

Article

Cumulative Rewards for Birth–Death Processes with Catastrophes

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Abstract

This paper investigates cumulative reward functionals for birth–death processes with Poisson-driven total catastrophes. Using an auxiliary killed Markov process, we derive the integral equations for the distribution and expectation of the reward accumulated up to the first catastrophe, and establish the existence and uniqueness of bounded solutions. Explicit expressions are provided for constant birth–death rates. The asymptotic limits of the scaled expected reward are analyzed for vanishing and diverging catastrophe rates, with limiting behavior classified by the recurrence of the underlying catastrophe-free birth–death process.

Keywords: birth–death process; catastrophe; cumulative reward; Killed Markov process; Laplace transform

MSC: 60J80; 60J05

1. Introduction

During the theoretical development of continuous-time countable Markov chains, two fundamental concepts, the decay parameter and quasi-stationary distributions, have occupied a central research position for decades. Quasi-stationary distributions describe the long-run conditional behavior of Markovian systems prior to absorption, and their origin can be traced back to Yaglom [1], who investigated asymptotic properties for subcritical Galton–Watson branching processes. The decay parameter quantifies the exponential convergence rate of transition probabilities for infinite-state Markov chains and was pioneered by Kingman in the early 1960s. Starting with the seminal contributions of Kingman [2] and Vere-Jones [3], substantial theoretical advances have been achieved. Darroch and Seneta [4] examined quasi-stationary behavior for finite absorbing continuous-time Markov chains; Flaspohler [5] extended these results to countably infinite state spaces. Pollett [6,7] developed practical methodologies to compute quasi-stationary distributions directly from generator matrices. Further important contributions concerning μ -invariant measures, skip-free absorbing chains, exponential ergodicity and decay-parameter bounds were delivered by Kelly [8], Kijima [9], Nair & Pollett [10], Tweedie [11], and Van Doorn [12], among many other researchers. Most classical literature focuses on stationary, quasi-stationary and ergodic properties of ordinary birth–death systems, while sample-path behaviors conditioned on pre-absorption epochs deserve further exploration.

Birth–death processes subject to exogenous catastrophes form an important subclass of Markov processes, where external Poisson-driven catastrophic shocks force the system to jump abruptly from positive states toward lower states, most commonly state zero, which



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is distinct from local endogenous birth–death transitions. Such stochastic models find broad practical applications across queuing theory, population biology, risk theory and reliability engineering: catastrophes may represent sudden server breakdowns clearing all customers in queuing networks, mass-extinction environmental hazards in ecological population modeling, large-scale market crashes or catastrophic claim arrivals in insurance risk theory, and sudden total equipment failures in reliability systems.

Existing literature on Markov chains with catastrophes can be roughly divided into several branches. A large body of work concentrates on stationary and transient state distributions, ergodicity criteria, the extinction probability and the mean extinction time; see [13–20]. Another stream of research targets quasi-stationary and quasi-limiting distributions for catastrophe-bearing Markov chains, establishing existence conditions and spectral representations conditioned on non-absorption. Various generalizations have been developed towards binomial, multi-type, and state-dependent catastrophe mechanisms, where catastrophes may only partially reduce population sizes instead of performing a complete reset to state zero. Many of these results are expressed via generating functions, Laplace transforms or spectral decompositions, though explicit closed-form solutions are mostly restricted to special rate structures such as linear birth–death rates.

Within this broad body of literature, several representative papers are particularly close to our setting of Poisson-driven total catastrophes for birth–death processes. Cairns and Pollett [21] and Pollett et al. [22] derived explicit formulae for the expected extinction times of general birth–death-catastrophe models. Their scope is limited to moments of the extinction stopping time, and they do not consider path-integrated cumulative reward functionals before the first catastrophe. Chao and Zheng [23] investigated immigration–birth–death processes subject to total catastrophes and characterized transient and stationary state distributions, but additive reward integrals along sample paths are outside their scope. Di Crescenzo et al. [15,24] studied queuing and birth–death models with catastrophes, focusing on busy-period quantities and the distribution of the first catastrophe epoch. Their analyses are built upon hitting-time and queue-length statistics, without treating general state-dependent cumulative reward functionals.

Taken together, these closely related works mainly explore state distributions and stopping-time statistics for catastrophe-perturbed systems. Comparatively little attention has been paid to cumulative reward functionals evaluated over the random time horizon up to the first exogenous catastrophe. Most notably, how the recurrence–transience classification of the underlying catastrophe-free birth–death process Q_0 shapes the asymptotic behavior of scaled expected reward $\xi V(i)$ has not been systematically treated in existing references.

In many real-world scenarios, it is essential to evaluate cumulative quantities accumulated over the random interval up to the arrival of the first disaster: for instance, cumulative system throughput before server crash, total population-weighted sojourn time prior to mass extinction, or aggregated risk exposure before a market crash. Questions concerning the integral-equation characterization of the reward distribution, its uniqueness, and the asymptotic behavior of expected reward as the catastrophe rate ξ varies over $(0, \infty)$ remain largely under-explored. This paper aims to fill these research gaps.

Definition 1. A matrix $Q = (q_{i,j} : i, j \in \mathbb{Z}_+)$ is called a birth–death q -matrix with catastrophe if

$$q_{i,j} = \begin{cases} \beta_i, & i \geq 0, j = i + 1, \\ \alpha_1 + \xi, & i = 1, j = 0, \\ \xi, & i \geq 2, j = 0, \\ \alpha_i, & i \geq 2, j = i - 1, \\ -\beta_0, & i = j = 0, \\ -(\alpha_i + \beta_i + \xi), & i = j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\xi > 0, \quad \alpha_i > 0 \ (i \geq 1), \quad \beta_i > 0 \ (i \geq 0).$$

Here $\xi > 0$ denotes the exogenous catastrophe rate. The state transition diagram of Q is given as Figure 1 below:

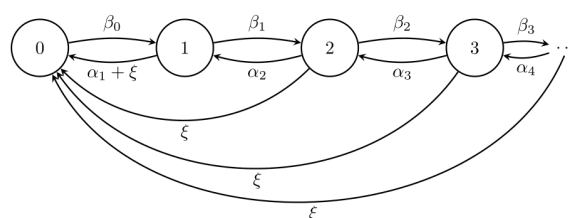


Figure 1. State transition diagram of Q .

When $\xi = 0$, Q reduces to the ordinary birth–death q -matrix denoted by Q_0 . Throughout this paper, we assume that Q_0 is regular, which guarantees the regularity of Q . Write $q_i := -q_{i,i}$ for $i \in \mathbb{Z}_+$, so $q_0 = \beta_0$ and $q_i = \alpha_i + \beta_i + \xi$ for $i \geq 1$. Let $\{X_t : t \geq 0\}$ be the Q -process and $P(t) = (p_{i,j}(t) : i, j \in \mathbb{Z}_+)$ be its transition probability function. For initial state $X_0 = i$, P_i denotes the corresponding probability measure and $\mathbb{E}_i$ denotes the expectation.

Denote by $\{X_t^{(0)} : t \geq 0\}$ the underlying catastrophe-free birth–death process governed by Q_0 . Conditional on the sample path of $\{X_t^{(0)} : t \geq 0\}$, exogenous catastrophe events form an inhomogeneous Poisson point process with intensity $\xi \cdot \mathbf{1}_{\{X_t^{(0)} \geq 1\}}$. That is, catastrophes would arrive at rate ξ during periods when the underlying catastrophe-free process occupies non-zero states, while catastrophe intensity vanishes whenever $X_t^{(0)} = 0$. This conditional-Poisson perspective is probabilistically equivalent to the q -matrix specification for the full Q -process $\{X_t : t \geq 0\}$, and captures the Markovian conditional-independence property used in our subsequent analysis.

Define the first exogenous catastrophe time

$$\tau = \inf\{t > 0 : \text{exogenous catastrophe occurs at time } t\}.$$

Given a bounded non-negative instantaneous reward function $r(\cdot)$ satisfying $0 \leq r(\cdot) \leq R$, the total accumulated reward up to the first catastrophe and its conditional expectation are given by

$$R(\tau) := \int_0^\tau r(X_t) dt, \quad V(i) := \mathbb{E}_i[R(\tau)], \quad i \in \mathbb{Z}_+.$$

The primary objective of this paper is to investigate the probabilistic properties of the total reward $R(\tau)$ and its mean value $V(i)$. We construct an auxiliary killed process

augmented with an absorbing cemetery state -1 to faithfully represent system trajectories prior to the first exogenous catastrophe. Recursive formulae for the distribution and expectation of the partial reward evaluated at each jump time are established. We derive the integral equation and Laplace transform equation for the reward distribution $F(i, x) := P_i(R(\tau) \leq x)$ and derive the equation for the expected reward $V(i)$. Furthermore, we prove the uniqueness of the solution for the corresponding equation. Under the condition $\{i \in \mathbb{Z}_+ : r(i) = 0\} = \emptyset$, the existence of a strictly positive probability density for $F(i, x)$ is also proved.

Moreover, we thoroughly analyze the limiting behavior of the scaled expected reward $\xi V(i)$ as the catastrophe rate ξ tends to zero and infinity. For small $\xi \downarrow 0$, a complete trichotomy is presented: when the underlying ordinary birth–death process Q_0 is positive-recurrent, the limit is expressed via the stationary distribution of Q_0 ; when Q_0 is null-recurrent or transient, the limit is controlled by the lim-inf and lim-sup bounds of reward function $r(\cdot)$. For large $\xi \rightarrow \infty$, explicit state-dependent limiting results are established.

We remark that although the original research motivation lists several further potential directions, including the full distribution of τ , exponential tail bound for $F_i(x)$, moment analysis of $F_i(x)$, and conditional state distribution under linear-rate parametrization $\alpha_i = i\alpha, \beta_i = i\beta$, those problems are not completely resolved within the present work and are left as open topics for future investigation.

The rest of the paper is organized as follows. Section 2 introduces an auxiliary killed process by adding an absorbing cemetery state -1 , and establishes recursive formulae for the distribution and expectation of the partial reward accumulated up to each jump time. Section 3 derives the full distribution and expectation of $R(\tau)$, together with uniqueness theorems for the associated equations. Section 4 is devoted to limiting behaviors of $V(i)$ as $\xi \downarrow 0$ and $\xi \rightarrow \infty$. For the small ξ case, we establish a trichotomy: for positive-recurrent Q_0 , limits are expressed in terms of the stationary distribution of Q_0 ; for null-recurrent and transient Q_0 , limits are controlled by the asymptotic behavior of reward function $r(\cdot)$. For large ξ , explicit state-dependent limiting results are obtained.

2. Preliminaries and Auxiliary Killed Process

By the definition of Q , we see that the exogenous catastrophe drives the system directly from state $i \geq 1$ to state 0 and the endogenous transition rate α_1 also drives the system from state 1 to state 0. In order to discuss the behavior of $\{X_t : t \geq 0\}$ before the exogenous catastrophe, we first need to distinguish the exogenous catastrophe time. For this purpose, we extend the state space to $\tilde{\mathbb{Z}}_+ := \mathbb{Z}_+ \cup \{-1\}$ by adding an extra state -1 and construct another q -matrix $\tilde{Q} := (\tilde{q}_{i,j} : i, j \in \tilde{\mathbb{Z}}_+)$ as follows:

$$\tilde{q}_{i,j} = \begin{cases} \beta_i, & i \geq 0, j = i + 1, \\ \alpha_i, & i \geq 1, j = i - 1, \\ \xi, & i \geq 1, j = -1, \\ -\beta_0, & i = j = 0, \\ -(\alpha_i + \beta_i + \xi), & i = j \geq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Denote

$$\hat{q}_{i,j} := \begin{cases} \tilde{q}_{i,j}, & i \neq j, i, j \in \tilde{\mathbb{Z}}_+, \\ 0, & i = j \in \tilde{\mathbb{Z}}_+. \end{cases}$$

Let $\{\tilde{X}_t : t \geq 0\}$ denote the $\tilde{Q}$ -process and $\tilde{P}(t) := (\tilde{p}_{ij}(t) : i, j \in \mathbb{Z}_+)$ denote the corresponding transition probability function. For any $i \in \mathbb{Z}_+$, the probability measure provided $\tilde{X}_0 = i$ is denoted by $\tilde{P}_i$ and the corresponding mathematical expectation is denoted by $\tilde{\mathbb{E}}_i$. Let

$$\tilde{\tau} := \inf\{t \geq 0 : \tilde{X}_t = -1\} \quad (2)$$

be the extinction time of $\{\tilde{X}_t : t \geq 0\}$ and $\tilde{\sigma}_n$ be the n th jumping time of $\{\tilde{X}_t : t \geq 0\}$. Further extend the reward function $r(\cdot)$ on to $\mathbb{Z}_+$ by defining $r(-1) = 0$. Define

$$\tilde{R}(\tilde{\tau}) := \int_0^{\tilde{\tau}} r(\tilde{X}_t) dt = \int_0^{\infty} r(\tilde{X}_t) dt, \quad (3)$$

$$\tilde{F}(i, x) := \tilde{P}_i(\tilde{R}(\tilde{\tau}) \leq x), \quad x \geq 0, i \in \mathbb{Z}_+. \quad (4)$$

and

$$\tilde{V}(i) := \tilde{\mathbb{E}}_i[\tilde{R}(\tilde{\tau})], \quad i \in \mathbb{Z}_+. \quad (5)$$

Lemma 1. The process $\{X_t : t \in [0, \tau)\}$ and the auxiliary killed process $\{\tilde{X}_t : t \in [0, \tilde{\tau})\}$ are equal in law. Hence,

$$P_i(\tau \leq t) = \tilde{P}_i(\tilde{\tau} \leq t) = \tilde{p}_{i,-1}(t), \quad t \geq 0, i \in \mathbb{Z}_+, \quad (6)$$

$$P_i(X_t = j | t < \tau) = \tilde{P}_i(\tilde{X}_t = j | t < \tilde{\tau}) = \tilde{p}_{ij}(t), \quad t \geq 0, i, j \in \mathbb{Z}_+. \quad (7)$$

Therefore,

$$F(i, x) = \tilde{F}(i, x), \quad x \geq 0, i \in \mathbb{Z}_+ \text{ and } V(i) = \tilde{V}(i), \quad i \in \mathbb{Z}_+. \quad (8)$$

Proof. By construction of $\tilde{Q}$, the transition rates of $\{\tilde{X}_t : t \geq 0\}$ before hitting the absorbing state -1 coincide exactly with those of $\{X_t : t \geq 0\}$ before the arrival of an exogenous catastrophe. The only difference is that an exogenous catastrophe drives $\{\tilde{X}_t : t \geq 0\}$ into the absorbing state -1 , whereas $\{X_t : t \geq 0\}$ transfers to state 0. In this way, $\tilde{Q}$ distinguishes the abrupt jumps triggered by exogenous catastrophes from intrinsic transitions in Q . Consequently, the transition rates of $\{\tilde{X}_t : t \in [0, \tilde{\tau})\}$ and $\{X_t : t \in [0, \tau)\}$ are the same. It follows that the finite-dimensional distributions of $\{\tilde{X}_t : t \in [0, \tilde{\tau})\}$ and $\{X_t : t \in [0, \tau)\}$ agree. Therefore, Equations (6)–(8) hold. $\square$

By Lemma 1, in order to discuss the behavior of $\{X_t : t \geq 0\}$ before exogenous catastrophe, we only need to consider the behavior of $\{\tilde{X}_t : t \geq 0\}$ before extinction. For this purpose, we define

$$\tilde{R}(\tilde{\sigma}_n) := \int_0^{\tilde{\sigma}_n} r(\tilde{X}_t) dt, \quad n \geq 0, \quad (9)$$

$$\tilde{F}_n(i, x) := \tilde{P}_i(\tilde{R}(\tilde{\sigma}_n) \leq x), \quad x \geq 0, i \in \mathbb{Z}_+, n \geq 0, \quad (10)$$

and

$$\tilde{V}_n(i) := \tilde{\mathbb{E}}_i[\tilde{R}(\tilde{\sigma}_n)], \quad i \in \mathbb{Z}_+, n \geq 0. \quad (11)$$

Since $\tilde{R}(\tilde{\sigma}_n) < \infty$ almost surely, $\tilde{F}_n(i, x)$ is the probability distribution function of $\tilde{R}(\tilde{\sigma}_n)$ under $\tilde{P}_i$.

Denote

$$H_0 := \{i \in \mathbb{Z}_+ : r(i) = 0\} \text{ and } H_+ := \mathbb{Z}_+ \setminus H_0.$$

Lemma 2. $\tilde{F}_n(i, x)$ satisfies the following recursive formula:

$$\begin{cases} \tilde{F}_0(i, x) = 1, & x \geq 0, i \in \mathbb{Z}_+, \\ \tilde{F}_{n+1}(0, x) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{q_0} \cdot \tilde{F}_n(j, x), & x \geq 0, 0 \in H_0, n \geq 0, \\ \tilde{F}_{n+1}(0, x) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{r(0)} \cdot \int_0^x e^{-\frac{q_0 u}{r(0)}} \tilde{F}_n(j, x-u) du, & x \geq 0, 0 \in H_+, n \geq 0, \\ \tilde{F}_{n+1}(i, x) = \frac{\zeta}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{q_i} \cdot \tilde{F}_n(j, x), & x \geq 0, i \in H_0, n \geq 0, \\ \tilde{F}_{n+1}(i, x) = \frac{\zeta}{q_i} (1 - e^{-\frac{q_i x}{r(i)}}) + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{r(i)} \int_0^x e^{-\frac{q_i u}{r(i)}} \tilde{F}_n(j, x-u) du, & x \geq 0, i \in H_+, n \geq 0. \end{cases} \quad (12)$$

i.e.,

$$\begin{cases} \tilde{\mathcal{F}}_0(i, \lambda) = \frac{1}{\lambda}, & \lambda > 0, i \in \mathbb{Z}_+, \\ \tilde{\mathcal{F}}_{n+1}(0, \lambda) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{\lambda r(0) + q_0} \cdot \tilde{\mathcal{F}}_n(j, \lambda), & \lambda > 0, n \geq 0, \\ \tilde{\mathcal{F}}_{n+1}(i, \lambda) = \frac{\zeta}{\lambda(\lambda r(i) + q_i)} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{\lambda r(i) + q_i} \cdot \tilde{\mathcal{F}}_n(j, \lambda), & \lambda > 0, i \geq 1, n \geq 0, \end{cases} \quad (13)$$

where $\tilde{\mathcal{F}}_n(i, \lambda) := \int_0^\infty e^{-\lambda x} \tilde{F}_n(i, x) dx$ ($\lambda > 0$).

Proof. Let $\tilde{\sigma}_n^{(1)}$ be the n th jumping time of $\tilde{X}_{\tilde{\sigma}_1+t}$. By the Markov property, if $i = 0 \in H_+$, then

$$\begin{aligned} & \tilde{F}_{n+1}(0, x) \\ &= \tilde{P}_0 \left(\int_0^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x \right) \\ &= \tilde{P}_0 \left(r(0)\tilde{\sigma}_1 + \int_{\tilde{\sigma}_1}^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x \right) \\ &= \tilde{P}_0 \left(r(0)\tilde{\sigma}_1 + \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x \right) \\ &= \sum_{j \in \mathbb{Z}_+} \tilde{P}_0 \left(\tilde{X}_{\tilde{\sigma}_1} = j, \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x - r(0)\tilde{\sigma}_1 \right) \\ &= \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{q_0} \int_0^{\frac{x}{r(0)}} q_0 e^{-q_0 s} \tilde{P}_j \left(\int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x - r(0)s \right) ds \\ &= \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{r(0)} \int_0^x e^{-\frac{q_0 u}{r(0)}} \tilde{F}_n(j, x-u) du. \end{aligned}$$

If $i = 0 \in H_0$, then

$$\begin{aligned}
& \tilde{F}_{n+1}(0, x) \\
&= \tilde{P}_0\left(\int_0^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x\right) \\
&= \tilde{P}_0\left(\int_{\tilde{\sigma}_1}^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x\right) \\
&= \tilde{P}_0\left(\int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x\right) \\
&= \sum_{j \in \mathbb{Z}_+} \tilde{P}_0\left(\tilde{X}_{\tilde{\sigma}_1} = j, \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x\right) \\
&= \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{q_0} \tilde{P}_j\left(\int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x\right) \\
&= \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{q_0} \tilde{F}_n(j, x).
\end{aligned}$$

If $i \geq 1$ and $i \in H_+$, then

$$\begin{aligned}
& \tilde{F}_{n+1}(i, x) \\
&= \tilde{P}_i\left(\int_0^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x\right) \\
&= \tilde{P}_i(r(i)\tilde{\sigma}_1 + \int_{\tilde{\sigma}_1}^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x) \\
&= \tilde{P}_i(r(i)\tilde{\sigma}_1 + \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x) \\
&= \tilde{P}_i(\tilde{X}_{\tilde{\sigma}_1} = -1, r(i)\tilde{\sigma}_1 \leq x) + \sum_{j \in \mathbb{Z}_+} \tilde{P}_i(\tilde{X}_{\tilde{\sigma}_1} = j, \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x - r(i)\tilde{\sigma}_1) \\
&= \frac{\xi}{q_i} (1 - e^{-\frac{q_i x}{r(i)}}) + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{q_i} \int_0^{\frac{x}{r(i)}} q_i e^{-q_i s} \tilde{P}_j\left(\int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x - r(i)s\right) ds \\
&= \frac{\xi}{q_i} (1 - e^{-\frac{q_i x}{r(i)}}) + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{r(i)} \int_0^x e^{-\frac{q_i u}{r(i)}} \tilde{F}_n(j, x - u) du.
\end{aligned}$$

If $i \geq 1$ and $i \in H_0$, then

$$\begin{aligned}
& \tilde{F}_{n+1}(i, x) \\
&= \tilde{P}_i\left(\int_0^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x\right) \\
&= \tilde{P}_i\left(\int_{\tilde{\sigma}_1}^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \leq x\right) \\
&= \tilde{P}_i\left(\int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x\right) \\
&= \tilde{P}_i(\tilde{X}_{\tilde{\sigma}_1} = -1) + \sum_{j \in \mathbb{Z}_+} \tilde{P}_i(\tilde{X}_{\tilde{\sigma}_1} = j, \int_0^{\tilde{\sigma}_n^{(1)}} r(\tilde{X}_{\tilde{\sigma}_1+t}) dt \leq x) \\
&= \frac{\xi}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{q_i} \tilde{F}_n(j, x).
\end{aligned}$$

Equation (12) is proved. Taking the Laplace transform on (12) yields (13). $\square$

Lemma 3. (i) $\tilde{V}_n(i)$ satisfies the following recursive formula:

$$\begin{cases} \tilde{V}_0(i) = 0, i \in \mathbb{Z}_+, \\ \tilde{V}_{n+1}(i) = \frac{r(i)}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{q_i} \cdot \tilde{V}_n(j), i \in \mathbb{Z}_+, n \geq 0. \end{cases} \quad (14)$$

(ii) $\{\tilde{V}_n(\cdot) : n \geq 0\}$ are bounded functions. More specifically,

$$\tilde{V}_n(0) \leq C + \frac{R}{\beta_0}, \quad \tilde{V}_n(i) \leq C, \quad i \geq 1$$

for all $n \geq 1$, where

$$C := \frac{R}{\xi} \left(1 + \frac{\alpha_1}{\beta_0}\right).$$

Proof. First prove (i). Since $\tilde{X}_t = \tilde{X}_0$ for $t \in [0, \tilde{\sigma}_1)$, we know that

$$\tilde{V}_1(i) = \tilde{\mathbb{E}}_i \left[\int_0^{\tilde{\sigma}_1} r(\tilde{X}_t) dt \right] = \tilde{\mathbb{E}}_i [r(\tilde{X}_0) \tilde{\sigma}_1] = \frac{r(i)}{q_i}, \quad i \in \mathbb{Z}_+.$$

Recursively, for $n \geq 1$, by the Markov property

$$\begin{aligned} & \tilde{V}_{n+1}(i) \\ &= \tilde{\mathbb{E}}_i \left[\int_0^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \right] \\ &= \tilde{\mathbb{E}}_i [r(\tilde{X}_0) \tilde{\sigma}_1] + \tilde{\mathbb{E}}_i \left[\int_{\tilde{\sigma}_1}^{\tilde{\sigma}_{n+1}} r(\tilde{X}_t) dt \mid \tilde{\sigma}_1, \tilde{X}_{\tilde{\sigma}_1} \right] \\ &= \frac{r(i)}{q_i} + \sum_{j \in \mathbb{Z}_+ \setminus \{i\}} \int_0^\infty \tilde{q}_{i,j} e^{-q_i s} \tilde{\mathbb{E}}_j \left[\int_0^{\tilde{\sigma}_n} r(\tilde{X}_t) dt \right] ds \\ &= \frac{r(i)}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{q_i} \cdot \tilde{V}_n(j). \end{aligned}$$

Therefore, (i) is proved.

Next prove (ii). Define

$$m_i := \begin{cases} C + \frac{R}{\beta_0}, & i = 0, \\ C, & i \geq 1. \end{cases}$$

We only need to prove that

$$\tilde{V}_n(i) \leq m_i, \quad i \in \mathbb{Z}_+, \quad n \geq 0. \quad (15)$$

Obviously, $\tilde{V}_0(i) = 0 \leq m_i$. For $n = 1$, we have

$$\tilde{V}_1(0) = \frac{r(0)}{q_0} \leq \frac{R}{\beta_0} \leq m_0$$

and

$$\tilde{V}_1(i) = \frac{r(i)}{q_i} \leq \frac{R}{\xi} \leq m_i, \quad i \geq 1.$$

Suppose that (15) holds for n ; then by the Markov property,

$$\begin{aligned} \tilde{V}_{n+1}(0) &\leq \frac{r(0)}{q_0} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0,j}}{q_0} \cdot m_j \\ &= \frac{r(0)}{q_0} + C \\ &\leq C + \frac{R}{\beta_0} = m_0, \end{aligned}$$

$$\begin{aligned}
\tilde{V}_{n+1}(1) &\leq \frac{r(1)}{q_1} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{1,j}}{q_1} \cdot m_j \\
&= \frac{r(1)}{q_1} + \frac{\alpha_1 m_0 + \beta_1 m_1}{q_1} \\
&\leq \frac{R}{q_1} + \frac{1}{q_1} [\alpha_1 (C + \frac{R}{\beta_0}) + \beta_1 C] \\
&= \frac{R}{q_1} + C(1 - \frac{\xi}{q_1}) + \frac{R\alpha_1}{q_1 \beta_0} \\
&= C - \frac{1}{q_1} [C\xi - R - \frac{R\alpha_1}{\beta_0}] \\
&= C = m_1
\end{aligned}$$

and

$$\begin{aligned}
\tilde{V}_{n+1}(i) &\leq \frac{R}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{i,j}}{q_i} \cdot m_j \\
&= \frac{R}{q_i} + \frac{C}{q_i} (\alpha_i + \beta_i) \\
&= C - \frac{1}{q_i} (C\xi - R) \\
&\leq C = m_i, \quad i \geq 2.
\end{aligned}$$

By mathematical induction, Equation (15) is proved. $\square$

3. Distribution and Expectation of Cumulative Reward

In this section, we consider the distribution of the total reward and its expectation.

Lemma 4. Suppose that Q is regular. Then

$$\tilde{R}(\tilde{\sigma}_n) \uparrow \tilde{R}(\tilde{\tau}), \quad a.s.$$

Moreover, $\tilde{V}(i) = \tilde{\mathbb{E}}_i[\tilde{R}(\tilde{\tau})] < \infty$ ($i \in \mathbb{Z}_+$) and hence, $\tilde{P}_i(\tilde{\tau} < \infty) = 1$ ($i \in \mathbb{Z}_+$).

Proof. Since -1 is absorbing and $r(-1) = 0$, we know that $\tilde{R}(\tilde{\sigma}_n) = \int_0^{\tilde{\sigma}_n} r(\tilde{X}_t) dt = \int_0^{\tilde{\tau} \wedge \tilde{\sigma}_n} r(\tilde{X}_t) dt$. Noting that $\tilde{Q}$ (since Q is regular) is regular, we further know that $\tilde{\tau} \wedge \tilde{\sigma}_n \uparrow \tilde{\tau}$ as $n \rightarrow \infty$. Therefore, $\tilde{R}(\tilde{\sigma}_n) \uparrow \tilde{R}(\tilde{\tau})$ as $n \rightarrow \infty$. On the other hand, by Lemma 3 and the monotone convergence theorem, we obtain that $\tilde{\mathbb{E}}_i[\tilde{R}(\tilde{\tau})] \leq C + \frac{R}{\beta_0} < \infty$ ($i \in \mathbb{Z}_+$) and hence $\tilde{P}_i(\tilde{\tau} < \infty) = 1$ ($i \in \mathbb{Z}_+$). $\square$

Theorem 1. (i) $\tilde{F}(i, x) := \lim_{n \rightarrow \infty} \tilde{F}_n(i, x)$ exists for all $x \geq 0$ and $i \in \mathbb{Z}_+$. $\tilde{F}(i, x)$ is the probability distribution function of $\tilde{R}(\tilde{\tau})$ starting from state $i \in \mathbb{Z}_+$.

(ii) $(\tilde{F}(i, x) : i \in \mathbb{Z}_+)$ is the unique solution of

$$\begin{cases} h(0, x) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{q_0} h(j, x), & x \geq 0, 0 \in H_0, \\ h(0, x) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{r(0)} \int_0^x e^{-\frac{q_0 u}{r(0)}} h(j, x-u) du, & x \geq 0, 0 \in H_+, \\ h(i, x) = \frac{\xi}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{q_i} h(j, x), & x \geq 0, i \geq 1, i \in H_0, \\ h(i, x) = \frac{\xi}{q_i} (1 - e^{-\frac{q_i x}{r(i)}}) + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{r(i)} \int_0^x e^{-\frac{q_i u}{r(i)}} h(j, x-u) du, & x \geq 0, i \geq 1, i \in H_+, \\ 0 \leq h(i, x) \leq 1, & x \geq 0, i \in \mathbb{Z}_+. \end{cases} \quad (16)$$

i.e., for any $\lambda > 0$, $(\tilde{\mathcal{F}}(i, \lambda) : i \in \mathbb{Z}_+)$ is the unique solution of

$$\begin{cases} \mathcal{H}(0, \lambda) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{\lambda r(0) + q_0} \cdot \mathcal{H}(j, \lambda), \\ \mathcal{H}(i, \lambda) = \frac{\xi}{\lambda(\lambda r(i) + q_i)} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{\lambda r(i) + q_i} \cdot \mathcal{H}(j, \lambda), & i \geq 1, \\ 0 \leq \lambda \mathcal{H}(i, \lambda) \leq 1, & i \in \mathbb{Z}_+, \end{cases} \quad (17)$$

where $\tilde{\mathcal{F}}(i, \lambda) := \int_0^\infty e^{-\lambda x} \tilde{F}(i, x) dx$.

Proof. By the definition of $\tilde{R}(\tilde{\sigma}_n)$, we see that $\tilde{R}(\tilde{\sigma}_{n+1}) \geq \tilde{R}(\tilde{\sigma}_n)$ and $\tilde{R}(\tilde{\sigma}_n) \uparrow \tilde{R}(\tilde{\tau})$ as $n \rightarrow \infty$. Hence, $\lim_{n \rightarrow \infty} \tilde{F}_n(i, x) = \lim_{n \rightarrow \infty} \tilde{P}_i(\tilde{R}(\tilde{\sigma}_n) \leq x) = \tilde{P}_i(\tilde{R}(\tilde{\tau}) \leq x) = \tilde{F}(i, x)$. By Lemma 4, $\tilde{R}(\tilde{\tau}) < \infty$, a.s.; therefore, (i) is proved.

Next prove (ii). By Lemma 2, we know that for any $\lambda > 0$, $\tilde{\mathcal{F}}_0(i, \lambda) = \frac{1}{\lambda}$ ($i \in \mathbb{Z}_+$),

$$\tilde{\mathcal{F}}_{n+1}(0, \lambda) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{\lambda r(0) + q_0} \cdot \tilde{\mathcal{F}}_n(j, \lambda), \quad n \geq 0,$$

and

$$\tilde{\mathcal{F}}_{n+1}(i, \lambda) = \frac{\xi}{\lambda(\lambda r(i) + q_i)} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{\lambda r(i) + q_i} \cdot \tilde{\mathcal{F}}_n(j, \lambda), \quad i \geq 1, n \geq 0,$$

where $\tilde{\mathcal{F}}_n(i, \lambda) := \int_0^\infty e^{-\lambda x} \tilde{F}_n(i, x) dx$ ($\lambda > 0$). Since $\tilde{\mathcal{F}}_n(i, \lambda)$ is decreasing in n and $\lim_{n \rightarrow \infty} \tilde{\mathcal{F}}_n(i, \lambda) = \tilde{\mathcal{F}}(i, \lambda)$, letting $n \rightarrow \infty$ in the above equality and using monotone convergence theorem yields

$$\tilde{\mathcal{F}}(0, \lambda) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{\lambda r(0) + q_0} \cdot \tilde{\mathcal{F}}(j, \lambda)$$

and

$$\tilde{\mathcal{F}}(i, \lambda) = \frac{\xi}{\lambda(\lambda r(i) + q_i)} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{\lambda r(i) + q_i} \cdot \tilde{\mathcal{F}}(j, \lambda), \quad i \geq 1.$$

On the other hand, it is obvious that $\lambda \tilde{\mathcal{F}}(i, \lambda) \leq 1$. Thus, $(\tilde{\mathcal{F}}(i, \lambda) : i \in \mathbb{Z}_+)$ is a solution of (17).

We now prove the uniqueness of the solution for (17). Suppose that $(\hat{\mathcal{F}}(i, \lambda) : i \in \mathbb{Z}_+)$ is another solution of (17) for some $\lambda > 0$. Denote $\Delta_i(\lambda) := \hat{\mathcal{F}}(i, \lambda) - \tilde{\mathcal{F}}(i, \lambda)$ ($i \in \mathbb{Z}_+$). Then

$$\Delta_i(\lambda) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{\lambda r(i) + q_i} \cdot \Delta_j(\lambda), \quad i \in \mathbb{Z}_+.$$

Hence,

$$|\Delta_i(\lambda)| \leq \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{q_i} \cdot |\Delta_j(\lambda)|, \quad i \in \mathbb{Z}_+. \quad (18)$$

By (18), we can derive that

$$|\Delta_i(\lambda)| \leq \mathbb{E}_i[|\Delta_{\tilde{X}_{\tilde{\tau}}}(\lambda)| \cdot \mathbf{1}_{\{\tilde{\tau} > n\}}], \quad i \in \mathbb{Z}_+$$

for all $n \geq 1$. Note that $|\Delta_{\tilde{X}_{\tilde{\tau}}}(\lambda)| \leq \frac{1}{\lambda}$; by Lemma 4, we know that

$$|\Delta_i(\lambda)| \leq \lim_{n \rightarrow \infty} \lambda^{-1} \tilde{P}_i(\tilde{\tau} > n) = 0, \quad i \in \mathbb{Z}_+.$$

Therefore, $\Delta_i(\lambda) = 0$ ($i \in \mathbb{Z}_+$) for any $\lambda > 0$ and hence, $\hat{\mathcal{F}}(i, \lambda) = \tilde{\mathcal{F}}(i, \lambda)$ ($i \in \mathbb{Z}_+$) for any $\lambda > 0$; i.e., Equation (17) has a unique solution. By the property of Laplace transforms, Equation (16) has a unique solution. $\square$

Corollary 1. Suppose that $H_0 = \emptyset$. Then, for any $i \in \mathbb{Z}_+$, $f(i, x) := \tilde{F}'(i, x)$ exists and $f(i, x) > 0$ for all $x \geq 0$.

Proof. Recall from Theorem 1 that $\tilde{F}(i, x)$ satisfies the system of Volterra-type integral Equation (16). Since $H_0 = \emptyset$, we have $r(i) > 0$ for every $i \in \mathbb{Z}_+$. The right-hand sides of these integral equations consist of continuous functions and convolution integrals against continuous kernels $e^{-\frac{q_i u}{r(i)}}$. The standard properties of Volterra integral equations imply that $\tilde{F}(i, x)$ is differentiable with respect to $x \geq 0$.

We first treat the case $i \geq 1$. Differentiating both sides of (16) yields

$$\tilde{F}'(i, x) = \frac{\xi}{r(i)} \cdot e^{-\frac{q_i x}{r(i)}} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{r(i)} \left[\tilde{F}(j, x) - \frac{q_i}{r(i)} \int_0^x e^{-\frac{q_i(x-u)}{r(i)}} \tilde{F}(j, u) du \right].$$

We simplify the integral term inside the sum. Observe that

$$\frac{q_i}{r(i)} \int_0^x e^{-\frac{q_i(x-u)}{r(i)}} du = 1 - e^{-\frac{q_i x}{r(i)}}.$$

Because $\tilde{F}(j, \cdot)$ is a non-decreasing distribution function, we have

$$\int_0^x e^{-\frac{q_i(x-u)}{r(i)}} \tilde{F}(j, u) du \leq \tilde{F}(j, x) \int_0^x e^{-\frac{q_i(x-u)}{r(i)}} du.$$

Substituting this inequality into the expression for $\tilde{F}'(i, x)$ gives

$$\begin{aligned} \tilde{F}'(i, x) &\geq \frac{\xi}{r(i)} \cdot e^{-\frac{q_i x}{r(i)}} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{r(i)} \left[\tilde{F}(j, x) - \tilde{F}(j, x) \cdot \frac{q_i}{r(i)} \int_0^x e^{-\frac{q_i(x-u)}{r(i)}} du \right] \\ &= \frac{\xi}{r(i)} \cdot e^{-\frac{q_i x}{r(i)}} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{r(i)} \cdot \tilde{F}(j, x) e^{-\frac{q_i x}{r(i)}}. \end{aligned}$$

Every term on the right-hand side is non-negative. Moreover, the first term $\frac{\xi}{r(i)}e^{-\frac{q_i x}{r(i)}}$ is strictly positive for all $x \geq 0$; hence, $\tilde{F}'(i, x) > 0$ ($x \geq 0$) when $i \geq 1$.

Now consider $i = 0$. Differentiating the corresponding integral equation for $\tilde{F}(0, x)$, we obtain

$$\tilde{F}'(0, x) = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{r(0)} \left[\tilde{F}(j, x) - \frac{q_0}{r(0)} \int_0^x e^{-\frac{q_0(x-u)}{r(0)}} \tilde{F}(j, u) du \right].$$

Applying the same estimate used above,

$$\tilde{F}'(0, x) \geq \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{0j}}{r(0)} \cdot \tilde{F}(j, x) e^{-\frac{q_0 x}{r(0)}}.$$

Note that $\hat{q}_{0j} > 0$ for at least one $j \in \mathbb{Z}_+$, and $\tilde{F}(j, x) > 0$, $e^{-\frac{q_0 x}{r(0)}} > 0$. Consequently the right-hand side is strictly positive, so $\tilde{F}'(0, x) > 0$ ($x \geq 0$).

Therefore, $f(i, x) := \tilde{F}'(i, x)$ exists and satisfies $f(i, x) > 0$ ($x \geq 0$) for all $i \in \mathbb{Z}_+$. $\square$

Theorem 2. $(\tilde{V}(i) : i \in \mathbb{Z}_+)$ is the unique solution of

$$\begin{cases} \tilde{V}(i) = \frac{r(i)}{q_i} + \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{q_i} \cdot \tilde{V}(j), & i \in \mathbb{Z}_+, \\ \tilde{V}(0) \leq C + \frac{R}{\beta_0}, \quad \tilde{V}(i) \leq C, \quad i \geq 1, \end{cases} \quad (19)$$

where

$$C := \frac{R}{\xi} \left(1 + \frac{\alpha_1}{\beta_0} \right).$$

Proof. By Lemma 3 and the monotone convergence theorem, we know that $(\tilde{V}(i) : i \in \mathbb{Z}_+)$ is a solution of (19). It suffices to prove the uniqueness of the solution for (19). Suppose that $(\hat{V}(i) : i \in \mathbb{Z}_+)$ is another solution of (19). Denote $\Delta_i := \hat{V}(i) - \tilde{V}(i)$ ($i \in \mathbb{Z}_+$). Then

$$\Delta_i = \sum_{j \in \mathbb{Z}_+} \frac{\hat{q}_{ij}}{q_i} \cdot \Delta_j, \quad i \in \mathbb{Z}_+.$$

As in the proof of Theorem 1, we can derive that

$$|\Delta_i| \leq \lim_{n \rightarrow \infty} \left(C + \frac{R}{\beta_0} \right) \cdot \tilde{P}_i(\tilde{\tau} > n) = 0, \quad i \in \mathbb{Z}_+.$$

Therefore, $\Delta_i = 0$ ($i \in \mathbb{Z}_+$) and hence, $\hat{V}(i) = \tilde{V}(i)$ ($i \in \mathbb{Z}_+$); i.e., Equation (19) has a unique solution. $\square$

Corollary 2. Suppose that $\alpha_i = \alpha$ ($i \geq 1$) and $\beta_i = \beta$ ($i \geq 0$). Then

$$\tilde{V}(i) = \left[\frac{r(0)}{\beta(1-\lambda_-)} + \frac{(\lambda_+ - 1)\mathcal{L}_0^+(r)}{(1-\lambda_-)\sqrt{\Delta}} \right] \cdot \lambda_-^i + \frac{\mathcal{L}_i^+(r) + \mathcal{L}_i^-(r)}{\sqrt{\Delta}}, \quad i \in \mathbb{Z}_+, \quad (20)$$

where

$$\begin{aligned} \Delta &:= (\alpha + \beta + \xi)^2 - 4\alpha\beta, \\ \lambda_+ &:= \frac{\alpha + \beta + \xi + \sqrt{\Delta}}{2\beta}, \quad \lambda_- := \frac{\alpha + \beta + \xi - \sqrt{\Delta}}{2\beta}, \\ \mathcal{L}_i^+(r) &:= \lambda_+^i \sum_{k=i+1}^{\infty} \lambda_+^{-k} r(k), \quad \mathcal{L}_i^-(r) := \lambda_-^i \sum_{k=1}^i \lambda_-^{-k} r(k). \end{aligned}$$

In particular,

$$\mathbb{E}_i[\tilde{\tau}] = \frac{\lambda_-^i}{\beta(1-\lambda_-)} + \frac{1}{\xi}, \quad i \in \mathbb{Z}_+. \quad (21)$$

Proof. Since $\alpha_i = \alpha$ ($i \geq 1$) and $\beta_i = \beta$ ($i \geq 0$), Equation (19) becomes

$$\begin{cases} u_0 - u_1 = \beta^{-1}r(0) \\ \beta u_{i+1} - (\alpha + \beta + \xi)u_i + \alpha u_{i-1} = -r(i), & i \geq 1, \\ u_0 \leq \frac{R}{\xi}(1 + \frac{\alpha}{\beta}) + \frac{R}{\beta}, \quad u_i \leq \frac{R}{\xi}(1 + \frac{\alpha}{\beta}), & i \geq 1, \end{cases} \quad (22)$$

Consider the homogeneous difference equation

$$\beta u_{i+1} - (\alpha + \beta + \xi)u_i + \alpha u_{i-1} = 0, \quad i \geq 1 \quad (23)$$

with characteristic equation

$$\beta \lambda^2 - (\alpha + \beta + \xi)\lambda + \alpha = 0.$$

We get two characteristic values:

$$\lambda_+ := \frac{\alpha + \beta + \xi + \sqrt{\Delta}}{2\beta}, \quad \lambda_- := \frac{\alpha + \beta + \xi - \sqrt{\Delta}}{2\beta}$$

where $\Delta := (\alpha + \beta + \xi)^2 - 4\alpha\beta$. Furthermore, it can be investigated that

$$u_i^* := \frac{\mathcal{L}_i^+(r) + \mathcal{L}_i^-(r)}{\sqrt{\Delta}}, \quad i \geq 0$$

is a solution of

$$\beta u_{i+1} - (\alpha + \beta + \xi)u_i + \alpha u_{i-1} = -r(i), \quad i \geq 1,$$

where $\mathcal{L}_i^-(r) := \lambda_-^i \sum_{k=1}^i \lambda_-^{-k} r(k)$ and $\mathcal{L}_i^+(r) := \lambda_+^i \sum_{k=i+1}^{\infty} \lambda_+^{-k} r(k)$. Therefore, the solution of (22) can be expressed as

$$u_i = A\lambda_-^i + \frac{\mathcal{L}_i^+(r) + \mathcal{L}_i^-(r)}{\sqrt{\Delta}}, \quad i \geq 0.$$

By the boundary condition $u_0 = \frac{r(0)}{\beta} + u_1$, we obtain that

$$A = \frac{r(0)}{\beta(1-\lambda_-)} + \frac{(\lambda_+ - 1)\mathcal{L}_0^+(r)}{(1-\lambda_-)\sqrt{\Delta}}.$$

Equation (20) is proved.

In particular, taking $r(i) = 1$ ($i \in \mathbb{Z}_+$) and using some algebra yields (21). $\square$

4. Asymptotics of Scaled Expected Reward

We now consider the limit property of $\tilde{V}(i)$ as the exogenous catastrophe rate ξ tends to zero or to infinity.

Theorem 3. Suppose the Q_0 -process is positive recurrent. Then

$$\xi \sum_{i \geq 1} \pi_i \tilde{V}(i) = \sum_{i \geq 0} \pi_i r(i). \quad (24)$$

Hence,

$$\lim_{\xi \downarrow 0} \xi \tilde{V}(i) = (1 - \pi_0)^{-1} \sum_{k \in \mathbb{Z}_+} \pi_k r(k), \quad i \in \mathbb{Z}_+,$$

where $(\pi_k : k \in \mathbb{Z}_+)$ is the stationary probability measure of the Q_0 -process.

Proof. Denote $G_i(\xi) := \xi \tilde{V}(i)$. By Theorem 2,

$$\begin{cases} \beta_0 G_0(\xi) - \beta_0 G_1(\xi) = r(0)\xi, \\ (\alpha_i + \beta_i + \xi)G_i(\xi) - \alpha_i G_{i-1}(\xi) - \beta_i G_{i+1}(\xi) = r(i)\xi, & i \geq 1, \\ G_0(\xi) \leq R(1 + \frac{\alpha_1}{\beta_0}) + \frac{R\xi}{\beta_0}, \quad G_i(\xi) \leq R(1 + \frac{\alpha_1}{\beta_0}), & i \geq 1. \end{cases} \quad (25)$$

By (25),

$$(\alpha_i + \beta_i)G_i(\xi) - \alpha_i G_{i-1}(\xi) - \beta_i G_{i+1}(\xi) = [r(i) - G_i(\xi)]\xi.$$

Multiplying both sides of the above equality by π_i and summing over i yields that

$$\sum_{i \geq 1} [\pi_i (\alpha_i + \beta_i)G_i(\xi) - \pi_i \alpha_i G_{i-1}(\xi) - \pi_i \beta_i G_{i+1}(\xi)] = \xi \sum_{i \geq 1} \pi_i [r(i) - G_i(\xi)].$$

Combining this with $\pi_0 \beta_0 G_0(\xi) - \pi_0 \beta_0 G_1(\xi) = \xi \pi_0 r(0)$, we obtain that

$$\sum_{i \geq 1} \pi_i [r(i) - G_i(\xi)] + \pi_0 r(0) = 0.$$

Hence,

$$\sum_{i \geq 1} \pi_i G_i(\xi) = \sum_{i \in \mathbb{Z}_+} \pi_i r(i).$$

Equation (24) is proved. By the above equality,

$$\lim_{\xi \downarrow 0} \sum_{i \geq 1} \pi_i G_i(\xi) = \sum_{i \in \mathbb{Z}_+} \pi_i r(i). \quad (26)$$

On the other hand, it follows from (25) that for any ξ_k satisfying $\xi_k \rightarrow 0$, if $\lim_{k \rightarrow \infty} G_0(\xi_k) = C$, then $\lim_{k \rightarrow \infty} G_i(\xi_k) = C$. Since $G_0(\xi) \leq R(1 + \frac{\alpha_1}{\beta_0}) + \frac{R}{\beta_0}$ on $\xi \in (0, 1]$, we know that

$$\sum_{i \geq 1} \pi_i \lim_{k \rightarrow \infty} G_i(\xi_k) = \sum_{i \in \mathbb{Z}_+} \pi_i r(i)$$

which implies $\lim_{k \rightarrow \infty} G_j(\xi_k) = C = (1 - \pi_0)^{-1} \sum_{i \in \mathbb{Z}_+} \pi_i r(i)$ for $j \in \mathbb{Z}_+$. Furthermore,

$$\lim_{\xi \downarrow 0} \xi \tilde{V}(j) = C = (1 - \pi_0)^{-1} \sum_{i \in \mathbb{Z}_+} \pi_i r(i) \text{ for } j \in \mathbb{Z}_+. \quad \square$$

We next consider the limit behavior of $\xi \tilde{V}(i)$ ($i \in \mathbb{Z}_+$) as $\xi \rightarrow 0$. For this end, denote

$$r_* := \liminf_{k \rightarrow \infty} r(k) \quad \text{and} \quad r^* := \limsup_{k \rightarrow \infty} r(k).$$

Lemma 5. Let $g : [0, \infty) \rightarrow \mathbb{R}$ be bounded and measurable. Then

$$\liminf_{t \rightarrow \infty} g(t) \leq \liminf_{\xi \downarrow 0} \xi \int_0^\infty e^{-\xi t} g(t) dt \leq \limsup_{\xi \downarrow 0} \xi \int_0^\infty e^{-\xi t} g(t) dt \leq \limsup_{t \rightarrow \infty} g(t).$$

Proof. We only show the upper bound. The lower bound follows by considering $-g(t)$. Set $L^* = \limsup_{t \rightarrow \infty} g(t)$. For any $\varepsilon > 0$, there exists $T > 0$ such that $g(t) \leq L^* + \varepsilon$ for all $t > T$. Write

$$\xi \int_0^\infty e^{-\xi t} g(t) dt = \xi \int_0^T e^{-\xi t} g(t) dt + \xi \int_T^\infty e^{-\xi t} g(t) dt.$$

Since g is bounded, the first term tends to 0 as $\xi \downarrow 0$. The second term satisfies

$$\xi \int_T^\infty e^{-\xi t} g(t) dt \leq (L^* + \varepsilon) \xi \int_T^\infty e^{-\xi t} dt = (L^* + \varepsilon) e^{-\xi T} \leq L^* + \varepsilon.$$

Taking $\limsup_{\xi \downarrow 0}$ and then letting $\varepsilon \rightarrow 0$ yields the desired upper inequality. The lower bound is obtained by applying the upper estimate to $-g(t)$. $\square$

Theorem 4. Suppose that the Q_0 -process is null recurrent or transient. Then, for every $i \in \mathbb{Z}_+$, the following assertion holds.

$$r_* \leq \liminf_{\xi \downarrow 0} \xi \tilde{V}(i) \leq \limsup_{\xi \downarrow 0} \xi \tilde{V}(i) \leq r^*.$$

In particular, if $r_\infty := \lim_{k \rightarrow \infty} r(k)$ exists, then

$$\lim_{\xi \downarrow 0} \xi \tilde{V}(i) = r_\infty, \quad i \in \mathbb{Z}_+.$$

Proof. Let $G_i(\xi) := \xi \tilde{V}(i)$. Since the boundary birth rate at state 0 is β_0 , the first equation in (25) gives

$$G_0(\xi) = G_1(\xi) + \frac{\xi r(0)}{\beta_0}. \quad (27)$$

Define a conservative birth–death q -matrix Q^+ on $\mathbb{N} := \{1, 2, \dots\}$ by

$$q_{1,2}^+ = \beta_1, \quad q_{1,1}^+ = -\beta_1,$$

and, for $i \geq 2$, by

$$q_{i,i-1}^+ = \alpha_i, \quad q_{i,i+1}^+ = \beta_i, \quad q_{i,i}^+ = -(\alpha_i + \beta_i).$$

This is the trace of the Q_0 -process on $\mathbb{N}$, obtained by deleting the sojourns at state 0. Let $P^+(t) = (p_{ij}^+(t))$ be its transition function, and put

$$\bar{r}(1) := r(1) + \frac{\alpha_1}{\beta_0} r(0), \quad \bar{r}(i) := r(i), \quad i \geq 2.$$

Using (27) in the equation for $i = 1$ in (25), we obtain

$$\begin{aligned} (\xi + \beta_1) G_1(\xi) - \beta_1 G_2(\xi) &= \xi \bar{r}(1), \\ (\xi + \alpha_i + \beta_i) G_i(\xi) - \alpha_i G_{i-1}(\xi) - \beta_i G_{i+1}(\xi) &= \xi \bar{r}(i), \quad i \geq 2. \end{aligned}$$

Equivalently, $G(\xi) := (G_i(\xi) : i \geq 1)$ is the bounded solution of the resolvent equation

$$(\xi I - Q^+)G(\xi) = \xi \bar{r}.$$

Consequently,

$$G_i(\xi) = \xi \int_0^\infty e^{-\xi t} \sum_{j \geq 1} p_{ij}^+(t) \bar{r}(j) dt, \quad i \geq 1. \quad (28)$$

Changing the behavior at the finite boundary does not change the recurrence classification of a birth–death process. Hence Q^+ is null recurrent or transient. In either case it is not positive recurrent, and therefore

$$p_{ij}^+(t) \longrightarrow 0, \quad t \rightarrow \infty, \quad i, j \geq 1.$$

It follows that, for every fixed K ,

$$\sum_{j=1}^K p_{ij}^+(t) \longrightarrow 0, \quad t \rightarrow \infty. \quad (29)$$

Since $\bar{r}$ is bounded and differs from r at only one state, (29) gives

$$r_* \leq \liminf_{t \rightarrow \infty} \sum_{j \geq 1} p_{ij}^+(t) \bar{r}(j) \leq \limsup_{t \rightarrow \infty} \sum_{j \geq 1} p_{ij}^+(t) \bar{r}(j) \leq r^*.$$

Since $G_i(\xi) = \xi \tilde{V}(i)$, by Lemma 5, we obtain the asserted bounds for $i \geq 1$. They also hold for $i = 0$ by (27). If $r(k) \rightarrow r_\infty$, the lower and upper bounds coincide. This yields the last assertion. $\square$

Remark 1. The assumption $0 \leq r(\cdot) \leq R$ alone does not, in general, ensure the existence of $\lim_{\xi \downarrow 0} \xi \tilde{V}(i)$. The bounds in Theorem 4 are valid under the assumptions made earlier in the paper, whereas the displayed equality follows under the natural additional condition $r(k) \rightarrow r_\infty$.

Theorem 5. (i) For every fixed $i \in \mathbb{Z}_+$,

$$\lim_{\xi \rightarrow \infty} \xi \tilde{V}(i) = \begin{cases} +\infty, & i = 0 \text{ and } r(0) > 0, \\ r(1), & i = 0 \text{ and } r(0) = 0, \\ r(1) + \frac{\alpha_1}{\beta_0} r(0), & i = 1, \\ r(i), & i \geq 2. \end{cases}$$

$$(ii) \lim_{\xi \rightarrow \infty} \tilde{V}(0) = \frac{r(0)}{\beta_0}.$$

Proof. Again write $G_i(\xi) = \xi \tilde{V}(i)$. By Lemma 3,

$$0 \leq G_i(\xi) \leq R \left(1 + \frac{\alpha_1}{\beta_0} \right), \quad i \geq 1, \quad \xi > 0. \quad (30)$$

For any fixed $i \geq 2$, the second equation in (25) can be written as

$$G_i(\xi) = \frac{\xi}{\xi + \alpha_i + \beta_i} r(i) + \frac{\alpha_i}{\xi + \alpha_i + \beta_i} G_{i-1}(\xi) + \frac{\beta_i}{\xi + \alpha_i + \beta_i} G_{i+1}(\xi).$$

The bound (30) therefore implies

$$\lim_{\xi \rightarrow \infty} G_i(\xi) = r(i), \quad i \geq 2.$$

For $i = 1$, eliminating $G_0(\xi)$ by (27) gives

$$G_1(\xi) = \frac{\xi}{\xi + \beta_1} \left(r(1) + \frac{\alpha_1}{\beta_0} r(0) \right) + \frac{\beta_1}{\xi + \beta_1} G_2(\xi).$$

Using (30) once more yields

$$\lim_{\xi \rightarrow \infty} G_1(\xi) = r(1) + \frac{\alpha_1}{\beta_0} r(0).$$

Finally, Equation (27) gives

$$G_0(\xi) = G_1(\xi) + \frac{\xi r(0)}{\beta_0}.$$

Thus $G_0(\xi) \rightarrow +\infty$ if $r(0) > 0$, whereas $G_0(\xi) = G_1(\xi) \rightarrow r(1)$ if $r(0) = 0$. Finally, since $\tilde{V}(0) = \frac{G_0(\xi)}{\xi} = \frac{G_1(\xi)}{\xi} + \frac{r(0)}{\beta_0}$, we know that (ii) holds. $\square$

At the end of this section, we provide numerical illustrations demonstrating how the catastrophe rate ξ influences the expected cumulative reward $\tilde{V}(i)$.

Example 1. Consider constant birth–death rates $\alpha_i \equiv \alpha = 0.6$, $\beta_i \equiv \beta = 0.5$, so the closed-form formula from Corollary 2 applies. We fix the reward function to be constant $r(k) \equiv 1$ for all $k \in \mathbb{Z}_+$.

The following Figure 2 plots $\tilde{V}(i)$ against the catastrophe rate $\xi > 0$ for several initial states i . This is consistent with our asymptotic theory developed in Theorems 3–5.

From Figure 2, one can see the following:

(i) For small $\xi \downarrow 0$, catastrophes are rare. The process typically evolves for a long time before a catastrophe arrives, yielding large values of the expected reward $\tilde{V}(i)$.

(ii) As ξ increases, catastrophes become more frequent; the process is terminated earlier, and $\tilde{V}(i)$ decreases monotonically.

(iii) For large $\xi \rightarrow \infty$, $\tilde{V}(i)$ decays towards finite state-dependent limits predicted by Theorem 5.

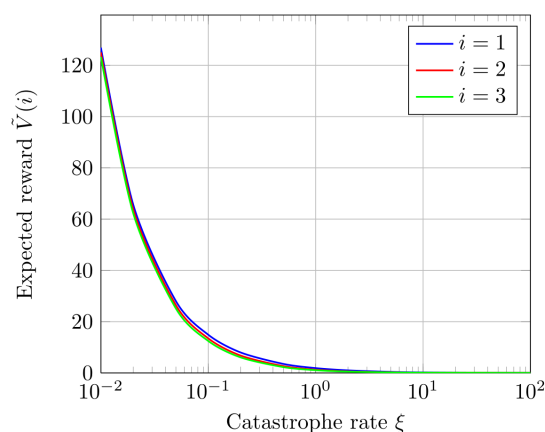


Figure 2. $\tilde{V}(i)$ versus ξ , for $\alpha = 0.6$, $\beta = 0.5$, $r \equiv 1$; curves for initial states $i = 1, 2, 3$.

5. Conclusions

This paper has investigated cumulative reward functionals for continuous-time birth–death processes subject to Poisson-driven total exogenous catastrophes. By constructing an auxiliary killed Markov process with an absorbing cemetery state, we transformed the

original model such that sample-path behaviors prior to the first catastrophe correspond to trajectories before absorption in the killed process. This technique enabled us to derive recursive relations for the distribution and expectation of partial rewards evaluated at successive jump times.

We established integral equations together with their Laplace transform counterparts governing the reward distribution $F_i(x) = P_i(R(\tau) \leq x)$ and the equation for the expected cumulative reward $V(i) = \mathbb{E}_i[R(\tau)]$. The existence and uniqueness of bounded solutions to these systems were proved. Under the condition $H_0 = \emptyset$, we further demonstrated the existence of strictly positive probability densities for the reward distribution. For constant birth and death rates, explicit closed-form expressions for the expected reward were obtained by solving the corresponding linear difference equations, which also yielded a concrete formula for the mean time to the first catastrophe as a special case.

We analyzed the asymptotic behavior of the scaled expected reward $\xi V(i)$ in two regimes: a vanishing catastrophe rate $\xi \downarrow 0$ and diverging catastrophe rate $\xi \rightarrow \infty$. A trichotomy of limiting results was presented depending on the recurrence type of the underlying catastrophe-free birth–death process Q_0 . When Q_0 is positive-recurrent, the limit can be expressed in terms of its stationary distribution. For null-recurrent or transient Q_0 , the limits are controlled by the $\liminf$ and $\limsup$ of the reward function $r(\cdot)$; a definite limit emerges if $r(k)$ converges as $k \rightarrow \infty$. For large catastrophe rates $\xi \rightarrow \infty$, state-dependent explicit limits for $\xi V(i)$ were derived.

Several interesting directions remain open for further research. These include a thorough investigation of the exponential tail bounds for the survival function $\bar{F}(i, x) := 1 - F(i, x)$, higher-order moment analysis for the cumulative reward functional, and conditional state distributions under linear-rate birth–death parameterizations. Extending the present framework to models with partial binomial catastrophes, state-dependent catastrophe intensities or multi-type disaster events would also be worthwhile. Such generalizations would further broaden the applicability of reward-related quantities for queuing theory, population biology and risk-modeling applications.

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