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Fixed Point for Some Polynomial Contractions

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Abstract

In this paper, we introduce the notion of Kannan polynomial contraction in the setting of metric spaces and establish the existence and uniqueness of fixed points for this class of mappings. Furthermore, we extend the study to b-metric spaces by investigating polynomial contractions and Kannan polynomial contractions in this generalized framework. Sufficient conditions are provided to guaranty the existence and uniqueness of fixed points in b-metric spaces. The results obtained extend several known fixed point theorems from metric spaces to b-metric spaces. Moreover, we illustrate our results with several examples and applications.

Keywords: metric space; b-metric space; polynomial contractions; Kannan polynomial contractions; fixed point

MSC: Primary 47H10; Secondary 54H25

1. Introduction

Fixed point theory constitutes one of the most productive and constructive frameworks in modern mathematics and is considered a cornerstone of nonlinear analysis. It has an important position in many branches of science, especially in the study of differential equations, dynamical systems, and nonlinear analysis. Furthermore, fixed-point results have been formulated in various types of metric space, b-metric spaces, extended b-metric spaces, and even more generalized metric systems. For topological background, see [1].

One of the most significant contributions to fixed point theory is the Banach Fixed Point Theorem, originally proved by S. Banach in 1922 [2]. This theorem represents one of the earliest and most fundamental results in the theory. It states that if (Ξ, δ) is a complete metric space and $\Gamma : \Xi \longrightarrow \Xi$ is a contraction mapping, that is, there exists a constant $0 < \rho < 1$ such that

$$\delta(\Gamma\xi, \Gamma\mu) \leq \rho\delta(\xi, \mu), \quad \text{for all } \xi, \mu \in \Xi,$$

then the mapping Γ admits a unique fixed point in Ξ.

Many researchers focusing on extensions of the classical Banach contraction principle have been drawn to fixed point theory, particularly those modifying or relaxing the contraction condition. These generalizations purport to widen the range of space types and mapping types beyond the usual metric spaces. For multivalued extensions, see [3–5].

Several researchers have studied fixed point theory under alternative contractive conditions, especially when the mapping is not continuous. A notable example is Kannan's Fixed Point Theorem, introduced by R. Kannan (1969) [6], which does not require continuity.

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Kannan proposed that for a complete metric space (Ξ, δ) and a mapping $\Gamma : \Xi \longrightarrow \Xi$, if there exists a constant $\rho < \frac{1}{2}$ such that:

$$\delta(\Gamma\xi, \Gamma\mu) \leq \rho[\delta(\xi, \Gamma\xi) + \delta(\mu, \Gamma\mu)] \quad \text{for all } \xi, \mu \in \Xi, \tag{1}$$

then Γ has a unique fixed point, and iterative sequences converge to it. This condition generalizes Banach's theorem. Related developments include Chatterjea contractions [7], Kannan-type results involving u-distances [8], and weak contractions [9].

While Kannan-type contractions provide an alternative to the classical Banach contractive condition, a complementary line of research extends fixed-point theory by relaxing the underlying metric structure. In particular, some distance functions satisfy symmetry and definiteness but fail to satisfy the triangle inequality. To accommodate such functions, Bakhtin [10] and Czerwik [11] developed the framework of b-metric spaces, in which the triangle inequality is relaxed by a constant factor.

Definition 1 (b-metric space [10,11]). *Let Ξ be a nonempty set and let $s \geq 1$. A mapping $\delta : \Xi \times \Xi \to [0, \infty)$ is called a b-metric on Ξ if, for all $\xi, \mu, \zeta \in \Xi$, the following conditions hold:*

(i) $\delta(\xi, \mu) = 0$ *if and only if* $\xi = \mu$;
(ii) $\delta(\xi, \mu) = \delta(\mu, \xi)$;
(iii) $\delta(\xi, \mu) \leq s[\delta(\xi, \zeta) + \delta(\zeta, \mu)]$.

The triple (Ξ, δ, s) is called a b-metric space.

When $s = 1$, condition (iii) becomes the ordinary triangle inequality. Thus, every metric space is a b-metric space, whereas the converse is not true, in general. For example,

$$\delta(\xi, \mu) = |\xi - \mu|^2$$

defines a b-metric on $\mathbb{R}$ with coefficient $s = 2$, but it is not a metric, since

$$\delta(0, 2) = 4 > \delta(0, 1) + \delta(1, 2) = 2.$$

This framework therefore permits the study of fixed points for distance functions beyond the classical metric setting. For multivalued results in b-metric spaces, see [12,13].

Czerwik [11] established extensions of Banach's contraction principle [2] in b-metric spaces. Kannan-type contractions were also investigated in this setting, including the following result of Kir and Kiziltunc [14].

Theorem 1 ([14]). *Let (Ξ, δ, s) be a nonempty complete b-metric space. Suppose that $\Gamma : \Xi \to \Xi$ satisfies*

$$\delta(\Gamma\xi, \Gamma\mu) \leq \rho[\delta(\xi, \Gamma\xi) + \delta(\mu, \Gamma\mu)]$$

for all $\xi, \mu \in \Xi$, where $0 \leq \rho < \frac{1}{2}$. Then Γ has a unique fixed point $\xi^* \in \Xi$. Moreover, for every $\xi_0 \in \Xi$, the Picard sequence $\xi_{n+1} = \Gamma\xi_n$ converges to ξ^*.

A further extension replaces the constant coefficient s by a control function $\theta : \Xi \times \Xi \to [1, \infty)$, yielding the extended b-metric inequality

$$\delta_\theta(\xi, \mu) \leq \theta(\xi, \mu)[\delta_\theta(\xi, \zeta) + \delta_\theta(\zeta, \mu)].$$

This structure was introduced by Kamran et al. [15]. Within this framework, Abulhamil [16] established the following Kannan-type result under continuity and control-function assumptions.

Theorem 2 ([16]). *Let (Ξ, δ_θ) be a nonempty complete extended b-metric space, and assume that δ_θ is continuous. Suppose that $\Gamma : \Xi \to \Xi$ satisfies*

$$\delta_\theta(\Gamma\xi, \Gamma\mu) \leq \rho[\delta_\theta(\xi, \Gamma\xi) + \delta_\theta(\mu, \Gamma\mu)]$$

for all $\xi, \mu \in \Xi$, where $0 < \rho < \frac{1}{2}$. Assume further that

$$\theta(\xi, \mu) < \frac{1}{\rho} \qquad \text{for all } \xi, \mu \in \Xi,$$

and that, for every $\xi_0 \in \Xi$, the sequence $\xi_n = \Gamma^n \xi_0$ satisfies

$$\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) < \frac{1-\rho}{\rho}.$$

Then Γ has a unique fixed point in Ξ.

Further fixed-point results have been established in several variants of *b*-metric spaces. Kiran et al. [17] obtained fixed-point results for single-valued and multi-valued mappings in complete extended *b*-metric spaces. Chen et al. [18] established fixed-point theorems in convex *b*-metric spaces, including a Kannan-type result. More recently, Alsulami and Alarfaj [19] studied α-ψ-contractive conditions for single-valued and multi-valued operators in complete strong *b*-metric spaces.

Very recently, Jleli et al. [20] introduced polynomial contractions in metric spaces and established fixed-point results for this class. Hamza et al. [21] subsequently considered polynomial *F*-contractions. We recall the original definition and the corresponding fixed-point theorem in the following.

Definition 2 ([20]). *Let (Ξ, δ) be a metric space and $\Gamma : \Xi \to \Xi$ a mapping. The mapping Γ is called a polynomial contraction if there exist a constant $\rho \in [0, 1)$, a natural number $\kappa \geq 1$, and mappings $\Theta_i : \Xi \times \Xi \to [0, \infty)$, $i = 0, \ldots, \kappa$, such that*

$$\sum_{i=0}^{\kappa} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \leq \rho \sum_{i=0}^{\kappa} \Theta_i(\xi, \mu)\delta^i(\xi, \mu)$$

for all $\xi, \mu \in \Xi$.

Theorem 3 ([20], Theorem 2.1). *Let (Ξ, δ) be a complete metric space and $\Gamma : \Xi \to \Xi$ a polynomial contraction. Assume that the following conditions hold:*

(i) Γ is continuous;
(ii) there exist $j \in \{1, \ldots, \kappa\}$ and $A_j > 0$ such that

$$\Theta_j(\xi, \mu) \geq A_j, \qquad \xi, \mu \in \Xi.$$

Then Γ admits a unique fixed point $\xi^* \in \Xi$. Moreover, for every $\xi_0 \in \Xi$, the Picard sequence $\{\xi_n\} \subset \Xi$, defined by $\xi_{n+1} = \Gamma\xi_n$ for all $n \geq 0$, converges to ξ^*.

We observe that the term corresponding to $i = 0$ in the definition can be problematic: when $\delta(\Gamma\xi, \Gamma\mu)$ approaches zero, this term can become undefined or uninformative.

Moreover, the treatment of the term corresponding to $i = 0$ in the proof requires clarification. Unlike the terms involving positive powers of the distance, the zeroth-order term does not vanish as the distance tends to zero. Consequently, any step that treats all terms in the sum as vanishing requires separate justification for $i = 0$. In the present work, we restrict the summation to $i = 1, \ldots, \kappa$, thereby excluding the zeroth-order term.

Furthermore, during the convergence proof, the authors applied the triangle inequality in the following form:

$$\delta_j(\zeta_n, \zeta_{n+m}) \leq \delta_j(\zeta_n, \zeta_{n+1}) + \delta_j(\zeta_{n+1}, \zeta_{n+2}) + \cdots + \delta_j(\zeta_{n+m-1}, \zeta_{n+m}), \quad \text{for all } n \geq 0, m \geq 1.$$

However, this estimation is not universally accurate and may not hold in all metric contexts or for every class of mappings.

In this article, we address the issues discussed above by modifying the definition so that the summation starts at $i = 1$, thus excluding the problematic zeroth-order term. We also employ an alternative technique to establish convergence and the existence of fixed points, rather than relying on repeated applications of the triangle inequality.

We introduce Kannan polynomial contractions and establish fixed-point theorems for this class in complete metric and b-metric spaces. In addition, we establish a fixed-point theorem for polynomial contractions in complete b-metric spaces. We support our results with examples and applications.

2. Fixed Point for Kannan Polynomial Contractions

In this section, we develop the notion of Kannan polynomial contractions in metric spaces and establish fixed point results that extend and generalize several existing theorems in the literature.

Definition 3. *Let (Ξ, δ) be a metric space, and let $\Gamma : \Xi \longrightarrow \Xi$ satisfy*

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi, \Gamma\xi)\delta^i(\xi, \Gamma\xi) + \Theta_i(\mu, \Gamma\mu)\delta^i(\mu, \Gamma\mu)\big], \tag{2}$$

where $0 < \rho < \frac{1}{2}$; $\Theta_i(\xi, \mu) : \Xi \times \Xi \to (0, \infty)$ for all $i = 1, \ldots \kappa$.

Then, we say Γ satisfies Kannan Polynomial Contraction.

Theorem 4. *Let (Ξ, δ) be a complete metric space $\Gamma : \Xi \longrightarrow \Xi$ satisfies (2) and assume:*

(i) Γ is continuous,

(ii) There exists $j \in \{1, \ldots \kappa\}$ and $\mathfrak{A}_j > 0$ such that $\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu)$ for all $\xi, \mu \in \Xi$.

Then Γ has a unique fixed point.

Proof. Let (ξ_0) be an element in Ξ, then the iteration sequence defined by $\xi_n = \Gamma^n(\xi_0)$

Using inequality (2), we deduce that

$$\begin{aligned}
\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0, \Gamma\xi_1)\delta^i(\Gamma\xi_0, \Gamma\xi_1) &\leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi_0, \Gamma\xi_0)\delta^i(\xi_0, \Gamma\xi_0) + \Theta_i(\xi_1, \Gamma\xi_1)\delta^i(\xi_1, \Gamma\xi_1)\big] \\
\sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2) &\leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1) + \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2)\big] \\
&\leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1) + \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2). \\
\text{So, } \sum_{i=1}^{\kappa} \Theta_1(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2) &\leqslant \frac{\rho}{1-\rho} \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1).
\end{aligned} \tag{3}$$

By repeated application of the inequality (2) for (ξ_1, ξ_2), we derive

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_1, \Gamma\xi_2)\delta^i(\Gamma\xi_1, \Gamma\xi_2) \leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi_1, \Gamma\xi_1)\delta^i(\xi_1, \Gamma\xi_1) + \Theta_i(\xi_2, \Gamma\xi_2)\delta^i(\xi_2, \Gamma\xi_2)\big],$$

$$\sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3) \leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2) + \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3)\big]$$

$$\leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2) + \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3),$$

$$\sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3) \leqslant \frac{\rho}{1-\rho} \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2).$$

By using (3), we have:

$$\sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3) \leqslant \left(\frac{\rho}{1-\rho}\right)^2 \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{4}$$

By the same process and by applying (2) for (ξ_{n-1}, ξ_n), we get the following:

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_{n-1}, \Gamma\xi_n)\delta^i(\Gamma\xi_{n-1}, \Gamma\xi_n) \leqslant \rho \sum_{i=1}^{\kappa} \big[\Theta_i(\xi_{n-1}, \Gamma\xi_{n-1})\delta^i(\xi_{n-1}, \Gamma\xi_{n-1}) + \Theta_i(\xi_n, \Gamma\xi_n)\delta^i(\xi_n, \Gamma\xi_n)\big].$$

Therefore, $$\sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1}) \leqslant \left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{5}$$

From (ii), there exist $j \in \{1, 2, \ldots \kappa\}$ and $\mathfrak{A}_j > 0$ such that

$$0 < \mathfrak{A}_j \leqslant \Theta_i(\xi_n, \xi_{n+1}),$$

$$\mathfrak{A}_j \delta^j(\xi_n, \xi_{n+1}) \leqslant \Theta_j(\xi_n, \xi_{n+1})\delta^j(\xi_n, \xi_{n+1}) \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1})$$

$$\leqslant \left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1),$$

as a result, $$\delta^j(\xi_n, \xi_{n+1}) \leqslant \mathfrak{A}_j^{-1} \left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1).$$

Assume $\Omega_{j,0} = \mathfrak{A}_j^{-1} \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1)$.
So,

$$\delta(\xi_n, \xi_{n+1}) \leqslant \left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \big(\Omega_{j,0}\big)^{\frac{1}{j}}. \tag{6}$$

Now, we will prove (ξ_n) is a Cauchy sequence. Take $n, m \in \mathbb{N}$

$$\delta(\xi_n, \xi_{n+m}) \leqslant \delta(\xi_n, \xi_{n+1}) + \delta(\xi_{n+1}, \xi_{n+2}) + \cdots + \delta(\xi_{n+m-1}, \xi_{n+m})$$

$$\leqslant \left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \big(\Omega_{j,0}\big)^{\frac{1}{j}} + \left(\frac{\rho}{1-\rho}\right)^{\frac{n+1}{j}} \big(\Omega_{j,0}\big)^{\frac{1}{j}} + \cdots + \left(\frac{\rho}{1-\rho}\right)^{\frac{n+m-1}{j}} \big(\Omega_{j,0}\big)^{\frac{1}{j}}$$

$$\leqslant \left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \big(\Omega_{j,0}\big)^{\frac{1}{j}} \left[1 + \left(\frac{\rho}{1-\rho}\right)^{\frac{1}{j}} + \left(\frac{\rho}{1-\rho}\right)^{\frac{2}{j}} + \cdots + \left(\frac{\rho}{1-\rho}\right)^{\frac{m-1}{j}}\right].$$

Because $\rho < \frac{1}{2}$ and $j > 1$, then $\left(\frac{\rho}{1-\rho}\right)^{\frac{1}{j}} < 1$. Therefore, as $n, m \to \infty$ $\delta(\xi_n, \xi_{n+m})$ converges to 0. By completeness there exists $\xi_0 \in \Xi$ such that $\delta(\xi_n, \xi_0) \to 0$ as $n \longrightarrow \infty$.

Now, we will prove ξ_0 is a fixed point.

The continuity of Γ gives $\lim_{n\to\infty} \delta(\Gamma\xi_n, \Gamma\xi_0) = \lim_{n\to\infty} \delta(\xi_{n+1}, \Gamma\xi_0) = 0$.

By the uniqueness of the limit, we have $\Gamma(\xi_0) = \xi_0$.

To establish the uniqueness of the fixed point, suppose that $\mu_0 \in \Xi$ is another fixed point of Γ, i.e., $\Gamma(\mu_0) = \mu_0$.

Applying the contraction inequality (2), for (ξ_0, μ_0) and using the identities $\delta(\xi_0, \Gamma\xi_0) = 0$ and $\delta(\mu_0, \Gamma\mu_0) = 0$, we obtain

$$0 \leqslant \sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0, \Gamma\mu_0)\delta^i(\Gamma\xi_0, \Gamma\mu_0) \leqslant \rho\left[\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \Gamma\xi_0)\delta^i(\xi_0, \Gamma\xi_0) + \sum_{i=1}^{\kappa} \Theta_i(\mu_0, \Gamma\mu_0)\delta^i(\mu_0, \Gamma\mu_0)\right]$$

$$0 \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) \leqslant \rho\left[\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \Gamma\xi_0)\delta^i(\xi_0, \Gamma\xi_0) + \sum_{i=1}^{\kappa} \Theta_i(\mu_0, \Gamma\mu_0)\delta^i(\mu_0, \Gamma\mu_0))\right] = 0.$$

Since there exist $j \in \{1, 2, \cdots \kappa\}$ and $\mathfrak{A}_j > 0$ such that

$$\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu), \quad \forall \xi, \mu \in \Xi,$$

Therefore,

$$\mathfrak{A}_j \cdot \delta^j(\xi_0, \mu_0) \leqslant \Theta_j(\xi_0, \mu_0)\delta^j(\xi_0, \mu_0) \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) = 0,$$

it follows that $\delta^j(\xi_0, \mu_0) = 0$. Hence, $\delta(\xi_0, \mu_0) = 0$.

Which proves the uniqueness of the fixed point. □

From Theorem (4), we deduce the following result.

Corollary 1. *Let (Ξ, δ) be a complete metric space and $\Gamma : \Xi \to \Xi$ be a given mapping. Assume that there exist $\rho \in (0, \frac{1}{2})$, a natural number $\kappa \geq 1$ and a finite sequence $\{\Theta_i\}_{i=1}^{\kappa} \subset (0, \infty)$ such that:*

$$\sum_{i=1}^{\kappa} \Theta_i\delta^i(\Gamma(\xi), \Gamma(\mu)) \leqslant \rho \sum_{i=1}^{\kappa}\left[\Theta_i\delta^i(\xi, \Gamma(\xi)) + \Theta_i\delta^i(\mu, \Gamma(\mu))\right], \tag{7}$$

for every $\xi, \mu \in \Xi$. Also, it is satisfies:

(i) There exist $j \in \{1, \cdots, \kappa\}$ and $\mathfrak{A}_j > 0$ such that

$$\Theta_j \geq \mathfrak{A}_j,$$

(ii) Γ is continuous.

Then Γ admits a unique fixed point $\zeta^* \in \Xi$. Moreover, for every $\zeta_0 \in \Xi$, the Picard sequence $\{\zeta_n\} \subset \Xi$ defined by $\zeta_{n+1} = \Gamma\zeta_n$ for all $n \geq 0$, converges to ζ^*.

In addition, following from the result obtained earlier, in the case when $\kappa = 1$ and $\Theta_1 = 1$, the inequality simplifies to the classical Kannan contraction condition. Therefore, the result of the fixed point theorem is in agreement with the often cited Kannan Fixed Point Theorem in 1969 [6].

The following example illustrates the applicability of the theorem to Kannan polynomial contractions in metric spaces.

Example 1. *Let (Ξ, δ) be the metric space $\Xi = \mathbb{R}$ with $\delta(\xi, \mu) = |\xi - \mu|$. Define the mapping*

$$\Gamma : \Xi \to \Xi, \qquad \Gamma(\xi) = \frac{\xi}{4}.$$

Let $\kappa = 1$ and choose the coefficient

$$\Theta_1(\xi, \mu) = 1, \qquad \forall \xi, \mu \in \Xi.$$

Then the Kannan polynomial contraction condition (2) becomes

$$\delta(\Gamma\xi, \Gamma\mu) \leq \rho\big(\delta(\xi, \Gamma\xi) + \delta(\mu, \Gamma\mu)\big), \qquad 0 < \rho < \frac{1}{2}.$$

We compute

$$\delta(\Gamma\xi,\Gamma\mu)=\frac{1}{4}\,\delta(\xi,\mu),\qquad \delta(\xi,\Gamma\xi)=\left|\xi-\frac{\xi}{4}\right|=\frac{3}{4}|\xi|,\qquad \delta(\mu,\Gamma\mu)=\frac{3}{4}|\mu|.$$

Since $\delta(\xi,\mu)\le|\xi|+|\mu|$, *we obtain*

$$\frac{1}{4}\delta(\xi,\mu)\le\frac{1}{4}(|\xi|+|\mu|)=\frac{1}{3}\cdot\frac{3}{4}(|\xi|+|\mu|)=\rho\big(\delta(\xi,\Gamma\xi)+\delta(\mu,\Gamma\mu)\big)$$

with $\rho=\frac{1}{3}<\frac{1}{2}$. *Thus,* Γ *satisfies (2), and by Theorem (4) it admits a unique fixed point, namely* 0.

The next example considers a Kannan polynomial contraction in a metric space with nonconstant coefficient mappings and $\kappa>1$.

Example 2. *Let* (Ξ,δ) *be the metric space* $\Xi=\mathbb{R}$ *with* $\delta(x,y)=|x-y|$. *Define*

$$\Gamma(\xi)=\frac{\xi}{4}.$$

Take $\kappa=2$ *and*

$$\Theta_1(\xi,\mu)=1+|\xi|+|\mu|,\qquad \Theta_2(\xi,\mu)=1+|\xi-\mu|.$$

Then

$$\delta(\Gamma\xi,\Gamma\mu)=\frac{1}{4}\,|\xi-\mu|,\qquad \delta^2(\Gamma\xi,\Gamma\mu)=\frac{1}{16}\,|\xi-\mu|^2.$$

Also,

$$\Theta_1(\Gamma\xi,\Gamma\mu)=1+|\Gamma\xi|+|\Gamma\mu|=1+\left|\frac{\xi}{4}\right|+\left|\frac{\mu}{4}\right|=1+\frac{|\xi|+|\mu|}{4},$$

and

$$\Theta_2(\Gamma\xi,\Gamma\mu)=1+|\Gamma\xi-\Gamma\mu|=1+\left|\frac{\xi}{4}-\frac{\mu}{4}\right|=1+\frac{|\xi-\mu|}{4}.$$

Using the following inequalities, valid for all $\xi,\mu\in\mathbb{R}$,

$$(|\xi|+|\mu|)^2\le 2\big(|\xi|^2+|\mu|^2\big),\qquad (|\xi|+|\mu|)^3\le 4\big(|\xi|^3+|\mu|^3\big),$$

together with the triangle inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\Theta_i(\Gamma\xi,\Gamma\mu)\,\delta^i(\Gamma\xi,\Gamma\mu)\\
&=\left(1+\frac{|\xi|+|\mu|}{4}\right)\frac{|\xi-\mu|}{4}+\left(1+\frac{|\xi-\mu|}{4}\right)\frac{|\xi-\mu|^2}{16}\\
&\le\frac{|\xi|+|\mu|}{4}+\frac{(|\xi|+|\mu|)^2}{8}+\frac{(|\xi|+|\mu|)^3}{64}\\
&\le\frac{|\xi|+|\mu|}{4}+\frac{|\xi|^2+|\mu|^2}{4}+\frac{|\xi|^3+|\mu|^3}{16}\\
&\le\frac{|\xi|+|\mu|}{4}+\frac{|\xi|^2+|\mu|^2}{2}+\frac{9}{64}\big(|\xi|^3+|\mu|^3\big).
\end{aligned}$$

On the other hand, direct computation gives

$$\begin{aligned}
&\sum_{i=1}^{2}\Theta_i(\xi,\Gamma\xi)\,\delta^i(\xi,\Gamma\xi)\\
&=\left(1+\frac{5|\xi|}{4}\right)\frac{3|\xi|}{4}+\left(1+\frac{3|\xi|}{4}\right)\frac{9|\xi|^2}{16}\\
&=\frac{3|\xi|}{4}+\frac{3}{2}|\xi|^2+\frac{27}{64}|\xi|^3.
\end{aligned}$$

The analogous identity holds with ξ replaced by μ. Take $\rho = \frac{1}{3}$. Combining the above estimates, we obtain

$$\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\, \delta^i(\Gamma\xi, \Gamma\mu)$$
$$\leq \rho \sum_{i=1}^{2} \left[\Theta_i(\xi, \Gamma\xi)\delta^i(\xi, \Gamma\xi) + \Theta_i(\mu, \Gamma\mu)\delta^i(\mu, \Gamma\mu)\right].$$

Thus, Γ is a Kannan polynomial contraction with $\kappa > 1$ and non-constant coefficients. Thus, Γ satisfies (2), and by Theorem (4) it admits a unique fixed point, namely 0.

3. Polynomial Contractions on *b*-Metric Spaces

In this section, we examine the notion of polynomial contractions with respect to *b*-metric spaces. The purpose of this investigation is to broaden the traditional fixed point theorems by applying more general contractive conditions suitable for *b*-metric spaces.

We now proceed with the definition of polynomial contractions in relation to *b*-metric spaces.

Definition 4. *Let (Ξ, δ) be a b-metric space with $s \geq 1$ and $\Gamma : \Xi \longrightarrow \Xi$ satisfies*

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi, \mu)\delta^i(\xi, \mu), \tag{8}$$

where $0 < \rho < 1, \rho s \leq 1$ and $\Theta_i(\xi, \mu) : \Xi \times \Xi \to (0, \infty)$ for all $i = 1, \ldots \kappa$. Then, we say Γ satisfies the Polynomial Contraction in b-metric space.

In the next theorem, we will prove the existence and uniqueness of a fixed point on *b*-metric space which satisfies (8).

Theorem 5. *Let (Ξ, δ) be a complete b-metric space and $\Gamma : \Xi \longrightarrow \Xi$ satisfies (8). Assume that the following conditions hold:*

(i) *Γ is continuous,*

(ii) *There exists $j \in \{1, \ldots \kappa\}$ and $\mathfrak{A}_j > 0$ such that $\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu)$ for all $\xi, \mu \in \Xi$ and $s\rho^{1/j} < 1$.*

Then Γ has a unique fixed point.

Proof. Let (ξ_0) be an element in Ξ, then the iteration sequence defined by $\xi_n = \Gamma^n(\xi_0)$

Now, apply the inequality (8) for (ξ_0, ξ_1), we get

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0, \Gamma\xi_1)\delta^i(\Gamma\xi_0, \Gamma\xi_1) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1),$$
$$\text{then, } \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{9}$$

Applying inequality (8) for (ξ_1, ξ_2), we get:

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_1, \Gamma\xi_2)\delta^i(\Gamma\xi_1, \Gamma\xi_2) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2),$$
$$\text{hence, } \sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1, \xi_2)\delta^i(\xi_1, \xi_2).$$

By using (9), we have:

$$\sum_{i=1}^{\kappa} \Theta_i(\xi_2, \xi_3)\delta^i(\xi_2, \xi_3) \leqslant \rho^2 \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{10}$$

By the same process and by applying (8) for (ξ_{n-1}, ξ_n), we get the following:

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_{n-1}, \Gamma\xi_n)\delta^i(\Gamma\xi_{n-1}, \Gamma\xi_n) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_{n-1}, \xi_n)\delta^i(\xi_{n-1}, \xi_n),$$

$$\text{hence, } \sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1}) \leqslant \rho^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{11}$$

Also, from (i), there exists $\mathfrak{A}_j > 0$ such that

$$0 < \mathfrak{A}_j \leqslant \Theta_i(\xi_n, \xi_{n+1})$$

This implies:

$$\mathfrak{A}_j\delta^j(\xi_n, \xi_{n+1}) \leqslant \Theta_j(\xi_n, \xi_{n+1})\delta^j(\xi_n, \xi_{n+1}) \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1})$$
$$\leqslant \rho^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1).$$

Assume $\Omega_{j,0} = \mathfrak{A}_j^{-1} \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1)$.
So,

$$\delta(\xi_n, \xi_{n+1}) \leqslant \rho^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}}. \tag{12}$$

Now, we will prove (ξ_n) is a Cauchy sequence.
Take $n, m \in \mathbb{N}$

$$\begin{aligned}
\delta(\xi_n, \xi_{n+m}) &\leqslant s[\delta(\xi_n, \xi_{n+1}) + \delta(\xi_{n+1}, \xi_{n+m})] \\
&\leqslant s\delta(\xi_n, \xi_{n+1}) + s\delta(\xi_{n+1}, \xi_{n+m}) \\
&\leqslant s\delta(\xi_n, \xi_{n+1}) + s^2[\delta(\xi_{n+1}, \xi_{n+2}) + \delta(\xi_{n+2}, \xi_{n+m})] \\
&\leqslant s\delta(\xi_n, \xi_{n+1}) + s^2\delta(\xi_{n+1}, \xi_{n+2}) + \cdots + s^m\delta(\xi_{n+m-1}, \xi_{n+m}) \\
&\leqslant s\rho^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} + s^2\rho^{\frac{n+1}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} + \cdots + s^m\rho^{\frac{n+m-1}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} \\
&\leqslant s\rho^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} \left[1 + s\rho^{\frac{1}{j}} + s^2\rho^{\frac{2}{j}} + \cdots + s^{m-1}\rho^{\frac{m-1}{j}}\right].
\end{aligned}$$

Since $s\rho^{1/j} < 1$, the geometric sum is bounded above by $(1-s\rho^{1/j})^{-1}$. Hence, $\delta(\xi_n, \xi_{n+m}) \to 0$ as $n \to \infty$, uniformly in $m \geq 1$, which shows that (ξ_n) is a Cauchy sequence.
By completeness there exists $\xi_0 \in \Xi$ such that $\delta(\xi_n, \xi_0) \to 0$ as $n \longrightarrow \infty$.
Now. we will show that ξ_0 is a fixed point.
By the continuity of Γ, then $\lim_{n\to\infty} \delta(\Gamma\xi_n, \Gamma\xi_0) = \lim_{n\to\infty} \delta(\xi_{n+1}, \Gamma\xi_0) = 0$.
By the uniqueness of the limit, we have $\Gamma(\xi_0) = \xi_0$.
To prove the uniqueness, let $\mu_0 \in \Xi$ where $\Gamma\mu_0 = \mu_0$.
Now, apply (8) for (ξ_0, μ_0),

$$0 \leqslant \sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0, \Gamma\mu_0)\delta^i(\Gamma\xi_0, \Gamma\mu_0) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0),$$
$$\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) \leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0).$$

$$\text{Hence, } (1-\rho) \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) \leqslant 0.$$

Thus, $(1-\rho) \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) = 0$. As $(1-\rho > 0)$, we conclude that $\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) = 0$.
Since there exist $j \in \{1, 2, \cdots \kappa\}$ and $\mathfrak{A}_j > 0$ such that:

$$\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu), \quad \text{for all } \xi, \mu \in \Xi.$$

Therefore,

$$\mathfrak{A}_j \cdot \delta^j(\xi_0, \mu_0) \leqslant \Theta_j(\xi_0, \mu_0)\delta^j(\xi_0, \mu_0) \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) = 0,$$
$$\text{therefore, } \delta^j(\xi_0, \mu_0) = 0 \text{ for some } j, \text{ which implies } \delta(\xi_0, \mu_0) = 0,$$

which proves the uniqueness. □

Following the same approach mentioned in [20] throughout this work, we will show that the mapping is continuous as long as it satisfies a polynomial contraction on b-metric space with extra conditions. This extends well known continuity results related to mappings of contractive type and shows the continuity of such operators in the class of polynomial contractions and the stability of these operators in their structural nature.

Proposition 1. *Let (Ξ, δ) be a b-metric space and $\Gamma : \Xi \to \Xi$ be a polynomial contraction satisfied (8). Assume that the following conditions hold:*

(i) For all $i \in \{1, \cdots, \kappa\}$, there exists $\mathfrak{B}_i > 0$ such that

$$\Theta_i(\xi, \mu) \leq \mathfrak{B}_i, \quad \xi, \mu \in \Xi$$

(ii) There exists $j \in \{1, \ldots \kappa\}$ and $\mathfrak{A}_j > 0$ such that $\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu)$ for all $\xi, \mu \in \Xi$ and $s\rho^{1/j} < 1$.

Then Γ is continuous.

Proof. Let $\{u_n\} \subset \Xi$ be a sequence such that

$$\lim_{n\to\infty} \delta(u_n, u) = 0, \quad \text{for some } u \in \Xi. \tag{13}$$

Applying the polynomial contraction (8) with

$$(\xi, \mu) = (u_n, u),$$

we get

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma u_n, \Gamma u)\delta^i(\Gamma u_n, \Gamma u) \leq \rho \sum_{i=1}^{\kappa} \Theta_i(u_n, u)\delta^i(u_n, u),$$

which implies by (i) and (ii) that

$$\mathfrak{A}_j\delta^j(\Gamma u_n, \Gamma u) \leq \rho \sum_{i=1}^{k} \mathfrak{B}_i\delta^i(u_n, u), \quad n \geq 0. \tag{14}$$

Then, using (13) and passing to the limit as $n \to \infty$ in (14), we get:

$$\lim_{n\to\infty} \delta^j(\Gamma u_n, \Gamma u) = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \delta(\Gamma u_n, \Gamma u) = 0.$$

This shows that Γ is a continuous mapping. □

From Theorem (5) and Proposition (1), we deduce the following result.

Corollary 2. *Let (Ξ, δ) be a complete b-metric space and $\Gamma : \Xi \to \Xi$ be a polynomial contraction. Assume that the following conditions hold:*

(i) For all $i \in \{1, \cdots, \kappa\}$, there exists $\mathfrak{B}_i > 0$ such that

$$\Theta_i(\xi, \mu) \leq \mathfrak{B}_i, \quad \xi, \mu \in \Xi.$$

(ii) There exist $j \in \{1, \cdots, \kappa\}$ *and* $\mathfrak{A}_j > 0$ *such that:*

$$\Theta_j(\xi, \mu) \geq \mathfrak{A}_j, \quad \xi, \mu \in \Xi,$$

and $s\rho^{1/j} < 1$.

Then Γ *admits a unique fixed point* $\zeta^* \in \Xi$. *Moreover, for every* $\zeta_0 \in \Xi$, *the Picard sequence* $\{\zeta_n\} \subset \Xi$ *defined by* $\zeta_{n+1} = \Gamma\zeta_n$ *for all* $n \geq 0$, *converges to* ζ^*.

The following corollary can be made from the result above.

Corollary 3. *Let* (Ξ, δ) *be a complete b-metric space and* $\Gamma : \Xi \to \Xi$ *be a given mapping. Assume that there exist* $\rho \in (0, 1)$ *such that and* $s\rho^{1/j} < 1$ *for some* $j \in \{1, \cdots, \kappa\}$, *a natural number* $\kappa \geq 1$ *and a finite sequence* $\{\Theta_i\}_{i=1}^{\kappa} \subset (0, \infty)$ *such that:*

$$\sum_{i=1}^{\kappa} \Theta_i \delta^i(\Gamma\xi, \Gamma\mu) \leq \rho \sum_{i=1}^{\kappa} \Theta_i \delta^i(\xi, \mu), \tag{15}$$

for every $\xi, \mu \in \Xi$. *Then* Γ *admits a unique fixed point* $\zeta^* \in \Xi$. *Moreover, for every* $\zeta_0 \in \Xi$, *the Picard sequence* $\{\zeta_n\} \subset \Xi$ *defined by* $\zeta_{n+1} = \Gamma\zeta_n$ *for all* $n \geq 0$, *converges to* ζ^*.

More specifically, in the case where $\kappa = 1$ and $\Theta_1 = 1$, the polynomial contraction condition becomes degenerative in nature, as previously dealt with in Swati Agrawal, K. Qureshi and Jyoti Nema's A Fixed Point Theorem For *b*-Metric Space ([22], 2016). Thus, their work is a particular case of the result we are providing.

The following example illustrates the applicability of the theorem (5) to polynomial contractions in *b*-metric spaces.

Example 3. *Let* $\Xi = \mathbb{R}$ *and define the b-metric*

$$\delta(\xi, \mu) = |\xi - \mu|^2.$$

Then (Ξ, δ) *is a b-metric space with coefficient* $s = 2$, *since*

$$\delta(\xi, \zeta) = |\xi - \zeta|^2 \leq 2\big(|\xi - \mu|^2 + |\mu - \zeta|^2\big) = s\big(\delta(\xi, \mu) + \delta(\mu, \zeta)\big).$$

Define

$$\Gamma(\xi) = \frac{\xi}{4}.$$

Let

$$\kappa = 1$$

and choose $\Theta_1(\xi, \mu) = 1$.

Then

$$\delta(\Gamma\xi, \Gamma\mu) = \left|\frac{\xi - \mu}{4}\right|^2 = \frac{1}{16}\,\delta(\xi, \mu).$$

The polynomial contraction condition (8) becomes

$$\Theta_1(\Gamma\xi, \Gamma\mu)\,\delta(\Gamma\xi, \Gamma\mu) \leq \rho\,\Theta_1(\xi, \mu)\,\delta(\xi, \mu),$$

which holds with $\rho = \frac{1}{16} < \frac{1}{s} = \frac{1}{2}$. *Thus* Γ *satisfies (8), and by Theorem (5) it has a unique fixed point, namely* 0.

While the preceding example uses constant coefficient mappings, the following examples presents a polynomial contraction in a *b*-metric space with nonconstant coefficient mappings and $\kappa > 1$.

Example 4. *Let* (Ξ, δ) *be the metric space* $\Xi = \mathbb{R}$ *with* $\delta(x, y) = |x - y|$. *Define*

$$\Gamma(\xi) = \frac{\xi}{4}.$$

Take $\kappa = 2$ *and*

$$\Theta_1(\xi, \mu) = 1 + |\xi| + |\mu|, \qquad \Theta_2(\xi, \mu) = 1 + |\xi - \mu|.$$

Then

$$\delta(\Gamma\xi, \Gamma\mu) = \frac{1}{4}\,\delta(\xi, \mu), \qquad \delta^2(\Gamma\xi, \Gamma\mu) = \frac{1}{16}\,\delta^2(\xi, \mu).$$

Hence,

$$\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\,\delta^i(\Gamma\xi, \Gamma\mu) = \Theta_1(\Gamma\xi, \Gamma\mu)\,\delta(\Gamma\xi, \Gamma\mu) + \Theta_2(\Gamma\xi, \Gamma\mu)\,\delta^2(\Gamma\xi, \Gamma\mu).$$

We estimate

$$\Theta_1(\Gamma\xi, \Gamma\mu) = 1 + \left|\tfrac{\xi}{4}\right| + \left|\tfrac{\mu}{4}\right| \le 1 + \tfrac{1}{4}(|\xi| + |\mu|) \le \Theta_1(\xi, \mu),$$

and

$$\Theta_2(\Gamma\xi, \Gamma\mu) = 1 + \left|\tfrac{\xi}{4} - \tfrac{\mu}{4}\right| = 1 + \tfrac{1}{4}|\xi - \mu| \le 1 + |\xi - \mu| = \Theta_2(\xi, \mu).$$

Therefore

$$\Theta_1(\Gamma\xi, \Gamma\mu)\,\delta(\Gamma\xi, \Gamma\mu) + \Theta_2(\Gamma\xi, \Gamma\mu)\,\delta^2(\Gamma\xi, \Gamma\mu) \le \frac{1}{4}\Theta_1(\xi, \mu)\,\delta(\xi, \mu) + \frac{1}{16}\Theta_2(\xi, \mu)\,\delta^2(\xi, \mu).$$

If we choose

$$\rho = \frac{1}{4},$$

then

$$\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\,\delta^i(\Gamma\xi, \Gamma\mu) \le \rho \sum_{i=1}^{2} \Theta_i(\xi, \mu)\,\delta^i(\xi, \mu),$$

Hence, Γ *is a polynomial contraction in the b-metric space with* $\kappa > 1$ *and nonconstant coefficient mappings. Since* Γ *satisfies (8), Theorem (5) ensures that* Γ *has a unique fixed point, namely* 0.

Example 5. *Let* $\Xi = \mathbb{R}$ *and*

$$\delta(\xi, \mu) = |\xi - \mu|^2,$$

so (Ξ, δ) *is a b-metric space with coefficient* $s = 2$.

Define

$$\Gamma(\xi) = \frac{\xi}{5}.$$

Take $\kappa = 3$ *and*

$$\Theta_1(\xi, \mu) = 1 + |\xi|, \qquad \Theta_2(\xi, \mu) = 1 + |\mu|, \qquad \Theta_3(\xi, \mu) = 1 + |\xi - \mu|.$$

Then

$$\delta(\Gamma\xi, \Gamma\mu) = \left|\tfrac{\xi - \mu}{5}\right|^2 = \frac{1}{25}\delta(\xi, \mu), \quad \delta^2(\Gamma\xi, \Gamma\mu) = \frac{1}{625}\delta^2(\xi, \mu), \quad \delta^3(\Gamma\xi, \Gamma\mu) = \frac{1}{15625}\delta^3(\xi, \mu).$$

Moreover,

$$\Theta_1(\Gamma\xi, \Gamma\mu) = 1 + \left|\tfrac{\xi}{5}\right| \le 1 + |\xi| = \Theta_1(\xi, \mu),$$

$$\Theta_2(\Gamma\xi, \Gamma\mu) = 1 + \left|\tfrac{\mu}{5}\right| \le 1 + |\mu| = \Theta_2(\xi, \mu),$$

$$\Theta_3(\Gamma\xi, \Gamma\mu) = 1 + \left|\tfrac{\xi}{5} - \tfrac{\mu}{5}\right| = 1 + \tfrac{1}{5}|\xi - \mu| \le 1 + |\xi - \mu| = \Theta_3(\xi, \mu).$$

Thus,

$$\sum_{i=1}^{3} \Theta_i(\Gamma\xi,\Gamma\mu)\,\delta^i(\Gamma\xi,\Gamma\mu) \leq \frac{1}{25}\Theta_1(\xi,\mu)\delta(\xi,\mu) + \frac{1}{625}\Theta_2(\xi,\mu)\delta^2(\xi,\mu) + \frac{1}{15625}\Theta_3(\xi,\mu)\delta^3(\xi,\mu).$$

Choosing

$$\rho = \frac{1}{25} < \frac{1}{s} = \frac{1}{2},$$

we obtain

$$\sum_{i=1}^{3} \Theta_i(\Gamma\xi,\Gamma\mu)\,\delta^i(\Gamma\xi,\Gamma\mu) \leq \rho \sum_{i=1}^{3} \Theta_i(\xi,\mu)\,\delta^i(\xi,\mu),$$

Hence, Γ is a polynomial contraction in the b-metric space with $\kappa > 1$ and nonconstant coefficient mappings. Since Γ satisfies (8), Theorem 5 ensures that Γ has a unique fixed point, namely 0.

Next, we define the Kannan Polynomial Contraction in *b*-metric spaces and establish the existence and uniqueness of a fixed point for mappings satisfying such contractions.

Definition 5. *Let (Ξ,δ) be a b-metric space, and let $\Gamma : \Xi \longrightarrow \Xi$ satisfy*

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi,\Gamma\mu)\delta^i(\Gamma\xi,\Gamma\mu) \leqslant \rho \sum_{i=1}^{\kappa}\big[\Theta_i(\xi,\Gamma\xi)\delta^i(\xi,\Gamma\xi) + \Theta_i(\mu,\Gamma\mu)\delta^i(\mu,\Gamma\mu)\big], \tag{16}$$

where $0 < \rho < \frac{1}{2}$; $s\rho < 1$; $\Theta_i(\xi,\mu) : \Xi \times \Xi \to (0,\infty)$ for all $i = 1, \dots \kappa$.

Then, we say Γ satisfies Kannan Polynomial Contraction in b-metric spaces.

Theorem 6. *Let (Ξ,δ) be a complete b-metric space $\Gamma : \Xi \longrightarrow \Xi$ satisfies (16): and assume:*

(i) *Γ is continuous,*

(ii) *There exist $j \in \{1, \dots \kappa\}$ and $\mathfrak{A}_j > 0$ such that $\mathfrak{A}_j \leqslant \Theta_j(\xi,\mu)$ for all $\xi,\mu \in \Xi$ and $s^j\frac{\rho}{1-\rho} < 1$.*

Then Γ has a unique fixed point.

Proof. Let (ξ_0) be an element in Ξ, then the iteration sequence defined by $\xi_n = \Gamma^n(\xi_0)$

Now, apply the inequality (16) for (ξ_0,ξ_1), we get

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0,\Gamma\xi_1)\delta^i(\Gamma\xi_0,\Gamma\xi_1) \leqslant \rho \sum_{i=1}^{\kappa}[\Theta_i(\xi_0,\Gamma\xi_0)\delta^i(\xi_0,\Gamma\xi_0) + \Theta_i(\xi_1,\Gamma\xi_1)\delta^i(\xi_1,\Gamma\xi_1)].$$

$$\text{Therefore, } \sum_{i=1}^{\kappa} \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2) \leqslant \rho \sum_{i=1}^{\kappa}[\Theta_i(\xi_0,\xi_1)\delta^i(\xi_0,\xi_1) + \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2)]$$

$$\leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_0,\xi_1)\delta^i(\xi_0,\xi_1) + \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2),$$

$$\text{so, } \sum_{i=1}^{\kappa} \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2) \leqslant \frac{\rho}{1-\rho} \sum_{i=1}^{\kappa} \Theta_i(\xi_0,\xi_1)\delta^i(\xi_0,\xi_1). \tag{17}$$

Applying inequality (16) for (ξ_1,ξ_2), we get:

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_1,\Gamma\xi_2)\delta^i(\Gamma\xi_1,\Gamma\xi_2) \leqslant \rho \sum_{i=1}^{\kappa}[\Theta_i(\xi_1,\Gamma\xi_1)\delta^i(\xi_1,\Gamma\xi_1) + \Theta_i(\xi_2,\Gamma\xi_2)\delta^i(\xi_2,\Gamma\xi_2)]$$

$$\text{so, } \sum_{i=1}^{\kappa} \Theta_i(\xi_2,\xi_3)\delta^i(\xi_2,\xi_3) \leqslant \rho \sum_{i=1}^{\kappa}[\Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2) + \Theta_i(\xi_2,\xi_3)\delta^i(\xi_2,\xi_3)]$$

$$\leqslant \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2) + \rho \sum_{i=1}^{\kappa} \Theta_i(\xi_2,\xi_3)\delta^i(\xi_2,\xi_3)$$

$$\text{Therefore, } \sum_{i=1}^{\kappa} \Theta_i(\xi_2,\xi_3)\delta^i(\xi_2,\xi_3) \leqslant \frac{\rho}{1-\rho} \sum_{i=1}^{\kappa} \Theta_i(\xi_1,\xi_2)\delta^i(\xi_1,\xi_2).$$

By using (17), we have:

$$\text{Then, } \sum_{i=1}^{\kappa} \Theta_i(\xi_2,\xi_3)\delta^i(\xi_2,\xi_3) \leqslant \left(\frac{\rho}{1-\rho}\right)^2 \sum_{i=1}^{\kappa} \Theta_i(\xi_0,\xi_1)\delta^i(\xi_0,\xi_1). \tag{18}$$

By the same process and by applying (16) for (ξ_{n-1}, ξ_n), we get the following:

$$\sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_{n-1}, \Gamma\xi_n)\delta^i(\Gamma\xi_{n-1}, \Gamma\xi_n) \leqslant \rho \sum_{i=1}^{\kappa} \left[\Theta_i(\xi_{n-1}, \Gamma(\xi_{n-1}))\delta^i(\xi_{n-1}, \Gamma(\xi_{n-1})) + \Theta_i(\xi_n, \Gamma(\xi_n))\delta^i(\xi_n, \Gamma(\xi_n))\right],$$

$$\text{hence, } \sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1}) \leqslant \left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1). \tag{19}$$

From (ii), there exists $\mathfrak{A}_j > 0$ such that

$$0 < \mathfrak{A}_j \leqslant \theta_i(\xi_n, \xi_{n+1}).$$

Hence,

$$\begin{aligned}
\mathfrak{A}_j\delta^j(\xi_n, \xi_{n+1}) &\leqslant \Theta_j(\xi_n, \xi_{n+1})\delta(\xi_n, \xi_{n+1}) \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_n, \xi_{n+1})\delta^i(\xi_n, \xi_{n+1}) \\
&\leqslant \left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1), \\
\text{so, } \delta^j(\xi_n, \xi_{n+1}) &\leqslant \mathfrak{A}_j^{-1}\left(\frac{\rho}{1-\rho}\right)^n \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1).
\end{aligned}$$

Assume $\Omega_{j,0} = \mathfrak{A}_j^{-1} \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_1)\delta^i(\xi_0, \xi_1)$.

So,

$$\delta(\xi_n, \xi_{n+1}) \leqslant \left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} \tag{20}$$

Now, we will prove (ξ_n) is a Cauchy sequence. Take $n, m \in \mathbb{N}$

$$\begin{aligned}
\delta(\xi_n, \xi_{n+m}) &\leqslant s\delta(\xi_n, \xi_{n+1}) + s\delta(\xi_{n+1}, \xi_{n+m}) \\
&\leqslant s\delta(\xi_n, \xi_{n+1}) + s^2\delta(\xi_{n+1}, \xi_{n+2}) + s^2\delta(\xi_{n+2}, \xi_{n+m}) \\
&\leqslant s\delta(\xi_n, \xi_{n+1}) + s^2\delta(\xi_{n+1}, \xi_{n+2}) + s^3\delta(\xi_{n+1}, \xi_{n+3}) \cdots + s^m\delta(\xi_{n+m-1}, \xi_{n+m}) \\
&\leqslant s\left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} + s^2\left(\frac{\rho}{1-\rho}\right)^{\frac{n+1}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} + s^3\left(\frac{\rho}{1-\rho}\right)^{\frac{n+2}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} + \cdots + \left(\frac{\rho}{1-\rho}\right)^{\frac{n+m-1}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} \\
&\leqslant s\left(\frac{\rho}{1-\rho}\right)^{\frac{n}{j}} \left(\Omega_{j,0}\right)^{\frac{1}{j}} \left[1 + s\left(\frac{\rho}{1-\rho}\right)^{\frac{1}{j}} + s^2\left(\frac{\rho}{1-\rho}\right)^{\frac{2}{j}} + \cdots + s^{m-1}\left(\frac{\rho}{1-\rho}\right)^{\frac{m-1}{j}}\right].
\end{aligned}$$

Since $s^j\frac{\rho}{1-\rho} < 1$, taking the root j-th yields $s\left(\frac{\rho}{1-\rho}\right)^{1/j} < 1$.

Therefore, the geometric sum is bounded above by $\left(1 - s\left(\frac{\rho}{1-\rho}\right)^{1/j}\right)^{-1}$.

Hence, $\delta(\xi_n, \xi_{n+m}) \to 0$ as $n \to \infty$, uniformly in $m \geq 1$, which shows that (ξ_n) is a Cauchy sequence.

By completeness there exists $\xi_0 \in \Xi$ such that $\delta(\xi_n, \xi_0) \to 0$ as $n \longrightarrow \infty$

Now, we will prove ξ_0 to be a fixed point.

By the continuity of Γ, then $\lim_{n\to\infty} \delta(\Gamma\xi_n, \Gamma\xi_0) = \lim_{n\to\infty} \delta(\xi_{n+1}, \Gamma\xi_0) = 0$. By the uniqueness of the limit, we have $\Gamma\xi_0 = \xi_0$.

To prove the uniqueness, assume $\mu_0 \in \Xi$ where $\Gamma\mu_0 = \mu_0$.

Now, apply (16) for (ξ_0, μ_0):

$$0 \leqslant \sum_{i=1}^{\kappa} \Theta_i(\Gamma\xi_0, \Gamma\mu_0)\delta^i(\Gamma\xi_0, \Gamma\mu_0) \leqslant \rho\left[\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \Gamma\xi_0)\delta^i(\xi_0, \Gamma\xi_0) + \sum_{i=1}^{\kappa} \Theta_i(\mu_0, \Gamma\mu_0)\delta^i(\mu_0, \Gamma\mu_0)\right],$$

$$\text{therefore, } 0 \leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) \leqslant \rho\left[\sum_{i=1}^{\kappa} \Theta_i(\xi_0, \xi_0)\delta^i(\xi_0, \xi_0) + \sum_{i=1}^{\kappa} \Theta_i(\mu_0, \mu_0)\delta^i(\mu_0, \mu_0)\right] = 0.$$

Since there exist $j \in \{1, 2, \cdots \kappa\}$ and $\mathfrak{A}_j > 0$ such that:

$$\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu), \quad \text{for all } \xi, \mu \in \Xi.$$

Therefore,

$$\begin{aligned}\mathfrak{A}_j \cdot \delta^j(\xi_0, \mu_0) &\leqslant \Theta_j(\xi_0, \mu_0)\delta^j(\xi_0, \mu_0) \\ &\leqslant \sum_{i=1}^{\kappa} \Theta_i(\xi_0, \mu_0)\delta^i(\xi_0, \mu_0) = 0.\end{aligned}$$

Then, $\delta^j(\xi_0, \mu_0) = 0$ which implies, $\delta(\xi_0, \mu_0) = 0$.

which proves the uniqueness. □

From Theorem (6), we deduce the following result.

Corollary 4. *Let (Ξ, δ) be a complete b-metric space and $\Gamma : \Xi \to \Xi$ be a given mapping. Assume that there exist $\rho \in (0, \frac{1}{2})$, a natural number $\kappa \geq 1$ and a finite sequence $\{\theta_i\}_{i=1}^{\kappa} \subset (0, \infty)$ such that:*

$$\sum_{i=1}^{\kappa} \Theta_i \delta^i(\Gamma\xi, \Gamma\mu) \leqslant \rho \sum_{i=1}^{\kappa} \left[\Theta_i \delta^i(\xi, \Gamma\xi) + \Theta_i \delta^i(\mu, \Gamma\mu)\right], \text{ for every } \xi, \mu \in \Xi. \tag{21}$$

In addition, it satisfies the following:

(i) *Γ is continuous.*

(ii) *There exist $j \in \{1, \ldots \kappa\}$ and $\mathfrak{A}_j > 0$ such that $\mathfrak{A}_j \leqslant \Theta_j(\xi, \mu)$ for all $\xi, \mu \in \Xi$ and $s^j \frac{\rho}{1-\rho} < 1$.*

Then Γ admits a unique fixed point $\zeta^* \in \Xi$. Moreover, for every $\zeta_0 \in \Xi$, the Picard sequence $\{\zeta_n\} \subset \Xi$ defined by $\zeta_{n+1} = \Gamma\zeta_n$ for all $n \geq 0$, converges to ζ^*.

Also, following the previous result, in the specific case where $\kappa = 1$ and $\Theta_1 = 1$, the polynomial contraction condition becomes a Kannan-type contraction framework in *b*-metric spaces.

The following example illustrates the applicability of Theorem (6) to Kannan polynomial contractions in *b*-metric spaces.

Example 6. *Let $\Xi = \mathbb{R}$ with the b-metric $\delta(\xi, \mu) = |\xi - \mu|^2$ (coefficient $s = 2$). Define*

$$\Gamma(\xi) = \frac{\xi}{5}.$$

Let $\kappa = 1$ and $\Theta_1(\xi, \mu) = 1$. Then

$$\delta(\Gamma\xi, \Gamma\mu) = \left|\frac{\xi - \mu}{5}\right|^2 = \frac{1}{25}\, d(\xi, \mu).$$

The Kannan-type polynomial contraction (16) becomes

$$\delta(\Gamma\xi, \Gamma\mu) \leq \rho\big(\delta(\xi, \Gamma\xi) + \delta(\mu, \Gamma\mu)\big), \qquad 0 < \rho < \tfrac{1}{2}, \quad s\rho < 1.$$

We compute

$$\delta(\xi, \Gamma\xi) = |\xi - \tfrac{\xi}{5}|^2 = \left(\tfrac{4}{5}|\xi|\right)^2 = \tfrac{16}{25}|\xi|^2,$$

and similarly for $\delta(\mu, \Gamma\mu)$. Since $|\xi - \mu|^2 \leq 2(|\xi|^2 + |\mu|^2)$, we obtain

$$\frac{1}{25}|\xi - \mu|^2 \leq \frac{2}{25}(|\xi|^2 + |\mu|^2) = \rho\big(\tfrac{16}{25}|\xi|^2 + \tfrac{16}{25}|\mu|^2\big)$$

with $\rho = \frac{1}{8}$, which satisfies $\rho < \frac{1}{2}$ and $s\rho = 2 \cdot \frac{1}{8} = \frac{1}{4} < 1$. Thus, Γ satisfies (16), and Theorem (6) guaranties a unique fixed point 0.

4. Applications of Polynomial Contractions

In this section, we present several applications of the fixed point results established in Sections 2 and 3. These applications demonstrate how Kannan polynomial contractions and polynomial contractions in b-metric spaces naturally arise in nonlinear analysis, integral equations, differential equations, and iterative methods.

Application to Nonlinear Integral Equations.

Consider the nonlinear integral equation

$$\xi(\tau) = \int_0^1 K(\tau, \eta)\, g(\xi(\eta))\, d\eta, \qquad \tau \in [0,1],$$

where $K : [0,1]^2 \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$|g(\omega) - g(\nu)| \leq \alpha|\omega - \nu| + \beta|\omega - \nu|^2, \qquad \omega, \nu \in \mathbb{R},$$

for some constants $\alpha, \beta \geq 0$.

Let $\Xi = C([0,1], \mathbb{R})$ and define

$$\delta(\xi, \mu) = \|\xi - \mu\|_\infty = \sup_{\tau \in [0,1]} |\xi(\tau) - \mu(\tau)|.$$

Since Ξ is a Banach space under the supremum norm, (Ξ, δ) is a complete metric space. Define the integral operator $\Gamma : \Xi \to \Xi$ by

$$(\Gamma\xi)(\tau) = \int_0^1 K(\tau, \eta)\, g(\xi(\eta))\, d\eta.$$

The assumed inequality implies that g is continuous. Consequently, the continuity of K ensures that $\Gamma\xi \in \Xi$ for every $\xi \in \Xi$.

Set

$$M = \sup_{\tau \in [0,1]} \int_0^1 |K(\tau, \eta)|\, d\eta$$

and assume that

$$M \max\{\alpha, \beta\} < 1.$$

For any $\xi, \mu \in \Xi$, we have

$$\begin{aligned} |(\Gamma\xi)(\tau) - (\Gamma\mu)(\tau)| &\leq \int_0^1 |K(\tau, \eta)|\, |g(\xi(\eta)) - g(\mu(\eta))|\, d\eta \\ &\leq \int_0^1 |K(\tau, \eta)| \big(\alpha|\xi(\eta) - \mu(\eta)| + \beta|\xi(\eta) - \mu(\eta)|^2\big) d\eta. \end{aligned}$$

Taking the supremum over $\tau \in [0,1]$ gives

$$\delta(\Gamma\xi, \Gamma\mu) \leq M\alpha\, \delta(\xi, \mu) + M\beta\, \delta^2(\xi, \mu).$$

To establish the required polynomial contraction, we first derive a sharper estimate for g. Fix $\omega, \nu \in \mathbb{R}$ and $n \in \mathbb{N}$. Divide the interval joining ν and ω into n equal parts by setting

$$z_k = \nu + \frac{k}{n}(\omega - \nu), \qquad k = 0, \ldots, n.$$

Then $z_0 = \nu$, $z_n = \omega$, and

$$|z_k - z_{k-1}| = \frac{|\omega - \nu|}{n}, \qquad k = 1, \ldots, n.$$

Since

$$g(\omega) - g(\nu) = \sum_{k=1}^{n} \big(g(z_k) - g(z_{k-1})\big),$$

the triangle inequality and the assumed condition on g give

$$\begin{aligned}|g(\omega) - g(\nu)| &\leq \sum_{k=1}^{n} |g(z_k) - g(z_{k-1})| \\ &\leq \sum_{k=1}^{n} \left(\alpha \frac{|\omega - \nu|}{n} + \beta \frac{|\omega - \nu|^2}{n^2} \right) \\ &= \alpha|\omega - \nu| + \frac{\beta}{n}|\omega - \nu|^2.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$|g(\omega) - g(\nu)| \leq \alpha|\omega - \nu|.$$

Hence

$$\delta(\Gamma\xi, \Gamma\mu) \leq q\,\delta(\xi, \mu), \qquad q = M\alpha < 1.$$

Choose $\rho \in (0, 1)$ such that $\rho = M \max\{\alpha, \beta\}$. Since $q \leq \rho < 1$,

$$\begin{aligned}\delta(\Gamma\xi, \Gamma\mu) + \delta^2(\Gamma\xi, \Gamma\mu) &\leq q\,\delta(\xi, \mu) + q^2\,\delta^2(\xi, \mu) \\ &\leq \rho\big(\delta(\xi, \mu) + \delta^2(\xi, \mu)\big).\end{aligned}$$

Thus, Γ satisfies the polynomial contraction condition with constant coefficient mappings $\Theta_1 = \Theta_2 = 1$. By Theorem (5), Γ has a unique fixed point $\xi^* \in \Xi$, which is the unique continuous solution of the integral equation.

The following example provides a further application of Theorem (5) to nonlinear differential equations through the associated Picard operator.

Application to an Initial Value Problem.

Consider the initial value problem

$$\xi'(\tau) = f(\tau, \xi(\tau)), \qquad \xi(0) = \xi_0, \qquad \tau \in [0, T],$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume that there exist constants $L_1, L_2 \geq 0$ such that

$$|f(\tau, \omega) - f(\tau, \nu)| \leq L_1|\omega - \nu| + L_2|\omega - \nu|^2$$

for all $\tau \in [0, T]$ and $\omega, \nu \in \mathbb{R}$, and choose $T > 0$ so that $TL_1 < 1$.

Let $\Xi = C([0, T], \mathbb{R})$, equipped with the supremum metric

$$\delta(\xi, \mu) = \|\xi - \mu\|_\infty = \max_{\tau \in [0,T]} |\xi(\tau) - \mu(\tau)|.$$

Then (Ξ, δ) is a complete metric space and, consequently, a complete b-metric space with coefficient $s = 1$.

Define the Picard operator $\Gamma : \Xi \to \Xi$ by

$$(\Gamma\xi)(\tau) = \xi_0 + \int_0^\tau f(\eta, \xi(\eta))\, d\eta.$$

The continuity of f ensures that Γ maps Ξ into itself.

For fixed $\tau \in [0, T]$, $\omega, \nu \in \mathbb{R}$, $n \in \mathbb{N}$, and by the triangle inequality,

$$\begin{aligned}|f(\tau, \omega) - f(\tau, \nu)| &\leq \sum_{k=1}^{n} |f(\tau, z_k) - f(\tau, z_{k-1})| \\ &\leq \sum_{k=1}^{n} \left(L_1 \frac{|\omega - \nu|}{n} + L_2 \frac{|\omega - \nu|^2}{n^2} \right) \\ &= L_1|\omega - \nu| + \frac{L_2}{n}|\omega - \nu|^2.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$|f(\tau,\omega) - f(\tau,\nu)| \le L_1|\omega - \nu|.$$

Hence, for all $\xi, \mu \in \Xi$,

$$\begin{aligned}|(\Gamma\xi)(\tau) - (\Gamma\mu)(\tau)| &\le \int_0^\tau |f(\eta,\xi(\eta)) - f(\eta,\mu(\eta))|\,d\eta \\ &\le L_1 \int_0^\tau |\xi(\eta) - \mu(\eta)|\,d\eta \\ &\le TL_1\delta(\xi,\mu).\end{aligned}$$

Taking the maximum over $\tau \in [0,T]$ gives

$$\delta(\Gamma\xi,\Gamma\mu) \le q\,\delta(\xi,\mu), \qquad q = TL_1 < 1.$$

Choose $\rho \in (0,1)$ with $q \le \rho$, and define $\Theta_1(\xi,\mu) = \Theta_2(\xi,\mu) = 1$ for all $\xi, \mu \in \Xi$. Since $q^2 \le q \le \rho$, we have

$$\begin{aligned}\sum_{i=1}^{2} \Theta_i(\Gamma\xi,\Gamma\mu)\delta^i(\Gamma\xi,\Gamma\mu) &= \delta(\Gamma\xi,\Gamma\mu) + \delta^2(\Gamma\xi,\Gamma\mu) \\ &\le q\,\delta(\xi,\mu) + q^2\delta^2(\xi,\mu) \\ &\le \rho \sum_{i=1}^{2} \Theta_i(\xi,\mu)\delta^i(\xi,\mu).\end{aligned}$$

Thus, Γ satisfies the polynomial contraction condition (8). Moreover, Θ_1 is bounded below by 1, and $s\rho < 1$ since $s = 1$.

By Theorem (5), Γ has a unique fixed point $\xi^* \in \Xi$. Therefore,

$$\xi^*(\tau) = \xi_0 + \int_0^\tau f(\eta,\xi^*(\eta))\,d\eta.$$

The continuity of the integrand implies that $\xi^* \in C^1([0,T],\mathbb{R})$ and satisfies the initial value problem. Conversely, every solution of the initial value problem is a fixed point of Γ.

Hence, the initial value problem has a unique solution on $[0,T]$.

The following application illustrates how the polynomial contraction framework ensures convergence of a gradient iteration to the unique minimizer of a strongly convex function.

Application in Optimization.

Let $\Xi = \mathbb{R}^n$, equipped with the Euclidean metric

$$\delta(\xi,\mu) = \|\xi - \mu\|.$$

Then (Ξ,δ) is a complete metric space and, consequently, a complete b-metric space with coefficient $s = 1$.

Let $\varphi : \Xi \to \mathbb{R}$ be differentiable and m-strongly convex for some $m > 0$, that is,

$$\|\nabla\varphi(\xi) - \nabla\varphi(\mu)\| \le L_1\|\xi - \mu\| + L_2\|\xi - \mu\|^2$$

for all $\xi, \mu \in \Xi$, where $L_1 > 0$ and $L_2 \ge 0$.

For $N \in \mathbb{N}$, set

$$z_k = \mu + \frac{k}{N}(\xi - \mu), \qquad k = 0,\dots,N.$$

Then

$$\begin{aligned}\|\nabla\varphi(\xi) - \nabla\varphi(\mu)\| &\le \sum_{k=1}^{N} \|\nabla\varphi(z_k) - \nabla\varphi(z_{k-1})\| \\ &\le \sum_{k=1}^{N} \left(\frac{L_1}{N}\|\xi - \mu\| + \frac{L_2}{N^2}\|\xi - \mu\|^2\right) \\ &= L_1\|\xi - \mu\| + \frac{L_2}{N}\|\xi - \mu\|^2.\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\|\nabla\varphi(\xi) - \nabla\varphi(\mu)\| \le L_1\|\xi - \mu\|.$$

Moreover, strong convexity implies

$$\langle\nabla\varphi(\xi) - \nabla\varphi(\mu), \xi - \mu\rangle \ge m\|\xi - \mu\|^2.$$

In particular, $m \le L_1$.
Define $\Gamma : \Xi \to \Xi$ by

$$\Gamma\xi = \xi - \gamma\nabla\varphi(\xi), \qquad 0 < \gamma < \frac{2m}{L_1^2}.$$

Therefore,

$$\begin{aligned}\delta^2(\Gamma\xi, \Gamma\mu) &= \|(\xi - \mu) - \gamma(\nabla\varphi(\xi) - \nabla\varphi(\mu))\|^2 \\ &= \|\xi - \mu\|^2 - 2\gamma\langle\xi - \mu, \nabla\varphi(\xi) - \nabla\varphi(\mu)\rangle \\ &\quad + \gamma^2\|\nabla\varphi(\xi) - \nabla\varphi(\mu)\|^2 \\ &\le (1 - 2\gamma m + \gamma^2 L_1^2)\|\xi - \mu\|^2.\end{aligned}$$

Set

$$q = \sqrt{1 - 2\gamma m + \gamma^2 L_1^2}.$$

Since $m \le L_1$ and $0 < \gamma < 2m/L_1^2$, we have $0 \le q < 1$. Hence,

$$\delta(\Gamma\xi, \Gamma\mu) \le q\delta(\xi, \mu).$$

Set $\kappa = 2$ and $\Theta_1(\xi, \mu) = \Theta_2(\xi, \mu) = 1$ for all $\xi, \mu \in \Xi$.
Choose

$$\rho = \frac{1 + q}{2} \in (0, 1).$$

Since $q^2 \le q \le \rho$, we obtain

$$\begin{aligned}&\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \\ &= \delta(\Gamma\xi, \Gamma\mu) + \delta^2(\Gamma\xi, \Gamma\mu) \\ &\le q\delta(\xi, \mu) + q^2\delta^2(\xi, \mu) \\ &\le \rho[\delta(\xi, \mu) + \delta^2(\xi, \mu)] \\ &= \rho\sum_{i=1}^{2} \Theta_i(\xi, \mu)\delta^i(\xi, \mu).\end{aligned}$$

Thus, Γ satisfies the polynomial contraction condition (8) with positive constant coefficient mappings.

Moreover, since $s = 1$ and $0 < \rho < 1$,

$$s\rho^{1/j} = \rho^{1/j} < 1$$

for every $j \in \{1, 2\}$.

By Theorem 5, Γ has a unique fixed point $\xi^* \in \Xi$, and the iteration

$$\xi_{k+1} = \Gamma\xi_k = \xi_k - \gamma\nabla\varphi(\xi_k)$$

converges to ξ^* for every $\xi_0 \in \Xi$.

Since $\gamma > 0$,

$$\Gamma\xi^* = \xi^* \quad \Longleftrightarrow \quad \nabla\varphi(\xi^*) = 0.$$

Strong convexity then implies that

$$\varphi(\xi) \geq \varphi(\xi^*) + \frac{m}{2}\|\xi - \xi^*\|^2, \qquad \xi \in \Xi.$$

Thus, ξ^* is the unique global minimizer of φ.

The following application illustrates the result of polynomial contraction in a complete b-metric space whose distance function, given by the squared norm, is not a metric.

Application in b-Metric Spaces.

Let $\Xi = \mathbb{R}^n$, equipped with the b-metric

$$\delta(\xi, \mu) = \|\xi - \mu\|^2.$$

Then (Ξ, δ) is complete, with coefficient $s = 2$, since

$$\delta(\xi, \mu) \leq 2[\delta(\xi, \zeta) + \delta(\zeta, \mu)].$$

Consider the mapping $\Gamma : \Xi \to \Xi$ defined by

$$\Gamma\xi = A\xi + g(\xi),$$

where $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes its induced operator norm, and $g : \Xi \to \Xi$ satisfies

$$\|g(\xi) - g(\mu)\| \leq L\|\xi - \mu\|, \qquad \xi, \mu \in \Xi,$$

for some $L \geq 0$. Assume that

$$0 < (\|A\| + L)^2 < \frac{1}{2}.$$

Consequently,

$$\begin{aligned}
\delta(\Gamma\xi, \Gamma\mu) &= \|A(\xi - \mu) + g(\xi) - g(\mu)\|^2 \\
&\leq \big(\|A(\xi - \mu)\| + \|g(\xi) - g(\mu)\|\big)^2 \\
&\leq \big(\|A\|\|\xi - \mu\| + L\|\xi - \mu\|\big)^2 \\
&= (\|A\| + L)^2\|\xi - \mu\|^2 \\
&= (\|A\| + L)^2\delta(\xi, \mu).
\end{aligned}$$

Here, the triangle inequality is applied before squaring, so the mixed term is retained in

$$(\|A\| + L)^2 = \|A\|^2 + 2\|A\|L + L^2.$$

Set $\kappa = 2$ and $\Theta_1 = \Theta_2 = 1$.

Choose $\rho = (\|A\| + L)^2$.

Since $0 < \rho < \frac{1}{2}$, we have $\rho^2 \leq \rho$.

Thus, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \\
&= \delta(\Gamma\xi, \Gamma\mu) + \delta^2(\Gamma\xi, \Gamma\mu) \\
&\leq \rho\delta(\xi, \mu) + \rho^2\delta^2(\xi, \mu) \\
&\leq \rho[\delta(\xi, \mu) + \delta^2(\xi, \mu)] \\
&= \rho \sum_{i=1}^{2} \Theta_i(\xi, \mu)\delta^i(\xi, \mu).
\end{aligned}$$

Thus, Γ satisfies (8).

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Since $\Theta_1 \equiv 1$, we may take $j = 1$, and hence

$$s\rho^{1/j} = 2\rho < 1.$$

By Theorem 5, Γ has a unique fixed point in Ξ.

Equilibrium states of recurrent neural networks can be characterized as fixed points of the associated network mapping. The following application provides sufficient conditions for the existence and uniqueness of an equilibrium and convergence of the corresponding fixed-point iteration.

Application to Neural Network Equilibria.

Let $\Xi = \mathbb{R}^n$, equipped with the Euclidean metric

$$\delta(\xi, \mu) = \|\xi - \mu\|.$$

Then (Ξ, δ) is a complete metric space and hence a complete b-metric space with coefficient $s = 1$.

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be an activation function satisfying

$$|\sigma(u) - \sigma(v)| \leq \alpha|u - v| + \beta|u - v|^2$$

for all $u, v \in \mathbb{R}$, where $\alpha, \beta \geq 0$.

For $N \in \mathbb{N}$, set

$$z_k = v + \frac{k}{N}(u - v), \qquad k = 0, \dots, N.$$

Then

$$\begin{aligned} |\sigma(u) - \sigma(v)| &\leq \sum_{k=1}^{N} |\sigma(z_k) - \sigma(z_{k-1})| \\ &\leq \sum_{k=1}^{N} \left(\frac{\alpha}{N}|u - v| + \frac{\beta}{N^2}|u - v|^2 \right) \\ &= \alpha|u - v| + \frac{\beta}{N}|u - v|^2. \end{aligned}$$

Letting $N \to \infty$, we obtain

$$|\sigma(u) - \sigma(v)| \leq \alpha|u - v|.$$

Define $\Psi : \Xi \to \Xi$ by applying σ to each coordinate:

$$\Psi(\xi) = \begin{pmatrix} \sigma(\xi_1) \\ \vdots \\ \sigma(\xi_n) \end{pmatrix}.$$

Since σ is α-Lipschitz, it follows that

$$\|\Psi(\xi) - \Psi(\mu)\| \leq \alpha\|\xi - \mu\|, \qquad \xi, \mu \in \Xi.$$

Consider the fixed-point iteration for a single-layer network, defined by the mapping $\Gamma : \Xi \to \Xi$,

$$\Gamma\xi = W\Psi(\xi) + b,$$

where $W \in \mathbb{R}^{n \times n}$ is a weight matrix and $b \in \mathbb{R}^n$ is a bias vector.

Assume that

$$\alpha\|W\| < 1,$$

where $\|W\|$ denotes the operator norm induced by the Euclidean norm.

Then, for all $\xi, \mu \in \Xi$,

$$\begin{aligned}\delta(\Gamma\xi, \Gamma\mu) &= \|W\Psi(\xi) + b - W\Psi(\mu) - b\| \\ &= \|W(\Psi(\xi) - \Psi(\mu))\| \\ &\leq \|W\|\,\|\Psi(\xi) - \Psi(\mu)\| \\ &\leq \alpha\|W\|\|\xi - \mu\| \\ &= \alpha\|W\|\delta(\xi, \mu).\end{aligned}$$

Set $\kappa = 2$ and $\Theta_1 = \Theta_2 = 1$. Choose $\rho = \frac{1+\alpha\|W\|}{2} < 1$ such that
Since $\rho^2 \leq \rho$, it follows that

$$\begin{aligned}&\sum_{i=1}^{2} \Theta_i(\Gamma\xi, \Gamma\mu)\delta^i(\Gamma\xi, \Gamma\mu) \\ &= \delta(\Gamma\xi, \Gamma\mu) + \delta^2(\Gamma\xi, \Gamma\mu) \\ &\leq \rho\delta(\xi, \mu) + \rho^2\delta^2(\xi, \mu) \\ &\leq \rho[\delta(\xi, \mu) + \delta^2(\xi, \mu)] \\ &= \rho\sum_{i=1}^{2} \Theta_i(\xi, \mu)\delta^i(\xi, \mu).\end{aligned}$$

Thus, Γ satisfies (8) with positive constant coefficient mappings. Moreover, since $s = 1$,

$$s\rho^{1/j} = \rho^{1/j} < 1, \qquad j \in \{1, 2\}.$$

By Theorem 5, Γ has a unique fixed point $\xi^* \in \Xi$, satisfying

$$\xi^* = W\Psi(\xi^*) + b.$$

Hence, the network admits a unique equilibrium state, and the iteration

$$\xi_{k+1} = W\Psi(\xi_k) + b$$

converges to ξ^* for every initial state $\xi_0 \in \Xi$.

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