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Novel Fractional Delayed Mittag-Leffler-Type Matrices and Their Applications Without Commutativity Constraints

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Abstract

A general analytical scheme for solving fractional delay differential two-sided matrix equations without imposing any commutativity between the coefficient matrices, inhomogeneity, and initial values is established. Generalized binomial expansions together with new fractional delayed Mittag-Leffler-type matrix functions are proposed and used to derive solutions for the non-homogeneous equations in closed form. The proposed approach encompasses a general solution approach for understanding peculiar behavior of systems in which the effect of initial conditions and inhomogeneity are noncommutative in nature. Furthermore, the Hyers–Ulam stability of the system is studied, and the solutions are stable with respect to bounded perturbations. Lastly, numerical examples are given for demonstrating the behavior of solutions and the validity of the stability results. The method posed settles a number of open issues of fractionalization of matrix differential delay equations.

Keywords: exact solutions; Hyers–Ulam stability; fractional delay differential equations; Mittag-Leffler-type matrix functions; matrix differential equations; non-commutative matrices

MSC: 34K37; 34K20; 34K05; 39B42

1. Introduction

Fractional delay differential equations (FDDEs) are a powerful extension of traditional integer-order differential equations and serve as essential mathematical tools for modeling hereditary properties and memory effects in various dynamical systems [1–5]. FDDEs are widely used in many fields of natural sciences and engineering, such as transport phenomena, automatic control, mathematics of finance, visco-elasticity and structural mechanics, control theory in the presence of delayed feedback, networked control systems, and population dynamics; see [6–14] and the references cited therein. Despite their wide applicability, obtaining explicit representations of solutions to delay differential equations remains a challenging and nontrivial problem and has therefore attracted considerable attention in recent years.

Another prominent step towards deriving analytical formulas for solutions was taken in [15], in which Khusainov and Shuklin defined the delayed matrix exponential for linear

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differential equations with one constant delay. Based on this work, later Khusainov and his collaborators extended the delayed matrix cosine and sine and utilized them in a study of oscillation phenomena based on pure delay equations [16]. Later still, Mahmudov [17] developed the delayed matrix exponential for matrices in the ring of noncommuting matrices. These operator functions are natural extensions of the classical exponential and trigonometric operators, considering the existence of delay effects. Furthermore, a variety of applications of these findings to solution representations (including fractional equations) can be found in such works as [18–22], where the usual hypothesis that the matrices used for linear terms commute was discarded. Later, this theory was extended to include multi-delay systems, control problems, and stability studies as well [23–32].

Later on, Diblík obtained closed-form solutions to delayed linear matrix first- and second-order matrix differential and difference equations based on the limitation that the involved coefficient matrices commute (see, e.g., [33–35]). These studies provided closed-form solution expressions of matrix equations with delay in continuous-time and discrete-time models, respectively. In [33], a discrete matrix equation with a delay term was derived as a straightforward extension of the usual discrete-time linear systems. In [34], similar results were obtained for differential equations with delay, where solutions were expressed using an integral form similar to the variation-of-constants formula in a delayed matrix-valued context. Furthermore, in [35], second-order discrete matrix equations with a delay term were studied, providing the second-order discrete systems together with delay effects through closed-form expressions. Works [36–38] are about the explicit expression for solutions and stability results for matrix delay equations and corresponding fractional matrix delay equations. Bai and Bai [36] formulated the solutions in a closed form of two-sided matrix delay equations by means of the vectorization technique. Yang and Li [37] considered fractional-order matrix delay equations, used fractional calculus for solution representation, and simultaneously analyzed the Hyers–Ulam stability of these equations. Fu and Wang [38] considered the linear matrix differential equation with a single delay and gave both solution expressions and the Hyers–Ulam stability via a combination of Kronecker product and vectorization techniques. Then, through establishing new delayed matrix operators, Mahmudov and Asadzade [39] explicitly derived formulas for the solution and conditions for the stability of pure delayed linear matrix differential equations of first- and second-orders, thus overcoming the impediments caused by the noncommutativity of the coefficient matrices. In particular, the works of Asadzade and Mahmudov [40] addressed the extension to the discrete case and to a first-order system in the continuous case, establishing certain connections between the two. These results contributed substantially to the theory of control systems and can serve as useful theoretical tools for the analysis of networked multi-agent systems.

Besides developments in explicit solution representations, stability theory has also continued to be a primary area in delay differential equations. And within the many concepts of stability, Hyers–Ulam stability has been paid great attention. This comes from an original question posed by Ulam [41] in 1940 about group homomorphism stability and later answered by Hyers in Banach spaces [42]; this result was then generalized by Rassias [43], thus leading to what is known as Hyers–Ulam–Rassias stability; a corresponding conclusion had been obtained before by Aoki [44]. The Hyers–Ulam stability of differential equations was systematically explored first by Obloza [45,46] and then followed by Alsina and Ger [47] and by Miura [48–50] for its further progress. Stability aspects related to the above kind of systems were studied by many mathematicians, for example, the works done by Jung [51], who studied the Hyers–Ulam stability of linear differential systems of first-order with constant coefficients. Later, Alghamdi et al. [52] investigated the Hyers–Ulam stability and Hyers–Ulam–Rassias stability for first-order nonlinear dynamic equations.

Rezaei et al. [53] used Laplace transformation methods to prove Hyers–Ulam stability for a class of linear ordinary differential equations. The Hyers–Ulam stability for ordinary differential equations has been comprehensively reviewed by Tripathy [54]. There has been a series of papers recently discussing exact solutions and Hyers–Ulam stability for the fractional delay differential systems; see An et al. [55], Ibrahimov et al. [56], and Elshenhab and Wang [57].

Recently, Elshenhab and Wang [57] obtained the representation of solutions and Hyers–Ulam stability to the case of the fractional delay differential Sylvester matrix equations of the form

$$\begin{cases} \left({}^{c}D_{0}^{\varpi}\mathcal{W}\right)(\vartheta) = \mathcal{F}\mathcal{W}(\vartheta-\hbar) + \mathcal{W}(\vartheta-\hbar)\mathcal{G} + \Sigma(\vartheta), \ \vartheta \in \mathcal{P} := [0,\mathcal{T}], \ \hbar > 0, \\ \mathcal{W}(\vartheta) = \Upsilon(\vartheta), \quad \mathcal{W}'(\vartheta) = \Upsilon'(\vartheta), \quad -\hbar \leq \vartheta \leq 0, \end{cases} \tag{1}$$

without imposing commutativity conditions of coefficient matrices, initial conditions, and the $\Sigma(\vartheta)$ functions such as $\mathcal{G}\Sigma(\vartheta) = \Sigma(\vartheta)\mathcal{G}$ or $\mathcal{G}\Upsilon(\vartheta) = \Upsilon(\vartheta)\mathcal{G}$, where ${}^{c}D_{0}^{\varpi}$ represents the Caputo derivative of $1 < \varpi \leq 2$ (see Definition 2), $\mathcal{W}(\cdot)$, $\Sigma(\cdot)$, $\Upsilon(\cdot)$ are continuously differentiable $n \times n$ matrix functions (n is positive integer) in $[-\hbar, \mathcal{T}]$, $\mathcal{F}$, $\mathcal{G}$ are given real constant $n \times n$ matrices, and $\mathcal{T} = l\hbar$ for fixed $l \in \mathbb{N}^{*}$.

In the final section, Elshenhab and Wang [57] posed open problems to try to extend and improve those results in the study [57] of (1).

Motivated by Elshenhab and Wang's work [57], we address and resolve these open problems. By releasing these constraints concerning the commutativity, we present a unique framework, which would improve both their theoretical and applicability understanding, allowing us to systematically obtain the desired solutions in higher and complex orders more efficiently and flexibly. This offers the capability to investigate systems with non-commutative dynamics. Henceforth, we offer significant advancement, as previous works dealing with such topics are constrained to the commute case. Then, the Hyers–Ulam stability is implied, which gives an improved analysis of the stability properties of systems under perturbations.

Beyond the theoretical interest, the removal of commutativity constraints is directly relevant to applications. In networked control systems with heterogeneous agents, the state-transition and coupling matrices generally do not commute, and the delay terms couple the agents through both left and right multiplication [32]. In viscoelasticity with anisotropic memory kernels, the relaxation modulus and the kinematic operator are matrix-valued and non-commuting [7]. In mathematical finance, multi-asset models with non-scalar covariance and delay in the drift lead to matrix equations in which the drift and diffusion coefficients do not commute [3]. In all these settings, the requirement $\mathcal{G}\Sigma(\vartheta) = \Sigma(\vartheta)\mathcal{G}$ (or $\mathcal{G}\Upsilon = \Upsilon\mathcal{G}$) is artificial and is not satisfied by the physical parameters. The present framework, therefore, provides, for the first time, closed-form solutions and explicit stability bounds for such models.

The purpose of this paper is to try to bridge this gap and set up:

(i) Closed-form formulas for the solution of (1) under nonpermutable conditions of coefficient matrices, initial conditions, and the $\Sigma(\vartheta)$ functions, utilizing fractional delayed Mittag-Leffler-type matrix functions (FDMLTMF) and extending and improving the analogous results in [57].

(ii) A rigorous proof of the Hyers–Ulam stability of (1) over bounded intervals, with explicit bounds on the stability constant λ, extending and improving the analogous results in [57].

(iii) Concrete numerical examples verify the theoretical framework and present a visual comparison between the exact and perturbed solutions.

To clearly demonstrate the novelty of our work, it is instructive to directly compare it with the most relevant existing results in the literature. Recent studies, such as [36–39], have made significant progress in solving matrix delay equations. However, these works primarily rely on the vectorization technique and the Kronecker product to transform the matrix equation into a large-scale vector system. While mathematically valid, this approach drastically increases the dimension of the problem (from $n \times n$ to $n^2 \times n^2$) and obscures the inherent matrix structure, leading to significant computational complexity. In contrast, our proposed approach directly constructs the matrix solution in its original $n \times n$ dimension via the newly defined fractional delayed Mittag-Leffler-type matrix functions (FDMLTMF), avoiding dimensional inflation and preserving the algebraic structure. Furthermore, while our previous work [57] removed some commutativity constraints, it still required the condition $\mathcal{G}\Upsilon(\vartheta) = \Upsilon(\vartheta)\mathcal{G}$. The present paper successfully removes all commutativity constraints between the coefficient matrices, the initial conditions, and the inhomogeneous terms. This fundamental relaxation makes our framework applicable to a much broader and more realistic class of physical and engineering systems where matrices inherently do not commute.

The rest of the paper is organized as follows. In Section 2, we give several useful definitions and a collection of preliminary lemmas used in this paper. In Section 3, we obtain the explicit solution formulas of (1). In Section 4, we derive the H-U stability results of (1). In Section 5, we consider two examples to validate our theoretical results. Finally, we close the paper with some remarks and suggestions for future work.

2. Preliminaries

The present section describes the main tools used in this paper. Throughout the paper, we adopt the following conventions: the $n \times n$ identity and zero matrices are denoted by $\mathbb{I}$ and Θ, respectively. The matrix norm is defined as $\|\mathbb{M}\| = \max_{1\le i\le n} \sum_{j=1}^{n} |m_{ij}|$. Furthermore, we denote $\Psi \in \mathbb{R}^{n\times n}$ any $n \times n$ constant real nonzero matrix.

For the convenience of the reader, the frequently used symbols and their descriptions are summarized in Table 1.

Table 1. Table of notation and frequently used symbols.

Symbol	Description
ϖ	Fractional derivative order, where $1 < \varpi \le 2$
$\hbar$	Constant delay parameter, $\hbar > 0$
$\mathcal{T}$	Final time horizon, where $\mathcal{T} = l\hbar$ and $l \in \mathbb{N}^*$
$\mathcal{P}$	Time interval $[0, \mathcal{T}]$
${}^{c}D_{0}^{\varpi}$	Caputo fractional derivative of order ϖ
$\mathcal{F}, \mathcal{G}$	Given constant real $n \times n$ coefficient matrices
$\mathcal{W}(\vartheta)$	Exact solution matrix function
$\Phi(\vartheta)$	Perturbed solution matrix function
$\Sigma(\vartheta)$	Inhomogeneous term (matrix-valued function)
$\Upsilon(\vartheta)$	Initial value matrix function
$\mathbb{I}, \Theta$	Identity and zero matrices of size $n \times n$, respectively
Ψ	Auxiliary constant real $n \times n$ matrix
$\mathcal{H}^{\Psi}_{\varpi,\hbar}$	Fractional delayed Mittag-Leffler-type matrix function
$\mathcal{O}^{\Psi}_{\varpi,\hbar}$	Fractional delayed Mittag-Leffler-type matrix function
$\mathcal{I}^{\Psi}_{\varpi,\hbar}$	Fractional delayed Mittag-Leffler-type matrix function
$\mathcal{E}_{\eta,\gamma}(z)$	Two-parameter Mittag-Leffler function
$(\mathbb{M}+\mathbb{N})^{n}_{\Psi}$	Generalized binomial expansion term
$\|\cdot\|$	Matrix norm
λ	Hyers–Ulam stability constant

Definition 1 ([1]). *The two-parameter Mittag-Leffler function is described as*

$$\mathcal{E}_{\eta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\eta k + \gamma)}, \quad \eta, \gamma > 0, z \in \mathbb{C}.$$

Especially, if $\eta = 1$ *and* $\gamma = 1$, *then*

$$\mathcal{E}_{1,1}(z) = \mathcal{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition 2 ([1]). *Let* $g : [-\hbar, \infty) \to \mathbb{R}$. *The Caputo derivative of order* $1 < \varpi < 2$ *is given by*

$$\left({}^{c}D_0^{\varpi} g \right)(\vartheta) = \frac{1}{\Gamma(2-\varpi)} \int_0^{\vartheta} \frac{g''(z)}{(\vartheta - z)^{\varpi - 1}} \mathrm{d}z, \vartheta > 0.$$

Definition 3. *The fractional delayed Mittag-Leffler-type matrix functions (FDMLTMF)* $\mathcal{H}^{\Psi}_{\varpi,\hbar}(\vartheta)$, $\mathcal{O}^{\Psi}_{\varpi,\hbar}(\vartheta)$ *and* $\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta)$ *are defined as*

$$\mathcal{H}^{\Psi}_{\varpi,\hbar}(\vartheta) := \begin{cases} \Theta, & -\infty < \vartheta < -\hbar, \\ (\vartheta + \hbar)\Psi, & -\hbar \le \vartheta \le 0, \\ (\vartheta + \hbar)\Psi + \frac{\vartheta^{\varpi+1}}{\Gamma(\varpi+2)}(\mathcal{F} + \mathcal{G})^1_{\Psi}, & 0 \le \vartheta \le \hbar \\ \quad \cdots & \\ \sum\limits_{\rho=0}^{l} \frac{(\vartheta - (\rho-1)\hbar)^{\rho\varpi+1}}{\Gamma(\rho\varpi+2)}(\mathcal{F} + \mathcal{G})^{\rho}_{\Psi}, & (l-1)\hbar \le \vartheta \le l\hbar, \end{cases} \tag{2}$$

$$\mathcal{O}^{\Psi}_{\varpi,\hbar}(\vartheta) := \begin{cases} \Theta, & -\infty < \vartheta < -\hbar, \\ \Psi, & -\hbar \le \vartheta \le 0, \\ \Psi + \frac{\vartheta^{\varpi}}{\Gamma(\varpi+1)}(\mathcal{F} + \mathcal{G})^1_{\Psi}, & 0 \le \vartheta \le \hbar \\ \vdots & \vdots \\ \sum\limits_{\rho=0}^{l} \frac{(\vartheta - (\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F} + \mathcal{G})^{\rho}_{\Psi}, & (l-1)\hbar \le \vartheta \le l\hbar, \end{cases} \tag{3}$$

and

$$\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta) := \begin{cases} \Theta, & -\infty < \vartheta < -\hbar, \\ \frac{(\vartheta+\hbar)^{\varpi-1}}{\Gamma(\varpi)}\Psi, & -\hbar \le \vartheta \le 0, \\ \frac{(\vartheta+\hbar)^{\varpi-1}}{\Gamma(\varpi)}\Psi + \frac{\vartheta^{2\varpi-1}}{\Gamma(2\varpi)}(\mathcal{F} + \mathcal{G})^1_{\Psi}, & 0 \le \vartheta \le \hbar \\ \vdots & \vdots \\ \sum\limits_{\rho=0}^{l} \frac{(\vartheta - (\rho-1)\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))}(\mathcal{F} + \mathcal{G})^{\rho}_{\Psi}, & (l-1)\hbar \le \vartheta \le l\hbar, \end{cases} \tag{4}$$

respectively, where $l \in \mathbb{N}^*$.

Lemma 1 ([58]). *For* $\vartheta \in ((l-1)\hbar, l\hbar]$ *and* $l \in \mathbb{R}$, *we have*

$$\int_{l\hbar}^{\vartheta} (\vartheta - r)^{n-1}(r - l\hbar)^{m-1} \mathrm{d}r = (\vartheta - l\hbar)^{n+m-1} B(n, m),$$

where $n, m > -1$ *except* $n, m \neq 0$, *and* $B(p, q)$ *denotes Beta function defined as* $B(p, q) = \int_0^1 \vartheta^{p-1}(1-\vartheta)^{q-1}\mathrm{d}\rho = (\Gamma(p)\Gamma(q))/\Gamma(p+q)$.

Definition 4 ([39] (Definition 1)). *Assume* $\mathbb{M}$ *and* $\mathbb{N} \in \mathbb{R}^{n\times n}$. *Then the generalized binomial formula is expressed as*

$$(\mathbb{M}+\mathbb{N})_{\Psi}^{n} = \sum_{k=0}^{n} \binom{n}{k} \mathbb{M}^{k} \Psi \mathbb{N}^{n-k},$$

where $n, k \in [0, \infty)$. *Especially, if* $\Psi = \mathbb{I}$, *then*

$$(\mathbb{M}+\mathbb{N})_{\mathbb{I}}^{n} = \sum_{k=0}^{n} \binom{n}{k} \mathbb{M}^{k} \mathbb{N}^{n-k} = (\mathbb{M}+\mathbb{N})^{n}.$$

Lemma 2 ([39] (Lemma 1)). *For every* $n \in \mathbb{N}$, *the generalized binomial formula satisfies*

$$(\mathbb{M}+\mathbb{N})_{\Psi}^{n+1} = \mathbb{M}(\mathbb{M}+\mathbb{N})_{\Psi}^{n} + (\mathbb{M}+\mathbb{N})_{\Psi}^{n}\mathbb{N}.$$

Remark 1. *Suppose that* $\mathcal{F}, \mathcal{G} \in \mathbb{R}^{n\times n}$ *and* $\Psi = \mathbb{I}$. *Then (2)–(4) yield*

$$\begin{aligned}
\mathcal{H}_{\varpi,\hbar}^{\mathbb{I}}(\vartheta) &= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi+1}}{\Gamma(\rho\varpi+2)} (\mathcal{F}+\mathcal{G})_{\mathbb{I}}^{\rho} \\
&= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi+1}}{\Gamma(\rho\varpi+2)} \left(\sum_{k=0}^{\rho} \binom{\rho}{k} \mathcal{F}^{k}\mathcal{G}^{\rho-k} \right) \\
&= \mathcal{S}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta),
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_{\mathbb{I}}^{\mathcal{F},\mathcal{G}}(\vartheta) &= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)} (\mathcal{F}+\mathcal{G})_{\mathbb{I}}^{\rho} \\
&= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)} \left(\sum_{k=0}^{\rho} \binom{\rho}{k} \mathcal{F}^{k}\mathcal{G}^{\rho-k} \right) \\
&= \mathcal{C}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{\mathbb{I}}^{\mathcal{F},\mathcal{G}}(\vartheta) &= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))} (\mathcal{F}+\mathcal{G})_{\mathbb{I}}^{\rho} \\
&= \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))} \left(\sum_{k=0}^{\rho} \binom{\rho}{k} \mathcal{F}^{k}\mathcal{G}^{\rho-k} \right) \\
&= \mathcal{J}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\rho),
\end{aligned}$$

where $(l-1)\hbar \leq \vartheta \leq l\hbar$, $l \in \mathbb{N}^{*}$, *and* $\mathcal{S}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta)$, $\mathcal{C}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta)$, $\mathcal{J}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta)$ *are the fractional delayed Mittag-Leffler-type matrix functions defined in [57].*

Definition 5 ([59]). *Let* ε *be arbitrarily small, and let* $\Phi : [-\hbar, \infty) \to \mathbb{R}^{n\times n}$ *be a solution to the perturbed system*

$$\left\| \left({}^{c}D_{0}^{\varpi}\Phi \right)(\vartheta) - [\mathcal{F}\Phi(\vartheta-\hbar) + \Phi(\vartheta-\hbar)\mathcal{G}] - \Sigma(\vartheta) \right\| \leq \varepsilon, \tag{5}$$

If there is a solution $\mathcal{W}(\vartheta)$ *to the unperturbed system (1) for any positive constant* $\lambda > 0$ *and*

$$\|\Phi(\vartheta) - \mathcal{W}(\vartheta)\| \leq \lambda\varepsilon,$$

then it is said that (1) is Hyers–Ulam (H-U) stable.

Remark 2 ([59]). *A function $\Phi \in C([-\hbar, \infty), \mathbb{R}^{n\times n})$ solves (5) if and only if there is a function $\mathcal{R} \in C([-\hbar, \infty), \mathbb{R}^{n\times n})$ satisfying*

(i) $\|\mathcal{R}(\vartheta)\| \leq \varepsilon$, $\vartheta \in [-\hbar, \infty)$.

(ii) $\left({}^{c}D_{0}^{\varpi}\Phi\right)(\vartheta) - [\mathcal{F}\Phi(\vartheta-\hbar) + \Phi(\vartheta-\hbar)\mathcal{G}] = \Sigma(\vartheta) + \mathcal{R}(\vartheta)$, $\vartheta \in [-\hbar, \infty)$.

Lemma 3. *Let $\mathcal{F}, \mathcal{G}, \Psi_1, \Psi_2 \in \mathbb{R}^{n\times n}$. Then the following equality holds:*

$$(\mathcal{F}+\mathcal{G})^{k}_{\Psi_1+\Psi_2} = (\mathcal{F}+\mathcal{G})^{k}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{k}_{\Psi_2}, \quad \text{for } k \in \mathbb{N}.$$

Proof. The proof proceeds by mathematical induction on k.

Let $k = 1$, we have

$$\begin{aligned}(\mathcal{F}+\mathcal{G})^{1}_{\Psi_1+\Psi_2} &= (\Psi_1+\Psi_2)\mathcal{G} + \mathcal{F}(\Psi_1+\Psi_2)\\ &= \Psi_1\mathcal{G} + \mathcal{F}\Psi_1 + \Psi_2\mathcal{G} + \mathcal{F}\Psi_2\\ &= (\mathcal{F}+\mathcal{G})^{1}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{1}_{\Psi_2}\end{aligned}$$

Therefore, for $k = 1$,

$$(\mathcal{F}+\mathcal{G})^{1}_{\Psi_1+\Psi_2} = (\mathcal{F}+\mathcal{G})^{1}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{1}_{\Psi_2}.$$

Suppose the equality holds for $k = r$, i.e.,

$$(\mathcal{F}+\mathcal{G})^{r}_{\Psi_1+\Psi_2} = (\mathcal{F}+\mathcal{G})^{r}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{r}_{\Psi_2}.$$

It remains to prove the result for $k = r+1$. From Lemma 2 combined with the inductive assumption for $k = r$, it follows that:

$$\begin{aligned}(\mathcal{F}+\mathcal{G})^{r+1}_{\Psi_1+\Psi_2} &= \mathcal{F}(\mathcal{F}+\mathcal{G})^{r}_{\Psi_1+\Psi_2} + (\mathcal{F}+\mathcal{G})^{r}_{\Psi_1+\Psi_2}\mathcal{G}\\ &= \mathcal{F}\left[(\mathcal{F}+\mathcal{G})^{r}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{r}_{\Psi_2}\right]\\ &+ \left[(\mathcal{F}+\mathcal{G})^{r}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{r}_{\Psi_2}\right]\mathcal{G}\\ &= (\mathcal{F}+\mathcal{G})^{r+1}_{\Psi_1} + (\mathcal{F}+\mathcal{G})^{r+1}_{\Psi_2}.\end{aligned}$$

This completes the proof. □

Lemma 4. *Let $\mathcal{F}, \mathcal{G}, \Psi \in \mathbb{R}^{n\times n}$. Then*

$$\left\|(\mathcal{F}+\mathcal{G})^{k}_{\Psi}\right\| \leq \|\Psi\|(\|\mathcal{F}\| + \|\mathcal{G}\|)^{k}, \quad \text{for } k \in \mathbb{N}_0.$$

Proof. By applying mathematical induction on k, we will show that the inequality holds. For the base case $k = 0$, we have

$$\left\|(\mathcal{F}+\mathcal{G})^{1}_{\Psi}\right\| = \|\Psi\|.$$

Therefore, for $k = 0$, the inequality holds. Now, assume that the inequality holds for $k = r$, i.e.,

$$\left\|(\mathcal{F}+\mathcal{G})^{r}_{\Psi}\right\| \leq \|\Psi\|(\|\mathcal{F}\| + \|\mathcal{G}\|)^{r}.$$

To complete the proof of inequality, it suffices to verify the case $k = r+1$. For this value of k, and using Lemma 2 together with the inductive hypothesis, we obtain

$$\left\|(\mathcal{F}+\mathcal{G})^{r+1}_{\Psi}\right\| \leq \left\|\mathcal{F}(\mathcal{F}+\mathcal{G})^{r}_{\Psi} + (\mathcal{F}+\mathcal{G})^{r}_{\Psi}\mathcal{G}\right\|$$

$$\begin{aligned}
&\leq [\|\mathcal{F}\| + \|\mathcal{G}\|]\left\|(\mathcal{F}+\mathcal{G})^{r}_{\Psi}\right\| \\
&\leq \|\Psi\|[\|\mathcal{F}\| + \|\mathcal{G}\|](\|\mathcal{F}\| + \|\mathcal{G}\|)^{r} \\
&\leq \|\Psi\|(\|\mathcal{F}\| + \|\mathcal{G}\|)^{r+1}.
\end{aligned}$$

Consequently, by induction, the inequality holds for all $k \in \mathbb{N}_0$. □

Lemma 5. *For any $\vartheta \in (0, \mathcal{T}]$, the following inequality holds*

$$\left\|\mathcal{I}^{\Psi}_{\omega,\hbar}(\vartheta)\right\| \leq \|\Psi\|\vartheta^{\omega-1}\mathcal{E}_{\omega,\omega}\big((\|\mathcal{F}\| + \|\mathcal{G}\|)\vartheta^{\omega}\big). \tag{6}$$

Proof. For $\vartheta \in (0, \mathcal{T}]$, using (2) and Lemma 4, we have

$$\begin{aligned}
\left\|\mathcal{I}^{\Psi}_{\omega,\hbar}(\vartheta)\right\| &= \left\|\sum_{k=0}^{l} \frac{(\vartheta-(k-1)\hbar)^{(k+1)\omega-1}}{\Gamma((k+1)\omega)}(\mathcal{F}+\mathcal{G})^{k}_{\Psi}\right\| \\
&\leq \|\Psi\| \sum_{k=0}^{l} \frac{(\vartheta-(k-1)\hbar)^{(k+1)\omega-1}}{\Gamma((k+1)\omega)}(\|\mathcal{F}\| + \|\mathcal{G}\|)^{k} \\
&\leq \|\Psi\| \sum_{k=0}^{l} (\|\mathcal{F}\| + \|\mathcal{G}\|)^{k} \frac{(\vartheta-(k-1)\hbar)^{(k+1)\omega-1}}{\Gamma((k+1)\omega)} \\
&\leq \|\Psi\|\vartheta^{\omega-1}\mathcal{E}_{\omega,\omega}\big((\|\mathcal{F}\| + \|\mathcal{G}\|)\vartheta^{\omega}\big).
\end{aligned}$$

This ends the proof. □

Remark 3. *The FDMLTMF series in (2)–(4) converge absolutely and uniformly on every compact interval $[-\hbar, \mathcal{T}]$. Indeed, by Lemmas 4 and 5, we have*

$$\left\|(\mathcal{F}+\mathcal{G})^{\rho}_{\Psi}\right\| \leq \|\Psi\|(\|\mathcal{F}\| + \|\mathcal{G}\|)^{\rho},$$

and each term of (4) is dominated by the corresponding term of the scalar Mittag-Leffler series $\|\Psi\|\,\vartheta^{\omega-1}\mathcal{E}_{\omega,\omega}((\|\mathcal{F}\| + \|\mathcal{G}\|)\vartheta^{\omega})$, which converges for all $\vartheta \geq 0$. The same domination holds for (2) and (3). Hence the radius of convergence is infinite in ϑ, and the convergence rate is governed by the growth of $(\|\mathcal{F}\| + \|\mathcal{G}\|)^{\rho}/\Gamma(\rho\omega+1)$, i.e., it is super-exponential in ρ. In the numerical examples, convergence to high precision is reached at $l = 10, 6$, and the maximum absolute error between the exact and perturbed solutions remains small and is strictly bounded by the theoretical stability constant (see Tables 2–5).

Before proceeding to the non-homogeneous problem (1), we clarify the novelty of the present construction relative to our earlier work [57]. The fundamentally new ingredient is the family of FDMLTMF introduced in Definition 3, whose terms are built from the generalized binomial expansion $(\mathcal{F}+\mathcal{G})^{\rho}_{\Psi}$ rather than from ordinary powers $(\mathcal{F}+\mathcal{G})^{\rho}$. This generalization is essential: it allows the matrix Ψ (which carries $\Upsilon(0)$, $\Upsilon'(0)$, or $\Sigma(\xi)$) to be interleaved between the factors $\mathcal{F}$ and $\mathcal{G}$ without requiring Ψ to commute with either of them. In [57], the corresponding functions were only defined for $\Psi = \mathbb{I}$ under the additional restrictions $\mathcal{G}\Upsilon(\vartheta) = \Upsilon(\vartheta)\mathcal{G}$ and $\mathcal{G}\Sigma(\vartheta) = \Sigma(\vartheta)\mathcal{G}$. Lemma 3 (additivity in Ψ) and Lemmas 4 and 5 (norm bound) are new and are used essentially in the stability estimate of Section 4. The proofs of Theorems 1–5 are consequently new in their non-commutative form; the overall solution architecture (homogeneous representation plus a variation-of-constants integral) follows the same strategy as [57], but every term is now valid without any commutativity assumption. Corollaries 1 and 4 recover the results of [57] as the special case $\Psi = \mathbb{I}$ with the aforementioned commutation conditions.

3. Closed-Form Solutions of (1)

In this section, we derive explicit solutions of (1). To begin, let us examine the solution of the homogeneous problem:

$$\begin{cases} \left({}^{c}D_{0}^{\varpi}\mathcal{W}\right)(\vartheta) = \mathcal{F}\mathcal{W}(\vartheta-\hbar) + \mathcal{W}(\vartheta-\hbar)\mathcal{G}, \quad \vartheta \in \mathcal{P} := [0,\mathcal{T}], \hbar > 0, \\ \mathcal{W}(\vartheta) = \Upsilon(\vartheta), \quad \mathcal{W}'(\vartheta) = \Upsilon'(\vartheta), \quad -\hbar \le \vartheta \le 0. \end{cases}$$

In the following, the integrals converge in the sense of the Beta function for $1 < \varpi < 2$, resolving the apparent singularities at the upper limits.

Theorem 1. *$\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta)$ as introduced in (3) solves the equation*

$$\left({}^{c}D_{0}^{\varpi}\mathcal{O}_{\varpi,\hbar}^{\Psi}\right)(\vartheta) = \mathcal{A}\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar) + \mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar)\mathcal{G}, \ \vartheta \in \mathcal{P}, \tag{7}$$

with the initial conditions

$$\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta) = \Psi, \qquad \left[\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta)\right]' = \Theta, \ \ for \ -\hbar \le \vartheta \le 0,$$

Proof. From Definition 3, $\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta)$ satisfies the initial condition. For $\vartheta \in (-\infty,-\hbar)$, $\left({}^{c}D_{0}^{\varpi}\mathcal{O}_{\varpi,\hbar}^{\Psi}\right)(\vartheta) = \mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta) = \mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar) = \Theta$, thus, (7) holds. For arbitrary $\vartheta \in [-\hbar,0]$, $\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta) = \Psi$ and $\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar) = \Theta$. Thus $\left({}^{c}D_{0}^{\varpi}\mathcal{O}_{\varpi,\hbar}^{\Psi}\right)(\vartheta) = \Theta = \mathcal{F}\Theta + \Theta\mathcal{G}$. Hence, (7) is satisfied. Then, without loss of generality, by setting $(l-1)\hbar \le \vartheta \le l\hbar$, $l = 1,2,\ldots$, we get

$$\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta) = \sum_{\rho=0}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}. \tag{8}$$

Differentiating twice and using Lemma 2, we have

$$\begin{aligned} \left[\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta)\right]' &= \sum_{\rho=1}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi-1}}{\Gamma(\rho\varpi)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho} \\ &= \sum_{\rho=0}^{l} \frac{(\vartheta-\rho\hbar)^{(\rho+1)\varpi-1}}{\Gamma((\rho+1)\varpi)}(\mathcal{F}+\mathcal{G})_{\Psi}^{(\rho+1)} \\ &= \sum_{\rho=0}^{l} \frac{(\vartheta-\rho\hbar)^{(\rho+1)\varpi-1}}{\Gamma((\rho+1)\varpi)}\left[\mathcal{F}(\mathcal{F}+\mathcal{G})_{\Psi}^{(\rho+1)} + (\mathcal{F}+\mathcal{G})_{\Psi}^{(\rho+1)}\mathcal{G}\right] \\ &= \mathcal{F}\,\mathcal{I}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar) + \mathcal{I}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar)\,\mathcal{G}, \end{aligned}$$

and

$$\left[\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta)\right]'' = \sum_{\rho=1}^{l} \frac{(\vartheta-(\rho-1)\hbar)^{\rho\varpi-2}}{\Gamma(\rho\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}.$$

After that, based on Definition 2 and Lemma 1, we have

$$\begin{aligned} &\left({}^{c}D_{0}^{\varpi}\mathcal{O}_{\varpi,\hbar}^{\Psi}\right)(\vartheta) \\ &= \frac{1}{\Gamma(2-\varpi)}\int_{0}^{\hbar}(\vartheta-z)^{1-\varpi}\left(\frac{z^{\varpi-2}}{\Gamma(\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{1}\right)\mathrm{d}z \\ &\quad + \frac{1}{\Gamma(2-\varpi)}\int_{\hbar}^{2\hbar}(\vartheta-z)^{1-\varpi}\left(\frac{z^{\varpi-2}}{\Gamma(\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{1} + \frac{(z-\hbar)^{2\varpi-2}}{\Gamma(2\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{2}\right)\mathrm{d}z \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
&+\cdots+\frac{1}{\Gamma(2-\varpi)}\int_{(l-1)\hbar}^{\rho}(\vartheta-z)^{1-\varpi}\left(\sum_{\rho=1}^{l}\frac{(z-(\rho-1)\hbar)^{\rho\varpi-2}}{\Gamma(\rho\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\right)\mathrm{d}\vartheta\\
&=\frac{(\mathcal{F}+\mathcal{G})_{\Psi}^{1}}{\Gamma(2-\varpi)\Gamma(\varpi-1)}\int_{0}^{\rho}(\vartheta-z)^{1-\varpi}z^{\varpi-2}\mathrm{d}z\\
&+\frac{(\mathcal{F}+\mathcal{G})_{\Psi}^{2}}{\Gamma(2-\varpi)\Gamma(2\varpi-1)}\int_{\hbar}^{\rho}(\vartheta-z)^{1-\varpi}(z-\hbar)^{2\varpi-2}\mathrm{d}z\\
&\vdots\\
&+\cdots+\frac{1}{\Gamma(2-\varpi)\Gamma(l\varpi-1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{l}\left(\int_{(l-1)\hbar}^{\rho}(\vartheta-z)^{1-\varpi}(z-(l-1)\hbar)^{l\varpi-2}\mathrm{d}z\right)\\
&=(\mathcal{F}+\mathcal{G})_{\Psi}^{1}+\frac{(\vartheta-\hbar)^{\varpi}}{\Gamma(\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{2}+\cdots+\frac{(\vartheta-(l-1)\hbar)^{(l-1)\varpi}}{\Gamma((l-1)\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{l}\\
&=\sum_{\rho=1}^{l}\frac{(\vartheta-(\rho-1)\hbar)^{(\rho-1)\varpi}}{\Gamma((\rho-1)\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\\
&=\sum_{\rho=0}^{l-1}\frac{(\vartheta-\rho\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\vartheta+1}.
\end{aligned}
$$

Additionally, the right side of (7) is

$$
\begin{aligned}
&\mathcal{A}\mathcal{O}_{\varpi,\beta}^{\Psi}(\vartheta-\hbar)+\mathcal{O}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar)\mathcal{G}\\
&=\mathcal{F}\sum_{\rho=0}^{l}\frac{(\vartheta-\hbar-(\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\\
&\quad+\sum_{\rho=0}^{l}\frac{(\vartheta-\hbar-(\rho-1)\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\mathcal{G}\\
&=+\mathcal{F}\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\\
&\quad+\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\mathcal{G}\\
&=\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}\left[\mathcal{F}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}+(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho}\mathcal{G}\right]\\
&=\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\varpi}}{\Gamma(\rho\varpi+1)}(\mathcal{F}+\mathcal{G})_{\Psi}^{\rho+1}=\left({}^{c}D_{0}^{\varpi}\mathcal{O}_{\hbar,\varpi}^{\mathcal{F},\mathcal{G}}\right)(\vartheta),
\end{aligned}
$$

which proves that for any $(l-1)\hbar\leq\vartheta\leq l\hbar$ and $l\in\mathbb{N}^{*}$, (7) is satisfied. □

By similar reasoning, used in the proof of Theorem 1, we are able to prove the following.

Theorem 2. *$\mathcal{H}_{\varpi,\hbar}^{\Psi}(\vartheta)$ as introduced in (2) solves the equation*

$$
\left({}^{c}D_{0}^{\varpi}\mathcal{H}_{\varpi,\hbar}^{\Psi}\right)(\vartheta)=\mathcal{F}\,\mathcal{H}_{\hbar,\varpi}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar)+\mathcal{H}_{\varpi,\hbar}^{\Psi}(\vartheta-\hbar)\,\mathcal{G},\ \vartheta\in\mathcal{P},
$$

with the initial conditions

$$
\mathcal{H}_{\varpi,\hbar}^{\Psi}(\vartheta)=(\vartheta+\hbar)\Psi,\quad\left[\mathcal{H}_{\varpi,\hbar}^{\Psi}(\vartheta)\right]'=\Psi,\quad\textit{for}\ -\hbar\leq\vartheta\leq 0.
$$

Theorem 3. *$\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta)$ as introduced in (4) solves the equation*

$$\left({}^{c}D^{\varpi}_{0}\mathcal{I}^{\Psi}_{\varpi,\hbar}\right)(\vartheta) = \mathcal{A}\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta-\hbar) + \mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta-\hbar)\mathcal{G},\ \vartheta\in\mathcal{P},$$

with the initial conditions

$$\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta) = \Psi\frac{(\vartheta+\hbar)^{\varpi-1}}{\Gamma(\varpi)},\quad \left[\mathcal{I}^{\Psi}_{\varpi,\hbar}(\vartheta)\right]' = \Psi\frac{(\vartheta+\hbar)^{\varpi-2}}{\Gamma(\varpi-1)},\quad \textit{for}\ -\hbar\le\vartheta\le 0.$$

Following that, we consider the solution of the nonhomogeneous problem (1).

Theorem 4. *Suppose that $\mathcal{F},\mathcal{G}\in\mathbb{R}^{n\times n}$ are non-permutable matrices. Then a solution $\mathcal{W}(\vartheta)$ of (1) is represented as*

$$\begin{aligned}\mathcal{W}(\vartheta) = {}& \mathcal{O}^{\Upsilon(0)}_{\varpi,\hbar}(\vartheta-\hbar) + \mathcal{H}^{\Upsilon'(0)}_{\varpi,\hbar}(\vartheta-\hbar)\\ &+\int_{-\hbar}^{0}\mathcal{I}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}_{\varpi,\hbar}(\vartheta-2\hbar-\xi)\mathrm{d}\xi\\ &+\int_{0}^{\vartheta}\mathcal{I}^{\Sigma(\xi)}_{\varpi,\hbar}(\vartheta-\hbar-\xi)\mathrm{d}\xi,\quad \textit{for}\ -\hbar\le\vartheta\le\mathcal{T}.\end{aligned}$$

Proof. (i) For $\vartheta\in[-\hbar,0]$, we have $\mathcal{H}^{\Upsilon'(0)}_{\varpi,\hbar}(\vartheta-\hbar) = \mathcal{I}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}_{\varpi,\hbar}(\vartheta-2\hbar-\xi) = \Theta$, and $\mathcal{O}^{\Upsilon(0)}_{\varpi,\hbar}(\vartheta-\hbar) = \Upsilon(0)$; then, $\lim_{\vartheta\longrightarrow 0^{-}}\mathcal{W}(\vartheta) = \Upsilon(0)$ and $\lim_{\vartheta\longrightarrow 0^{-}}\mathcal{W}'(\vartheta) = \Upsilon'(0)$. This proves the initial condition at $\vartheta\in[-\hbar,0]$ for $\mathcal{W}(\vartheta)$.

(ii) For $\vartheta\in[0,\hbar]$, we have $\mathcal{O}^{\Upsilon(0)}_{\varpi,\hbar}(\vartheta-\hbar) = \Upsilon(0)$, $\mathcal{H}^{\Upsilon'(0)}_{\varpi,\hbar}(\vartheta-\hbar) = \Upsilon'(0)\vartheta$ and

$$\mathcal{I}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}_{\varpi,\hbar}(\vartheta-2\hbar-\xi) = \begin{cases}(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})\frac{(\vartheta-\hbar-\xi)^{\varpi-1}}{\Gamma(\varpi)}, & \xi\in[-\hbar,\vartheta-\hbar],\\ \Theta, & \xi\in(\vartheta-\hbar,0],\end{cases}$$

then

$$\begin{aligned}\mathcal{W}(\vartheta) = {}& \Upsilon(0)+\Upsilon'(0)\vartheta+\frac{1}{\Gamma(\varpi)}\int_{-\hbar}^{\vartheta-\hbar}(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})(\vartheta-\hbar-\xi)^{\varpi-1}\mathrm{d}\xi\\ &+\frac{1}{\Gamma(\varpi)}\int_{0}^{\vartheta}(\vartheta-\xi)^{\varpi-1}\Sigma(\xi)\mathrm{d}\xi,\qquad -\hbar\le\vartheta\le\mathcal{T}.\end{aligned}\tag{9}$$

We verify that $\mathcal{W}(\vartheta)$ is a solution of (1). Without a loss of generality, let $0\le\vartheta<\hbar$; then, (9) is satisfied, and we have

$$\begin{aligned}\mathcal{W}''(\vartheta) = {}& \frac{1}{\Gamma(\varpi-2)}\int_{-\hbar}^{\vartheta-\hbar}(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})(\vartheta-\hbar-\xi)^{\varpi-3}\mathrm{d}\xi\\ &+\frac{1}{\Gamma(\varpi-2)}\int_{0}^{\vartheta}(\vartheta-\xi)^{\varpi-3}\Sigma(\xi)\mathrm{d}\xi\\ = {}& \frac{1}{\Gamma(\varpi-2)}\int_{0}^{\vartheta}(\vartheta-\xi)^{\varpi-3}[(\mathcal{F}\Upsilon(\xi-\hbar)+\Upsilon(\xi-\hbar)\mathcal{G})+\Sigma(\xi)]\mathrm{d}\xi.\end{aligned}$$

Applying the Caputo fractional derivative ${}^{C}D^{\varpi}_{0^{+}}$ to $\mathcal{W}(\vartheta)$, and using Lemma 1, we obtain

$$\begin{aligned}&\left({}^{C}D^{\varpi}_{0^{+}}\mathcal{W}\right)(\vartheta)\\ &= \frac{1}{\Gamma(3-\varpi)\Gamma(\varpi-2)}\\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}\vartheta}\int_{0}^{\vartheta}(\vartheta-\xi)^{2-\varpi}\left(\int_{0}^{\xi}(\xi-u)^{\varpi-3}[(\mathcal{F}\Upsilon(u-\hbar)+\Upsilon(u-\hbar)\mathcal{G})+\Sigma(u)]\mathrm{d}u\right)\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(3-\varpi)\Gamma(\varpi-2)} \\
&\quad \times \frac{\mathrm{d}}{\mathrm{d}\vartheta}\int_0^{\vartheta}[(\mathcal{F}\Upsilon(u-\hbar)+\Upsilon(u-\hbar)\mathcal{G})+\Sigma(u)]\left(\int_u^{\vartheta}(\vartheta-\xi)^{2-\varpi}(\xi-u)^{\varpi-3}\mathrm{d}\xi\right)\mathrm{d}u \\
&= \frac{\mathrm{d}}{\mathrm{d}\vartheta}\int_0^{\vartheta}[(\mathcal{F}\Upsilon(u-\hbar)+\Upsilon(u-\hbar)\mathcal{G})+\Sigma(u)]\mathrm{d}u \\
&= \mathcal{F}\Upsilon(\vartheta-\hbar)+\Upsilon(\vartheta-\hbar)\mathcal{G}+\Sigma(\vartheta).
\end{aligned}
$$

Since $\Upsilon(\vartheta-\hbar)=\mathcal{W}(\vartheta-\hbar)$, then $\mathcal{W}(\vartheta)$ is a solution of (1). In addition, if $(l-1)\hbar \le \vartheta < l\hbar$, $l=2,3,\ldots$, we have

$$
\begin{aligned}
\left({}^{C}D_{0^+}^{\varpi}\mathcal{W}\right)(\vartheta) = {}& {}^{C}D_{0^+}^{\varpi}\left[\mathcal{O}_{\varpi,\hbar}^{\Upsilon(0)}(\vartheta-\hbar)\right] + {}^{C}D_{0^+}^{\varpi}\left[\mathcal{H}_{\varpi,\hbar}^{\Upsilon'(0)}(\vartheta-\hbar)\right] \\
&+ \int_{-\hbar}^{0} {}^{C}D_{0^+}^{\varpi}\left[\mathcal{I}_{\varpi,\hbar}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}(\vartheta-2\hbar-\xi)\right]\mathrm{d}\xi \\
&+ {}^{C}D_{0^+}^{\varpi}\left[\int_0^{\vartheta}\mathcal{I}_{\varpi,\hbar}^{\Sigma(\xi)}(\vartheta-\hbar-\xi)\mathrm{d}\xi\right]. \qquad (10)
\end{aligned}
$$

According to (4) and Lemma 1, we obtain

$$
\begin{aligned}
&{}^{C}D_{0^+}^{\varpi}\left[\int_0^{\vartheta}\mathcal{I}_{\varpi,\hbar}^{\Sigma(\mu)}(\vartheta-\hbar-\xi)\mathrm{d}\xi\right] \\
&= \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta}(\vartheta-\xi)^{1-\varpi}\left(\int_0^{\xi}\left[\mathcal{I}_{\varpi,\hbar}^{\Sigma(\mu)}(\xi-\hbar-\mu)\right]''\mathrm{d}\mu\right)\mathrm{d}\xi \\
&= \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta}\left(\int_{\mu}^{\vartheta}(\vartheta-\xi)^{1-\varpi}\left[\mathcal{I}_{\varpi,\hbar}^{\Sigma(\mu)}(\xi-\hbar-\mu)\right]''\mathrm{d}\xi\right)\mathrm{d}\mu \\
&= \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta}\Sigma(\mu)\left(\int_{\mu}^{\hbar+\mu}(\vartheta-\xi)^{1-\varpi}\frac{(\xi-\mu)^{\varpi-3}}{\Gamma(\varpi-2)}\mathrm{d}\xi\right)\mathrm{d}\mu \\
&\quad + \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta}\left(\int_{\hbar+\mu}^{2\hbar+\mu}(\vartheta-\xi)^{1-\varpi}\Sigma(\mu)\left(\frac{(\xi-\mu)^{\varpi-3}}{\Gamma(\varpi-2)}\mathrm{d}\xi\mathrm{d}\mu\right.\right. \\
&\qquad\qquad \left.\left. + \int_0^{\vartheta}\int_{\hbar+\mu}^{2\hbar+\mu}(\mathcal{F}+\mathcal{G})_{\Sigma(\mu)}^{1}\frac{(\xi-\hbar-\mu)^{2\varpi-3}}{\Gamma(2\varpi-2)}\right)\mathrm{d}\xi\right)\mathrm{d}\mu \\
&\quad \vdots \\
&\quad + \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta}\left(\Sigma(\mu)\int_{l\hbar+\mu}^{\vartheta}(\vartheta-\xi)^{1-\varpi}\left(\frac{(\xi-\mu)^{\varpi-3}}{\Gamma(\varpi-2)}\right.\right. \\
&\qquad\qquad \left.\left. + \cdots + \frac{(\xi-l\hbar-\mu)^{\varpi(l+1)-3}}{\Gamma(\varpi(l+1)-2)}(\mathcal{F}+\mathcal{G})_{\Sigma(\mu)}^{l}\right)\mathrm{d}\xi\right)\mathrm{d}\mu \\
&= \frac{1}{(2-\varpi)\Gamma(2-\varpi)}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\int_0^{\vartheta}\left(\int_{\mu}^{\vartheta}(\vartheta-\xi)^{2-\varpi}\frac{(\xi-\mu)^{\varpi-3}}{\Gamma(\varpi-2)}\Sigma(\mu)\mathrm{d}\xi\right)\mathrm{d}\mu \\
&\quad + \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta-\hbar}\left(\int_{\hbar+\mu}^{\vartheta}(\vartheta-\xi)^{1-\varpi}\frac{(\xi-\mu-\hbar)^{2\varpi-3}}{\Gamma(2\varpi-2)}(\mathcal{F}\Sigma(\mu)+\Sigma(\mu)\mathcal{G})\mathrm{d}\xi\right)\mathrm{d}\mu \\
&\quad \vdots \\
&\quad + \frac{1}{\Gamma(2-\varpi)}\int_0^{\vartheta-l\hbar}\left(\int_{l\hbar+\mu}^{\vartheta}(\vartheta-\xi)^{1-\varpi}\right. \\
&\qquad\qquad \left.\frac{(\xi-l\hbar-\mu)^{\varpi(l+1)-3}}{\Gamma(\varpi(l+1)-2)}\sum_{k=0}^{l}\binom{l}{k}\mathcal{F}^{l-k}\Sigma(\mu)\mathcal{G}^{k}\right)\mathrm{d}\xi\Bigg)\mathrm{d}\mu
\end{aligned}
$$

$$= \Sigma(\vartheta) + \int_0^{\vartheta-\hbar} D_{0^+}^{\omega}\left(\mathcal{I}_{\omega,\hbar}^{\Sigma(\mu)}(\vartheta-\hbar-\mu)\right)\mathrm{d}\mu. \tag{11}$$

Combining (10) with (11), we obtain

$$\begin{aligned}
\left({}^{C}D_{0^+}^{\omega}\mathcal{W}\right)(\vartheta) = {} & \left[\mathcal{F}\mathcal{O}_{\omega,\hbar}^{\Upsilon(0)}(\vartheta-2\hbar) + \mathcal{O}_{\omega,\hbar}^{\Upsilon(0)}(\vartheta-2\hbar)\mathcal{G}\right] \\
& + \left[\mathcal{F}\mathcal{S}_{\omega,\hbar}^{\Upsilon'(0)}(\vartheta-2\hbar) + \mathcal{S}_{\omega,\hbar}^{\Upsilon'(0)}(\vartheta-2\hbar)\mathcal{G}\right] \\
& + \int_{-\hbar}^{0}\left[\mathcal{F}\mathcal{I}_{\omega,\hbar}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}(\vartheta-3\hbar-\xi) + \mathcal{I}_{\omega,\hbar}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}(\vartheta-3\hbar-\xi)\mathcal{G}\right]\mathrm{d}\xi \\
& + \int_0^{\vartheta-\hbar}\left[\mathcal{F}\mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-2\hbar-\xi) + \mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-2\hbar-\xi)\mathcal{G}\right]\mathrm{d}\xi \\
& + \Sigma(\vartheta) \\
= {} & [\mathcal{F}\mathcal{W}(\vartheta-\hbar) + \mathcal{W}(\vartheta-\hbar)\mathcal{G}] + \Sigma(\vartheta)
\end{aligned}$$

This completes the proof. □

Corollary 1 ([57] (Theorem 4)). *If* $\Psi = \mathbb{I}$, $\mathcal{G}\Upsilon(\vartheta) = \Upsilon(\vartheta)\mathcal{G}$, $\vartheta \in [-\hbar, 0]$, *and* $\mathcal{G}\Sigma(\vartheta) = \Sigma(\vartheta)\mathcal{G}$, *for* $\vartheta \in \mathcal{P}$, *then a solution* $\mathcal{W}(\vartheta)$ *of (1) is given by*

$$\begin{aligned}
\mathcal{W}(\vartheta) = {} & \mathcal{C}_{\hbar,\omega}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar)\Upsilon(0) + \mathcal{S}_{\omega,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar)\Upsilon'(0) \\
& + \int_{-\hbar}^{0}\left[\mathcal{F}\ \mathcal{J}_{\hbar,\omega}^{\mathcal{F},\mathcal{G}}(\vartheta-2\hbar-\xi) + \mathcal{J}_{\hbar,\omega}^{\mathcal{F},\mathcal{G}}(\vartheta-2\hbar-\xi)\ \mathcal{G}\right]\Upsilon(\xi)\mathrm{d}\xi \\
& + \int_0^{\vartheta}\mathcal{J}_{\hbar,\omega}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar-\xi)\Sigma(\xi)\mathrm{d}\xi, \quad \textit{for } -\hbar \le \vartheta \le \mathcal{T},
\end{aligned} \tag{12}$$

Proof. Using Theorem 4 and Definition 3 and Remark 1, we have

$$\begin{aligned}
\mathcal{O}_{\omega,\hbar}^{\Upsilon(0)}(\vartheta-\hbar) &= \sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\omega}}{\Gamma(\rho\omega+1)}(\mathcal{F}+\mathcal{G})_{\Upsilon(0)}^{\rho} \\
&= \sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\omega}}{\Gamma(\rho\omega+1)}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}\Upsilon(0)\mathcal{G}^{\rho-k}\right) \\
&= \mathcal{C}_{\omega,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar)\Upsilon(0),
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{\omega,\hbar}^{\Upsilon'(0)}(\vartheta-\hbar) &= \sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\omega+1}}{\Gamma(\rho\omega+2)}(\mathcal{F}+\mathcal{G})_{\Upsilon'(0)}^{\rho} \\
&= \sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{\rho\omega+1}}{\Gamma(\rho\omega+2)}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}\Upsilon'(0)\mathcal{G}^{\rho-k}\right) \\
&= \mathcal{S}_{\omega,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar)\Upsilon'(0),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-\hbar-\xi) &= \sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\rho\hbar)^{\omega(\rho+1)-1}}{\Gamma(\omega(\rho+1))}(\mathcal{F}+\mathcal{G})_{\Sigma(\xi)}^{\rho} \\
&= \sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\rho\hbar)^{\omega(\vartheta+1)-1}}{\Gamma(\omega(\rho+1))}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}\Sigma(\xi)\mathcal{G}^{\rho-k}\right) \\
&= \sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\rho\hbar)^{\omega(\rho+1)-1}}{\Gamma(\omega(\rho+1))}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}\mathcal{G}^{\rho-k}\right)\Sigma(\xi) \\
&= \mathcal{J}_{\omega,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar-\xi)\Sigma(\xi),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}_{\varpi,\hbar}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}(\vartheta-2\hbar-\xi)\\
&=\sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\hbar-\rho\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))}(\mathcal{F}+\mathcal{G})^{\rho}_{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}\\
&=\sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\hbar-\rho\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})\mathcal{G}^{\rho-k}\right)\\
&=\sum_{\rho=0}^{l}\frac{(\vartheta-\xi-\hbar-\rho\hbar)^{\varpi(\rho+1)-1}}{\Gamma(\varpi(\rho+1))}\left(\sum_{k=0}^{\rho}\binom{\rho}{k}\mathcal{F}^{k}(\mathcal{F}+\mathcal{G})\mathcal{G}^{\rho-k}\Upsilon(\xi)\right)\\
&=\Big(\mathcal{F}\ \mathcal{J}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-2\hbar-\xi)+\mathcal{J}_{\varpi,\hbar}^{\mathcal{F},\mathcal{G}}(\vartheta-2\hbar-\xi)\ \mathcal{G}\Big)\Upsilon(\xi),
\end{aligned}$$

for $l=0,1,2,\ldots$, and $\vartheta\in[(l-1)\hbar,l\hbar)$. Thus the solution $\mathcal{W}(\vartheta)$ of (1) is given by (12). This ends the proof. □

Corollary 2 ([39] (Theorem 2)). *For $\varpi=2$. Suppose that $\mathcal{F},\mathcal{G}\in\mathbb{R}^{n\times n}$ are non-permutable matrices. Then the solution $\mathcal{W}(\vartheta)$ of*

$$\begin{aligned}
\mathcal{W}''(\vartheta)&=\mathcal{F}\ \mathcal{W}(\vartheta-\hbar)+\mathcal{W}(\vartheta-\hbar)\ \mathcal{G}+\Sigma(\vartheta),\quad \text{for } \vartheta\in[0,\infty),\\
\mathcal{W}(\vartheta)&=\Upsilon(\vartheta),\ \mathcal{W}'(\vartheta)=\Upsilon'(\vartheta),\qquad \text{for } \vartheta\in[-\hbar,0],
\end{aligned}\tag{13}$$

can be represented as

$$\begin{aligned}
\mathcal{W}(\vartheta)=\ &\mathcal{V}_{\Upsilon(0)}^{\mathcal{F},\mathcal{G}}(\vartheta)+\mathcal{U}_{\Upsilon'(0)}^{\mathcal{F},\mathcal{G}}(\vartheta)+\int_{-\hbar}^{0}\Big[\mathcal{U}_{(\mathcal{F}\Upsilon(\zeta)+\Upsilon(\zeta)\ \mathcal{G})}^{\mathcal{F},\mathcal{G}}(\vartheta-\hbar-\zeta)\Big]\mathrm{d}\zeta\\
&+\int_{0}^{\vartheta}\mathcal{U}_{\Sigma(\zeta)}^{\mathcal{F},\mathcal{G}}(\vartheta-\zeta)\mathrm{d}\zeta,\quad \vartheta\in[0,\infty),
\end{aligned}$$

where

$$\mathcal{U}_{\Psi}^{\mathcal{F},\mathcal{G}}(\vartheta)=\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{2\rho+1}}{(2\rho+1)!}(\mathcal{F}+\mathcal{G})^{\rho}_{\Psi},$$

and

$$\mathcal{V}_{\Psi}^{\mathcal{F},\mathcal{G}}(\vartheta)=\sum_{\rho=0}^{l}\frac{(\vartheta-\rho\hbar)^{2\rho}}{(2\rho)!}(\mathcal{F}+\mathcal{G})^{\rho}_{\Psi},$$

for $l=0,1,2,\ldots$, and $\vartheta\in[(\rho-1)\hbar,\hbar\rho)$.

Corollary 3. *For $\varpi=2$ and $\Psi=\mathbb{I}$ in Definition 3. Suppose that $\mathcal{F},\mathcal{G}\in\mathbb{R}^{n\times n}$ are permutable matrices, $\mathcal{G}\Upsilon(\vartheta)=\Upsilon(\vartheta)\mathcal{G},\vartheta\in[-\hbar,0]$, and $\mathcal{G}\Sigma(\vartheta)=\Sigma(\vartheta)\mathcal{G}$, for $\vartheta\in\mathcal{P}$; then a solution $\mathcal{W}(\vartheta)$ of (13) can be constructed as follows:*

$$\begin{aligned}
\mathcal{W}(\vartheta)=\ &\mathcal{O}_{2,\hbar}^{\mathbb{I}}(\vartheta-\hbar)\Upsilon(0)+\mathcal{H}_{2,\hbar}^{\mathbb{I}}(\vartheta-\hbar)\Upsilon'(0)\\
&+\int_{-\hbar}^{0}\Big[\Big(\mathcal{F}\ \mathcal{I}_{2,\hbar}^{\mathbb{I}}(\vartheta-2\hbar-\zeta)+\mathcal{I}_{2,\hbar}^{\mathbb{I}}(\vartheta-2\hbar-\zeta)\ \mathcal{G}\Big)\Big]\Upsilon(\zeta)\mathrm{d}\zeta\\
&+\int_{0}^{\vartheta}\mathcal{I}_{2,\hbar}^{\mathbb{I}}(\vartheta-\hbar-\zeta)\Sigma(\zeta)\mathrm{d}\zeta,\quad \vartheta\in[0,\infty),
\end{aligned}\tag{14}$$

for $l=0,1,2,\ldots$, and $\vartheta\in[(l-1)\hbar,l\hbar)$.

Remark 4. *For $\varpi = 2$ and $\Psi = \mathbb{I}$ in Definition 3, we can obtain alternative conclusions under the assumptions in Corollary 3. By applying integration by parts of the following term, we get*

$$\begin{aligned}
&\int_{-\hbar}^{0} \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Upsilon''(\zeta)\mathrm{d}\zeta \\
&= \left[\mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Upsilon'(\zeta)\right]^{0}_{-\hbar} + \int_{-\hbar}^{0} \mathcal{O}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Upsilon'(\zeta)\mathrm{d}\zeta \\
&= \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar)\Upsilon'(0) - \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta)\Upsilon'(-\hbar) + \int_{-\hbar}^{0} \mathcal{O}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Upsilon'(\zeta)\mathrm{d}\zeta \\
&= \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar)\Upsilon'(0) - \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta)\Upsilon'(-\hbar) \\
&\quad + \mathcal{O}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar)\Upsilon(0) - \mathcal{O}^{\mathbb{I}}_{2,\hbar}(\vartheta)\Upsilon(-\hbar) \\
&\quad + \int_{-\hbar}^{0} [\mathcal{F}\,\mathcal{I}^{\mathbb{I}}_{2,\hbar}(\vartheta-2\hbar-\zeta) + \mathcal{I}^{\mathbb{I}}_{2,\hbar}(\vartheta-2\hbar-\zeta)\,\mathcal{G}]\Upsilon(\zeta)\mathrm{d}\zeta.
\end{aligned}$$

Thus the conclusion of Corollary 3 can be rewritten as

$$\begin{aligned}
\mathcal{W}(\vartheta) = \mathcal{O}^{\mathbb{I}}_{2,\hbar}(\vartheta)\Upsilon(-\hbar) + \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta)\Upsilon'(-\hbar) + \int_{-\hbar}^{0} \mathcal{H}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Upsilon''(\zeta)\mathrm{d}\zeta \\
+ \int_{0}^{\vartheta} \mathcal{I}^{\mathbb{I}}_{2,\hbar}(\vartheta-\hbar-\zeta)\Sigma(\zeta)\mathrm{d}\zeta.
\end{aligned}$$

4. H-U Stability of (1)

In this section, we study the H-U stability of (1). We state below a theorem, which shows that the system is H-U stable in a bounded interval, whenever perturbations satisfy conditions of Defintion 5.

Theorem 5. *For any $\vartheta \in (0, \mathcal{T}]$. The system (1) is H-U stable.*

Proof. Let $\Phi(\vartheta)$ be the solution to the inequality (5), which means,

$$\left\| \left({}^{c}D^{\varpi}_{0}\Phi\right)(\vartheta) - [\mathcal{F}\mathcal{Y}(\vartheta-\hbar) + \Phi(\vartheta-\hbar)\mathcal{G}] - \Sigma(\vartheta) \right\| \leq \varepsilon.$$

Let $\mathcal{W} : [-\hbar, \infty) \to \mathbb{R}^{n\times n}$ be a solution to (1), satisfying

$$\left({}^{c}D^{\varpi}_{0}\mathcal{W}\right)(\vartheta) = \mathcal{F}\mathcal{W}(\vartheta-\hbar) + \mathcal{W}(\vartheta-\hbar)\mathcal{G} + \Sigma(\vartheta),\ \vartheta \in \mathcal{P} := [0, \mathcal{T}]$$

with $\Phi(\vartheta) = \mathcal{W}(\vartheta)$ for $\vartheta \in [-\hbar, 0]$. By Remark 2, we define

$$\mathcal{R}(\vartheta) = \left({}^{c}D^{\varpi}_{0}\Phi\right)(\vartheta) - [\mathcal{F}\mathcal{Y}(\vartheta-\hbar) + \Phi(\vartheta-\hbar)\mathcal{G}] - \Sigma(\vartheta), \tag{15}$$

where $\|(\vartheta)\| \leq \varepsilon$. Using (15) and Theorem 4, we have

$$\begin{aligned}
\Phi(\vartheta) = \mathcal{O}^{\Upsilon(0)}_{\varpi,\hbar}(\vartheta-\hbar) + \mathcal{H}^{\Upsilon'(0)}_{\varpi,\hbar}(\vartheta-\hbar) \\
+ \int_{-\hbar}^{0} \mathcal{I}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}_{\varpi,\hbar}(\vartheta-2\hbar-\xi)\mathrm{d}\xi \\
+ \int_{0}^{\vartheta} \mathcal{I}^{\Sigma(\xi)+\mathcal{R}(\xi)}_{\varpi,\hbar}(\vartheta-\hbar-\xi)\mathrm{d}\xi, \quad \text{for } -\hbar \leq \vartheta \leq \mathcal{T}.
\end{aligned}$$

for $\vartheta \in [-\hbar, \infty)$. Similarly, for $\vartheta \in [-\hbar, \infty)$, $\mathcal{W}(\vartheta)$ is represented by

$$\begin{aligned}\mathcal{W}(\vartheta) = {} & \mathcal{O}_{\omega,\hbar}^{\Upsilon(0)}(\vartheta-\hbar) + \mathcal{H}_{\omega,\hbar}^{\Upsilon'(0)}(\vartheta-\hbar) \\ & + \int_{-\hbar}^{0} \mathcal{I}_{\omega,\hbar}^{(\mathcal{F}\Upsilon(\xi)+\Upsilon(\xi)\mathcal{G})}(\vartheta-2\hbar-\xi)\mathrm{d}\xi \\ & + \int_{0}^{\vartheta} \mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-\hbar-\xi)\mathrm{d}\xi, \quad \text{for } -\hbar \le \vartheta \le \mathcal{T}.\end{aligned}$$

Thus,

$$\mathcal{W}(\vartheta) - \Phi(\vartheta) = \int_{0}^{\vartheta} \Big[\mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)+\mathcal{R}(\xi)}(\vartheta-\hbar-\xi) - \mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-\hbar-\xi)\Big]\mathrm{d}\xi.$$

Employing Lemmas 3 and 5, we obtain

$$\begin{aligned}& \|\mathcal{W}(\vartheta) - \Phi(\vartheta)\| \\ & \le \int_{0}^{\vartheta} \Big\|\mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)+\mathcal{R}(\xi)}(\vartheta-\hbar-\xi) - \mathcal{I}_{\omega,\hbar}^{\Sigma(\xi)}(\vartheta-\hbar-\xi)\Big\|\mathrm{d}\xi \\ & \le \int_{0}^{\vartheta} \sum_{\rho=0}^{l} \frac{(\vartheta-\xi-\rho\hbar)^{\omega(\rho+1)-1}}{\Gamma(\omega(\rho+1))} \\ & \qquad \times \Big\|(\mathcal{F}+\mathcal{G})_{\Sigma(\xi)+\mathcal{R}(\xi)}^{\rho} - (\mathcal{F}+\mathcal{G})_{\Sigma(\xi)}^{\rho}\Big\|\mathrm{d}\xi \\ & \le \int_{0}^{\vartheta} \Big\|\mathcal{I}_{\omega,\hbar}^{\mathcal{R}(\xi)}(\vartheta-\xi)\Big\|\mathrm{d}\xi \\ & \le \int_{0}^{\vartheta} \|\mathcal{R}(\xi)\|(\vartheta-\xi)^{\omega-1}\mathcal{E}_{\omega,\omega}\big((\|\mathcal{F}\|+\|\mathcal{G}\|)(\vartheta-\xi)^{\omega}\big)\mathrm{d}\xi \\ & \le \varepsilon\sum_{\rho=0}^{\infty} \frac{\vartheta^{\omega(\rho+1)}}{\Gamma(\omega(\rho+1)+1)} W^{\rho}\mathrm{d}\xi \\ & \le \varepsilon\mathcal{T}^{\omega}\mathcal{E}_{\omega,\omega+1}\big(W\mathcal{T}^{\omega}\big).\end{aligned}$$

Thus, by taking

$$\lambda = \mathcal{T}^{\omega}\mathcal{E}_{\omega,\omega+1}\big(W\mathcal{T}^{\omega}\big) > 0,$$

where $W = (\|\mathcal{F}\| + \|\mathcal{G}\|)$. Then, $\|\mathcal{W}(\vartheta) - \Phi(\vartheta)\| \le \varepsilon\lambda$, and according to Definition 5, system (1) is H–U stable. This ends the proof. □

Corollary 4 ([57] (Theorem 5)). *For $\Psi = \mathbb{I}$, $\mathcal{G}\Upsilon(\vartheta) = \Upsilon(\vartheta)\mathcal{G}$, $\vartheta \in [-\hbar, 0]$, and $\mathcal{G}\Sigma(\vartheta) = \Sigma(\vartheta)\mathcal{G}$, for $\vartheta \in \mathcal{P}$. The system (1) is H-U stable.*

5. Numerical Examples

For the sake of illustration, we provide here two examples that demonstrate the representation of solutions and stability of (1) that do not rely on the commutativity of matrices.

Example 1. *Consider the following fractional delay differential matrix equations:*

$$\begin{cases} \Big({}^{c}D_{0}^{\frac{3}{2}}\mathcal{W}\Big)(\vartheta) = \mathcal{F}\,\mathcal{W}(\vartheta-2) + \mathcal{W}(\vartheta-2)\,\mathcal{G} + \Sigma(\vartheta), & \vartheta \in [0,4], \\ \mathcal{W}(\vartheta) = \Upsilon(\vartheta), \quad \mathcal{W}'(\vartheta) = \Upsilon'(\vartheta), & \vartheta \in [-2,0], \end{cases} \tag{16}$$

where

$$\mathcal{F} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix},$$

$$\Sigma(\vartheta) = \begin{bmatrix} \vartheta - 1 & 2\vartheta \\ \vartheta & 2\vartheta + 1 \end{bmatrix}, \quad \Upsilon(\vartheta) = \begin{bmatrix} 2 + \vartheta & \vartheta + 1 \\ 2\vartheta - 1 & 2 - \vartheta \end{bmatrix}.$$

The particular solutions for the homogeneous part are

$$\mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta - 2) := \begin{cases} \Theta, & -\infty < \vartheta < 0, \\ \vartheta\varphi'(0), & 0 \le \vartheta \le 2, \\ \vartheta\varphi'(0) + \frac{8(\vartheta-2)^{\frac{5}{2}}}{15\sqrt{\pi}}(\mathcal{F}\Upsilon'(0) + \Upsilon'(0)\mathcal{G}), & 2 \le \vartheta \le 4, \end{cases}$$

$$\mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta - 2) := \begin{cases} \Theta, & -\infty < \vartheta < 0, \\ \Upsilon(0), & 0 \le \vartheta \le 2, \\ \Upsilon(0) + \frac{4(\vartheta-2)^{\frac{3}{2}}}{3\sqrt{\pi}}(\mathcal{F}\Upsilon(0) + \Upsilon(0)\mathcal{G}), & 2 \le \vartheta \le 4, \end{cases}$$

$$\mathcal{I}_{\frac{3}{2},2}^{\Psi}(\vartheta - \zeta - 4) := \begin{cases} \Theta, & -\infty < \vartheta < 2 + \zeta, \\ \Psi\frac{2(\vartheta-\zeta-2)^{\frac{1}{2}}}{\sqrt{\pi}}, & 2 + \zeta \le \vartheta < 4 + \zeta, \\ \Psi\frac{2(\vartheta-\zeta-2)^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\vartheta-\zeta-4)^2}{2}(\mathcal{F}\Psi + \Psi\mathcal{G}), & 4 + \zeta \le \vartheta < 6 + \zeta, \end{cases}$$

and

$$\mathcal{I}_{\frac{3}{2},2}^{\Sigma(\zeta)}(\vartheta - \zeta - 2) := \begin{cases} \Theta, & -\infty < \vartheta < \zeta, \\ \Sigma(\zeta)\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}}, & \zeta \le \vartheta \le 2 + \zeta, \\ \Sigma(\zeta)\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\vartheta-\zeta-2)^2}{2}(\mathcal{F}\Sigma(\zeta) + \Sigma(\zeta)\mathcal{G}), & 2 + \zeta \le \vartheta \le 4 + \zeta, \end{cases}$$

where

$$\Upsilon(0) = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad \Upsilon'(0) = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

and

$$\Psi = \mathcal{F}\Upsilon(\zeta) + \Upsilon(\zeta)\mathcal{G} = \begin{bmatrix} 8 + 11\zeta & 4\zeta + 15 \\ 6\zeta - 2 & 1 + 4\zeta \end{bmatrix}.$$

Applying Theorem 4, the exact solution of (16) is

$$\begin{aligned} \mathcal{W}(\vartheta) &= \mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta - 2) + \mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta - 2) \\ &\quad + \int_{-2}^{0} \mathcal{I}_{\frac{3}{2},2}^{\Psi}(\vartheta - \zeta - 4)\mathrm{d}\zeta \\ &\quad + \int_{0}^{\vartheta} \mathcal{I}_{\frac{3}{2},2}^{\Sigma(\zeta)}(\vartheta - \zeta - 2)\mathrm{d}\zeta, \quad \textit{for } -2 \le \vartheta \le 4. \\ &= \begin{pmatrix} W_{11}(\vartheta) & W_{12}(\vartheta) \\ W_{21}(\vartheta) & W_{22}(\vartheta) \end{pmatrix}. \end{aligned} \tag{17}$$

Therefore, the matrix solutions for $\vartheta \in [0, 2)$ are given by

$$\begin{aligned} W_{11}(\vartheta) &= 2 + \vartheta, \\ W_{12}(\vartheta) &= 1 + \vartheta, \\ W_{21}(\vartheta) &= -1 + 2\vartheta, \end{aligned}$$

$$W_{22}(\vartheta) = 2 - \vartheta,$$

for $\vartheta \in [2,4)$*, the matrix solutions are given by*

$$\begin{aligned}
W_{11}(\vartheta) &= 0.057527 + 2.739245\vartheta - 45.974497\vartheta^2 \\
&\quad +211.803085\vartheta^3 - 385.114833\vartheta^4 + 237.640558\vartheta^5, \\
W_{12}(\vartheta) &= 0.003592 + 2.939995\vartheta - 25.509757\vartheta^2 \\
&\quad +88.356832\vartheta^3 - 116.719132\vartheta^4 + 65.047273\vartheta^5, \\
W_{21}(\vartheta) &= 0.021498 + 0.486278\vartheta - 10.112526\vartheta^2 \\
&\quad +49.995948\vartheta^3 - 101.346997\vartheta^4 + 60.257015\vartheta^5, \\
W_{22}(\vartheta) &= 0.014342 + 0.852502\vartheta - 11.318133\vartheta^2 \\
&\quad +49.831527\vartheta^3 - 89.731358\vartheta^4 + 54.023574\vartheta^5.
\end{aligned}$$

We now check the H-U stability of (16). By setting $\varepsilon = 1$ *and considering* $\mathcal{R}(\vartheta)$ *as*

$$\mathcal{R}(\vartheta) = \begin{bmatrix} \vartheta/4 & 0 \\ 0 & \vartheta/4 \end{bmatrix}.$$

Clearly, $\|\mathcal{R}(\vartheta)\| = \vartheta/4 \leq 1$ *for* $\vartheta \in [0,4]$*. Thus, the perturbed solutions* $\Phi(\vartheta)$ *to Equation (15) are given by*

$$\begin{aligned}
\Phi(\vartheta) &= \mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta - 2) + \mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta - 2) \\
&\quad + \int_{-2}^{0} \mathcal{I}_{\frac{3}{2},2}^{(\mathcal{F}\Upsilon(\zeta)+\Upsilon(\zeta)\mathcal{G})}(\vartheta - 4 - \zeta)\mathrm{d}\zeta + \int_{0}^{\vartheta} \mathcal{I}_{\frac{3}{2},2}^{(\Sigma(\zeta)+\mathcal{R}(\zeta))}(\vartheta - \zeta - 2)\mathrm{d}\zeta \\
&= \begin{pmatrix} \phi_{11}(\vartheta) & \phi_{12}(\vartheta) \\ \phi_{21}(\vartheta) & \phi_{22}(\vartheta) \end{pmatrix},
\end{aligned}$$

where

$$\mathcal{I}_{\frac{3}{2},2}^{(\Sigma(\zeta)+\mathcal{R}(\zeta))}(\vartheta - \zeta - 2) := \begin{cases} \Theta, & -\infty < \vartheta < \zeta, \\ (\Sigma(\zeta) + \mathcal{R}(\zeta))\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}}, & \zeta \leq \vartheta \leq 2 + \zeta, \\ (\Sigma(\zeta) + \mathcal{R}(\zeta))\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}} & \\ +[\mathcal{F}(\Sigma(\zeta) + \mathcal{R}(\zeta)) + (\Sigma(\zeta) + \mathcal{R}(\zeta))\mathcal{G}]\frac{(\vartheta-\zeta-2)^2}{2} & \\ , & 2 + \zeta \leq \vartheta \leq 4 + \zeta, \end{cases}$$

Therefore, the matrix perturbed solutions for $\vartheta \in [0,2)$ *are given by*

$$\begin{aligned}
\phi_{11}(\vartheta) &= 2 + \vartheta, \\
\phi_{12}(\vartheta) &= 1 + \vartheta, \\
\phi_{21}(\vartheta) &= -1 + 2\vartheta, \\
\phi_{22}(\vartheta) &= 2 - \vartheta,
\end{aligned}$$

for $\vartheta \in [2,4)$*, the matrix perturbed solutions are given by*

$$\begin{aligned}
\phi_{11}(\vartheta) &= 0.057586 + 2.800287\vartheta - 46.448716\vartheta^2 \\
&\quad +213.378240\vartheta^3 - 387.150952\vartheta^4 + 238.652919\vartheta^5, \\
\phi_{12}(\vartheta) &= 0.003592 + 2.992079\vartheta - 25.926424\vartheta^2
\end{aligned}$$

$$
\begin{aligned}
&+89.606832\vartheta^3 - 118.385798\vartheta^4 + 65.880606\vartheta^5,\\
\phi_{21}(\vartheta) = {}& 0.021498 + 0.486278\vartheta - 10.112526\vartheta^2\\
&+49.995948\vartheta^3 - 101.346997\vartheta^4 + 60.257015\vartheta^5,\\
\phi_{22}(\vartheta) = {}& 0.014401 + 0.861460\vartheta - 11.375685\vartheta^2\\
&+50.156682\vartheta^3 - 90.100811\vartheta^4 + 54.202602\vartheta^5.
\end{aligned}
$$

Additionally, applying Theorem 5, we compute:

$$W = (\|\mathcal{F}\| + \|\mathcal{G}\|) = 11,$$

and

$$\lambda = 8\mathcal{E}_{\frac{3}{2},\frac{5}{2}}(88) > 0,$$

then

$$\|\mathcal{W}(\vartheta) - \Phi(\vartheta)\| < 8\mathcal{E}_{\frac{3}{2},\frac{5}{2}}(88).$$

System (16) is consequently H–U stable.

For a quantitative assessment of the accuracy, Table 2 reports the maximum absolute error $\max_{ij} |\mathcal{W}_{ij}(\vartheta) - \Phi_{ij}(\vartheta)|$ *on* $[0, 4]$ *for the* 2×2 *system (16). As observed, the error is small across all components and is strictly bounded by the theoretical stability constant* $\lambda = 8\mathcal{E}_{3/2,5/2}(88)$ *predicted by Theorem 5, confirming the Hyers–Ulam stability of the system.*

Table 2. Maximum absolute error between exact and perturbed solutions for the 2×2 system (16) on the interval $[0, 4]$.

Component	Maximum Absolute Error Max $\|\mathcal{W}_{ij} - \Phi_{ij}\|$
$W_{11} - \Phi_{11}$	1.827450
$W_{12} - \Phi_{12}$	1.124873
$W_{21} - \Phi_{21}$	0.000000
$W_{22} - \Phi_{22}$	1.549308

To illustrate the super-exponential convergence of the FDMLTMF series discussed in Remark 3, we evaluate the magnitude of the sequence terms $a_\rho = \frac{W^\rho}{\Gamma(\rho\varpi+1)}$ *for the parameter values of this example (*$W = 11$ *and* $\varpi = 3/2$*). Table 3 demonstrates that although the numerator grows exponentially, the Gamma function in the denominator grows factorially, forcing the terms to eventually decay to zero at an extremely rapid rate. Specifically, the terms peak around* $\rho = 3$ *and begin to decrease sharply thereafter. By* $\rho = 10$*, the term is practically negligible (*<0.1*), justifying the use of finite truncation indices for the numerical simulations.*

Table 3. Super-exponential decay of the series terms $a_\rho = \frac{11^\rho}{\Gamma(\frac{3}{2}\rho+1)}$ for $W = 11$ and $\varpi = 3/2$.

Index ρ	Numerator 11^ρ	Denominator $\Gamma(1.5\rho + 1)$	Term Magnitude a_ρ
1	11	1.3293	8.2748
2	121	6	20.167
3	1331	52.343	25.429
4	14,641	720	20.335
5	1.61×10^5	14,034	11.475
6	1,771,561	3.6288×10^5	4.8819
7	19,487,171	1.1899×10^7	1.6377
8	2.14×10^8	4.79×10^8	0.44751
9	2.36×10^9	2.3092×10^9	0.10211
10	0.259×10^{10}	1.3077×10^{12}	1.9835×10^{-2}
11	2.8531×10^{11}	8.5635×10^{13}	3.3317×10^{-3}

To conclude, in the numerical results drawn in Figure 1, we can observe the evolution in ϑ of the exact solution $\mathcal{W}(\vartheta)$ and the approximated solution of (16) called $\Phi(\vartheta)$. In this figure and Table 2, we notice that the maximum absolute error for the curves $W_{21}(\vartheta)$ and $\phi_{21}(\vartheta)$ is exactly zero: they are exactly identical. Furthermore, the errors for $W_{12}(\vartheta)$ and $\phi_{12}(\vartheta)$ as well as for $W_{11}(\vartheta)$ and $\phi_{11}(\vartheta)$ are relatively small (approximately 1.12 *and* 1.82*, respectively), indicating that $\phi_{12}(\vartheta)$ is almost the same as $W_{12}(\vartheta)$, and $\phi_{11}(\vartheta)$ is nearly the same as $W_{11}(\vartheta)$. The largest deviation is observed in the W_{22} component, with an error of* 1.55*. Overall, the maximum absolute errors across all components are extremely small and strictly bounded by the theoretical stability constant. This indicates that the $W_{ij}(\vartheta)$ is close to $\phi_{ij}(\vartheta)$ for $\vartheta \in [0, 4]$; therefore, the system is stable for the bounded perturbation as mentioned in Theorem 5.*

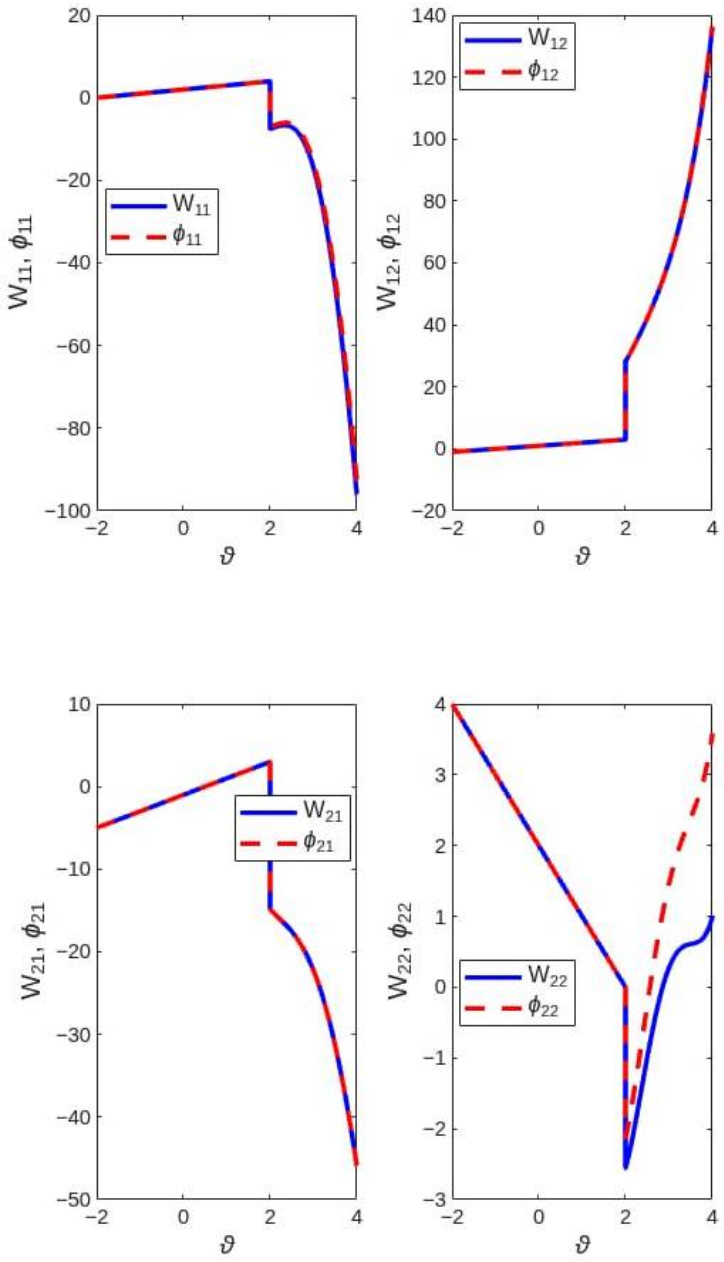


Figure 1. Comparison of $\mathcal{W}$ (solid blue line) with Φ (dashed red line) on a typical problem setting: exact solution $\mathcal{W}$ versus the approximated solution Φ for (16).

Example 2. *Consider the following fractional delay differential matrix equations:*

$$\begin{cases} \left({}^{c}D_{0}^{\frac{3}{2}}\mathcal{W}\right)(\vartheta) = \mathcal{F}\,\mathcal{W}(\vartheta-2) + \mathcal{W}(\vartheta-2)\,\mathcal{G} + \Sigma(\vartheta), & \vartheta \in [0,4], \\ \mathcal{W}(\vartheta) = \Upsilon(\vartheta), \quad \mathcal{W}'(\vartheta) = \Upsilon'(\vartheta), & \vartheta \in [-2,0], \end{cases} \tag{18}$$

where

$$\mathcal{F} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Sigma(\vartheta) = \begin{bmatrix} \vartheta & 0 & 1 \\ 0 & \vartheta & 0 \\ 1 & 0 & \vartheta \end{bmatrix}, \quad \Upsilon(\vartheta) = \begin{bmatrix} 1+\vartheta & 0 & 0 \\ 0 & 1-\vartheta & 0 \\ 0 & 0 & 1+\vartheta \end{bmatrix}.$$

First, we verify that the matrices are non-commutative. A direct calculation yields:

$$\mathcal{F}\mathcal{G} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \neq \mathcal{G}\mathcal{F} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$$

Furthermore, $\mathcal{G}\Upsilon(\vartheta) \neq \Upsilon(\vartheta)\mathcal{G}$ and $\mathcal{G}\Sigma(\vartheta) \neq \Sigma(\vartheta)\mathcal{G}$. Thus, the conditions of Theorems 4 and 5 are perfectly satisfied.

The particular solutions for the homogeneous part are

$$\mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta-2) := \begin{cases} \Theta, & -\infty < \vartheta < 0, \\ \vartheta\varphi'(0), & 0 \le \vartheta \le 2, \\ \vartheta\varphi'(0) + \frac{8(\vartheta-2)^{\frac{5}{2}}}{15\sqrt{\pi}}(\mathcal{F}\Upsilon'(0) + \Upsilon'(0)\mathcal{G}), & 2 \le \vartheta \le 4, \end{cases}$$

$$\mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta-2) := \begin{cases} \Theta, & -\infty < \vartheta < 0, \\ \Upsilon(0), & 0 \le \vartheta \le 2, \\ \Upsilon(0) + \frac{4(\vartheta-2)^{\frac{3}{2}}}{3\sqrt{\pi}}(\mathcal{F}\Upsilon(0) + \Upsilon(0)\mathcal{G}), & 2 \le \vartheta \le 4, \end{cases}$$

$$\mathcal{I}_{\frac{3}{2},2}^{\Psi}(\vartheta-\zeta-4) := \begin{cases} \Theta, & -\infty < \vartheta < 2+\zeta, \\ \Psi\frac{2(\vartheta-\zeta-2)^{\frac{1}{2}}}{\sqrt{\pi}}, & 2+\zeta \le \vartheta < 4+\zeta, \\ \Psi\frac{2(\vartheta-\zeta-2)^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\vartheta-\zeta-4)^2}{2}(\mathcal{F}\Psi + \Psi\mathcal{G}), & 4+\zeta \le \vartheta < 6+\zeta, \end{cases}$$

and

$$\mathcal{I}_{\frac{3}{2},2}^{\Sigma(\zeta)}(\vartheta-\zeta-2) := \begin{cases} \Theta, & -\infty < \vartheta < \zeta, \\ \Sigma(\zeta)\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}}, & \zeta \le \vartheta \le 2+\zeta, \\ \Sigma(\zeta)\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{(\vartheta-\zeta-2)^2}{2}(\mathcal{F}\Sigma(\zeta) + \Sigma(\zeta)\mathcal{G}), & 2+\zeta \le \vartheta \le 4+\zeta, \end{cases}$$

where

$$\Upsilon(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbb{I}, \qquad \Upsilon'(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$\Upsilon'(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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and

$$\Psi(\zeta) = \mathcal{F}\Upsilon(\zeta) + \Upsilon(\zeta)\mathcal{G} = \begin{bmatrix} 1+\zeta & 2 & 0 \\ 0 & 2-2\zeta & 2 \\ 1+\zeta & 0 & 3+3\zeta \end{bmatrix}.$$

Applying Theorem 4, the exact solution of (18) is

$$\begin{aligned} \mathcal{W}(\vartheta) &= \mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta-2) + \mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta-2) \\ &\quad + \int_{-2}^{0} \mathcal{I}_{\frac{3}{2},2}^{\Psi(\zeta)}(\vartheta-\zeta-4)\mathrm{d}\zeta \\ &\quad + \int_{0}^{\vartheta} \mathcal{I}_{\frac{3}{2},2}^{\Sigma(\zeta)}(\vartheta-\zeta-2)\mathrm{d}\zeta, \quad \textit{for } -2 \le \vartheta \le 4. \\ &= \begin{pmatrix} W_{11}(\vartheta) & W_{12}(\vartheta) & W_{13}(\vartheta) \\ W_{21}(\vartheta) & W_{22}(\vartheta) & W_{23}(\vartheta) \\ W_{31}(\vartheta) & W_{32}(\vartheta) & W_{33}(\vartheta) \end{pmatrix}. \end{aligned} \tag{19}$$

Therefore, the matrix solutions for $\vartheta \in [0,2)$ are given by the initial extension:

$$\begin{aligned} &W_{11}(\vartheta) = 1+\vartheta, \quad W_{12}(\vartheta) = 0, \quad W_{13}(\vartheta) = 0, \\ &W_{21}(\vartheta) = 0, \quad W_{22}(\vartheta) = 1-\vartheta, \quad W_{23}(\vartheta) = 0, \\ &W_{31}(\vartheta) = 0, \quad W_{32}(\vartheta) = 0, \quad W_{33}(\vartheta) = 1+\vartheta. \end{aligned}$$

For $\vartheta \in [2,4]$, the matrix solutions are given by:

$$\begin{aligned} W_{11}(\vartheta) &= 0.000744\vartheta^5 + 0.066355\vartheta^4 - 0.412321\vartheta^3 + 2.218414\vartheta^2 - 2.448844\vartheta + 2.513434, \\ W_{12}(\vartheta) &= -0.000080\vartheta^5 + 0.042836\vartheta^4 - 0.006743\vartheta^3 - 0.980567\vartheta^2 + 2.638739\vartheta - 1.984073, \\ W_{13}(\vartheta) &= 0.000299\vartheta^5 + 0.284853\vartheta^4 - 2.724149\vartheta^3 + 10.281541\vartheta^2 - 15.447180\vartheta + 8.696352, \\ W_{21}(\vartheta) &= -0.000634\vartheta^5 + 0.053331\vartheta^4 + 0.732635\vartheta^3 - 5.225848\vartheta^2 + 13.648334\vartheta - 8.831256, \\ W_{22}(\vartheta) &= -0.002003\vartheta^5 - 0.044245\vartheta^4 + 2.269912\vartheta^3 - 11.997929\vartheta^2 + 28.776895\vartheta - 16.883361, \\ W_{23}(\vartheta) &= -0.000012\vartheta^5 + 0.041838\vartheta^4 - 0.334338\vartheta^3 + 1.002935\vartheta^2 - 1.337596\vartheta + 0.669119, \\ W_{31}(\vartheta) &= -0.000472\vartheta^5 + 0.008138\vartheta^4 + 0.769426\vartheta^3 - 4.573406\vartheta^2 + 10.433732\vartheta - 6.716328, \\ W_{32}(\vartheta) &= -0.000634\vartheta^5 + 0.136664\vartheta^4 + 0.065968\vartheta^3 - 3.225848\vartheta^2 + 10.981667\vartheta - 7.497923, \\ W_{33}(\vartheta) &= 0.002104\vartheta^5 + 0.455753\vartheta^4 - 4.914529\vartheta^3 + 20.971378\vartheta^2 - 34.938851\vartheta + 21.373695 \end{aligned}$$

We now check the H-U stability of (18). By setting $\varepsilon = 1$ and considering the perturbation matrix

$$\mathcal{R}(\vartheta) = \begin{bmatrix} \vartheta/4 & 0 & 0 \\ 0 & \vartheta/4 & 0 \\ 0 & 0 & \vartheta/4 \end{bmatrix}.$$

Clearly, $\|\mathcal{R}(\vartheta)\| = \vartheta/4 \le 1$ for $\vartheta \in [0,4]$. Thus, the perturbed solutions $\Phi(\vartheta)$ to Equation (15) are given by

$$\begin{aligned} \Phi(\vartheta) &= \mathcal{O}_{\frac{3}{2},2}^{\Upsilon(0)}(\vartheta-2) + \mathcal{H}_{\frac{3}{2},2}^{\Upsilon'(0)}(\vartheta-2) \\ &\quad + \int_{-2}^{0} \mathcal{I}_{\frac{3}{2},2}^{(\mathcal{F}\Upsilon(\zeta)+\Upsilon(\zeta)\mathcal{G})}(\vartheta-4-\zeta)\mathrm{d}\zeta \end{aligned}$$

$$+\int_0^{\vartheta}\mathcal{I}_{\frac{3}{2},2}^{(\Sigma(\zeta)+\mathcal{R}(\zeta))}(\vartheta-\zeta-2)\mathrm{d}\zeta$$
$$=\begin{pmatrix}\phi_{11}(\vartheta) & \phi_{12}(\vartheta) & \phi_{13}(\vartheta)\\ \phi_{21}(\vartheta) & \phi_{22}(\vartheta) & \phi_{23}(\vartheta)\\ \phi_{31}(\vartheta) & \phi_{32}(\vartheta) & \phi_{33}(\vartheta)\end{pmatrix},$$

where

$$\mathcal{I}_{\frac{3}{2},2}^{(\Sigma(\zeta)+\mathcal{R}(\zeta))}(\vartheta-\zeta-2):=\begin{cases}\Theta, & -\infty<\vartheta<\zeta,\\ (\Sigma(\zeta)+\mathcal{R}(\zeta))\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}}, & \zeta\le\vartheta\le 2+\zeta,\\ (\Sigma(\zeta)+\mathcal{R}(\zeta))\frac{2(\vartheta-\zeta)^{\frac{1}{2}}}{\sqrt{\pi}} & \\ +[\mathcal{F}(\Sigma(\zeta)+\mathcal{R}(\zeta))+(\Sigma(\zeta)+\mathcal{R}(\zeta))\mathcal{G}]\frac{(\vartheta-\zeta-2)^2}{2} & \\ , & 2+\zeta\le\vartheta\le 4+\zeta,\end{cases}$$

Therefore, the matrix perturbed solutions for $\vartheta\in[0,2)$ *are given by*

$$\begin{aligned}&\phi_{11}(\vartheta)=1+\vartheta,\quad \phi_{12}(\vartheta)=0,\quad \phi_{13}(\vartheta)=0,\\ &\phi_{21}(\vartheta)=0,\quad \phi_{22}(\vartheta)=1-\vartheta,\quad \phi_{23}(\vartheta)=0,\\ &\phi_{31}(\vartheta)=0,\quad \phi_{32}(\vartheta)=0,\quad \phi_{33}(\vartheta)=1+\vartheta.\end{aligned}$$

for $\vartheta\in[2,4)$, *the matrix perturbed solutions are given by the degree 5 polynomial approximations:*

$$\begin{aligned}\phi_{11}(\vartheta)&=0.000803\vartheta^5+0.075310\vartheta^4-0.469854\vartheta^3+2.543516\vartheta^2-2.818228\vartheta+2.692428,\\ \phi_{12}(\vartheta)&=-0.000080\vartheta^5+0.042836\vartheta^4-0.006743\vartheta^3-0.980567\vartheta^2+2.638739\vartheta-1.984073,\\ \phi_{13}(\vartheta)&=0.000299\vartheta^5+0.295269\vartheta^4-2.807482\vartheta^3+10.531541\vartheta^2-15.780513\vartheta+8.863019,\\ \phi_{21}(\vartheta)&=-0.000634\vartheta^5+0.074164\vartheta^4+0.565968\vartheta^3-4.725848\vartheta^2+12.981667\vartheta-8.497923,\\ \phi_{22}(\vartheta)&=-0.001944\vartheta^5-0.024873\vartheta^4+2.129045\vartheta^3-11.422827\vartheta^2+28.074178\vartheta-16.537701,\\ \phi_{23}(\vartheta)&=-0.000012\vartheta^5+0.041838\vartheta^4-0.334338\vartheta^3+1.002935\vartheta^2-1.337596\vartheta+0.669119,\\ \phi_{31}(\vartheta)&=-0.000472\vartheta^5+0.008138\vartheta^4+0.769426\vartheta^3-4.573406\vartheta^2+10.433732\vartheta-6.716328,\\ \phi_{32}(\vartheta)&=-0.000634\vartheta^5+0.157497\vartheta^4-0.100699\vartheta^3-2.725848\vartheta^2+10.315001\vartheta-7.164589,\\ \phi_{33}(\vartheta)&=0.002163\vartheta^5+0.485542\vartheta^4-5.138728\vartheta^3+21.796480\vartheta^2-35.974900\vartheta+21.886022\end{aligned}$$

Additionally, applying Theorem 5, we compute:

$$W=(\|\mathcal{F}\|+\|\mathcal{G}\|)=3+1=4,$$

and

$$\lambda=4^{3/2}\mathcal{E}_{\frac{3}{2},\frac{5}{2}}\left(4\cdot 4^{3/2}\right)=8\mathcal{E}_{\frac{3}{2},\frac{5}{2}}(32)>0,$$

then

$$\|\mathcal{W}(\vartheta)-\Phi(\vartheta)\|<8\mathcal{E}_{\frac{3}{2},\frac{5}{2}}(32).$$

System (18) is consequently H-U stable.

For a quantitative assessment of the accuracy, Table 4 reports the maximum absolute error $\max_{ij}|\mathcal{W}_{ij}(\vartheta)-\Phi_{ij}(\vartheta)|$ *on* $[0,4]$ *for the* 3×3 *system (18). As observed, several components (such as* W_{13}, W_{21}, *and* W_{32}*) exhibit zero error, meaning the perturbation matrix* $\mathcal{R}(\vartheta)$ *does not affect them. For the affected components, the error is small and strictly bounded by the theoretical stability constant* $\lambda=8\mathcal{E}_{3/2,5/2}(32)$ *predicted by Theorem 5, confirming the Hyers–Ulam stability of the system.*

Table 4. Maximum absolute error between exact and perturbed solutions for the 3×3 system (18) on the interval $[0, 4]$.

Component	Maximum Absolute Error Max $\lvert \mathcal{W}_{ij} - \Phi_{ij} \rvert$
$W_{11} - \Phi_{11}$	2.573873
$W_{12} - \Phi_{12}$	3.333333×10^{-1}
$W_{13} - \Phi_{13}$	0
$W_{21} - \Phi_{21}$	0
$W_{22} - \Phi_{22}$	2.740540
$W_{23} - \Phi_{23}$	3.333333×10^{-1}
$W_{31} - \Phi_{31}$	1.666667×10^{-1}
$W_{32} - \Phi_{32}$	0
$W_{33} - \Phi_{33}$	2.907206

To illustrate the super-exponential convergence of the FDMLTMF series discussed in Remark 3, we evaluate the magnitude of the sequence terms $a_\rho = \frac{W^\rho}{\Gamma(\rho\varpi+1)}$ *for the parameter values of this example (*$W = 4$ *and* $\varpi = 3/2$*). Table 5 demonstrates that although the numerator grows exponentially, the Gamma function in the denominator grows factorially, forcing the terms to decay to zero at an extremely rapid rate. Specifically, the terms become smaller than* 1 *starting at* $\rho = 4$*, and by* $\rho = 6$ *the term is negligible (*0.01*). Therefore, to ensure a high degree of accuracy while maintaining computational efficiency, we truncate the FDMLTMF series at* $l = 6$ *for the numerical simulations in this section.*

Table 5. Super-exponential decay of the series terms $a_\rho = \frac{4^\rho}{\Gamma(\frac{3}{2}\rho+1)}$ for the 3×3 system.

Index ρ	Numerator 4^ρ	Denominator $\Gamma(1.5\rho + 1)$	Term Magnitude a_ρ
1	4	1.3293	3.009
2	16	6	2.6667
3	64	52.343	1.2227
4	256	720	0.35556
5	1024	14,034	7.2964×10^{-2}
6	4096	3.6288×10^5	1.1287×10^{-2}
7	16,384	1.1899×10^7	1.3769×10^{-3}

To conclude, in the numerical results drawn in Figure 2, we can observe the evolution in ϑ *of the exact solution* $\mathcal{W}(\vartheta)$ *and the approximated solution of (18) called* $\Phi(\vartheta)$*. In this figure and Table 4, we notice that the maximum absolute error for the curves* $W_{13}(\vartheta)$ *and* $\phi_{13}(\vartheta)$*,* $W_{21}(\vartheta)$ *and* $\phi_{21}(\vartheta)$*, and* $W_{32}(\vartheta)$ *and* $\phi_{32}(\vartheta)$ *is exactly zero: they are exactly identical. Furthermore, the errors for the remaining off-diagonal components,* $W_{12}(\vartheta)$ *and* $\phi_{12}(\vartheta)$ *as well as* $W_{23}(\vartheta)$ *and* $\phi_{23}(\vartheta)$*, are relatively small (approximately* 0.33*), indicating that* $\phi_{12}(\vartheta)$ *and* $\phi_{23}(\vartheta)$ *are almost the same as their exact counterparts. The largest deviations are observed in the diagonal components* W_{33}*,* W_{22}*, and* W_{11}*, with errors of approximately* 2.91*,* 2.74*, and* 2.57*, respectively. Overall, the maximum absolute errors across all components are small and strictly bounded by the theoretical stability constant. This indicates that the* $W_{ij}(\vartheta)$ *is close to* $\phi_{ij}(\vartheta)$ *for* $\vartheta \in [0, 4]$*; therefore, the system is stable for the bounded perturbation, as mentioned in Theorem 5.*

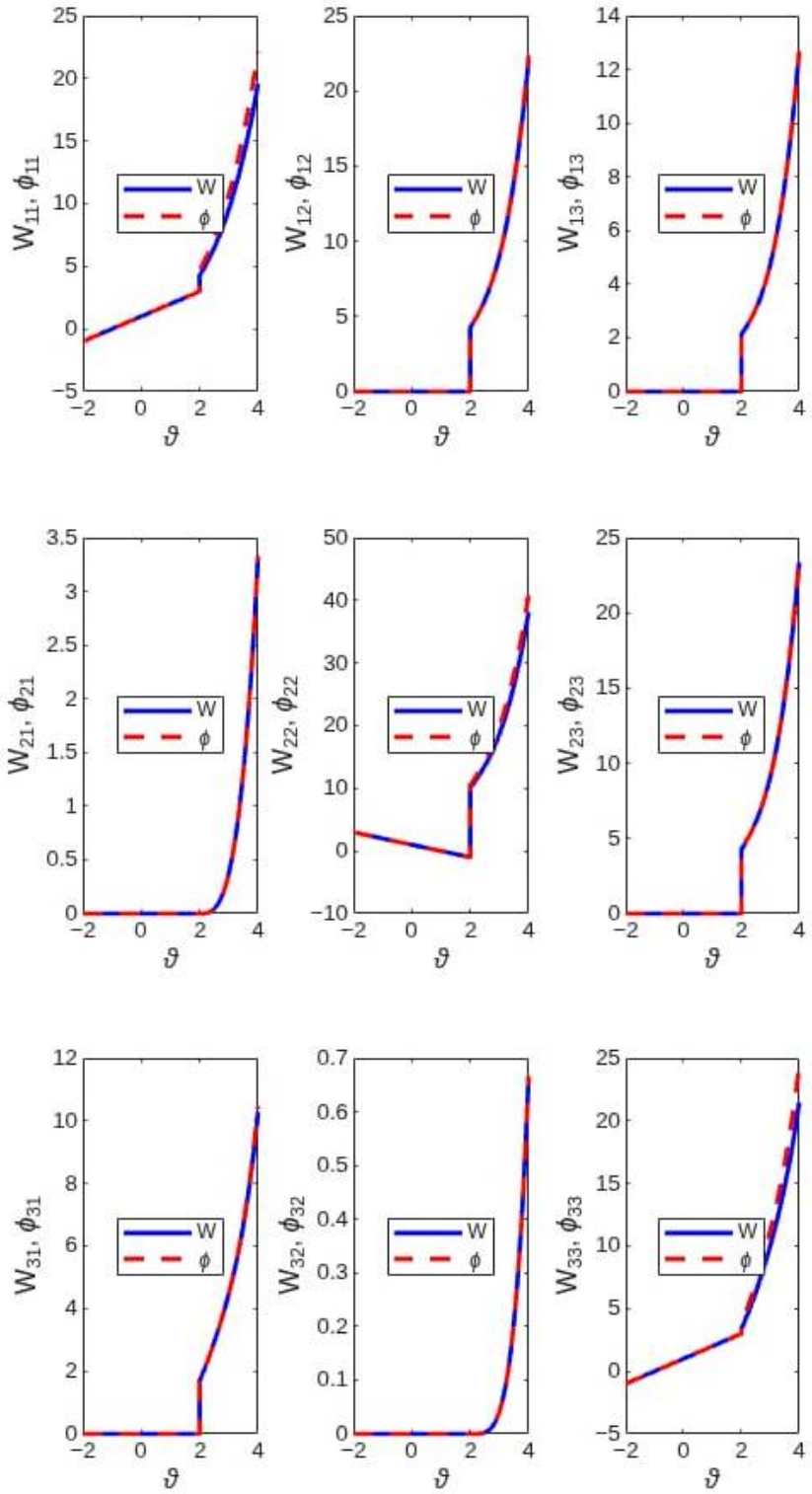


Figure 2. Comparison of $\mathcal{W}$ (solid blue line) with Φ (dashed red line) on a typical problem setting: exact solution $\mathcal{W}$ versus the approximated solution Φ for (18).

6. Conclusions

In this paper, we present exact expressions of solutions for fractional delayed two-sided matrix equations without requiring commutativity among the coefficient matrices and the inhomogeneous term. To establish a straightforward, unified framework for constructing fundamental solutions under non-commutative conditions, we formulate a novel approach using generalized binomial expansions alongside new fractional delayed Mittag-Leffler-

type matrices. Furthermore, we prove Hyers–Ulam stability for the non-homogeneous case. Our analytical findings are complemented and validated by numerical examples. In summary, these results extend and strengthen existing literature in the field. From an application viewpoint, the non-commutative formulation developed here is directly applicable to networked control systems, anisotropic viscoelastic models, and multi-asset financial models with delay, where the coefficient matrices are generically non-commuting.

Multiple promising directions for follow-up work stem from this study. Primary among them is generalizing the framework to systems governed by fractional-order models with multiple time delays. Beyond theoretical extensions, applications are expected in diverse areas, including control theory, finite-time stability analysis, and signal processing, where both delay effects and non-commutative dynamics play a central role in governing stability.

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