if $\beta = 0$.

5.1. An Exact Criterion

Theorem 4. *For arbitrary $0 < s_j < 1$, $\kappa_j > 0$ and $\gamma_j > 0$,*

$$F \in \mathrm{CBF} \iff \Psi(t) = \sum_{j=1}^{N} \gamma_j \theta_j(t) \le \pi \quad \textit{for every } t > 0. \tag{36}$$

Whenever these conditions hold, $\beta \le 1$ and $\sigma \le 1$. The representation (31) has density (33), drift (26), and

$$Q = \begin{cases} 0, & \textit{if at least one } s_j \ge \tfrac{1}{2}, \\ \displaystyle\prod_{j=1}^{N} q_{\kappa_j, s_j}^{\gamma_j}, & \textit{if every } s_j < \tfrac{1}{2}, \end{cases} \tag{37}$$

where q_{κ_j,s_j} is given explicitly in (7). For $0 < t \le h_j^2$, the modulus is now

$$r_j(t) = \kappa_j^{2s_j} \left| \frac{\Gamma(a_j + x_j(t))}{\Gamma(b_j + x_j(t))} \right|, \qquad x_j(t) = \sqrt{b_j^2 - t/\kappa_j^2}; \tag{38}$$

above the threshold, it is still (30). The representing measure is absolutely continuous.

Proof. Sufficiency. All individual phases are non-negative. Moreover, Lemma 5 and the expansions near zero give

$$\lim_{t \downarrow 0} \Psi(t) = \pi\beta, \qquad \lim_{t \to \infty} \Psi(t) = \pi\sigma.$$

Thus, $\Psi \le \pi$ forces $\beta, \sigma \le 1$. The normalized logarithm (L_F) is holomorphic, with continuous boundary phase Ψ, and (35) yields the angular conditions with coefficients β and σ. Proposition 2 applies and proves the complete Bernstein property.

Necessity. Suppose $F \in$ CBF. It is nonconstant according to (35), since $\sigma > 0$. The harmonic function ($\operatorname{Im} F$) is non-negative in $\mathbb{H}$ and cannot vanish at an interior point. The strong minimum principle would otherwise make it identically zero, and the open mapping theorem would force F to be constant. Hence, $F(\mathbb{H}) \subset \mathbb{H}$. Its logarithm with argument in $(0, \pi)$ agrees with L_F by normalization on the positive axis. Therefore, $0 < \operatorname{Im} L_F(z) < \pi$ for $z \in \mathbb{H}$. Passing to the continuous finite boundary phase at each $-t$ gives $0 \le \Psi(t) \le \pi$. In particular, replacing Ψ by an angle modulo 2π would lose necessary information.

The measure. The limits in (35) determine Q and D exactly as before. At every $t > 0$, all factors have finite nonzero upper boundary values, including at their positive thresholds. Their logarithms and powers converge locally uniformly away from those thresholds. Proposition 3 therefore gives (33) on each remaining interval and excludes atoms at the thresholds. This proves absolute continuity on $(0, \infty)$. □

Example 2. *Take*

$$s_1 = \tfrac{1}{4}, \qquad s_2 = \tfrac{3}{4}, \qquad \gamma_1 = \gamma_2 = 1, \qquad \kappa_1 = 1, \quad \kappa_2 = 2.$$

The two endpoint exponents satisfy $\sigma = s_1 + s_2 = 1$ *and* $\beta = \delta(s_1) + \delta(s_2) = 1$. *Nevertheless, the two threshold values are* $h_1^2 = 1/16$ *and* $h_2^2 = 1/4$. *For* $1/16 < t \le 1/4$, *Lemma 5 gives*

$$\theta_{1,1/4}(t) > 0, \qquad \theta_{2,3/4}(t) = \pi,$$

so $\Psi(t) > \pi$. *Theorem 4 shows that* $m_{1,1/4} m_{2,3/4}$ *is not a complete Bernstein function, even though* $\beta \le 1$ *and* $\sigma \le 1$. *In contrast, using the same scale for both complementary orders gives* $m_{\kappa,1/4}(\lambda) m_{\kappa,3/4}(\lambda) = \lambda$ *according to Proposition 7; this is a complete Bernstein function with unit drift and zero Lévy measure.*

Proposition 6. *Assume the phase criterion of Theorem 4 holds and* $\sigma = 1$, *without any restriction that the orders lie below one half. Then,* $D = 1$ *and, for some* $c > 0$,

$$0 \le \rho_F(t) = O(e^{-c\sqrt{t}}) \qquad (t \to \infty). \tag{39}$$

Consequently,

$$\int_0^\infty \rho_F(t)\, \mathrm{d}t = \int_0^\infty k_F(r)\, \mathrm{d}r < \infty. \tag{40}$$

In particular, the Lévy measure has finite total mass, and the corresponding subordinator consists of deterministic unit drift, a compound Poisson jump component (possibly absent), and killing at rate Q *when* $Q > 0$.

Proof. For each $s_j \ne 1/2$, set $A_j = \tan(\pi \min\{s_j, 1 - s_j\}) > 0$. Above all thresholds, (17) and (18) give

$$|\theta_j(t) - \pi s_j| \le 2A_j e^{-2\pi y_j(t)}, \qquad y_j(t) = \sqrt{t/\kappa_j^2 - b_j^2}.$$

Indeed, use the mean value theorem for arctan and $1 - \tanh u \le 2e^{-2u}$, both for $s_j < 1/2$ and for $s_j > 1/2$. For $s_j = 1/2$, the phase equals πs_j identically. Since there are finitely many factors and $y_j(t) \sim \sqrt{t}/\kappa_j$, it follows that

$$|\Psi(t) - \pi\sigma| = O(e^{-c\sqrt{t}})$$

for some $c > 0$. Here, $\sigma = 1$, and the exact phase criterion ensures $0 \le \Psi(t) \le \pi$. Thus, $0 \le \sin \Psi(t) \le |\pi - \Psi(t)| = O(e^{-c\sqrt{t}})$. The uniform Gamma ratio expansion gives $\prod_j r_j(t)^{\gamma_j} = t(1 + O(t^{-1}))$. Formula (33) proves the stated tail bound. The complete Bernstein representation ensures $\int_0^1 \rho_F(t)\, \mathrm{d}t < \infty$, while the exponential tail controls the integral over $(1, \infty)$. Finally, (44) and Tonelli's theorem give $\int_0^\infty k_F(r)\, \mathrm{d}r = \int_0^\infty \rho_F(t)\, \mathrm{d}t$. The final probabilistic description follows from the standard decomposition of Lévy and Khintchine for finite jump activity [7] (Chapters 3 and 5). □

The criterion involves only the explicit phase functions of one variable in Lemma 5, but the maximum of the total phase can depend on the relative scales. The following simpler condition separates the factors.

Theorem 5. *Define $c(s) = s$ for $0 < s \le 1/2$ and $c(s) = 1$ for $1/2 < s < 1$. If*

$$\sum_{j=1}^{N} \gamma_j c(s_j) \le 1, \tag{41}$$

then $F \in$ CBF. Under this sufficient condition, if at least one $s_j > 1/2$, then $\sigma < 1$, and the drift is zero.

Proof. Lemma 5 gives $0 \le \theta_j(t) \le \pi c(s_j)$. Hence, $\Psi(t) \le \pi \sum_j \gamma_j c(s_j) \le \pi$, so Theorem 4 applies. Since $s \le c(s)$, with strict inequality for $s > 1/2$, positive weights give $\sigma < \sum_j \gamma_j c(s_j) \le 1$ whenever such an order occurs. □

Corollary 3. *If every $s_j > 1/2$, then*

$$F \in \text{CBF} \iff \sum_{j=1}^{N} \gamma_j \le 1.$$

Proof. Sufficiency is Theorem 5. For necessity, if $0 < t < \min_j h_j^2$, then every phase is π, so Theorem 4 gives $\pi \sum_j \gamma_j \le \pi$. Alternatively, $F(\lambda) \sim C_0 \lambda^{\sum_j \gamma_j}$ near zero. If the exponent exceeded one, $F(\lambda)/\lambda \to 0$ as $\lambda \downarrow 0$, contradicting the positive nonincreasing ratio in Lemma 1. □

Example 3. *For $s_1 = 3/4$, $\gamma_1 = 1/2$, $s_2 = 1/4$ and $\gamma_2 = 1$, the left-hand side of (41) is $3/4$. Thus, $m^{1/2}_{\kappa_1,3/4} m_{\kappa_2,1/4}$ is a complete Bernstein function for arbitrary positive scales. It has $\sigma = 5/8$, $\beta = 1/2$, zero killing and zero drift. Below both positive thresholds the phase is $\pi/2$, so its representing density is already positive there. Fractional powers of negative gap values can therefore produce measures below the individual Gamma thresholds.*

5.2. Complementary Orders and the Limitation of the Sufficient Bound

Proposition 7. *For $0 < s < 1$ and $\kappa > 0$,*

$$m_{\kappa,s}(z) m_{\kappa,1-s}(z) = z, \qquad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{42}$$

Consequently, their normalized logarithms sum to Log z *and their boundary phases sum to π for every $t > 0$.*

Proof. For the order $1-s$, the two Gamma parameters are $1+b$ and $-b$, while the square-root argument is unchanged because $(-b)^2 = b^2$. Therefore,

$$\begin{aligned} m_{\kappa,s}(z)m_{\kappa,1-s}(z) &= \kappa^2 \frac{\Gamma(1-b+\nu)\Gamma(1+b+\nu)}{\Gamma(b+\nu)\Gamma(-b+\nu)} \\ &= \kappa^2(\nu-b)(\nu+b) = \kappa^2(\nu^2-b^2) = z. \end{aligned}$$

The Gamma recurrence proves the identity wherever all displayed Gamma values are finite, and holomorphic continuation gives it on the slit plane. Both logarithms are real on the positive axis, so their sum equals $\operatorname{Log} z$ there and, hence, throughout the domain. Taking upper boundary imaginary parts proves the phase identity. □

Remark 2. *If $0 < s < 1/2$, the unweighted complementary product is λ and, hence, is a complete Bernstein function with a drift of one and zero representing measure. Nevertheless, $c(s) + c(1-s) = s + 1 > 1$. Thus, (41) is not necessary for general mixed products. In particular, a mixed product can have $\sigma = 1$; its exclusion in Theorem 5 is specific to that theorem's sufficient hypothesis. The exact criterion (36) correctly includes this example.*

6. Lévy Densities and Their Asymptotic Behavior

The representing measure in (31) determines the Lévy measure by a Laplace transform. We give this step in detail, including the integrability argument, because the two measures have different meanings.

Theorem 6. *Let f be any multiplier or product proven above to be a complete Bernstein function and write*

$$f(\lambda) = q + d\lambda + \int_0^\infty \frac{\lambda}{\lambda + t}\rho_f(t)\,\mathrm{d}t.$$

Its Lévy and Khintchine representation is

$$f(\lambda) = q + d\lambda + \int_0^\infty (1 - e^{-\lambda r})k_f(r)\,\mathrm{d}r, \tag{43}$$

where

$$k_f(r) = \int_0^\infty te^{-tr}\rho_f(t)\,\mathrm{d}t, \qquad r > 0. \tag{44}$$

The k_f function is completely monotone and satisfies $\int_0^\infty (1 \wedge r)k_f(r)\,\mathrm{d}r < \infty$.

Proof. For $t, \lambda > 0$, direct integration gives

$$\int_0^\infty (1 - e^{-\lambda r})te^{-tr}\,\mathrm{d}r = 1 - \frac{t}{t+\lambda} = \frac{\lambda}{\lambda + t}.$$

All integrands are non-negative, so Tonelli's theorem permits interchange of the integrals with respect to r and t and gives (43).

For fixed $r > 0$, the $t(1+t)e^{-tr}$ function is bounded on $[0, \infty)$. Therefore, (44) is finite owing to the weighted integrability of ρ_f. On each compact subinterval of the range ($r > 0$), the same argument with $t^{n+1}(1+t)e^{-tr}$ justifies differentiation under the integral. Thus,

$$(-1)^n k_f^{(n)}(r) = \int_0^\infty t^{n+1}e^{-tr}\rho_f(t)\,\mathrm{d}t \geq 0.$$

Finally,

$$\int_0^\infty (1 - e^{-r})k_f(r)\,\mathrm{d}r = \int_0^\infty \frac{\rho_f(t)}{1+t}\,\mathrm{d}t < \infty.$$

The inequality expressed as $1 - e^{-r} \geq (1 - e^{-1})(1 \wedge r)$ proves the Lévy integrability condition. □

Corollary 4. *For the single multiplier,*

$$k_{\kappa,s}(r) = \frac{\kappa^{2s} \sin(\pi s)}{\pi^2} \int_{h^2}^{\infty} e^{-tr} \left| \Gamma\left(a + \mathrm{i}\sqrt{t/\kappa^2 - b^2}\right) \right|^2 \sinh\left(\pi\sqrt{t/\kappa^2 - b^2}\right) \mathrm{d}t. \tag{45}$$

Equivalently,

$$k_{\kappa,s}(r) = \frac{2\kappa^{2s+2} \sin(\pi s)}{\pi^2} e^{-h^2 r} \int_0^{\infty} y e^{-\kappa^2 r y^2} |\Gamma(a + \mathrm{i}y)|^2 \sinh(\pi y)\, \mathrm{d}y. \tag{46}$$

It satisfies the scaling relation

$$k_{\kappa,s}(r) = \kappa^{2s+2} k_{1,s}(\kappa^2 r). \tag{47}$$

Proof. Insert (8) into (44); the t factor cancels its denominator. Then, set $t = \kappa^2(b^2 + y^2)$, so $\mathrm{d}t = 2\kappa^2 y\, \mathrm{d}y$, to obtain (46). Comparison with the same formula at $\kappa = 1$ proves the scaling identity. □

Proposition 8. *For a single multiplier,*

$$k_{\kappa,s}(r) \sim \frac{s}{\Gamma(1-s)} r^{-1-s} \qquad (r \downarrow 0). \tag{48}$$

For $s \neq 1/2$, its behavior for large r values is

$$k_{\kappa,s}(r) \sim \frac{\kappa^{2s-1} \sin(\pi s) \Gamma(a)^2}{2\sqrt{\pi}} e^{-h^2 r} r^{-3/2} \qquad (r \to \infty). \tag{49}$$

For $s = 1/2$, $k_{\kappa,1/2}(r) = (2\sqrt{\pi})^{-1} r^{-3/2}$ exactly. More generally, for any product satisfying Theorem 4 with $0 < \sigma < 1$,

$$k_F(r) \sim \frac{\sigma}{\Gamma(1-\sigma)} r^{-1-\sigma} \qquad (r \downarrow 0). \tag{50}$$

Proof. For a single multiplier, Proposition 4 gives $t\rho_{\kappa,s}(t) \sim At^s$ with $A = \sin(\pi s)/\pi$. The same assertion holds for a product with s replaced by σ. The upper boundary asymptotic in (35) gives $\rho_F(t) \sim \sin(\pi\sigma) t^{\sigma-1}/\pi$ whenever $\sigma < 1$.

We give a direct argument for the asymptotic behavior of the Laplace transform in both cases. Write $p = s$ or $p = \sigma$, and $v(t) = t\rho_f(t) \sim At^p$. For every $\varepsilon > 0$, choose T with $(A - \varepsilon)t^p \leq v(t) \leq (A + \varepsilon)t^p$ for $t \geq T$. The integral of v on $(0, T)$ is finite and contributes $O(1)$ to $k_f(r)$ as $r \downarrow 0$. On (T, ∞), set $u = rt$ and squeeze between

$$(A \pm \varepsilon) r^{-p-1} \int_{rT}^{\infty} e^{-u} u^p\, \mathrm{d}u.$$

First let $r \downarrow 0$, then $\varepsilon \downarrow 0$. This yields $k_f(r) \sim A\Gamma(p+1) r^{-p-1}$. The reflection identity gives $\Gamma(p+1)\sin(\pi p)/\pi = p/\Gamma(1-p)$, proving the formulas for small r values.

For $s \neq 1/2$, (22) implies

$$t\rho_{\kappa,s}(t) \sim B\sqrt{t - h^2}, \qquad B = \frac{\kappa^{2s-1} \sin(\pi s) \Gamma(a)^2}{\pi}.$$

Put $u = t - h^2$ in (44) and factor out $e^{-h^2 r}$. On $0 < u < \varepsilon$, compare the remaining factor with $(B \pm \varepsilon) u^{1/2}$, after decreasing the interval if necessary. The $u \geq \varepsilon$ part is exponentially

smaller because the integrand has, at most, polynomial growth and contains e^{-ru}. Rescaling $v = ru$ on the first interval gives the leading coefficient of $B\Gamma(3/2) = B\sqrt{\pi}/2$, proving (49). At $s = 1/2$, integrate $t\rho(t) = \sqrt{t}/\pi$ directly. □

Remark 3. *The conclusion about finite total mass at $\sigma = 1$ holds for every admissible product in Theorem 4, including mixed orders, according to Proposition 6. Thus, $D = 1$, and both measures have the same finite total mass, as stated in (40).*

The pure complementary product ($F(\lambda) = \lambda$) has $k_F \equiv 0$. More general products with drift coefficient one may have a nonzero but finite jump intensity. This contrasts with the infinite Lévy mass in (50) when $0 < \sigma < 1$.

7. Functional Calculus and Subordination

Let A be a non-negative self-adjoint operator on a Hilbert space ($\mathcal{H}$), and let E_A be its spectral resolution. Let f be a multiplier or admissible product from the preceding sections. Its spectral realization is

$$f(A) = \int_{[0,\infty)} f(\lambda)\, E_A(\,\mathrm{d}\lambda), \qquad D(f(A)) = \left\{ u : \int_{[0,\infty)} f(\lambda)^2\, \mathrm{d}\langle E_A(\lambda)u, u\rangle < \infty \right\}. \tag{51}$$

It is non-negative and self-adjoint. These statements do not require e^{-rA} to preserve positivity; that additional property is used only for the Markov conclusion below.

Proposition 9. *For $u \in D(A)$,*

$$f(A)u = qu + dAu + \int_0^\infty A(A + tI)^{-1} u\, \rho_f(t)\, \mathrm{d}t, \tag{52}$$

and

$$f(A)u = qu + dAu + \int_0^\infty (I - e^{-rA})u\, k_f(r)\, \mathrm{d}r. \tag{53}$$

Both integrals converge in the norm of $\mathcal{H}$. If $f = m_{\kappa,s}$, then $D(f(A)) = D(A^s)$. If $f = F$ is a product with total order σ, then $D(F(A)) = D(A^\sigma)$. In each case, the corresponding graph norms are equivalent.

Proof. The standard spectral calculus argument [7] (Chapters 12–13) applies to the scalar formulas proven above. For completeness, if $u \in D(A)$,

$$\|A(A + tI)^{-1}u\| \le \min\{\|u\|, t^{-1}\|Au\|\}, \qquad \|(I - e^{-rA})u\| \le \min\{2\|u\|, r\|Au\|\}.$$

The Stieltjes integrability condition and the Lévy condition ensure that both operator integrals converge in norm. Spectral truncation then identifies them with $f(A)u$. Finally, the expansion of $f(\lambda) \sim \lambda^p$ at infinity, $p = s$ or $p = \sigma$ implies $1 + f(\lambda)^2 \asymp 1 + \lambda^{2p}$ on $[0, \infty)$; the spectral theorem gives the domain and graph norm assertions. □

Theorem 7. *There is a convolution semigroup of subprobability measures $((\zeta_\tau)_{\tau \ge 0})$ on $[0, \infty)$ characterized by*

$$\int_{[0,\infty)} e^{-\lambda r} \zeta_\tau(\,\mathrm{d}r) = e^{-\tau f(\lambda)}, \qquad \lambda \ge 0. \tag{54}$$

It has a total mass of $e^{-\tau q}$, and

$$e^{-\tau f(A)}u = \int_{[0,\infty)} e^{-rA} u\, \zeta_\tau(\,\mathrm{d}r), \qquad u \in \mathcal{H}. \tag{55}$$

If $\mathcal{H} = L^2(X, \mathfrak{m})$ and e^{-rA} is submarkovian, then $e^{-\tau f(A)}$ is submarkovian. If the original semigroup is conservative, and $q = 0$, the subordinate semigroup is also conservative.

Proof. The subordination theorem for Bernstein functions [7] (Chapters 5 and 13) supplies ζ_τ with a Laplace transform (54). Its mass is $e^{-\tau q}$. The integral in (55) is well defined, since $\|e^{-rA}\| \leq 1$; spectral calculus and Fubini's theorem identify its scalar spectral multiplier with $e^{-\tau f(\lambda)}$. The submarkovian property follows by integrating positive contractions against subprobability measures. In the conservative case with $q = 0$, the measures have total mass one, so the subordinate semigroup preserves the constant function, using the usual extensions of positive semigroups when $1 \notin L^2$. □

For example, $A = -\Delta$ on $L^2(\mathbb{R}^n)$ yields a Fourier multiplier with symbol $f(|\xi|^2)$. The semigroup can be written as

$$e^{-\tau f(-\Delta)}u = \int_{[0,\infty)} e^{r\Delta}u\, \zeta_\tau(\,\mathrm{d}r).$$

The atom at $r = 0$, if present, contributes the identity operator. Thus, no density for ζ_τ is assumed. For a single Gamma multiplier, the operator domain is the Sobolev space $(H^{2s}(\mathbb{R}^n))$; for an admissible product, it is $H^{2\sigma}(\mathbb{R}^n)$, according to Proposition 9.

7.1. Possible Links with Stochastic Control and Numerical Simulation

The examples of Yan et al. [11] and Huo [12] suggest two possible directions for applications. However, these applications require additional results beyond our scalar complete Bernstein results.

7.1.1. Wiener and Poisson Noise

Yan et al. studied control with a guaranteed upper bound on the cost for uncertain systems whose dynamics depend on mean values and are driven by Wiener and Poisson noise. In a different setting, if $-A$ generates a conservative Markov semigroup, the $f \in$ CBF multiplier specifies a positive random clock through (54). The subordinated generator is $-f(A)$, and its jump activity is described by the explicit density (k_f) in (44). In particular, when $\sigma = 1$ and the product is admissible, Proposition 6 gives a random clock with drift coefficient one and a compound Poisson component with finite jump intensity. This provides an explicit source of models that may contain both continuous diffusion and jumps. However, a diffusion evaluated at a random time does not automatically have the structure of the controlled systems considered in [11], which have finitely many state variables and dependence on mean values. Applying their control analysis would require a suitable representation in terms of state variables, admissible feedback controls, bounds on moments, and verification of their conditions over a finite time interval. Our paper does not establish these additional control estimates.

7.1.2. Finite Differences and Monte Carlo with Least Squares Regression

Huo [12] uses an approximation by finite differences to help construct an ansatz based on control variates for Monte Carlo with least squares regression. For a Markov generator $(-A)$, our semigroup $(P_\tau^f = e^{-\tau f(A)})$ provides an operator for conditional expectations in a subordinated model. Thus, the identity of

$$P_\tau^f g = \int_{[0,\infty)} e^{-rA} g\, \zeta_\tau(\,\mathrm{d}r)$$

could provide a reference continuation value or help construct a regression ansatz, provided that the payoff satisfies the required integrability conditions and the Markov assumptions

hold. A discrete method would still need to approximate the generally nonlocal operator $(f(A))$ or the subordination integral above. Consistency of standard local finite differences, variance reduction, and convergence of a Monte Carlo scheme based on least squares regression do not follow from our spectral formulas. These properties would require separate numerical analysis.

8. Scope and Concluding Remarks

The symmetry of $a + b = 1$ is the key computational feature of the family. It converts the boundary Gamma quotient into a positive modulus times an elementary complex expression. The resulting phase is bounded by πs below order one half and reaches π on the gap above order one half. These two behaviors explain both the sharp product theorem for orders of, at most, one half and the different sufficient condition for mixed orders.

The general Pick characterization, Stieltjes inversion, and subordination correspondence are standard. The role of the present calculations is to make these tools explicit for the family of Gamma ratios with square-root arguments, including the full representing densities, the endpoint drift, and the distinction between exact phase conditions and sufficient bounds applied to each factor. The earlier literature on Gamma functions cited in the introduction provides related monotonicity and representation methods; the results here concern the specific substitution (1) and the products (3). We make no broader priority claim for all complete Bernstein functions constructed from Gamma functions. The discussion following Theorem 4 demonstrates why the two endpoint exponents do not suffice when thresholds are unequal. Proposition 6 completes the analysis of products with drift coefficient one for all admissible mixed products. The applications discussed above describe possible uses of the explicit Lévy and semigroup formulas. They do not establish new results in stochastic control or on the convergence of numerical methods.

The preceding results also give an immediate but partial result for the replacement of the square-root argument by $\nu_\alpha(z) = \sqrt{b^2 + z^\alpha/\kappa^2}$, where $0 < \alpha \leq 1$ and z^α denotes the principal power. Indeed, the $g_\alpha(z) = z^\alpha$ function is a complete Bernstein function, and complete Bernstein functions are closed under composition [7] (Chapter 7). Consequently,

$$m_{\kappa,s}^{(\alpha)}(z) := \kappa^{2s} \frac{\Gamma\left(a + \sqrt{b^2 + z^\alpha/\kappa^2}\right)}{\Gamma\left(b + \sqrt{b^2 + z^\alpha/\kappa^2}\right)} = m_{\kappa,s}(z^\alpha)$$

is a complete Bernstein function for every $0 < s < 1$. More generally, for any product (F) that satisfies the complete Bernstein criterion established in this paper, $F_\alpha(z) := F(z^\alpha)$ is, again, a complete Bernstein function. For $0 < \alpha < 1$, Stieltjes inversion gives its representing density on the entire positive half line.

$$\rho_{F_\alpha}(t) = \frac{1}{\pi t} \operatorname{Im} F\left(t^\alpha e^{i\pi\alpha}\right), \qquad t > 0. \tag{56}$$

Here, the boundary values are finite and locally uniform because $t^\alpha e^{i\pi\alpha}$ stays in the open upper half plane; thus, the representing measure has no singular component on $(0, \infty)$. Also, $F_\alpha(0+) = F(0+)$, and its drift is zero for $0 < \alpha < 1$, since the admissible products satisfy $F(\lambda) \sim \lambda^\sigma$ with $0 < \sigma \leq 1$. In particular, the case of $\alpha = 1/3$ is covered by this composition argument. This observation does not classify products that fail the original criterion but become complete Bernstein functions after the power substitution; obtaining a sharp classification and similarly explicit densities in terms of Gamma functions for that larger family remains a separate problem.

For more general Gamma products without the relation of $a + b = 1$, Proposition 2 still applies whenever the absence of zeros, continuity of the boundary phases, and uniform

angular estimates near zero and infinity can be established. The obstacle is then the explicit control of the total phase. Within the present family, Theorem 4 also leaves a concrete parameter problem. It remains to characterize the relative scales for which the maximum of Ψ does not exceed π. Complementary orders show that exact algebraic relations can be decisive. The explicit Lévy densities also provide a starting point for studying the kernels of the resulting subordinate semigroups and their behavior as time tends to infinity.

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