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A Mathematical Extremum Method to Determine the Critical Slip Surface Using the Swedish Slice Method

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Abstract

To obtain the critical slip surface (CSS) and the factor of safety (FOS) using the Swedish slice method (SSM), a mathematical extremum method (MEM) is proposed. In the MEM, the complete mathematical expressions of essential equations determining the CSS and the FOS are derived. It is discovered that a set of approximate explicit solutions of the equations exists. The equations are written into an EXCEL spreadsheet and solved using the solver tool. During the solution process, the discovered approximate explicit solutions are used as iterative initial values. Results are obtained with just one iteration. Compared with the empirical method and the computer search method, it is verified that the MEM has good applicability. Compared with the graphical trial procedure, the commercial software, and the stability charts, it is further demonstrated that the MEM has acceptable generality. Additionally, assessments are made on some of the existing research based on the MEM. It is noted that some existing findings are either inaccurate or only applicable under specific conditions. These findings provide valuable references for engineers and researchers.

Keywords: slope; critical slip surface; factor of safety; Swedish slice method; mathematical extremum method

MSC: 65H10

1. Introduction

Slope stability is a classic problem in soil mechanics. For methods of evaluating slope stability, there are mainly qualitative and quantitative methods. As one of the quantitative methods, the limit equilibrium method has received more attention. In this method, the critical slip surface (CSS) and the factor of safety (FOS) of a slope can be obtained. In 1916, Petterson first determined the CSS and FOS of a slope based on an arc [1]. However, cohesive soil was not considered. In 1927, Fellenius extended the slip circle method to cohesive soils and soils with both friction and cohesion [2]. The above method is called the Swedish slice method (SSM), the Fellenius method, or the ordinary method of slices. It was the beginning of commonly used limit equilibrium methods.

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Duncan [3] pointed out that the FOS of the SSM is completely accurate in the analysis of ' $\varphi = 0^\circ$ '. For the total stress method of circular sliding surfaces, the basic correct results can be obtained. There is no problem with the numerical analysis of this method. Although codes from Europe [4] have recommended the use of limit equilibrium methods that consider both force and moment balance (the Bishop simplified method, etc.), some codes or manuals from China [5,6], America [7,8], and Britain [9] still recommend using the SSM or conservative methods in certain cases. The current research on the SSM mainly focuses on the following aspects: (a) the characteristics of the SSM itself [10,11]; (b) the revised SSM [12,13]; (c) the new academic or engineering problems based on the SSM [14–20]; and (d) the calculation results of the SSM, which are used as comparative variables [21,22]. Overall, the SSM still has value.

The key to applying the SSM is determining the CSS and FOS. The methods of determining the CSS and the FOS mainly include the empirical method [15,23,24], the computer search method [25–28], and the analytical method [29–32]. For the empirical method, the advantage is that the application is simple. The disadvantage is that accuracy may not be guaranteed. For the computer search method, the advantage is that the computational speed is fast. The disadvantage is that complex programming in a computer language is required, reasonable iterative initial value trials are unclear, and the calculation process is time-consuming. The analytical method has high accuracy. It is a promising method. However, for this method, there are some advantages: (i) it is not complete for the provided mathematical expressions of the necessary equations determining the CSS and the FOS, (ii) reasonable iterative initial values for solving the above equations are unclear, (iii) there is a lack of simple programs for solving the above equations, and (iv) assessments of some of the existing research are lacking. To address the above shortcomings, a mathematical extremum method (MEM) is proposed in this paper.

Specifically, although Cao and Zaman [31] and Niu [32] presented analytical formulations for the FOS of a slope using the SSM, they did not derive the mathematical equations for determining the minimum FOS. The FOS was obtained through repeated trial calculations implemented in computer codes, which require multiple iterations. This method lacks reasonable initial values for iteration and requires complex computer-language programming, thus limiting its computational efficiency. Although Baker [33] further validated Taylor's stability chart from an analytical perspective, the outcome remained in the form of charts. Charts inevitably involve plotting and reading errors, leading to limited accuracy. Therefore, the proposed MEM can remedy the aforementioned drawbacks and also help explore the mathematical feasibility of deriving an analytical solution for the minimum FOS when the SSM is used.

In the MEM, the complete mathematical expressions of the necessary equations determining the CSS and FOS are derived. A set of approximate explicit solutions of the equations is discovered. The equations are written into an EXCEL spreadsheet. They are solved using the solver tool. During the solution process, the discovered approximate explicit solutions are used as iterative initial values. The results are obtained with just one iteration. In Section 2, several methods, including the MEM, for determining the CSS and FOS of the SSM are introduced. In Section 3, the applicability of the MEM is verified by comparing it to other methods. In Section 4, the generality of the MEM is further tested by comparisons with other tools. In Section 5, assessments are made on some of the existing research. Section 6 outlines the discussion. In Section 7, conclusions are drawn.

2. Theories

2.1. Empirical Methods

2.1.1 The 4.5 H Method

The 4.5 H method is a type of empirical method [15,23]. The main characteristics of this method are as follows. As shown in Figure 1, when $\varphi \neq 0$ (φ is the soil's effective internal friction angle), the values of β_1 and β_2 can be selected from Table 1. According to the slope angle, the segments CO and BO are drawn. The points (denoted as $O_1 \sim O_4, O_{1'} \sim O_{4'}$) are selected on the extension line of segment EO . The corresponding FOSs (denoted as $FOS_1 \sim FOS_4, FOS_{1'} \sim FOS_{4'}$) are calculated. The minimal value of the FOS is found. The circle with the center O_{min} and the value of FOS_{min} are confirmed as the CSS and the FOS of a slope.

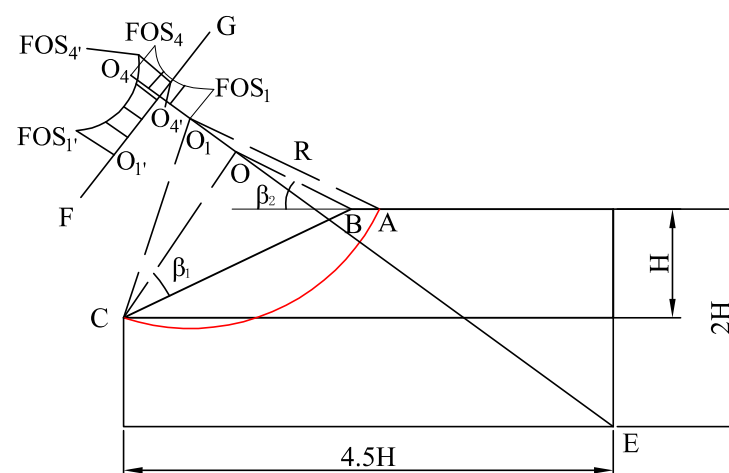


Figure 1. Determination of the center of the CSS using the empirical method. Note: the red line indicates the slip surface for center O_1 .

Table 1. β_1 and β_2 for different slope ratios in the empirical method.

Slope Ratio	Slope Angle (°)	β_1 (°)	β_2 (°)
1:0.58	60	29	40
1:1	45	28	37
1:1.5	33.79	26	35
1:2	26.57	25	35
1:3	18.43	25	35
1:5	11.32	25	37

2.1.2 The 36-Degree Line Method

The 36-degree line method is another type of empirical method [24]. In this method, the center point is assumed to be located on the line of BO . This means that β_1 is unnecessary, and β_2 equals 36 degrees.

2.2. Computer Search Methods

With the development of computer technology, searching for the CSS and FOS using computers and complex mathematical methods has become a trend. For the slope, several metaheuristic algorithms have been applied. They are mainly categorized as genetic algorithms (GAs) [34], particle swarm optimization (PSO) [35], biogeography-based optimization (BBO) [36], the gravitational search algorithm [37], ant colony optimization [38], the black hole algorithm [39], and the prey–predator algorithm [28].

2.3. The Proposed Mathematical Extremum Method

As a type of analytical method, the MEM is proposed in this paper. As shown in Figure 2, it is assumed that there is a homogeneous slope. For the slope, there are parameters such as the slope angle (denoted as α), the slope height (denoted as H), the unit weight (denoted as γ), the effective cohesion (denoted as c), and the effective internal friction angle (denoted as φ). A coordinate system is established. In the system, the toe of the slope (denoted as point P) is the origin. As stated, for relatively steep homogeneous soil slopes without special reinforcement equipment, the CSS mostly passes through the toe of the slope [2]. Additionally, hundreds of runs have been made using SLOPE/W to locate the critical circles of slopes with $c' - \varphi'$ soil. The failure circles are found to be mostly toe circles [40]. Therefore, this is mainly considered for toe circles.

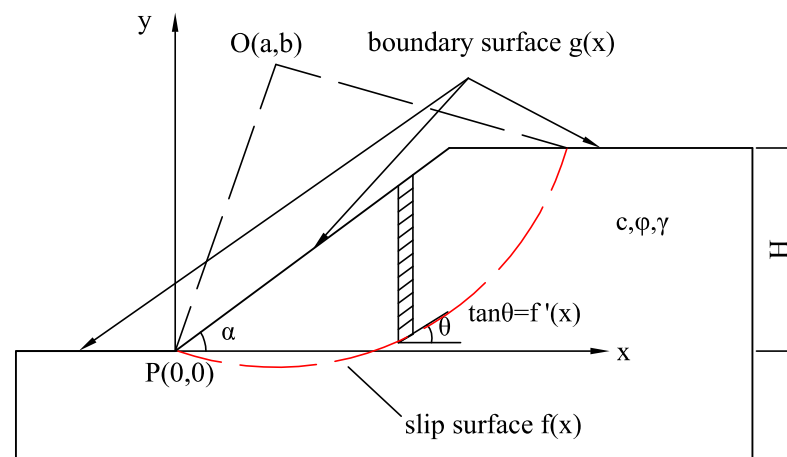


Figure 2. Sketch of the slope. Note: the red line indicates the slip surface for center $O(a,b)$.

2.3.1. Derivation of Mathematical Extremum Equations for Determining the Critical Slip Surface

The slip surface is assumed to be a circle. The coordinates of the center (denoted as point O) of the circle are denoted as (a, b) . As shown in Equation (1), the mathematical function expressing the slip surface can be obtained.

$$f(x) = b - \sqrt{a^2 + b^2 - (x - a)^2} \quad (1)$$

$$\text{When } x \geq a, f'(x) = \frac{(x-a)}{(a^2+b^2-(x-a)^2)^{\frac{1}{2}}} \geq 0 \text{ and } f''(x) = \frac{(a^2+b^2)}{(a^2+b^2-(x-a)^2)^{\frac{3}{2}}} \geq 0.$$

The function $f(x)$ is a concave increasing function. It is consistent with the general law of sliding. Consequently, Equation (1) has been shown to be feasible in expressing the slip surface.

The coordinates of the crest of the slope are $(H/\tan \alpha, H)$. To ensure that the crest of the slope is within the sliding area, relative parameters should satisfy Equation (2).

$$\left(\frac{H}{\tan \alpha} - a\right)^2 + (H - b)^2 \leq a^2 + b^2 \quad (2)$$

For a specific slope, α , H , γ , c , and φ are constants, and a and b are variables. The function expressing the boundary surface of the slope is denoted as $g(x)$. The function $g(x)$ can be divided into two segments. The first segment is denoted as $g(x) = (\tan \alpha) * x$ ($0 \leq x \leq \frac{H}{\tan \alpha}$). The second segment is denoted as $g(x) = H$ ($x > \frac{H}{\tan \alpha}$).

Then, the sliding area is divided into slices. For any soil slice, the angle between the opposite direction of the sliding soil slice and the direction of the X-axis is denoted as θ . As a result, the relative parameters satisfy Equation (3).

$$\sin \theta = \frac{x-a}{\sqrt{a^2+b^2}}, \cos \theta = \frac{\sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}}, \tan \theta = f'(x) = \frac{x-a}{\sqrt{a^2+b^2-(x-a)^2}} \quad (3)$$

The length of the sliding arc of the soil slice is denoted as dS . According to the trigonometric relations in Equation (3), dS can be calculated using Equation (4).

$$dS = \sqrt{1 + \tan^2 \theta} * dx = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2-(x-a)^2}} * dx \quad (4)$$

The gravity of the soil slice is denoted as dG . It can be calculated using Equation (5).

$$dG = \gamma * (g(x) - f(x)) * dx \quad (5)$$

The sliding force for the soil slice is denoted as dF . The normal force applied on the sliding surface by the soil slice is denoted as dN . dF and dN can be calculated using Equation (6).

$$\begin{cases} dF = dG * \sin \theta = \frac{\gamma * (g(x) - f(x)) * (x-a)}{\sqrt{a^2+b^2}} * dx \\ dN = dG * \cos \theta = \frac{\gamma * (g(x) - f(x)) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx \end{cases} \quad (6)$$

When the coordinates of the center of the slip surface are (a, b) , the calculated FOS of a slope using the SSM is denoted as $F_s(a, b)$. $F_s(a, b)$ can be calculated using Equation (7).

$$\begin{aligned} F_s(a, b) &= \frac{c * \int dS + \tan \varphi * \int dN}{\int dF} \\ &= \frac{c * \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2-(x-a)^2}} * dx + \gamma * \tan \varphi * \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{(g(x) - f(x)) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx}{\gamma * \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{(g(x) - f(x)) * (x-a)}{\sqrt{a^2+b^2}} * dx} = \\ &= \frac{c * [\int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2-(x-a)^2}} * dx] + \gamma * \tan \varphi * [\int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{g(x) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx - \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{f(x) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx]}{\gamma * [\int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{g(x) * (x-a)}{\sqrt{a^2+b^2}} * dx - \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{f(x) * (x-a)}{\sqrt{a^2+b^2}} * dx]} \end{aligned} \quad (7)$$

To simplify Equation (7), the following assumptions are made.

$$i_1 = \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2-(x-a)^2}} * dx \quad (8)$$

$$i_2 = \int_0^{\frac{H}{\tan \alpha}} \frac{(\tan \alpha) * x * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx \quad (9)$$

$$i_3 = \int_{\frac{H}{\tan \alpha}}^{a+\sqrt{a^2+2bH-H^2}} \frac{H * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx \quad (10)$$

$$i_4 = \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{f(x) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx$$

$$= \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{(b - \sqrt{a^2+b^2-(x-a)^2}) * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} * dx \quad (11)$$

$$i_5 = \int_0^{\frac{H}{\tan \alpha}} \frac{(\tan \alpha) * x * (x-a)}{\sqrt{a^2+b^2}} * dx \quad (12)$$

$$i_6 = \int_{\frac{H}{\tan \alpha}}^{a+\sqrt{a^2+2bH-H^2}} \frac{H * (x-a)}{\sqrt{a^2+b^2}} * dx \quad (13)$$

$$i_7 = \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{f(x) * (x-a)}{\sqrt{a^2+b^2}} * dx$$

$$= \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{(b - \sqrt{a^2+b^2-(x-a)^2}) * (x-a)}{\sqrt{a^2+b^2}} * dx \quad (14)$$

According to Equations (8)–(14), Equation (7) can be simplified into Equation (15).

$$F_s(a, b) = \frac{c * i_i + \gamma * \tan \varphi * (i_2 + i_3 - i_4)}{\gamma * (i_5 + i_6 - i_7)} = \left(\frac{i_1}{i_5 + i_6 - i_7} \right) * \frac{c}{\gamma} + \left(\frac{i_2 + i_3 - i_4}{i_5 + i_6 - i_7} \right) * \tan \varphi \quad (15)$$

According to the principle of integration, $i_1, i_2, i_3, i_4, i_5, i_6, i_7$ can be calculated using Equations (16)–(22).

$$i_1 = \int_0^{a+\sqrt{a^2+2bH-H^2}} \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2-(x-a)^2}} dx$$

$$= \sqrt{a^2+b^2} * \left[\sin^{-1} \left(\frac{\sqrt{a^2+2bH-H^2}}{\sqrt{a^2+b^2}} \right) + \sin^{-1} \left(\frac{a}{\sqrt{a^2+b^2}} \right) \right] \quad (16)$$

$$i_2 = \int_0^{\frac{H}{\tan \alpha}} \frac{(\tan \alpha) * x * \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} dx$$

$$= \frac{\tan \alpha}{\sqrt{a^2+b^2}} * \left[\frac{1}{3} * (b^2)^{\frac{3}{2}} - \frac{1}{3} * \left(b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2 \right)^{\frac{3}{2}} + \frac{1}{2} * a * \left(\frac{H}{\tan \alpha} - a \right) \right.$$

$$* \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \frac{1}{2} * a * (a^2 + b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2+b^2}} \right) + \frac{1}{2} * a^2$$

$$* b + \frac{1}{2} * a * (a^2 + b^2) * \sin^{-1} \left(\frac{a}{\sqrt{a^2+b^2}} \right) \left. \right] \quad (17)$$

$$\begin{aligned}
 i_3 &= \int_{\frac{H}{\tan \alpha}}^{a+\sqrt{a^2+2bH-H^2}} \frac{H}{\sqrt{a^2+b^2}} * \sqrt{a^2+b^2-(x-a)^2} dx \\
 &= \frac{H}{2 * \sqrt{a^2+b^2}} \\
 &\quad * \left[\sqrt{a^2+2bH-H^2} * \sqrt{(b-H)^2} + (a^2+b^2) * \sin^{-1} \left(\frac{\sqrt{a^2+2bH-H^2}}{\sqrt{a^2+b^2}} \right) \right. \\
 &\quad \left. - \left(\frac{H}{\tan \alpha} - a \right) * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} - (a^2+b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2+b^2}} \right) \right]
 \end{aligned} \quad (18)$$

$$\begin{aligned}
 i_4 &= \int_0^{a+\sqrt{a^2+2bH-H^2}} \left(b - \sqrt{a^2+b^2-(x-a)^2} \right) * \frac{\sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} dx \\
 &= \frac{b}{\sqrt{a^2+b^2}} \\
 &\quad * \left[\frac{1}{2} * (a^2+b^2) * \sin^{-1} \left(\frac{\sqrt{a^2+2bH-H^2}}{\sqrt{a^2+b^2}} \right) + \frac{1}{2} * \sqrt{a^2+2bH-H^2} * \sqrt{(b-H)^2} + \frac{1}{2} \right. \\
 &\quad * (a^2+b^2) * \sin^{-1} \left(\frac{a}{\sqrt{a^2+b^2}} \right) + \frac{ab}{2} \Big] \\
 &\quad - \left[\sqrt{a^2+b^2} * \left(a + \sqrt{a^2+2bH-H^2} \right) - \frac{1}{3\sqrt{a^2+b^2}} * (a^2+2bH-H^2)^{\frac{3}{2}} \right. \\
 &\quad \left. - \frac{a^3}{3\sqrt{a^2+b^2}} \right]
 \end{aligned} \quad (19)$$

$$i_5 = \int_0^{\frac{H}{\tan \alpha}} (\tan \alpha) * x * \frac{x-a}{\sqrt{a^2+b^2}} dx = \frac{\tan \alpha}{\sqrt{a^2+b^2}} * \left[\frac{1}{3} * \left(\frac{H}{\tan \alpha} - a \right)^3 + \frac{a}{2} * \left(\frac{H}{\tan \alpha} - a \right)^2 - \frac{a^3}{6} \right] \quad (20)$$

$$i_6 = \int_{\frac{H}{\tan \alpha}}^{a+\sqrt{a^2+2bH-H^2}} H * \frac{(x-a)}{\sqrt{a^2+b^2}} dx = \frac{H}{2\sqrt{a^2+b^2}} * \left[a^2+2bH-H^2 - \left(\frac{H}{\tan \alpha} - a \right)^2 \right] \quad (21)$$

$$\begin{aligned}
 i_7 &= \int_0^{a+\sqrt{a^2+2bH-H^2}} \left(\frac{b - \sqrt{a^2+b^2-(x-a)^2}}{\sqrt{a^2+b^2}} \right) * (x-a) dx \frac{b * (2bH-H^2)}{2\sqrt{a^2+b^2}} + \frac{1}{3\sqrt{a^2+b^2}} \\
 &\quad * \left[(b^2-2bH+H^2)^{\frac{3}{2}} - (b^2)^{\frac{3}{2}} \right]
 \end{aligned} \quad (22)$$

For a special slope, under the CSS, the value of $F_s(a, b)$ takes the local minimum. This means that the first partial derivatives of the expression $F_s(a, b)$ with respect to a and b are equal to zero. According to the previous text, a and b should satisfy Equation (2). The coordinates of the center of the CSS can be obtained directly by solving Equation (23).

$$\begin{cases} \frac{\partial F_s(a, b)}{\partial a} = 0 \\ \frac{\partial F_s(a, b)}{\partial b} = 0 \\ \left(\frac{H}{\tan \alpha} - a \right)^2 + (H - b)^2 \leq a^2 + b^2 \end{cases} \quad (23)$$

After substituting the relevant equations, Equation (23) can be changed into Equation

(24).

$$\begin{cases} \frac{\partial i_1}{\partial a} * (i_5 + i_6 - i_7) - i_1 * \left(\frac{\partial i_5}{\partial a} + \frac{\partial i_6}{\partial a} - \frac{\partial i_7}{\partial a} \right) * \frac{c}{\gamma} + \frac{\left(\frac{\partial i_2}{\partial a} + \frac{\partial i_3}{\partial a} - \frac{\partial i_4}{\partial a} \right) * (i_5 + i_6 - i_7) - \left(\frac{\partial i_5}{\partial a} + \frac{\partial i_6}{\partial a} - \frac{\partial i_7}{\partial a} \right) * (i_2 + i_3 - i_4)}{(i_5 + i_6 - i_7)^2} * \tan \varphi = 0 \\ \frac{\partial i_1}{\partial b} * (i_5 + i_6 - i_7) - i_1 * \left(\frac{\partial i_5}{\partial b} + \frac{\partial i_6}{\partial b} - \frac{\partial i_7}{\partial b} \right) * \frac{c}{\gamma} + \frac{\left(\frac{\partial i_2}{\partial b} + \frac{\partial i_3}{\partial b} - \frac{\partial i_4}{\partial b} \right) * (i_5 + i_6 - i_7) - \left(\frac{\partial i_5}{\partial b} + \frac{\partial i_6}{\partial b} - \frac{\partial i_7}{\partial b} \right) * (i_2 + i_3 - i_4)}{(i_5 + i_6 - i_7)^2} * \tan \varphi = 0 \\ \left(\frac{H}{\tan \alpha} - a \right)^2 + (H - b)^2 \leq a^2 + b^2 \end{cases} \quad (24)$$

To solve the above equations, $i_1, i_2, i_3, i_4, i_5, i_6, i_7, \frac{\partial i_1}{\partial a}, \frac{\partial i_2}{\partial a}, \frac{\partial i_3}{\partial a}, \frac{\partial i_4}{\partial a}, \frac{\partial i_5}{\partial a}, \frac{\partial i_6}{\partial a}, \frac{\partial i_7}{\partial a}$ and $\frac{\partial i_1}{\partial b}, \frac{\partial i_2}{\partial b}, \frac{\partial i_3}{\partial b}, \frac{\partial i_4}{\partial b}, \frac{\partial i_5}{\partial b}, \frac{\partial i_6}{\partial b}, \frac{\partial i_7}{\partial b}$ need to be calculated. It is easy to see from the mathematical forms of $i_1, i_2, i_3, i_4, i_5, i_6, i_7$ that, using the general rules of differentiation, the partial derivatives of these formulas with respect to a and b can be obtained. These partial derivatives are shown in Equations (25)–(38).

$$\frac{\partial i_1}{\partial a} = \frac{a}{\sqrt{a^2 + b^2}} * \left[\sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \right] + \frac{a * |b - H|}{\sqrt{a^2 + b^2} * \sqrt{a^2 + 2bH - H^2}} + \frac{|b|}{\sqrt{a^2 + b^2}} \quad (24)$$

$$\begin{aligned} \frac{\partial i_2}{\partial a} = \tan \alpha * & \left\{ \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} \right. \\ & * \left[\frac{1}{3} * (b^2)^{\frac{3}{2}} - \frac{1}{3} * \left(b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2 \right)^{\frac{3}{2}} + \frac{1}{2} * a * \left(\frac{H}{\tan \alpha} - a \right) * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \frac{1}{2} * a \right. \\ & * (a^2 + b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) + \frac{1}{2} * a * b^2 + \frac{1}{2} * a * (a^2 + b^2) * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \left. \right] + \frac{1}{\sqrt{a^2 + b^2}} \\ & * \left[\frac{(-H)}{\tan \alpha} * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \left(\frac{H}{2 \tan \alpha} - a \right) * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \left(\frac{aH}{2 \tan \alpha} - \frac{a^2}{2} \right) \right. \\ & * \frac{H}{\tan \alpha} * \frac{1}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} + \left(\frac{3a^2}{2} + \frac{b^2}{2} \right) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) + \left(\frac{a^3}{2} + \frac{ab^2}{2} \right) * (-1) \\ & * \frac{1}{a^2 + b^2} * \frac{\left(b^2 + \frac{aH}{\tan \alpha} \right)}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} + ab + \left(\frac{3a^2}{2} + \frac{b^2}{2} \right) * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \left(\frac{a^3}{2} + \frac{ab^2}{2} \right) \\ & \left. * \frac{|b|}{a^2 + b^2} \right\} \end{aligned} \quad (25)$$

$$\begin{aligned}
\frac{\partial i_3}{\partial a} = & \frac{H}{2} * \left\{ \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} \right. \\
& * \left[|b - H| * \sqrt{a^2 + 2bH - H^2} + (a^2 + b^2) * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) \right. \\
& - \left(\frac{H}{\tan \alpha} - a \right) * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} - (a^2 + b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) \Big] \\
& + \frac{1}{\sqrt{a^2 + b^2}} \\
& * \left[\frac{2 * |b - H| * a}{\sqrt{a^2 + 2bH - H^2}} + 2a * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} \right. \\
& - \left(\frac{H}{\tan \alpha} \right) * \frac{\left(\frac{H}{\tan \alpha} - a \right)}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} - 2a * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) \\
& \left. \left. + \frac{\left(b^2 + \frac{aH}{\tan \alpha} \right)}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} \right] \right\}
\end{aligned} \tag{26}$$

$$\begin{aligned}
\frac{\partial i_4}{\partial a} = & b * \left\{ \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} \right. \\
& * \left[\frac{1}{2} * (a^2 + b^2) * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \frac{1}{2} * \sqrt{a^2 + 2bH - H^2} * |b - H| + \frac{1}{2} \right. \\
& * (a^2 + b^2) * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \frac{ab}{2} \Big] + \frac{1}{\sqrt{a^2 + b^2}} \\
& * \left[a * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \frac{a * |b - H|}{\sqrt{a^2 + 2bH - H^2}} + a * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \frac{|b|}{2} \right. \\
& \left. \left. + \frac{b}{2} \right] \right\} - \frac{a}{\sqrt{a^2 + b^2}} * \left(a + \sqrt{a^2 + 2bH - H^2} \right) - \sqrt{a^2 + b^2} * \left(1 + \frac{a}{\sqrt{a^2 + 2bH - H^2}} \right) \\
& + a * \frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} - \frac{1}{3} * a * \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right)^3 + \frac{a^2}{\sqrt{a^2 + b^2}} - \frac{1}{3} \\
& * \frac{a^4}{(a^2 + b^2)^{\frac{3}{2}}}
\end{aligned} \tag{27}$$

$$\frac{\partial i_5}{\partial a} = \tan \alpha * \left\{ \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} * \left[\frac{1}{3} * \left(\frac{H}{\tan \alpha} - a \right)^3 + \frac{a}{2} * \left(\frac{H}{\tan \alpha} - a \right)^2 - \frac{a^3}{6} \right] + \frac{1}{\sqrt{a^2 + b^2}} \right. \\ \left. * \left[- \left(\frac{H}{\tan \alpha} - a \right)^2 + \frac{1}{2} * \left(\frac{H}{\tan \alpha} - a \right)^2 + a * \left(\frac{H}{\tan \alpha} - a \right) - \frac{a^2}{2} \right] \right\} \quad (28)$$

$$\frac{\partial i_6}{\partial a} = \frac{H}{2} * \left\{ \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} * \left[a^2 + 2bH - H^2 - \left(\frac{H}{\tan \alpha} - a \right)^2 \right] + \frac{1}{\sqrt{a^2 + b^2}} * \left(\frac{2H}{\tan \alpha} \right) \right\} \quad (29)$$

$$\frac{\partial i_7}{\partial a} = \frac{1}{2} * b * (2bH - H^2) * \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} + \frac{1}{3} * \left[(b^2 - 2bH + H^2)^{\frac{3}{2}} - (b^2)^{\frac{3}{2}} \right] * \frac{(-a)}{(a^2 + b^2)^{\frac{3}{2}}} \quad (30)$$

$$\frac{\partial i_1}{\partial b} = \frac{b}{\sqrt{a^2 + b^2}} * \left[\sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \right] + \sqrt{a^2 + b^2} \\ * \left[\frac{(b - H)}{|b - H|} * \frac{(-1) * (a^2 + bH)}{(a^2 + b^2) * \sqrt{a^2 + 2bH - H^2}} - \frac{ab}{|b| * (a^2 + b^2)} \right] \quad (31)$$

$$\frac{\partial i_2}{\partial b} = \tan \alpha * \left\{ \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} \right. \\ * \left[\frac{1}{3} * (b^2)^{\frac{3}{2}} - \frac{1}{3} * \left(b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2 \right)^{\frac{3}{2}} + \frac{1}{2} * a * \left(\frac{H}{\tan \alpha} - a \right) \right. \\ * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \frac{1}{2} * a * (a^2 + b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) + \frac{a^2 b}{2} + \frac{a}{2} * (a^2 + b^2) \\ \left. * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \right] + \frac{1}{\sqrt{a^2 + b^2}} \\ * \left[b * |b| - b * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} + \frac{a}{2} * \left(\frac{H}{\tan \alpha} - a \right) * \frac{b}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} \right. \\ + ab * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) + \left(\frac{a^3}{2} + \frac{ab^2}{2} \right) * \frac{1}{a^2 + b^2} * \frac{(-b) * \left(\frac{H}{\tan \alpha} - a \right)}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} + \frac{1}{2} * a^2 \\ \left. + ab * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \left(\frac{a^3}{2} + \frac{ab^2}{2} \right) * \frac{(-ab)}{|b| * (a^2 + b^2)} \right] \left. \right\} \quad (32)$$

$$\begin{aligned}
\frac{\partial i_3}{\partial b} = & \frac{H}{2} * \left\{ \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} \right. \\
& * \left[|b - H| * \sqrt{a^2 + 2bH - H^2} + (a^2 + b^2) * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) \right. \\
& \left. - \left(\frac{H}{\tan \alpha} - a \right) * \sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2} - (a^2 + b^2) * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) \right] \\
& + \frac{1}{\sqrt{a^2 + b^2}} \\
& * \left[\frac{H * |b - H|}{\sqrt{a^2 + 2bH - H^2}} + \frac{\sqrt{a^2 + 2bH - H^2} * (b - H)}{|b - H|} + 2b * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) \right. \\
& + \frac{(b - H)}{|b - H|} * \frac{(-1) * (a^2 + bH)}{\sqrt{a^2 + 2bH - H^2}} - \left(\frac{H}{\tan \alpha} - a \right) * \frac{b}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} - 2b \\
& \left. * \sin^{-1} \left(\frac{\frac{H}{\tan \alpha} - a}{\sqrt{a^2 + b^2}} \right) + \frac{b * \left(\frac{H}{\tan \alpha} - a \right)}{\sqrt{b^2 + \frac{2aH}{\tan \alpha} - \left(\frac{H}{\tan \alpha} \right)^2}} \right] \left. \right\}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\frac{\partial i_4}{\partial b} = & \frac{a^2}{(a^2 + b^2)^{\frac{3}{2}}} * \left[\frac{1}{2} * (a^2 + b^2) * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \frac{1}{2} * \sqrt{a^2 + 2bH - H^2} * |b - H| + \frac{1}{2} * (a^2 + b^2) \right. \\
& * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \frac{ab}{2} \left. \right] + \frac{b}{\sqrt{a^2 + b^2}} \\
& * \left[b * \sin^{-1} \left(\frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} \right) + \frac{1}{2} * \frac{(b - H)}{|b - H|} * \frac{(-1) * (a^2 + bH)}{\sqrt{a^2 + 2bH - H^2}} + \frac{1}{2} * \frac{H * |b - H|}{\sqrt{a^2 + 2bH - H^2}} + \frac{1}{2} \right. \\
& * \frac{\sqrt{a^2 + 2bH - H^2} * (b - H)}{|b - H|} + b * \sin^{-1} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + \frac{(-ab)}{2|b|} + \frac{a}{2} \left. \right] - \frac{b}{\sqrt{a^2 + b^2}} \\
& * \left(a + \sqrt{a^2 + 2bH - H^2} \right) - \sqrt{a^2 + b^2} * \frac{H}{\sqrt{a^2 + 2bH - H^2}} + H * \frac{\sqrt{a^2 + 2bH - H^2}}{\sqrt{a^2 + b^2}} - \frac{1}{3} * b \\
& * \left(\frac{a^2 + 2bH - H^2}{a^2 + b^2} \right)^{\frac{3}{2}} - \frac{1}{3} * \frac{a^3 b}{(a^2 + b^2)^{\frac{3}{2}}}
\end{aligned} \tag{34}$$

$$\frac{\partial i_5}{\partial b} = \tan \alpha * \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} * \left[\frac{1}{3} * \left(\frac{H}{\tan \alpha} - a \right)^3 + \frac{a}{2} * \left(\frac{H}{\tan \alpha} - a \right)^2 - \frac{a^3}{6} \right] \tag{35}$$

$$\frac{\partial i_6}{\partial b} = \frac{H}{2} * \left\{ \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} * \left[a^2 + 2bH - H^2 - \left(\frac{H}{\tan \alpha} - a \right)^2 \right] + \frac{2H}{\sqrt{a^2 + b^2}} \right\} \tag{36}$$

$$\begin{aligned} \frac{\partial i_7}{\partial b} = & \frac{1}{2} * \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} * (2b^2H - bH^2) + \frac{1}{2} * \frac{(4bH - H^2)}{\sqrt{a^2 + b^2}} + \frac{1}{3} * \frac{(-b)}{(a^2 + b^2)^{\frac{3}{2}}} \\ & * \left[(b^2 - 2bH + H^2)^{\frac{3}{2}} - (b^2)^{\frac{3}{2}} \right] + \frac{1}{\sqrt{a^2 + b^2}} * [(b - H) * |b - H| - b * |b|] \end{aligned} \quad (37)$$

After substituting the relative equations into Equation (24), equations can be obtained that contain only the two unknowns: a and b . Note that, in the equations, a and b need to satisfy Equation (23). In this way, obtaining the CSS and the FOS is transformed into solving the equations.

2.3.2. The Discovery of Approximately Explicit Solutions for the Derived Mathematical Extremum Equations

As shown in the previous text, the equations are complex. It is not possible to obtain accurate explicit solutions at this time. Solutions can only be obtained through iterative methods. However, it is necessary to set reasonable iterative initial values (denoted as a_0 and b_0) for the two unknowns. Otherwise, reliable convergence solutions may not be obtained. Although research [26,27,31] has indicated the region where the center of the CSS may lie, the specific location of this point is not provided. The clear iterative initial values for searching for the center are not offered. As shown in Equation (39) and Figure 3, it has been discovered that there are approximate explicit solutions for the equations. These solutions may be selected as the iterative initial values for solving the equations.

$$\begin{cases} a_0 = \frac{H}{2} \left[\frac{1}{\tan \alpha} - \tan(\alpha - 15^\circ) - \tan \varphi \right] \\ b_0 = \frac{H}{4} [2 + \cot^2 \beta - \cot \beta \cot \alpha + 2 \tan(\alpha - 15^\circ) + 2 \tan \varphi] \\ \cot \beta = \cot \alpha + \csc \alpha \sqrt{\frac{2c}{2c + \gamma H \tan \varphi}} \end{cases} \quad (38)$$

The procedure for determining this approximate solution is as follows. According to Pan [41], for purely cohesive soils, the center of the CSS lies on the vertical line passing through the midpoint of the slope face, denoted as line MN in Figure 3. According to Jiang et al. [42], if the slip surface is assumed to be planar, the CSS of the slope admits an analytical solution, which is a straight line inclined at an angle β to the X-axis (see Figure 3). If this planar CSS is treated as a special circular arc, a vertical line drawn through the midpoint of the slip surface intersects line MN at point M. It can be observed that taking the midpoint of MN and shifting it upward and left by an equal distance, respectively, which is Δx or Δy in Figure 3 (obtained by trial calculation and empirical estimation), yields a point located near the center of the critical slip circle.

2.3.3. Solving Derived Mathematical Extremum Equations Using EXCEL

For the convenience of engineers and researchers, the solver tool in the EXCEL spreadsheet can be used to obtain the solutions of the derived mathematical extremum equations. When Equations (1)–(39) are written in an EXCEL spreadsheet, the solutions of the equations can be obtained. This is the proposed MEM.

To preliminarily verify the feasibility of the approximate explicit solution and MEM, 480 general slope examples are designed. These slopes cover a range of cohesive forces (10 kPa–100 kPa), internal friction angles (5° – 45°), slope heights (5 m–20 m), and slope angles (15° – 85°). The calculation results are presented in Table 2. It is shown that the convergent solution can be obtained with a single iteration for most cases, except for

scenarios where both the slope angle and internal friction angle are very small, as well as cases where both parameters are very large.

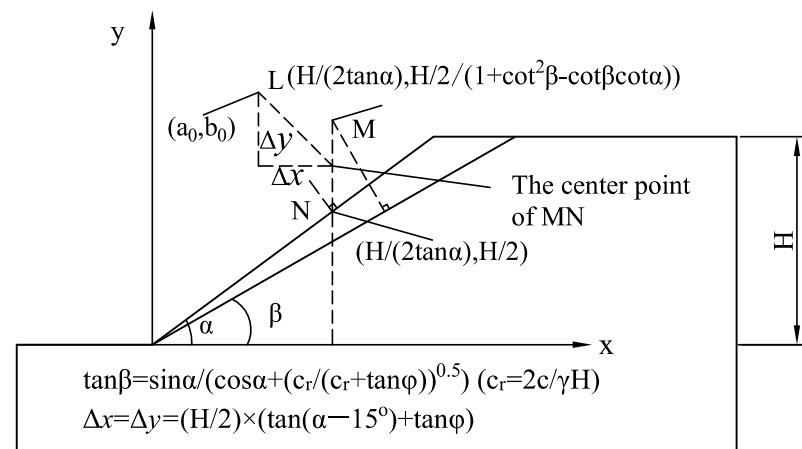


Figure 3. Sketch showing the determination of initial or approximate values for the parameters a and b .

Table 2. Results under the MEM for the preliminary verification.

α (°)	φ (°)	c (kPa)				c (kPa)				c (kPa)			
		10				50				100			
		H (m)				H (m)				H (m)			
		5	10	15	20	5	10	15	20	5	10	15	20
15	5	×	×	√	√	×	×	×	×	×	×	×	×
	15	√	√	√	√	×	×	×	√	×	×	×	×
	25	√	√	√	√	×	×	√	√	×	×	×	×
	35	√	√	√	√	×	×	√	√	×	×	√	√
	45	√	√	√	√	×	√	√	√	×	×	√	√
25	5	√	√	√	√	√	√	√	√	√	√	√	√
	15	√	√	√	√	√	√	√	√	√	√	√	√
	25	√	√	√	√	√	√	√	√	×	√	√	√
	35	√	√	√	√	√	√	√	√	×	√	√	√
	45	√	√	√	√	√	√	√	√	×	√	√	√
35	5	√	√	√	√	√	√	√	√	√	√	√	√
	15	√	√	√	√	√	√	√	√	√	√	√	√
	25	√	√	√	√	√	√	√	√	√	√	√	√
	35	√	√	√	√	√	√	√	√	√	√	√	√
	45	√	√	×	×	√	√	√	√	√	√	√	√
45	5	√	√	√	√	√	√	√	√	√	√	√	√
	15	√	√	√	√	√	√	√	√	√	√	√	√
	25	√	√	√	√	√	√	√	√	√	√	√	√
	35	√	√	√	√	√	√	√	√	√	√	√	√
	45	√	√	√	√	√	√	√	√	×	√	√	√
55	5	√	√	√	√	√	√	√	√	√	√	√	√
	15	√	√	√	√	√	√	√	√	√	√	√	√
	25	√	√	√	√	√	√	√	√	√	√	√	√
	35	√	√	√	√	√	√	√	√	√	√	√	√
	45	√	√	√	×	√	√	√	√	×	√	√	√
65	5	√	√	√	√	√	√	√	√	√	√	√	√
	15	√	√	√	√	√	√	√	√	√	√	√	√
	25	√	√	√	√	√	√	√	√	√	√	√	√

75	35	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	45	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	5	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	15	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	25	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	35	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	45	✓	×	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	5	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
85	15	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	25	✓	×	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	35	×	×	✓	×	✓	✓	✓	✓	✓	✓	✓	✓
	45	×	×	×	×	✓	✓	✓	✓	✓	✓	✓	✓
	5	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Note: “✓” denotes that a convergent solution is obtained in one iteration, and “×” denotes that a convergent solution cannot be obtained in one iteration.

3. Verification

To verify the applicability of the proposed MEM, a homogeneous slope is selected [28]. The slope has a slope angle of 26.6° , a slope height of 5 m, a unit weight of 17.64 kN/m^3 , an effective cohesion of 9.84 kPa, and an effective internal friction angle of 10° .

3.1. Empirical Methods

For the empirical method, the sliding body is divided into 16 slices. The FOSs under the 4.5 H method and the 36-degree line method are 1.290 and 1.308, respectively. The coordinates of the center and radius of the CSS under the 4.5 H method and the 36-degree line method are (8.75,14.55,10.26) and (8.81,14.50,10.24), respectively.

3.2. Computer Search Methods

For the computer search method, Leonetti et al. [28] have provided several results using methods such as genetic algorithms (GAs), biogeography-based optimization (BBO), grid search, and the prey–predator algorithm (PPA) for the slope. These results are directly adopted.

3.3. The Proposed Mathematical Extremum Method

As shown in Figure 4, the FOS is confirmed to be 1.283 based on the EXCEL spreadsheet when the MEM is applied. The coordinates of the center and radius of the CSS are given as (8.980, 13.075, 9.003). As shown in Figure 5, the obtained FOS of 1.283 is not only a local minimum near this slope, but also the global minimum.

Basic parameters		Process parameters		Process first-order derivatives				Governing equations	
c=	9,840	i ₁ =	15.12509102	δi ₁ /δa=	1.80031944	δi ₁ /δb=	0.3269697	δFs/δa=	-8.3308×10^{-10}
φ=	10	i ₂ =	22.84216504	δi ₂ /δa=	1.19485003	δi ₂ /δb=	0.4624419	δFs/δb=	-1.2962×10^{-8}
γ=	17,640	i ₃ =	7.113425806	δi ₃ /δa=	5.26656939	δi ₃ /δb=	2.5069858	Equation (2)=	-35.5304029
H=	5	i ₄ =	-0.08516675	δi ₄ /δa=	-4.19400178	δi ₄ /δb=	1.661342	Aim=	1.3796×10^{-8}
α=	26.6	i ₅ =	7.422131849	δi ₅ /δa=	-3.13711660	δi ₅ /δb=	-0.7395034	Fs(a,b)=	1.283036045
a=	3.979637	i ₆ =	9.866486346	δi ₆ /δa=	5.0609166	δi ₆ /δb=	1.7938673		
b=	8.075451	i ₇ =	6.584225485	δi ₇ /δa=	-0.32329047	δi ₇ /δb=	0.7324382		

Figure 4. Calculation process of the MEM.

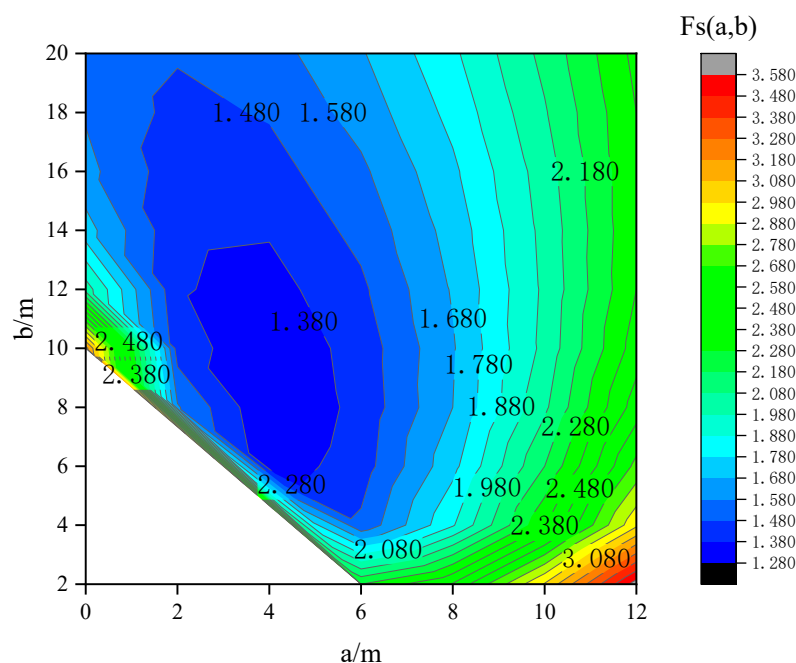


Figure 5. Contour map of the $F_s(a,b)$.

3.4. Comparisons

To enable comparison, results under different methods are summarized in Table 3. Schematic diagrams of the CSSs under different methods are drawn in Figure 6. As shown in Table 3, in terms of the FOS, in the empirical method, the 4.5 H method is more accurate than the 36-degree line method. The MEM is basically consistent with other methods. According to Figure 6, when referring to the CSS, except for the grid search method, the CSSs of various methods are similar to each other.

Table 3. Results under different methods.

Method	Solver	FOS (Number of Slices)	Coordinates of the Center and Radius of the CSS [X_c , Y_c , r_c]	Relative Error of the FOS
Empirical method	4.5 H method (this paper) (1)	1.290 (16)	[8.75, 14.55, 10.26]	0.5% (17–11/1)
	36-degree line method (this paper) (2)	1.308 (16)	[8.81, 14.50, 10.24]	1.9% (17–21/2)
Computer search method	GA [28] (3)	1.280 (52)	[9.12, 11.56, 8.06]	0.2% (17–31/3)
	BBO [28] (4)	1.272 (45)	[9.02, 12.27, 8.76]	0.9% (17–41/4)
	Grid search [28] (5)	1.314 (57)	[8.98, 12.00, 9.94]	2.4% (17–51/5)
	PPA [28] (6)	1.276 (54)	[8.799, 13.005, 9.339]	0.5% (17–61/6)
Analytical method	MEM (this paper) (7)	1.283 (∞)	[8.980, 13.075, 9.003]	–

Note: The coordinate system in the empirical method and the MEM is different from that in the reference [28]. By changing the center coordinates (a, b) calculated based on the empirical method and the MEM to $(a + 5, b + 5)$, the coordinates in the reference [28] are obtained.

strictly smaller than results calculated by the GeoStudio software. The average values of the absolute and relative errors between the two are 0.01 and 0.8%, respectively.

Table 5. Calculation results using the GeoStudio software and the MEM.

(Results from using the GeoStudio Software, Results from using the MEM)										
H (m)	α (°)	c (kPa)								
		10			12.38			15		
		φ (°)			φ (°)			φ (°)		
		15	20	25	15	20	25	15	20	25
6	25	(1.39, 1.37)	(1.64, 1.63)	(1.91, 1.90)	(1.53, 1.52)	(1.81, 1.79)	(2.07, 2.06)	(1.69, 1.68)	(1.97, 1.95)	(2.26, 2.23)
	35	(1.14, 1.13)	(1.32, 1.31)	(1.52, 1.50)	(1.29, 1.27)	(1.47, 1.46)	(1.67, 1.65)	(1.45, 1.42)	(1.63, 1.61)	(1.83, 1.81)
	45	(0.98, 0.97)	(1.12, 1.11)	(1.27, 1.26)	(1.11, 1.10)	(1.26, 1.25)	(1.41, 1.40)	(1.25, 1.24)	(1.41, 1.39)	(1.56, 1.55)
8	25	(1.22, 1.21)	(1.47, 1.46)	(1.72, 1.72)	(1.34, 1.33)	(1.59, 1.58)	(1.86, 1.85)	(1.46, 1.45)	(1.73, 1.71)	(2.00, 1.99)
	35	(0.98, 0.97)	(1.16, 1.15)	(1.35, 1.34)	(1.09, 1.08)	(1.27, 1.27)	(1.46, 1.45)	(1.21, 1.20)	(1.40, 1.39)	(1.60, 1.58)
	45	(0.83, 0.83)	(0.97, 0.96)	(1.11, 1.10)	(0.93, 0.93)	(1.07, 1.07)	(1.22, 1.22)	(1.04, 1.04)	(1.19, 1.18)	(1.34, 1.33)
10	25	(1.11, 1.11)	(1.36, 1.35)	(1.61, 1.61)	(1.22, 1.20)	(1.46, 1.45)	(1.72, 1.71)	(1.32, 1.31)	(1.57, 1.56)	(1.84, 1.83)
	35	(0.88, 0.88)	(1.06, 1.05)	(1.24, 1.23)	(0.97, 0.97)	(1.16, 1.15)	(1.34, 1.33)	(1.07, 1.06)	(1.25, 1.25)	(1.45, 1.44)
	45	(0.74, 0.74)	(0.88, 0.87)	(1.01, 1.01)	(0.83, 0.82)	(0.97, 0.96)	(1.11, 1.10)	(0.92, 0.91)	(1.06, 1.05)	(1.21, 1.20)

4.2.2. Rocscience Software

Rotaru and Bejan [44] provided the results of several slopes calculated using the Fellenius method and Slide2 in the Rocscience software. Based on these slopes, the MEM is applied. Two kinds of results are summarized in Table 6. As shown in Table 6, the average values of the absolute and relative errors between the two are 0.00 and 0.1%, respectively.

Table 6. Calculation results using the Rocscience software and the MEM.

Parameters of the Slope					Results of the Rocscience Software [44]	Results of the MEM
Slope Ratio	Slope Height (m)	Bulk Density (kN/m ³)	Effective Cohesion (kPa)	Effective Internal Friction Angle (°)	FOS	FOS
1:1.5	8	18.0	10	18	1.17	1.16
1:1.5	8	19.0	15	20	1.46	1.46
1:1.5	8	19.5	20	24	1.84	1.84
1:1.5	8	20.0	40	20	2.51	2.51
1:1	8	18.0	10	18	0.96	0.96
1:1	8	19.0	15	20	1.22	1.22
1:1	8	19.5	20	24	1.54	1.54
1:1	8	20.0	40	20	2.19	2.19
2:1	8	18.0	10	18	0.74	0.74

2:1	8	19.0	15	20	0.96	0.96
2:1	8	19.5	20	24	1.22	1.22
2:1	8	20.0	40	20	1.80	1.80

A comparison between the MEM and the two commercial software packages reveals differences in deviation patterns. For GeoStudio, FOSs obtained using the MEM are generally lower than those calculated by GeoStudio. A plausible explanation is that the software discretizes the sliding mass into a finite number of soil slices (32 in this paper) for numerical solution, and its accuracy is limited by the slice discretization scheme. In contrast, the MEM corresponds to an analytical formulation with the number of soil slices approaching infinity and thus possesses superior theoretical accuracy. For Rocscience, results from the MEM agree well with the software outputs without systematic bias. Two factors may account for this observation. First, the Rocscience data adopted herein are extracted from published papers rather than generated by direct numerical modeling in this study. The authors of the published paper did not state the number of soil slices used in the solution; the number may have been larger (it may be far greater than 32); therefore, the calculated results are closer to those of the MEM. Second, Rocscience and GeoStudio adopt different underlying solving frameworks, and inherent discrepancies may exist between their outputs.

4.3. Stability Charts

4.3.1. Taylor's Charts

Taylor [43] proposed the friction circle method and provided stability charts based on this method. Due to their accuracy, Taylor's charts have been widely used by researchers and engineers [33,45–47]. According to Taylor's charts, 21 examples with different slope angles, critical cohesion ratios ($c_d/\gamma H$), and internal friction angles (φ_d) are selected. FOSs of these examples are all equal to unity. Assuming that $\gamma = 20 \text{ kN/m}^3$ and $H = 10 \text{ m}$, the results of the proposed MEM are obtained. The relative results are summarized in Table 7. According to Table 7, results calculated using the MEM are close to results offered by Taylor's stability charts. The average values of the relative errors between the two are 1.4%, 4.1%, and 1.7% for angle-1, angle-2, and the FOS, respectively.

Table 7. Calculation results using Taylor's charts and the MEM.

Slope Angle (°)	Internal Friction Angle φ_d	Results using Taylor's Stability Charts [43]			Results using the MEM		
		$c_d/(\gamma H)$	Angle-1 (°)	Angle-2 (°)	Angle-1 (°)	Angle-2 (°)	FOS
15	10	0.023	14	34	14.21	36.86	0.96
	5	0.110	20	53	22.92	49.57	0.97
	10	0.075	25	44	25.11	45.42	0.96
	20	0.025	28	31	28.16	34.03	0.96
	25	0.009	29	25	29.16	25.02	0.98
30	5	0.136	31.2	42.1	31.33	42.95	0.99
	10	0.108	34	39.7	33.99	40.86	0.98
	15	0.083	36.1	37.2	36.37	38.67	0.97
	20	0.062	38	34.5	38.46	36.28	0.96
	25	0.044	40	31	40.27	33.47	0.96
45	0	0.191	35.3	35.4	35.34	35.9	1.00
	5	0.162	38.5	34.5	38.54	34.18	1.00
	10	0.138	41	33	41.24	32.7	1.00
	15	0.116	44	31.5	43.85	31.33	0.99
	20	0.097	46.5	30.2	46.28	30.06	0.99

	25	0.079	50	30	48.64	28.76	0.99
	0	0.219	41.8	25.9	41.72	25.81	1.00
	5	0.195	45	25	44.56	24.11	1.01
75	10	0.173	47.5	23.5	47.21	22.63	1.01
	15	0.152	50	23	49.78	21.32	1.01
	20	0.134	53	22	52.23	20.18	1.01
	25	0.117	56	22	54.65	19.17	1.01

4.3.2. Janbu's Charts

Taylor [43] cannot provide a method for determining the position of the center of the CSS of a slope with general cohesion and an internal friction angle ($FOS \neq 1$). To solve this problem, Janbu [48] proposed a chart method. In this method, the Swedish method is used under the assumption that the slip surface is a toe circle [49,50]. The method is suggested in some codes or manuals [51–53]. Meanwhile, this method is also used to study new scientific problems [54]. In the code [51], there are two detailed examples. Based on these examples, comparisons are made between the results of Janbu's charts and the results of the MEM. As shown in Table 8, there is good consistency between the two.

Table 8. Results using Janbu's charts and the MEM.

Condition	Parameters	Results using Janbu's Charts [51]	Results using the MEM
	(γ/pcf , H/ft , $\alpha/^\circ$, c/psf , $\varphi/^\circ$)	(X_0/ft , Y_0/ft , FOS_{jc})	(a_u/ft , b_u/ft , FOS_{mem})
$\varphi = 0$	(100, 20, 70, 260, 0)	(−6, 32, 0.62)	(−5.79, 31.52, 0.62)
$\varphi > 0$	(120, 23, 30, 280, 17)	(12, 38, 1.42)	(12.13, 36.18, 1.45)

5. Assessing Some of the Existing Research

Additionally, based on the proposed MEM, some of the existing research on the determination of the CSS using the SSM can be assessed [26,27,29,31]. The same examples as above are selected. As shown in Table 9, the recommended iterative initial values (a_0 , b_0) or stated ultimate values (a_u , b_u) in each of the items of literature are expressed by unified parameters.

Table 9. Recommended iterative initial values or stated ultimate values in each of the items of literature.

Resources	Recommended Iterative Initial Values (a_0, b_0) or Stated Ultimate Values for the Center of the CSS (a_u, b_u)
Leclerc and Hansen [29]	$a_0 = \frac{H}{\tan \alpha} - H$ $b_0 = H + \frac{H \left(\frac{1}{\tan \alpha} \right)^2 + 2 \frac{H}{\tan \alpha} - 3H}{8}$
Cao and Zaman [31]	$-\frac{H}{\tan \alpha} \leq a_u \leq \frac{H}{\tan \alpha}$ $b_u \geq H$
Duncan et al. [26]	$a_u < \frac{H}{2 \tan \alpha}$ $b_u = \cot \varphi \left(\frac{H}{2 \tan \alpha} - a_u \right) + \frac{H}{2}$
Kostic et al. [27]	$\{a_u\}_{\max, \min} = 23.43 + 2.61 \times H - 1.43 \times \alpha + 0.257 \times c - 0.349 \times \varphi - 4.074 \times (d/H) - 0.076 \times H \times \alpha + 0.142 \times \alpha \times (d/H)$ $+ 0.023 \times \alpha^2 - 0.0036 \times c^2 + 0.0054 \times \varphi^2 \pm 3.16$ $\{b_u\}_{\max, \min} = 18.86 + 4.843 \times H - 1.834 \times \alpha - 0.258 \times c + 0.673 \times \varphi - 1.251 \times (d/H) - 0.097 \times H \times \alpha + 0.0066 \times \alpha \times c$ $- 0.0193 \times \alpha \times \varphi + 0.0395 \times \alpha^2 \pm 5.3$

5.1. Research by Leclerc and Hansen [29]

It is found that the iterative initial values recommended by Leclerc and Hansen [29] may not be applicable for certain situations. As shown in Table 10, when the slope angle is small ($\alpha \leq 30^\circ$), the results under the iterative initial value suggested by Leclerc and Hansen [29] are equal to the results under the iterative initial value suggested in this paper. The relative error between them and the reference answer is relatively small, with a maximum value of -4.0% . When the slope angle is large ($\alpha \geq 60^\circ$), the results under the iterative initial value suggested by Leclerc and Hansen [29] are much smaller than the results under the iterative initial value suggested in this paper. The relative error between the former and the reference answer is relatively large, with a maximum value of -15.3% . The relative error between the latter and the reference answer is relatively small, with a maximum value of 1.7% . It can be seen that the iterative initial value recommended by Leclerc and Hansen [29] is not applicable to situations with large slope angles. The iterative initial values suggested in this paper (Equation (39)) have good applicability.

Table 10. Examination of the formulas recommended by Leclerc and Hansen [29].

References	Parameters	Iterative Initial Values by Leclerc and Hansen [29]	Ultimate Values Using the MEM Under (a_{lh0}, b_{lh0})	Iterative Initial Values Using Equation (39)	Ultimate Values Using the MEM Under (a_{eq0}, b_{eq0})	Reference Answer (FOS)
	$(H/m, \alpha/^\circ, c/kPa, \varphi/^\circ)$	(a_{lh0}, b_{lh0})	$(a_{lu}, b_{lu}, FOS_{lu})$	(a_{eq0}, b_{eq0})	$(a_{equ}, b_{equ}, FOS_{equ})$	FOS_{ra}
Leonetti et al. [28]	(5, 26.6, 9.84, 10)	(2.49, 8.11)	(3.98, 8.08, 1.283)	(4.04, 11.10)	(3.98, 8.08, 1.283)	1.276
	(8.5, 26.6, 14.71, 20)	(4.24, 13.79)	(5.26, 14.66, 1.640)	(6.07, 15.76)	(5.26, 14.66, 1.640)	1.642
Taylor [43]	(10, 30, 9.6, 15)	(3.66, 14.33)	(4.06, 16.55, 0.960)	(5.98, 14.77)	(4.06, 16.55, 0.960)	1.000
	(10, 60, 33, 5)	(-2.11, 8.11)	(1.09, 7.38, 0.887)	(-2.55, 14.55)	(-1.05, 14.28, 1.011)	1.000
	(10, 75, 30.8, 15)	(-3.66, 7.01)	(-2.69, 7.12, 0.847)	(-8.66, 16.94)	(-8.55, 15.84, 1.017)	1.000

5.2. Research by Cao and Zaman [31]

The location of the center of the CSS is not always accurate for the region $(-\frac{H}{\tan \alpha} \leq a_u \leq \frac{H}{\tan \alpha}, b_u \geq H)$ stated by Cao and Zaman [31]. As shown in Table 11, when the slope angle is small ($\alpha \leq 60^\circ$), the center of the CSS determined using the MEM is located in the area stated by Cao and Zaman [31]. However, when the slope angle is large ($\alpha = 75^\circ$), the center of the CSS determined using the MEM is not within the area stated by Cao and Zaman [31]. It can be seen that the formula stated by Cao and Zaman [31] is not applicable to situations with large slope angles.

Table 11. Examination of the formulas stated by Cao and Zaman [31].

Reference	Parameters	Ultimate Values Stated by Cao and Zaman [31]	Ultimate Values Calculated Using the MEM	Whether the Two Are Consistent
	$(H/m, \alpha/^\circ, c/kPa, \varphi/^\circ)$	a_{gzu} and b_{gzu}	(a_{memu}, b_{memu})	
Leonetti et al. [28]	(5, 26.6, 9.84, 10)	$-10.0 \leq a_{gzu} \leq 10.0$ $b_{gzu} \geq 5.0$	(3.98, 8.08)	Yes
	(8.5, 26.6, 14.71, 20)	$-17.0 \leq a_{gzu} \leq 17.0$ $b_{gzu} \geq 8.5$	(5.26, 14.66)	Yes
Taylor [43]	(10, 30, 9.6, 15)	$-17.3 \leq a_{gzu} \leq 17.3$ $b_{gzu} \geq 10.0$	(4.06, 16.55)	Yes
	(10, 60, 33, 5)	$-5.8 \leq a_{gzu} \leq 5.8$ $b_{gzu} \geq 10.0$	(−1.05, 14.28)	Yes
	(10, 75, 30.8, 15)	$-2.7 \leq a_{gzu} \leq 2.7$ $b_{gzu} \geq 10.0$	(−8.55, 15.84)	No

5.3. Research by Duncan et al. [26]

The possible position of the center of the CSS pointed out by Duncan et al. [26] is not always accurate. As shown in Table 12, when the slope height and slope angle are both small ($H = 5$ m, $\alpha = 26.6^\circ$), the center of the CSS calculated using the MEM is almost located on the straight line pointed out by Duncan et al. [26]. Specifically, let $a_{du} = a_{memu} = 3.98$, substitute this formula into $b_{du} = 5.7 \times (5.0 - a_{du}) + 2.5$, and obtain $b_{du} = 8.314$. Then, calculate the relative error between b_{du} and b_{memu} as $(b_{du} - b_{memu})/b_{memu} = (8.314 - 8.08)/8.08 = 2.9\%$. This error is relatively small, within an acceptable range of $-5\% \sim +5\%$. When the slope angle and slope angle are large ($H = 10$ m, $\alpha = 75^\circ$), according to the above calculation process, the relative error between b_{du} and b_{memu} can be confirmed as 162%. This error is significant and far beyond the acceptable range. It can be seen that the possible location of the center of the CSS pointed out by Duncan et al. [26] is only applicable to certain special situations. In these situations, the slope height and slope angle are small.

Table 12. Examination of the formulas stated by Duncan et al. [26].

References	Parameters	Ultimate Values Stated by Duncan et al. [26]	Ultimate Values Calculated Using the MEM	Whether the Two Are Consistent
	$(H/m, \alpha/^\circ, c/kPa, \varphi/^\circ)$	a_{du} and b_{du}	(a_{memu}, b_{memu})	
Leonetti et al. [28]	(5, 26.6, 9.84, 10)	$a_{du} \leq 5.0$ $b_{du} = 5.7(5.0 - a_{du}) + 2.5$	(3.98, 8.08)	Nearly yes
	(5, 26.6, 9.84, 10)	$a_{du} \leq 8.5$ $b_{du} = 2.7(8.5 - a_{du}) + 4.25$	(5.26, 14.66)	No
Taylor [43]	(10, 30, 9.6, 15)	$a_{du} \leq 8.7$ $b_{du} = 3.7(8.7 - a_{du}) + 5.0$	(4.06, 16.55)	No
	(10, 60, 33, 5)	$a_{du} \leq 2.9$ $b_{du} = 11.4(2.9 - a_{du}) + 5.0$	(−1.05, 14.28)	No
	(10, 75, 30.8, 15)	$a_{du} \leq 1.3$ $b_{du} = 3.7(1.3 - a_{du}) + 5.0$	(−8.55, 15.84)	No

5.4. Research by Kostic et al. [27]

The formulas stated by Kostic et al. [27] for determining the area where the center of the CSS is located lack clear physical significance and broad applicability. The shortcomings of these formulas are as follows. (i) It does not take into account the factor of bulk density γ . (ii) There is the problem of inconsistent dimensions between the left and right sides of the formulas. Taking the formula for calculating $\{a_u\}_{max,min}$ as an example, the dimension of $\{a_u\}_{max,min}$ is m , while the dimensions of β , c , φ , d/H , $\beta \times (d/H)$, β^2 , c^2 , and φ^2 are not m . (iii) When the slope angle is small, the formulas are applicable. When the slope angle is large, the formulas are not applicable. As shown in Table 13, when $\alpha \leq 30^\circ$, $\{a_u\}_{min} < a_{memu} < \{a_u\}_{max}$ and $\{b_u\}_{min} < b_{memu} < \{b_u\}_{max}$ exist. When $\alpha \geq 60^\circ$, a_{memu} and b_{memu} do not fall within the range of $[\{a_u\}_{min}, \{a_u\}_{max}]$ and $[\{b_u\}_{min}, \{b_u\}_{max}]$. It can be found that the above shortcomings can be overcome in the suggested iterative initial values (Equation (39)). (a) The γ is included in the suggested formulas for determining the center of the CSS. (b) The dimensions are consistent between the left and right sides of the formulas. (c) No matter whether the slope angle is large or small, the initial values (a_{eq0} or b_{eq0}) calculated by Equation (39) are always close to the ultimate values (a_{memu} or b_{memu}) calculated using the MEM. Thus, the suggested iterative initial values exhibit clear physical significance and broad applicability. It effectively addresses the limitations of the formulas stated by Kostic et al. [27].

Table 13. Examination of the formulas stated by Kostic et al. [27].

References	Parameters	Range of a_u [27]	Range of b_u [27]	Iterative Initial	Ultimate Values
				Values Calculated Using Equation (39)	Calculated Using the MEM
	$(H/m, \alpha/^\circ, c/kPa, \varphi/^\circ, d/H)$	$(\{a_u\}_{min}, \{a_u\}_{max})$	$(\{b_u\}_{min}, \{b_u\}_{max})$	(a_{eq0}, b_{eq0})	(a_{memu}, b_{memu})
Leonetti et al. [28]	(5, 26.6, 9.84, 10, 2)	(0.08, 6.40)	(2.32, 12.92)	(4.04, 11.10)	(3.98, 8.08)
	(5, 26.6, 9.84, 10, 2)	(1.10, 7.42)	(11.44, 22.04)	(6.07, 15.76)	(5.26, 14.66)
Taylor [43]	(10, 30, 9.6, 15, 2)	(−0.22, 6.10)	(11.78, 22.38)	(5.98, 14.77)	(4.06, 16.55)
	(10, 60, 33, 5, 2)	(8.68, 15.00)	(33.70, 44.30)	(−2.55, 14.55)	(−1.05, 14.28)
	(10, 75, 30.8, 15, 2)	(25.21, 31.53)	(67.96, 78.56)	(−8.66, 16.94)	(−8.55, 15.84)
	(10, 75, 30.8, 15, 2)	(25.21, 31.53)	(67.96, 78.56)	(−8.66, 16.94)	(−8.55, 15.84)

6. Discussion

According to the above research, it can be seen that the proposed MEM has good accuracy, efficiency, and generality. Additionally, the MEM can offer a unified criterion for assessing some of the existing research.

6.1. The Higher Computational Accuracy of the Proposed Mathematical Extremum Method

For one type of empirical method, the 4.5H method, the size of the divided slice determines calculation precision. In the MEM, the slice is converted into integral form. Compared with the conventional method of slicing, computational accuracy is improved.

Specifically, the defects of the 4.5 H method are as follows. (i) The determination of angle β_1 , angle β_2 , and point E is not accurate. According to Table 1, the values of the two angles β_1 and β_2 are only related to the slope angle. The position of point E is only

related to the slope height. The influence of other parameters is neglected. (ii) The accuracy of the calculated FOS depends on the distance between several points from O_1 to O_4 and from $O_{1'}$ to $O_{4'}$. In other words, the smaller the distance, the more accurate the result. (iii) Under each chosen center of the slip surface, the sliding mass is divided into several slices. In theory, the more slices there are, the more accurate the result is. However, the number of slices is finite. (iv) The gravity of each slice is calculated by multiplying the length of the vertical centerline of the slice, the width of the slice, and the unit weight of the slice. It ignores the geometrical irregularity of the slice. In the proposed MEM, the above defects are improved. (a) The mathematical equations include the slope angle, the slope height, the unit weight, the effective cohesion, and the effective friction angle. All parameters are considered. (b) The coordinates of the center of the CSS can be given directly. This avoids selecting the trial points. (c) The slice is considered in integral form. This means that the number of divided slices is infinite. (d) The gravity of each slice is calculated using the integral method. This considers the geometrical irregularity of the slice.

6.2. The Higher Computational Efficiency of the Proposed Mathematical Extremum Method

For computer search methods, obtaining the CSS and FOS requires complex programming and trials. In the MEM, this process is simplified. Compared with the conventional method of slicing, computational efficiency is improved.

Specifically, the drawbacks of the computer search methods are as follows. (i) Complex computer language programming is required. This poses significant challenges for researchers and engineers. (ii) It requires setting reasonable iterative initial values for searching. However, there is a lack of formulas to define these values well. (iii) It often requires multiple iterations and takes a long time. In the proposed MEM, the above shortcomings are improved. (a) Mathematical equations are written into an EXCEL 2016 spreadsheet. With the solver tool in EXCEL, solutions can be directly obtained. There is no need for complex computer language programming. It is convenient for engineers and researchers to master. (b) Reasonable iterative initial values are discovered and suggested. In the expressions, all slope parameters, including the slope angle, the slope height, the unit weight, the effective cohesion, and the effective internal friction angle, are considered. (c) Only one iteration is required, which saves computation time.

6.3. The Good Generality of the Proposed Mathematical Extremum Method

Compared with several other tools, the MEM maintains good accuracy in the calculation results. Relative errors between the two range from -5% to 5% . This indicates the good generality of the MEM.

In contrast to each of these tools, the MEM exhibits unique merits. For the graphical trial procedure, it is necessary to use soil strip partitioning and multiple trials. Based on the proposed MEM, accurate and reliable calculation results can be quickly obtained. For commercial software, researchers or engineers need to obtain a license. Based on the proposed MEM, an EXCEL spreadsheet can provide the calculation results directly. For the stability charts, the correctness lacks theoretical verification, and the application scope needs to be expanded for Janbu's charts. Based on the proposed MEM, these problems can be solved. In Janbu's charts, there are the FOS of a slope (denoted as F_s , $F_s = \frac{N_{cf} \times c}{\gamma H}$), the horizontal coordinate (denoted as X_0 , $X_0 = x_0 H$), and the vertical coordinate (denoted as Y_0 , $Y_0 = y_0 H$) of the center of the CSS of a slope. N_{cf} , x_0 , and y_0 are all determined by two independent variables of the slope angle (denoted as α) and the dimensionless parameter (denoted as $\lambda_{c\varphi}$, $\lambda_{c\varphi} = \frac{\gamma H \tan \varphi}{c}$). Assuming that $a = mH$ and $b = nH$, and

substituting the above formulas into Equation (15), $F_s = [K(m, n, \alpha) + L(m, n, \alpha)\lambda_{c\varphi}] \frac{c}{\gamma H}$ can be obtained, where K and L represent some functions. For certain values of m and n , $K(m, n, \alpha) + L(m, n, \alpha)\lambda_{c\varphi}$ is only controlled by two independent variables, α and $\lambda_{c\varphi}$. By substituting $a = mH$ and $b = nH$ into Equation (23), $\left\{ \begin{array}{l} \frac{\partial K(m, n, \alpha)}{\partial m} + \frac{\partial L(m, n, \alpha)}{\partial m} \lambda_{c\varphi} = 0 \\ \frac{\partial K(m, n, \alpha)}{\partial n} + \frac{\partial L(m, n, \alpha)}{\partial n} \lambda_{c\varphi} = 0 \end{array} \right\}$ can be obtained. For the given α and $\lambda_{c\varphi}$, these equations can be solved. When α and $\lambda_{c\varphi}$ are given, it is assumed that the roots of the equations are m_0 and n_0 , respectively. $\left\{ \begin{array}{l} x_0 = m_0 \\ y_0 = n_0 \\ N_{cf} = K(m_0, n_0, \alpha) + L(m_0, n_0, \alpha)\lambda_{c\varphi} \end{array} \right\}$ can be obtained. Using this method, the mechanical mechanism of Janbu's charts is verified. Additionally, the dimensionless parameter $\lambda_{c\varphi}$ ($\lambda_{c\varphi} = 0, 2, 5, 10, 20, 100$) is discontinuous in Janbu's charts. It limits the application scope. However, in the MEM, α and $\lambda_{c\varphi}$ are free. This indicates that the MEM can be adopted under any soil parameters. Therefore, the MEM can appropriately complement Janbu's charts.

6.4. The Unified Criterion for Assessing Existing Research Offered by the Proposed Mathematical Extremum Method

In the MEM, the complete mathematical expressions of essential equations determining the CSS and FOS are derived. A set of approximate explicit solutions of the equations has been discovered. Using the EXCEL spreadsheet, the exact solutions of the equations have been obtained. This offers a unified and precise criterion for some other studies [26,27,29,31]. As shown above, assessments can be made, and limitations can be identified for some of the existing research based on the criterion. This provides valuable references for engineers and researchers.

Currently, the results of this paper are only applicable to the estimation of the FOS and the rapid approximation of the CSS for simple homogeneous engineering slopes. Moreover, the current MEM program can only calculate the FOS and determine the center of the CSS; it cannot yet provide a graphical display of the results. Such functions can be expanded in future development. Of course, the current study still has some limitations. For example, this paper only considers homogeneous single-layer soil, circular slip surfaces, and toe circles, without considering layered soil, surcharge on the slope top, or three-dimensional effects, and assumes zero pore water pressure. When these complex conditions are considered, the equations governing the minimum FOS of the slope may become more complex and thus unsolvable, and the approximate explicit solution obtained may lose its validity. However, for homogeneous slopes, whether one can add a geometric parameter to consider the CSS as a base circle, or add a pore water pressure coefficient to characterize the effect of pore water pressure, can be studied. This does not appear to excessively increase the difficulty of mathematical derivation, and reliable results may still be derived. For complex slopes, the feasibility of the following research process can be explored: (a) a circular slip surface is assumed and the slope is divided into a certain number of slices; (b) based on the equivalence principles of the total resisting cohesion moment, the total resisting friction moment, and the total driving force, a set of equivalent cohesion, internal friction angles, and unit weights is obtained; (c) adopting the above equivalent parameters, along with the slope angle and height of the complex slope, a new homogeneous slope is virtually designed; (d) according to the research of this paper, the CSS is obtained; (e) the obtained CSS is treated as a new assumed slip surface and the above work is repeated; and (f) when the errors in the CSS and FOS

calculated in two consecutive iterations are sufficiently small, the CSS and FOS of the complex slope are considered to have been obtained. Based on the relevant ACADS examples, the authors preliminarily verified the feasibility of the above scheme; owing to space limitations, the details are not presented here.

7. Conclusions

To determine the factor of safety (FOS) and the critical slip surface (CSS) of a slope based on the Swedish slice method (SSM), a mathematical extremum method (MEM) is proposed in this paper. The applicability of this MEM is verified through comparisons with the results of various other methods. The generality of the MEM is further tested through comparisons with the results of different tools. Additionally, assessments are made on some of the existing research. Conclusions are drawn as follows.

(i) The complete mathematical expressions of essential equations determining the CSS and the FOS are derived. The integral form is adopted. The computational accuracy is improved.

(ii) A set of approximate explicit solutions of the equations is discovered. The solutions are suggested as the iterative initial values. The equations are written into an EXCEL spreadsheet and solved by the solver tool. Results are obtained after a single iteration. This avoids complex programming and reduces iteration counts. Computational efficiency is improved.

(iii) Compared with other methods (the empirical method and computer search method), it is verified that the MEM has good applicability. Compared with other tools (the graphical trial procedure, commercial software, and the stability charts), it is further demonstrated that the MEM has acceptable generality.

(iv) Based on the MEM, assessments can be made on some of the existing research. It is noted that some existing findings are either inaccurate or only applicable under specific conditions.

(v) In the future, further research can be conducted. For homogeneous slopes, an additional geometric parameter or pore water pressure coefficient may be introduced to consider base circles or pore water pressure. For complex slopes, using the equivalence principle of the total resisting cohesion moment, the total resisting friction moment, and the total driving moment, the complex slope may be equivalently converted into a homogeneous slope with the same slope angle and height. Thus, based on the research in this paper, the CSS and FOS of the complex slope may be determined through several iterations.

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