mathematics

MDPI

Article

Characterization of Compact Riemannian and Spacelike Hypersurfaces in Einstein Manifolds

Mohammed Guediri

Department of Mathematics, College of Sciences, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; mguediri@ksu.edu.sa

Abstract

We explore rigidity results for compact Riemannian and spacelike hypersurfaces in Einstein manifolds. We show that for a compact Riemannian (or spacelike) hypersurface (M, g) in an Einstein Riemannian (or Lorentzian) manifold $(\overline{M}, \overline{g})$ which admits a closed conformal vector field $\overline{\xi}$ (assumed to be timelike in the Lorentzian case), the hypersurface (M, g) is an extrinsic sphere in $(\overline{M}, \overline{g})$ if and only if it is not totally geodesic and the integral of the Ricci curvature of the hypersurface, taken in the direction of the tangential component $\xi^{\top}$ of the restriction ξ of $\overline{\xi}$ to M, satisfies a certain lower bound. As an application of this result, we investigate compact spacelike hypersurfaces in generalized Robertson–Walker (GRW) spacetimes. Additionally, other related results are also provided.

Keywords: Einstein manifolds; spacelike hypersurfaces; conformal vector fields; Ricci curvature; extrinsic spheres; generalized Robertson–Walker (GRW) spacetimes

MSC: 53A10; 53C25; 53C40; 53C42; 53C50; 53C65

1. Introduction

Round spheres are a central focus in the study of manifolds with positive curvature, as they provide canonical examples of manifolds with constant positive curvature. A well-known conjecture asserts that spheres are the unique compact orientable Riemannian manifolds that can support a Riemannian metric with constant scalar curvature and a nontrivial conformal vector field. This conjecture has been pursued by many mathematicians in the last century, including [1–3]. These attempts have led to a rich body of literature that continues to generate significant interest in the mathematical community [4,5]. For an overview of these results, we refer the reader to [6], and for more results in the setting of the above conjecture, we refer to [7–9]. A counterexample to this conjecture was provided in [10], where metrics of constant scalar curvature are constructed on a warped product of the Euclidean circle and a compact Riemannian manifold of constant scalar curvature. In this counterexample, the conformal vector field is created by multiplying the local coordinate vector field on the circle by the warping function.

In a different context, another problem seeks to use specific conditions on the mean curvature to characterize spheres among compact Riemannian or spacelike hypersurfaces in Riemannian or Lorentzian manifolds, respectively. It is well known that a compact Riemannian hypersurface with constant mean curvature (CMC) in Euclidean space is a sphere (see [11]). Additionally, it is known that a CMC spacelike hypersurface (M, g) in a Lorentzian space form of positive constant curvature c (such as de Sitter space) is totally umbilical if the absolute value of its mean curvature is bounded above by a constant

Check for updates

Academic Editor: Stéphane Puechmorel

Received: 18 September 2026
Revised: 21 October 2026
Accepted: 22 October 2026
Published: 24 October 2026

Copyright: © 2026 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

depending on c (see [12]). More precisely, according to Lemma 4.35 in [13], the mean curvature H is constant, and the hypersurface (M, g) has constant sectional curvature equal to $c - H^2$, implying that (M, g) is a sphere. For further results on CMC spacelike hypersurfaces in Lorentzian manifolds (specifically in generalized Robertson–Walker GRW spacetimes), we refer to [14–16].

In this paper, we concentrate on the characterization of extrinsic spheres (i.e., totally umbilical Riemannian hypersurfaces with non-zero constant mean curvature) in Einstein Riemannian and Lorentzian manifolds that admit a closed conformal vector field, which we assume to be timelike in the Lorentzian case. This generalizes recent results on the characterization of round spheres in Riemannian and Lorentzian space forms (see [17,18]). See also [19] for the characterization of spacelike hypersurfaces in some special Lorentzian space forms. The main result of this paper asserts that, given a compact Riemannian hypersurface (M, g) in an $(n+1)$-dimensional Riemannian or Lorentzian manifold $(\overline{M}, \overline{g})$, where $n \geq 2$, if $\overline{\xi}$ is a closed conformal vector field on $\overline{M}$, and if $\xi^\top$ represents the tangential component of the restriction ξ of $\overline{\xi}$ to M, with N being a unit normal to M, so that $\vartheta = \overline{g}(\xi, N)$ is non-vanishing on a set of positive measure when $(\overline{M}, \overline{g})$ is Riemannian, then the Ricci curvature of (M, g) in the direction of $\xi^\top$ satisfies the inequality

$$\int_M Ric(\xi^\top, \xi^\top) dv \geq \left(\frac{n-1}{n}\right) \int_M \left(\operatorname{div}(\xi^\top)\right)^2 dv \tag{1}$$

if and only if (M, g) is totally umbilical. In particular, if $(\overline{M}, \overline{g})$ is Einstein, then the mean curvature H of (M, g) is constant, and (M, g) is an extrinsic sphere in $(\overline{M}, \overline{g})$.

The present work is closely related to [20] concerning Goddard's conjecture, which states that a complete spacelike hypersurface with constant mean curvature in de Sitter space should necessarily be totally umbilical. In [20], we extended Goddard's conjecture to compact spacelike hypersurfaces immersed in Lorentzian manifolds admitting a timelike conformal vector field. Assuming a natural integral condition involving the ambient Ricci curvature, together with a hypothesis on the variation in the mean curvature along the induced tangential flow, we established that such hypersurfaces must be totally umbilical. This result was then applied to derive rigidity results for Einstein Lorentzian manifolds and, as a particular case, to recover the classical characterization of compact spacelike constant mean curvature hypersurfaces in de Sitter space as spheres.

The principal distinction of the present approach is that the curvature condition (1) is formulated in terms of the intrinsic Ricci curvature of the hypersurface rather than the ambient Ricci curvature; however, rigidity results associated with Goddard's conjecture in [20] were established under an integral condition involving the ambient Ricci curvature. Furthermore, in the present paper, the Riemannian case of Proposition 1 is already known from [21] and is included for completeness, while its Lorentzian counterpart provides an extension to the Lorentzian setting. Propositions 2 and 3 provide extensions to the Lorentzian setting of Propositions 1 and 2 in [22]. These comparisons identify the precise scope and novelty of the present contributions.

The paper is structured as follows. Section 2 provides the necessary preliminaries, including relevant concepts and formulas, as well as some useful tensorial lemmas. Section 3 presents key curvature formulas for Riemannian hypersurfaces in Riemannian and Lorentzian manifolds. Section 4 is the core of the paper, containing the main results and their supporting lemmas. Finally, Section 5 applies the main results to spacelike hypersurfaces in generalized Robertson–Walker (GRW) spacetimes.

2. Preliminaries

Let (M, g) be a semi-Riemannian manifold of dimension $n \geq 2$, and let ∇ denote the Levi-Civita connection associated with the metric g. We denote by $C^\infty(M)$ the set of all smooth real-valued functions, and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M.

The curvature tensor of (M, g) is defined as the $(1,3)$-tensor field given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

For any $\xi \in \mathfrak{X}(M)$, define A_ξ to be the derivation $\nabla_\xi - L_\xi$, where L_ξ is the Lie derivative in the direction of ξ. In other words, A_ξ is given by

$$A_\xi(X) = \nabla_X \xi,$$

for all $X \in \mathfrak{X}(M)$. It is clear that A_ξ is a (1,1)-tensor, and that $\operatorname{div}(\xi) = \operatorname{tr}(A_\xi)$, where tr denotes the trace.

By setting

$$\nabla^2_{X,Y}\xi = \nabla_X(\nabla_Y \xi) - \nabla_{\nabla_X Y}\xi,$$

for all $X, Y \in \mathfrak{X}(M)$, the Ricci identity is given by

$$R(X,Y)\xi = \nabla^2_{X,Y}\xi - \nabla^2_{Y,X}\xi, \tag{2}$$

for all $X, Y \in \mathfrak{X}(M)$.

We point out the advantage that, given

$$\nabla^2_{X,Y}\xi = (\nabla_X A_\xi)(Y),$$

the Ricci identity can be rewritten as follows

$$R(X,Y)\xi = (\nabla_X A_\xi)(Y) - (\nabla_Y A_\xi)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

The Ricci tensor is obtained by contracting the curvature tensor. More precisely, for a local orthonormal frame $\{e_1, \ldots, e_n\}$, with $\epsilon_i = g(e_i, e_i)$, it is given by

$$\operatorname{Ric}(X,Y) = \sum_{i=1}^{n} \epsilon_i \, g(R(X, e_i)e_i, Y),$$

for arbitrary vector fields $X, Y \in \mathfrak{X}(M)$.

Taking the trace of the Ricci tensor gives the scalar curvature, defined by

$$\operatorname{Sc}(p) = \sum_{i=1}^{n} \epsilon_i \operatorname{Ric}(e_i, e_i), \qquad p \in M.$$

For a vector field $X \in \mathfrak{X}(M)$, its divergence is defined by

$$\operatorname{div} X = \sum_{i=1}^{n} \epsilon_i \, g(\nabla_{e_i} X, e_i).$$

If B is a tensor field of type $(1,1)$, its divergence is the vector field

$$\operatorname{div} B = \sum_{i=1}^{n} (\nabla_{e_i} B)(e_i),$$

where

$$(\nabla_X B)(Y) = \nabla_X(BY) - B(\nabla_X Y), \qquad X, Y \in \mathfrak{X}(M).$$

https://doi.org/10.3390/math14213881

For a smooth function $u \in C^\infty(M)$, the gradient ∇u is characterized by

$$g(\nabla u, X) = X(u)$$

for every $X \in \mathfrak{X}(M)$. The Laplacian of u is defined as

$$\Delta u = \operatorname{div}(\nabla u).$$

The Hessian of u is the symmetric covariant tensor

$$\operatorname{Hess} u(X, Y) = g(\nabla_X \nabla u, Y), \qquad X, Y \in \mathfrak{X}(M).$$

The corresponding Hessian endomorphism is given by

$$(\nabla^2 u)(X) = \nabla_X \nabla u.$$

For a Riemannian manifold, the Bochner formula relates the Hessian, the Laplacian, and the Ricci curvature. It reads

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess} u|^2 + g\big(\nabla u, \nabla(\Delta u)\big) + \operatorname{Ric}(\nabla u, \nabla u). \tag{3}$$

Here, $|\operatorname{Hess} u|$ denotes the norm induced by the metric g on the Hessian tensor.

In the following lemma, we provide a key formula that will be useful later. This result was originally stated without proof in [23] for Riemannian manifolds, and here we present a proof that applies to semi-Riemannian manifolds. In fact, the Bochner formula can be derived from this result (see [24]).

Lemma 1. *Let (M, g) be an n-dimensional semi-Riemannian manifold, $n \geq 2$. Then, for any vector field $\xi \in \mathfrak{X}(M)$, we have*

$$\xi \cdot \operatorname{div}(\xi) = \operatorname{div}(\nabla_\xi \xi) - Ric(\xi, \xi) - \operatorname{tr}(A_\xi^2) \tag{4}$$

Proof. Let $\{e_i\}_{1 \leq i \leq n}$ be a local orthonormal frame chosen to be geodesic, with $\epsilon_i = g(e_i, e_i) = \pm 1$. Thus, we obtain

$$\begin{aligned} \operatorname{div}(\nabla_\xi \xi) &= \sum_{i=1}^n \epsilon_i g(\nabla_{e_i} \nabla_\xi \xi, e_i), \\ \operatorname{tr}(A_\xi^2) &= \sum_{i=1}^n \epsilon_i g(A_\xi^2(e_i,)e_i) \\ &= \sum_{i=1}^n \epsilon_i g(\nabla_{\nabla_{e_i}\xi} \xi, e_i). \end{aligned}$$

By using the Ricci identity (2), we also obtain

$$\begin{aligned} Ric(\xi, \xi) &= \sum_{i=1}^n \epsilon_i g(R(e_i, \xi)\xi, e_i) \\ &= \sum_{i=1}^n \epsilon_i g\big(\nabla^2_{e_i, \xi}\xi - \nabla^2_{\xi, e_i}\xi, e_i\big). \end{aligned}$$

https://doi.org/10.3390/math14213881

Now, by virtue of the above equations and considering that $\{e_i\}_{1\leq i\leq n}$ is geodesic, we conclude that

$$\begin{aligned}
\operatorname{div}(\nabla_{\xi}\xi) - \operatorname{tr}(A_{\xi}^2) - Ric(\xi,\xi) &= \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i}\nabla\xi, e_i) - \sum_{i=1}^{n} \epsilon_i g(\nabla_{\nabla_{e_i}\xi}\xi, e_i) - Ric(\xi,\xi) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{e_i,\xi}\xi, e_i) - Ric(\xi,\xi) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla^2_{\xi,e_i}\xi, e_i) \\
&= \sum_{i=1}^{n} \epsilon_i g(\nabla_{\xi}\nabla_{e_i}\xi, e_i) \\
&= \sum_{i=1}^{n} \epsilon_i \xi \cdot g(\nabla_{e_i}\xi, e_i) \\
&= \xi \cdot \operatorname{div}(\xi).
\end{aligned}$$

□

3. Riemannian Hypersurfaces in Riemannian and Lorentzian Manifolds

From now on, $(\overline{M},\overline{g})$ will be a semi-Riemannian manifold of dimension $(n+1)$, $n \geq 2$, which may be either Riemannian or Lorentzian (that is, $\overline{g}$ has signature $(+,+,\dots,+)$ or $(-,+,\dots,+)$, respectively). Let (M,g) be a Riemannian manifold of dimension n, that is isometrically immersed into $(\overline{M},\overline{g})$. Namely, (M,g) will be treated as a Riemannian hypersurface in $(\overline{M},\overline{g})$, which will be considered spacelike if $(\overline{M},\overline{g})$ is Lorentzian. Let N be a unit normal vector field along M that is timelike in the Lorentzian case.

Let ∇ and $\overline{\nabla}$ represent the Levi-Civita connections on (M,g) and $(\overline{M},\overline{g})$, respectively. Let $\mathfrak{X}(M)$ and $\mathfrak{X}(\overline{M})$ denote the sets of all tangent vector fields on M and $\overline{M}$, respectively.

If A denotes the shape operator of the hypersurface M in $\overline{M}$, with respect to a unit vector field N normal to M, then the formulas of Gauss and Weingarten for M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon g(A(X), Y)N \tag{5}$$

$$A(X) = -\overline{\nabla}_X N, \tag{6}$$

for all $X, Y \in \mathfrak{X}(M)$, with $\epsilon = \overline{g}(N,N) = \pm 1$.

The Gauss equation relates the tangential component of the curvature tensor $\overline{R}$ of $(\overline{M},\overline{g})$ to the curvature tensor R of (M,g) and the shape operator A via the so-called Gauss equation

$$\big(\overline{R}(X,Y)Z\big)^{\top} = R(X,Y)Z + \epsilon\big(g(A(X),Z)A(Y) - g(A(Y),Z)A(X)\big),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The Codazzi equation for (M,g) provides a formula for the normal part of $\overline{R}(X,Y)Z$, which is given by the following expression

$$\big(\overline{R}(X,Y)Z\big)^{\perp} = \epsilon\Big(g\big((\nabla_X A)(Y), Z\big) - g\big((\nabla_Y A)(X), Z\big)\Big)N,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Alternatively, it can be expressed as

$$\overline{R}(X,Y)N = (\nabla_Y A)(X) - (\nabla_X A)(Y), \tag{7}$$

for all $X, Y \in \mathfrak{X}(M)$.

The connection between the Ricci curvature Ric of (M,g) and the Ricci curvature $\overline{Ric}$ of $\overline{M}$. This relationship is given by the following classical identity

$$Ric(X,Y) = \overline{Ric}(X,Y) - \epsilon\overline{g}(\overline{R}(N,X)Y,N) + g(A(X), nHY - \epsilon A(Y)), \tag{8}$$

for all $X, Y \in \mathfrak{X}(M)$.

Taking the trace of Equation (8) yields the following classical relation between the scalar curvatures Sc of M and $\overline{\mathrm{Sc}}$ of $\overline{M}$

$$\mathrm{Sc} = \overline{\mathrm{Sc}} - 2\epsilon\overline{Ric}(N,N) + \epsilon(n^2H^2 - |A|^2), \tag{9}$$

where $|A|^2$ denotes the trace of A^2.

The mean curvature of (M,g) is given by

$$H = \frac{\epsilon}{n}\mathrm{tr}(A). \tag{10}$$

When $H = 0$, the hypersurface (M,g) is known as minimal if $(\overline{M},\overline{g})$ is Riemannian, and maximal if $(\overline{M},\overline{g})$ is Lorentzian.

A hypersurface (M,g) is called totally umbilical when $A = \epsilon HI$, where I is the identity operator, and totally geodesic when $A = 0$. It is referred to as an extrinsic sphere in $(\overline{M},\overline{g})$ if it is totally umbilical with a non-zero constant mean curvature H. For example, if $(\overline{M},\overline{g})$ has constant sectional curvature $\overline{c}$, extrinsic spheres are isometric to round spheres.

4. Main Results

From now on, if $\overline{\xi} \in \mathfrak{X}(\overline{M})$, then in the case where $(\overline{M},\overline{g})$ is Lorentzian, we assume that $\overline{\xi}$ is timelike. In this case, we can choose a globally defined unit timelike vector field N that is normal to M and shares the same time orientation as $\overline{\xi}$. As a result, the condition $\overline{g}(\overline{\xi},N) < 0$ is satisfied everywhere on M. On the other hand, if $(\overline{M},\overline{g})$ is Riemannian, we assume that M is a two-sided hypersurface, which guarantees the existence of a globally defined unit vector field N normal to M.

Let ξ represent the restriction of $\overline{\xi}$ to M. Then, a smooth function ϑ on M, called the support function of ξ, is naturally defined by $\vartheta = \overline{g}(\xi,N)$.

In the case where $(\overline{M},\overline{g})$ is Lorentzian, using the reverse Cauchy-Schwarz inequality, the following inequality holds on M

$$\vartheta \leq -\sqrt{-\overline{g}(\xi,\xi)}. \tag{11}$$

Let $\xi^\top$ denote the tangential component of ξ to M. Then, we have

$$\xi = \xi^\top + \epsilon\vartheta N, \tag{12}$$

where $\epsilon = \overline{g}(N,N) = \pm 1$.

It follows from (12) that we have

$$\overline{\nabla}_X\xi = \overline{\nabla}_X\xi^\top + \epsilon(X\cdot\vartheta)N + \epsilon\vartheta\overline{\nabla}_X N,$$

for every $X \in \mathfrak{X}(M)$.

Applying the Weingarten Formula (6), we can deduce that

$$\nabla_X\xi^\top = \left(\overline{\nabla}_X\xi\right)^\top + \epsilon\vartheta A(X), \tag{13}$$

for every $X \in \mathfrak{X}(M)$.

Assume further that $\overline{\xi}$ is a conformal vector field on $\overline{M}$ with conformal factor φ, meaning that $\overline{\xi}$ satisfies

$$\mathrm{L}_{\overline{\xi}}\overline{g} = 2\varphi\overline{g}, \tag{14}$$

where $\mathrm{L}_{\overline{\xi}}\overline{g}$ denotes the Lie derivative of $\overline{g}$ with respect to $\overline{\xi}$, and φ is a smooth function on $\overline{M}$. In this case, if $\{e_i\}_{1\leq i\leq n}$ is a local orthonormal frame on M, then by applying (13) and (14), we find that

$$\begin{aligned}\mathrm{div}(\xi^\top) &= \sum_{i=1}^{n} g\left(\left(\overline{\nabla}_{e_i}\xi\right)^\top, e_i\right) + \epsilon\vartheta\sum_{i=1}^{n} g\big(A(e_i),e_i\big)\\ &= \sum_{i=1}^{n} g(\varphi e_i, e_i) + \epsilon\vartheta\mathrm{tr(A)}\end{aligned}$$

which, when considering (10), leads to

$$\operatorname{div}(\xi^{\top}) = n\big(\varphi + \vartheta H\big) \tag{15}$$

Let us start by presenting a helpful lemma that is essential for our main result.

Lemma 2. *Let (M, g) be an oriented Riemannian hypersurface in an $(n+1)$-dimensional manifold $(\overline{M}, \overline{g})$, which can be either Riemannian or Lorentzian, where $n \geq 2$. If N is a unit normal to M and $\epsilon = \overline{g}(N, N)$, then, for any arbitrary vector field $X \in \mathfrak{X}(M)$, we have*

$$\overline{Ric}(X, N) = \epsilon n X(H) - g(\operatorname{div}(A), X) \tag{16}$$

Proof. Let $\{e_i\}_{1 \leq i \leq n}$ be a local orthonormal frame on M that we assume to be geodesic, and let $X \in \mathfrak{X}(M)$. Then, by taking the trace in both sides of the Codazzi Equation (7) and taking into account that $\nabla_Y A$ is self-adjoint, for each $Y \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned}
\overline{Ric}(X, N) &= \sum_{i=1}^{n} \overline{g}\big(\overline{R}(X, e_i)e_i, N\big) \\
&= -\sum_{i=1}^{n} \overline{g}\big(\overline{R}(X, e_i)N, e_i\big) \\
&= \sum_{i=1}^{n} \Big(g\big((\nabla_X A)(e_i), e_i\big) - g\big((\nabla_{e_i} A)(X), e_i\big)\Big) \\
&= \sum_{i=1}^{n} \Big(g(\nabla_X A(e_i), e_i) - g(A(\nabla_X e_i), e_i) - g\big((\nabla_{e_i} A)(e_i), X\big)\Big) \\
&= \sum_{i=1}^{n} \Big(X \cdot g(A(e_i), e_i) - g(\operatorname{div}(A), X)\Big) \\
&= X \cdot \operatorname{tr}(A) - g\big(\operatorname{div}(A), X\big) \\
&= \epsilon n X(H) - g\big(\operatorname{div}(A), X\big),
\end{aligned}$$

for all $X \in \mathfrak{X}(M)$. □

The following proposition characterizes totally umbilical Einstein manifolds in both the Riemannian and Lorentzian settings. After completing this work, we learned that the result had already been proven in the case where the ambient manifold is Riemannian (see [21]). The novelty of this proposition in the present paper lies in its extension to the Lorentzian setting.

Proposition 1. *Let (M, g) be a totally umbilical, oriented Riemannian hypersurface in an $(n+1)$-dimensional Einstein manifold $(\overline{M}, \overline{g})$, which can be either Riemannian or Lorentzian, where $n \geq 2$. Then, the mean curvature H of (M, g) is constant, and (M, g) is either totally geodesic or an extrinsic sphere in $(\overline{M}, \overline{g})$.*

Proof. Let $\{e_i\}_{1 \leq i \leq n}$ be a local orthonormal frame on M. Since (M, g) be a totally umbilical, the shape operator A satisfies $A = \epsilon H I$, where I is the identity operator. Therefore, for each i, we have

$$\begin{aligned}
\big(\nabla_{e_i} A\big)(e_i) &= \nabla_{e_i} A(e_i) - A\big(\nabla_{e_i} e_i\big) \\
&= \epsilon \nabla_{e_i}(H e_i) - \epsilon H \nabla_{e_i} e_i \\
&= \epsilon e_i(H) e_i.
\end{aligned}$$

It follows that

$$\begin{aligned} g(\mathrm{div}(A), X) &= \sum_{i=1}^{n} \epsilon g\big(e_i(H)e_i, X\big) \\ &= \sum_{i=1}^{n} \epsilon e_i(H) g\big(e_i, X\big) \\ &= \sum_{i=1}^{n} \epsilon g\big(\nabla H, e_i\big) g\big(X, e_i\big) \\ &= \epsilon g\Big(\nabla H, \sum_{i=1}^{n} g\big(X, e_i\big) e_i\Big) \\ &= \epsilon g\big(\nabla H, X\big) \\ &= \epsilon X(H), \end{aligned}$$

for all $X \in \mathfrak{X}(M)$. By substituting the latter expression into Equation (16), we get

$$\overline{Ric}(X, N) = \epsilon(n-1)X(H),$$

for all $X \in \mathfrak{X}(M)$. Since $(\overline{M}, \overline{g})$ is an Einstein manifold, it follows that $X(H) = 0$ for all $X \in \mathfrak{X}(M)$, meaning H must be constant. Thus, if (M, g) is not totally geodesic, then, as it is totally umbilical, it is necessarily an extrinsic sphere. □

Remark 1. *From the proof of the previous proposition, it follows that if (M, g) is a totally umbilical Riemannian hypersurface in an $(n+1)$-dimensional manifold $(\overline{M}, \overline{g})$, which may be either Riemannian or Lorentzian and is not necessarily an Einstein manifold, then*

$$\overline{Ric}(X, N) = \epsilon(n-1)X(H),$$

for all $X \in \mathfrak{X}(M)$.

One of the main results of this paper is a characterization of compact Riemannian totally umbilical hypersurfaces in either Riemannian or Lorentzian manifolds that admit a closed conformal (i.e., concircular) vector field $\overline{\xi}$, i.e.,

$$\overline{\nabla}_X \overline{\xi} = \varphi X, \qquad \text{for all } X \in \mathfrak{X}(\overline{M}). \tag{17}$$

Theorem 1. *Let (M, g) be a compact, oriented Riemannian hypersurface in an $(n+1)$-dimensional manifold $(\overline{M}, \overline{g})$, which can be either Riemannian or Lorentzian, where $n \geq 2$. Let $\overline{\xi}$ be a closed conformal vector field on $\overline{M}$, and let $\xi^\top$ represent the tangential component of the restriction ξ of $\overline{\xi}$ to M. Let N be a unit normal to M, and assume that $\vartheta = \overline{g}(\xi, N)$ is non-vanishing on a set of positive measure when $(\overline{M}, \overline{g})$ is Riemannian. Then, the Ricci curvature of (M, g) in direction $\xi^\top$ satisfies*

$$\int_M Ric\big(\xi^\top, \xi^\top\big) dv \geq \left(\frac{n-1}{n}\right) \int_M \big(\mathrm{div}(\xi^\top)\big)^2 dv \tag{18}$$

if and only if (M, g) is totally umbilical.

Proof. Let $\{e_i\}_{1\le i\le n}$ be a local orthonormal frame on M. Given that $A_{\xi^\top} = \nabla_X \xi^\top$, applying (13) and (17), we obtain

$$\begin{aligned}
\mathrm{tr}(A^2_{\xi^\top}) &= \sum_{i=1}^{2} g\Big(A^2_{\xi^\top}(e_i), e_i\Big) \\
&= \sum_{i=1}^{n} g\Big(A_{\xi^\top}\big(\varphi e_i + \epsilon\vartheta A(e_i)\big), e_i\Big) \\
&= \sum_{i=1}^{n} g\Big(\varphi\big(\varphi e_i + \epsilon\vartheta A(e_i)\big) + \epsilon\vartheta\big(\varphi A(e_i) + \epsilon\vartheta A^2(e_i)\big), e_i\Big) \\
&= \sum_{i=1}^{n} g\Big(\varphi^2 e_i + 2\epsilon\vartheta\varphi A(e_i) + \vartheta^2 A^2(e_i), e_i\Big) \\
&= n\varphi^2 + 2\epsilon\vartheta\varphi\,\mathrm{tr}(A) + \vartheta^2|A|^2 \\
&= n\varphi^2 + 2n\vartheta H\varphi + \vartheta^2|A|^2.
\end{aligned}$$

Since

$$\xi^\top \cdot \mathrm{div}(\xi^\top) = \mathrm{div}\big(\mathrm{div}(\xi^\top)\xi^\top\big) - \big(\mathrm{div}(\xi^\top)\big)^2,$$

then, by considering $\xi^\top$ instead of ξ in Equation (4), we obtain

$$\begin{aligned}
\mathrm{div}\big((\mathrm{div}\xi^\top)\xi^\top\big) - (\mathrm{div}\xi^\top)^2 &= \mathrm{div}(\nabla_{\xi^\top}\xi^\top) - Ric(\xi^\top,\xi^\top) - n\varphi^2 - 2n\vartheta H^2 - \vartheta^2|A|^2 \\
&= \mathrm{div}(\nabla_{\xi^\top}\xi^\top) - Ric(\xi^\top,\xi^\top) - n(\varphi+\vartheta H)^2 - \vartheta^2(|A|^2 - nH^2) \\
&= \mathrm{div}(\nabla_{\xi^\top}\xi^\top) - Ric(\xi^\top,\xi^\top) - \frac{(div\xi^\top)^2}{n} - \vartheta^2(|A|^2 - nH^2),
\end{aligned}$$

or equivalently

$$\mathrm{div}\Big((\mathrm{div}\xi^\top)\xi^\top - \nabla_{\xi^\top}\xi^\top\Big) = -Ric(\xi^\top,\xi^\top) + \frac{n-1}{n}(div\xi^\top)^2 - \vartheta^2(|A|^2 - nH^2) \tag{19}$$

Since M is Riemannian and compact without boundary, then by integrating both sides of (19) on M, and using the divergence theorem, it follows that

$$\int_M \vartheta^2\big(|A|^2 - nH^2\big)dv = \int_M \Big(\frac{n-1}{n}\big(div\xi^\top\big)^2 - Ric(\xi^\top,\xi^\top)\Big)dv, \tag{20}$$

where dv is the volume form of M.

Given that, according to the hypothesis,

$$Ric(\xi^\top,\xi^\top) \ge \frac{n-1}{n}(div\xi^\top)^2,$$

we can derive from (20) that

$$\int_M \vartheta^2(|A|^2 - nH^2)dv \le 0 \tag{21}$$

By the Cauchy-Schwarz inequality, we know that $|A|^2 - nH^2 \ge 0$. Since $\vartheta \neq 0$, it follows that the left-hand side of (21) is non-negative, which implies that $|A|^2 = nH^2$. This leads to the conclusion that $A = HI$, and therefore M is totally umbilical.

In accordance with Formula (19), the converse statement is evident. □

The second main result of this paper is presented in the following theorem, which focuses on the case where $(\overline{M},\overline{g})$ is an Einstein manifold. This result arises from the combination of Proposition 1 and Theorem 1.

Theorem 2. *Let (M,g) be a compact, oriented Riemannian hypersurface in an $(n+1)$-dimensional Einstein manifold $(\overline{M},\overline{g})$, which can be either Riemannian or Lorentzian, where $n \ge 2$. Let $\overline{\xi}$ be a closed conformal vector field on $\overline{M}$, and let $\xi^\top$ represent the tangential component of the restriction*

ξ of $\overline{\xi}$ to M. Let N be a unit normal to M, and assume that $\vartheta = \overline{g}(\xi, N)$ is non-vanishing on a set of positive measure when $(\overline{M}, \overline{g})$ is Riemannian. If (M, g) is not totally geodesic, then the Ricci curvature of (M, g) in direction $\xi^\top$ satisfies

$$\int_M Ric(\xi^\top, \xi^\top) dv \geq \left(\frac{n-1}{n}\right) \int_M \left(\operatorname{div}(\xi^\top)\right)^2 dv \tag{22}$$

if and only if (M, g) is an extrinsic sphere in $(\overline{M}, \overline{g})$.

As consequence of Theorem 2, we have

Corollary 1. *Under the assumptions and notation of Theorem 2, if ξ is a closed conformal vector field with constant conformal factor φ, then the inequality*

$$\int_M Ric(\xi^\top, \xi^\top) dv \geq n(n-1) \int_M (\vartheta H^2 - \varphi^2) dv$$

is satisfied if and only if M is an extrinsic sphere in $\overline{M}$.

Proof. By using (15), we obtain

$$\begin{aligned} \left(\operatorname{div}(\xi^\top)\right)^2 &= n^2\left(\varphi^2 + \vartheta^2 H^2 + 2\varphi\vartheta H\right) \\ &= n^2\left(2\varphi^2 + 2\varphi\vartheta H + \vartheta^2 H^2 - \varphi^2\right) \\ &= n^2\left(2\varphi(\varphi + \vartheta H) + \vartheta^2 H^2 - \varphi^2\right) \\ &= 2n\varphi \operatorname{div}(\xi^\top) + n^2\left(\vartheta^2 H^2 - \varphi^2\right). \end{aligned}$$

As a result, the inequality (22) in Theorem 2 reduces to

$$\int_M Ric(\xi^\top, \xi^\top) dv \geq 2(n-1) \int_M \varphi \operatorname{div}(\xi^\top) dv + n(n-1) \int_M \left(\vartheta^2 H^2 - \varphi^2\right) dv \tag{23}$$

Since φ is constant, we conclude the corollary from Theorem 2 by applying the divergence theorem to the first right-hand side of (23). □

Remark 2. *Theorem 2 clearly applies to characterizing spheres in both Riemannian and Lorentzian space forms, i.e., spaces with constant curvature. In this context, Theorem 2 extends previously known results by introducing greater generality. For instance, a similar result to Theorem 2 is provided in [17], where round spheres are characterized among compact Riemannian hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$. This characterization relies on the position vector field ψ, which is regarded as a special closed conformal vector field on $\mathbb{R}^{n+1}$, assuming its tangential component ψ^T is a geodesic vector field. More recently, this result was demonstrated in [18] without imposing any conditions on the tangential component ψ^T. Additionally, an analogous characterization of spheres among compact spacelike hypersurfaces in the de Sitter space has been established using a constant (parallel) vector field in Minkowski space (cf. [19]).*

The following propositions estimate the scalar curvature of minimal (or maximal) and totally umbilical Riemannian hypersurfaces in Einstein Riemannian and Lorentzian manifolds. Additionally, they characterize totally geodesic Riemannian hypersurfaces in both Einstein Riemannian and Lorentzian manifolds (compare to [22]).

Proposition 2. *Let (M, g) be a maximal (resp. minimal) hypersurface of an $(n+1)$-dimensional Einstein Lorentzian (resp. Riemannian) manifold $(\overline{M}, \overline{g})$, with $\overline{Ric} = \overline{\lambda}\overline{g}$, where $n \geq 2$. Then, the scalar curvature Sc of (M, g) satisfies $\mathrm{Sc} \geq (n-1)\overline{\lambda}$ (resp. $\mathrm{Sc} \leq (n-1)\overline{\lambda}$), and the equality holds if and only if (M, g) is a totally geodesic hypersurface in $(\overline{M}, \overline{g})$.*

Proof. Since $\overline{Ric} = \overline{\lambda}\overline{g}$, we have $\overline{\mathrm{Sc}} = (n+1)\overline{\lambda}$ and $\overline{Ric}(N,N) = \epsilon\overline{\lambda}$. Substituting these values into Equation (9) yields

$$\mathrm{Sc} = (n-1)\overline{\lambda} - \epsilon(|A|^2 - n^2H^2).$$

Given that (M,g) is maximal (resp. minimal), meaning $H = 0$, we have

$$\mathrm{Sc} = (n-1)\overline{\lambda} - \epsilon|A|^2$$

This implies $\mathrm{Sc} \geq (n-1)\overline{\lambda}$ (resp. $\mathrm{Sc} \leq (n-1)\overline{\lambda}$) when $\epsilon = -1$, indicating (M,g) is maximal (resp. $\epsilon = 1$, indicating (M,g) is minimal).

It is evident that equality holds if and only if $|A|^2 = 0$ (i.e., $A = 0$), which means (M,g) is totally geodesic. □

Proposition 3. *Let (M,g) be a totally umbilical Riemannian hypersurface of an $(n+1)$-dimensional Einstein Riemannian (resp. Lorentzian) manifold $(\overline{M},\overline{g})$, where $n \geq 2$. Then, the mean curvature H of (M,g) is constant, and the scalar curvature satisfies $\mathrm{Sc} \geq (n-1)\overline{\lambda}$ (resp. $\mathrm{Sc} \leq (n-1)\overline{\lambda}$). Equality occurs if and only if (M,g) is totally geodesic in $(\overline{M},\overline{g})$*

Proof. Since (M,g) is totally umbilical, we have $A = \epsilon HI$. Substituting this expression into Equation (9) gives

$$\mathrm{Sc} = (n-1)\overline{\lambda} + \epsilon n(n-1)H^2.$$

If $\epsilon = 1$, then $\mathrm{Sc} \geq (n-1)\overline{\lambda}$. Conversely, if $\epsilon = -1$ then $\mathrm{Sc} \leq (n-1)\overline{\lambda}$. □

5. Application of the Rigidity Theorem to Spacelike Hypersurfaces in Generalized Robertson–Walker (GRW) Spacetimes

In this section, we apply the main theorem to compact spacelike hypersurfaces in generalized Robertson–Walker (GRW) spacetimes. In fact, the structure of these spacetimes, together with the existence of a distinguished closed conformal timelike vector field, enables the general rigidity theorem to be applied effectively.

Let (N^n, g_0) be a Riemannian manifold of dimension $n \geq 2$, and let (a,b) be an open interval in $\mathbb{R}$. Let $\omega : (a,b) \to \mathbb{R}$ be a positive smooth function defined on (a,b). Consider $\overline{M} = (a,b) \times N^n$ equipped with the Lorentzian metric $\bar{g} = -dt^2 + \omega(t)^2 g_0$.

The Lorentzian warped product manifold $(\overline{M},\overline{g})$ is called a generalized Robertson–Walker (GRW) spacetime. GRW spacetimes are well-known for their importance in physics, especially in the theory of general relativity.

The vector field $V = \omega\frac{\partial}{\partial t}$ on $\overline{M}$, which represents a natural symmetry for $(\overline{M},\overline{g})$, can be easily proven to be a closed conformal vector field of $(\overline{M},\overline{g})$, satisfying

$$\overline{\nabla}_X V = \omega' X,$$

for all vector fields $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is here the Levi-Civita connection of $(\overline{M},\overline{g})$ and $\mathfrak{X}(\overline{M})$ is the space of all vector fields on $\overline{M}$.

Let (M,g) be a spacelike hypersurface in $(\overline{M},\overline{g})$. Since $\frac{\partial}{\partial t}$ establishes an orientation for $\overline{M}$, we let N be a timelike unit normal vector field to M, assumed to be globally defined. We define θ, the support function of a $\frac{\partial}{\partial t}$, as a smooth function on M given by $\theta = \overline{g}\left(\frac{\partial}{\partial t}, N\right)$. Using the notation introduced at the start of Section 4, we note that if $\vartheta = \overline{g}(V,N)$, then $\vartheta = \omega\theta$. Since $\overline{g}(V,V) = -\omega^2$, the application of inequality (11) to V implies

$$\theta \leq -1 \tag{24}$$

Let T represent the component of $\frac{\partial}{\partial t}$ that lies tangent to M, then $\frac{\partial}{\partial t}$ can be written as

$$\frac{\partial}{\partial t} = T - \theta N.$$

Let $\sigma : M \to (a,b)$ be the height function defined by $\sigma(x) = (p \circ \iota)(x)$, where $\iota : M \to \overline{M}$ is the isometric immersion representing (M,g) as a spacelike hypersurface of $(\overline{M},\overline{g})$, and $p : \overline{M} \to (a,b)$ is the projection on the base (a,b).

It is clear that the gradient $\overline{\nabla} p$ of p satisfies $\overline{\nabla} p = -\frac{\partial}{\partial t}$. It follows that the gradient of σ is given by

$$\nabla \sigma = -T, \tag{25}$$

The next two formulas, which describe the norm (i.e., length) and Laplacian of σ, are established in [25].

$$|\nabla \sigma|^2 = \theta^2 - 1, \tag{26}$$

$$\Delta \sigma = -(n - 1 + \theta^2)\frac{\omega'}{\omega} - n\theta H. \tag{27}$$

In the GRW spacetime $(\overline{M},\overline{g})$, there is a special family of totally umbilical spacelike hypersurfaces, called slices. A slice in $\overline{M}$ takes the form $M_{t_0} = t_0 \times M$. It is a totally umbilical spacelike hypersurface of constant mean curvature

$$H(t_0) = \frac{\omega'(t_0)}{\omega(t_0)}.$$

Thus, if (M,g) is assumed to be geodesically complete, M_{t_0} is either a totally geodesic hypersurface or an extrinsic sphere of $(\overline{M},\overline{g})$, according to whether $\omega'(t_0)$ is zero or not, respectively.

Theorem 3. *Let (M,g) be a compact spacelike hypersurface in a GRW spacetime $(\overline{M},\overline{g})$. Then, (M,g) is totally umbilical if and only if the height function σ satisfies*

$$\int_M Ric(\nabla\sigma,\nabla\sigma)dv \geq \left(\frac{n-1}{n}\right)\int_M \left(\Delta\sigma + \frac{\omega'}{\omega}|\nabla\sigma|^2\right)^2 dv \tag{28}$$

Proof. Since $\nabla\sigma = -T$, the gradient of ω satisfies $\nabla\omega = -\omega' T = \omega'\nabla\sigma$. Consequently, the inequality (18) with $\xi^\top = \omega T$ becomes

$$\begin{aligned}
\omega^2 Ric(\nabla\sigma,\nabla\sigma) &\geq \left(\frac{n-1}{n}\right)(\mathrm{div}(\omega\nabla\sigma))^2 \\
&= \left(\frac{n-1}{n}\right)(\omega\Delta\sigma + \nabla\sigma(\omega))^2 \\
&= \left(\frac{n-1}{n}\right)\omega^2\left(\Delta\sigma + \frac{\omega'}{\omega}|\nabla\sigma|^2\right)^2,
\end{aligned}$$

which leads to (28), and Theorem 3 then follows from Theorem 1. □

By considering (26) and (27), the subsequent corollary is a direct consequence of Theorem 3.

Corollary 2. *Let (M,g) be a compact spacelike hypersurface in a GRW spacetime $(\overline{M},\overline{g})$. Then, (M,g) is totally umbilical if and only if*

$$\int_M Ric(T,T)dv \geq n(n-1)\int_M \left(\frac{\omega'}{\omega} + \theta H\right)^2 dv$$

We conclude with the following result, which can be obtained by combining Proposition 1 with Theorem 3.

Theorem 4. *Let (M, g) be a compact spacelike hypersurface in an Einstein GRW spacetime $(\overline{M}, \overline{g})$. If (M, g) is not totally geodesic, then it is an extrinsic sphere in $(\overline{M}, \overline{g})$ if and only if*

$$\int_M Ric(\nabla\sigma, \nabla\sigma)dv \geq \left(\frac{n-1}{n}\right)\int_M \left(\Delta\sigma + \frac{\omega'}{\omega}|\nabla\sigma|^2\right)^2 dv$$

Remark 3. *One can easily confirm that a GRW spacetime $(\overline{M}, \overline{g})$ is Einstein if and only if the fiber factor (M, g_0) is Einstein and the warping function ω satisfies specific differential equations (see [26]).*

6. Conclusions and Future Directions

We have provided rigidity characterizations for compact Riemannian and spacelike hypersurfaces in Einstein Riemannian and Lorentzian manifolds, respectively. Using the presence of a closed conformal vector field of the ambient Einstein manifold, we have shown that, through an integral inequality involving the Ricci curvature in the direction of the tangential component of the conformal vector field, the compact hypersurface is isometric to an extrinsic sphere. This provides a good characterization of extrinsic spheres in both the Riemannian and Lorentzian settings. As an important application, we have investigated compact spacelike hypersurfaces in generalized Robertson–Walker (GRW) spacetimes. Note that examples of Riemannian and Lorentzian Einstein (but not of constant curvature) manifolds admitting a timelike closed conformal vector field exist. A natural Lorentzian example is a GRW spacetime whose fiber is Einstein but does not have constant sectional curvature. An explicit example consists of taking the fiber (N^n, g_0) as the complex hyperbolic space with its standard metric, normalized so that

$$Ric_{N^n} = -(m-1)g_0,$$

where $n = 2m$. The warping function can be choosen as $\omega(t) = \cosh t$, and therefore the closed conformal vector field is $V = \cosh t \frac{\partial}{\partial t}$.

The results obtained in this paper extend and unify several rigidity results for hypersurfaces in the presence of a conformal vector field of the ambient space using curvature integral estimates. It would be interesting to generalize the results of this paper to compact spacelike submanifolds in semi-Riemannian manifolds such as those of constant sectional curvature and Einstein manifolds in general. It would also be interesting to investigate whether analogous characterizations remain valid under weaker curvature assumptions for noncompact hypersurfaces or in other classes of Lorentzian warped-product spacetimes.

Funding: This research was supported by the Ongoing Research Funding Program (ORF-2026-1053), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares he has no conflicts of interest.

References

1. Lichnerowicz, A. Transformations infinitésimales conformes de certaines variétés riemanniennes compactes. *C. R. Acad. Sci.* **1955**, *241*, 726–729.
2. Yano, K.; Bochner, S. *Curvature and Betti Numbers [Annals of Mathematics Studies Number 32]*; Princeton University Press: Princeton, NJ, USA, 1953.
3. Goldberg, S.I.; Kobayashi, S. The conformal transformation group of a compact Riemannian manifold. *Am. J. Math.* **1962**, *84*, 170–174. [CrossRef]
4. Obata, M. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Jpn.* **1962**, *14*, 333–340. [CrossRef]

5. Nagano, T.; Yano, K. Einstein spaces admitting a one-parameter group of conformal transformations. *Ann. Math.* **1959**, *69*, 451–461. [CrossRef]
6. Yano, K. *Integral Formulas in Riemannian Geometry*; Marcel Dekker Inc.: New York, NY, USA, 1970.
7. Obata, M.; Yano, K. Sur le groupe de transformations conformes d'une variété de Riemann dont le scalaire de courbure est constant. *C. R. Acad. Sci.* **1965**, *260*, 2698–2700.
8. Obata, M.; Yano, K. Conformal changes of Riemannian metrics. *J. Differ. Geom.* **1970**, *4*, 53–72. [CrossRef]
9. Tashiro, Y. Complete Riemannian manifolds and some vector fields. *Trans. Am. Math. Soc.* **1965**, *117*, 251–275. [CrossRef]
10. Ejiri, N. A negative answer to a conjecture of conformal transformations of Riemannian manifolds. *J. Math. Soc. Jpn.* **1981**, *33*, 261–266. [CrossRef]
11. Nomizu, K.; Smyth, B. A formula of Simons' type and hypersurfaces of constant mean curvature. *J. Differ. Geom.* **1969**, *3*, 367–377. [CrossRef]
12. Akutagawa, K. On spacelike hypersurfaces with constant mean curvature in the de Sitter space. *Math. Z.* **1987**, *196*, 13–19. [CrossRef]
13. O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
14. Alias, L.J.; Romero, A.; Sanchez, M. Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson–Walker spacetimes. *Gen. Relativ. Gravit.* **1995**, *27*, 71–84. [CrossRef]
15. Alias, L.J.; Romero, A.; Sanchez, M. Spacelike hypersurfaces of constant mean curvature and Calabi–Bernstein type problems. *Tohoku Math. J.* **1997**, *49*, 337–345. [CrossRef]
16. Alias, L.J.; Romero, A.; Sanchez, M. Spacelike hypersurfaces of constant mean curvature in certain spacetimes. *Nonlinear Anal.* **1997**, *30*, 655–661. [CrossRef]
17. Deshmukh, S.; Guediri, M. Characterization of Euclidean spheres. *AIMS Math.* **2021**, *6*, 7733–7740. [CrossRef]
18. Bin-Asfour, M.; Albalawi, K.S.; Guediri, M. Characterizations of Euclidean spheres in terms of the tangential part of the position vector field. *Int. J. Math. Math. Sci.* **2024**, *2024*, 9640957. [CrossRef]
19. Yanlin, L.; Bin-Asfour, M.; Albalawi, K.S.; Guediri, M. Spacelike hypersurfaces in de Sitter space. *Axioms* **2025**, *14*, 155. [CrossRef]
20. Guediri, M. Rigidity of extrinsic spheres in Einstein spacetimes and the Goddard conjecture. *AIMS Math.* **2026**, *11*, 9319–9333. [CrossRef]
21. Kowalski, O. Properties of hypersurfaces which are characteristic for spaces of constant curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1972**, *26*, 233–245.
22. Koiso, N. Hypersurfaces of Einstein manifolds. *Ann. Sci. Éc. Norm. Supér.* **1981**, *14*, 433–443. [CrossRef]
23. Palomo, F.J.; Romero, A. Certain actual topics on modern Lorentzian geometry. In *Handbook of Differential Geometry*; Elsevier: Amsterdam, The Netherlands, 2006; Volume II.
24. Alshehri, N.; Guediri, M. Characterizing affine vector fields on semi-Riemannian manifolds. *Axioms* **2024**, *13*, 835. [CrossRef]
25. Alessa, N.; Guediri, M. On spacelike hypersurfaces in generalized Robertson–Walker spacetimes. *Axioms* **2024**, *13*, 636. [CrossRef]
26. Besse, A.L. *Einstein Manifolds*; Springer: Berlin, Germany, 1987.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.