mathematics

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Hyperbolic-Type Solitons on Ricci Pseudo-Symmetric Riemannian Concircular Structure Manifolds

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Abstract

The study of hyperbolic Ricci solitons and hyperbolic Yamabe solitons on Riemannian concircular structure manifolds is the main focus of the current research. We derive the expression of the Ricci curvature and scalar curvature of Riemannian concircular structure manifolds admitting hyperbolic Ricci solitons. Also, we show that the rate of change of hyperbolic Ricci solitons on Riemannian concircular structure manifolds is shrinking. Moreover, we discuss the nature of hyperbolic Ricci solitons and critical value and singular point on Ricci pseudo-symmetric Riemannian concircular structure manifolds, concircular pseudo-symmetric Riemannian concircular structure manifolds, projective pseudo-symmetric Riemannian concircular structure manifolds and conharmonic pseudo-symmetric Riemannian concircular structure manifolds. Lastly, depending solely on the rate of these solitons, we investigate certain applications in terms of modified second hyperbolic Yamabe solitons and modified second hyperbolic Ricci solitons.

Keywords: Riemannian concircular structure manifolds; hyperbolic Ricci solitons; hyperbolic Yamabe solitons; Ricci pseudo-symmetric Riemannian concircular structure manifolds

MSC: 53C35; 53B30; 53C50; 53C80

1. Introduction

A well-known example of symmetry is the soliton, which is connected to the geometric motion of manifolds. One of the core concepts of geometry is the study of hyperbolic solitons and their symmetry in manifolds [1].

In ref. [2], the theory of hyperbolic geometric flow was introduced.

$$\frac{1}{2}\frac{\eth^2}{\eth t^2}\mathfrak{g}(t) = -\mathfrak{S}(t)\mathfrak{g}(t), \quad \mathfrak{g}_0 = \mathfrak{g}(0), \quad \frac{\eth}{\eth t}\mathfrak{g}_{ij} = \mathfrak{h}_{ij}, \tag{1}$$

wherein $\mathfrak{h}_{ij}$ is a $(0,2)$-type symmetric tensor.

Hyperbolic Ricci solitons (HRS) and gradient hyperbolic Ricci solitons (GHRS) on Riemannian manifolds $(\mathbb{M}^m, \mathfrak{g})$ were put forward in ref. [3]:

$$\frac{1}{2}\mathfrak{L}_\zeta\mathfrak{L}_\zeta\mathfrak{g} + \diagdown\mathfrak{L}_\zeta\mathfrak{g} + \mathfrak{S} = \Lambda\mathfrak{g}, \tag{2}$$

wherein $\mathfrak{S}$ is Ricci curvature. In (2), $\diagdown$ and Λ denote the classifications of solitons and the rate of the underlying type, respectively.

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Academic Editors: Shihshu Walter Wei and En-Bing Lin

Received: 30 August 2026
Revised: 4 October 2026
Accepted: 13 October 2026
Published: 22 October 2026

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Recently, in 2024 Blaga and Özgür presented the idea that hyperbolic Yamabe flow is an evolutionary equation corresponding to the scalar curvature $\mathcal{R}$ [4]

$$\frac{\eth^2}{\eth t^2}\mathfrak{g}(t) = -\mathcal{R}(t)\mathfrak{g}(t). \tag{3}$$

They obtain the equation for the hyperbolic Yamabe solitons [4] (HYS), which are the stationary solutions of the hyperbolic Yamabe flow (3)

$$\mathfrak{L}_\zeta \mathfrak{L}_\zeta \mathfrak{g} + \lambda \mathfrak{L}_\zeta \mathfrak{g} = (\Lambda - \mathcal{R})\mathfrak{g}, \tag{4}$$

where in (2) and (4), Λ displays the rate of change in the solutions and has geometric significance. The rate of change (or type of soliton) of HRS and HYS can be growing (increasing), stable, or decreasing dependent on the constant Λ (or λ), regardless of whether $\Lambda > 0$ (or $\lambda > 0$), $\Lambda = 0$ (or $\lambda = 0$), or $\Lambda < 0$ (or $\lambda < 0$) and $\mathfrak{L}_\zeta$ indicates the Lie derivative along the soliton vector field ζ.

Ahsan and Ali studied Ricci solitons [5] and Shwarzschild solitons [6] on spacetime for the first time. Blaga [7] used Einstein and Ricci solitons to illustrate the properties of these solitons. The authors of [8,9] additionally addressed Ricci solitons. Several researchers have analyzed a variety of research articles using HRS (see [5,10–13] for further details). Özgür and Blaga investigated the HRS in a variety of methods in (refs. [14–16] for more details).

In differential geometry, Ricci pseudo-symmetric manifolds are unique spaces. They expand on the traditional notion of a "symmetric space" in which geometric shapes have the same appearance everywhere. It is helpful to see how geometers modified their restrictions over time in order to grasp pseudo-symmetry. A Ricci pseudo-symmetric manifold results from applying this pseudo-symmetric rule exclusively to the Ricci tensor, which quantifies changes in volume [17].

When a geometric property such as curvature $\mathfrak{R}$ is shifted by parallel transport in symmetric spaces, it remains unchanged. The derivative of curvature tensor $\mathfrak{R}$ is zero, i.e., [18]

$$\nabla.\mathfrak{R} = 0.$$

In 1992, the rule was further reduced by Ryszard Deszcz. The change must be proportional to a simplified version of the space's own metric structure rather than being zero.This is called the Ricci-pseudo-symmetric condition, introduced by Ryszard Deszcz [19].

If the tensor $\mathfrak{R}.\mathfrak{S}$ and the Tachibana tensor $\mathfrak{Q}(\mathfrak{g},\mathfrak{S})$ are linearly dependent, then an n-dimensional pseudo-Riemannian manifold $(\mathbb{M}^m,\mathfrak{g})$ is referred to be Ricci pseudo-symmetric [19], where

$$(\mathfrak{R}(X_1,X_2)\mathfrak{S})(X_3,X_4) = -\mathfrak{S}(\mathfrak{R}(X_1,X_2)X_3,X_4) - \mathfrak{S}(X_3,\mathfrak{R}(X_1,X_2)X_4), \tag{5}$$

$$\mathfrak{Q}(\mathfrak{g},\mathfrak{S})(X_3,X_4;X_1,X_2) = -\mathfrak{S}((X_1 \wedge_\mathfrak{g} X_2)X_3,X_4) - \mathfrak{S}(X_3,(X_1 \wedge_\mathfrak{g} X_2)X_4), \tag{6}$$

and

$$(X_1 \wedge_\mathfrak{g} X_2)X_3 = \mathfrak{g}(X_2,X_3)X_1 - \mathfrak{g}(X_1,X_3)X_2 \tag{7}$$

for all vector fields $X_1,X_2;X_3,X_4 \in (M^n,\mathfrak{g})$, $\mathfrak{R}$ denotes the curvature tensor, $\mathfrak{S}$ is Ricci tensor and $\mathfrak{Q}$ is Ricci operator such that $\mathfrak{S}(X_1,X_2) = g(\mathfrak{Q}X_1,X_2)$ in $(\mathbb{M}^m,\mathfrak{g})$.

Then a pseudo-Riemannian manifold $(\mathbb{M}^m,\mathfrak{g})$ is Ricci pseudo-symmetric if and only if

$$(\mathfrak{R}(X_1,X_2)\mathfrak{S})(X_3,X_4) = \mathcal{L}_\mathfrak{S}\mathfrak{Q}(\mathfrak{g},\mathfrak{S})(X_3,X_4;X_1,X_2) \tag{8}$$

holds in a set

$$\mathbb{U}_{\mathfrak{S}} = \left\{ p \in \mathbb{M} \,\middle|\, \mathfrak{S} \neq \frac{\mathcal{R}}{m} \ at \ p \right\},$$

with a function $\mathcal{L}_{\mathfrak{S}}$ on set $\mathbb{U}_{\mathfrak{S}}$.

Moreover, $(\mathbb{M}^m, \mathfrak{g})$ is referred to as Ricci semi-symmetric if [20]

$$\mathfrak{R}.\mathfrak{S} = 0.$$

It is not true that all Ricci semi-symmetric manifolds are Ricci pseudo-symmetric [19]. Shaikh and Hui investigated Ricci pseudo-symmetric generalized quasi Einstein manifolds in [21]. Deszcz et al. [22,23] extensively discussed the Ricci pseudo-symmetric properties of manifolds.

On the other hand, Chaubey and Suh first presented the idea of Riemannian concircular structure manifolds in [24], which are frequently shortened to $(RCS)_m$-manifolds. In 2025, Chen et al. [25] studied almost Ricci solitons on Riemannian concircular structure 3-manifolds.

In this research, the hyperbolic Ricci solitons (HRS) and hyperbolic Yamabe solitons (HYS) on Riemannian concircular structure manifolds are denoted by $(RCS)_m$-manifolds. In Section 2, we provide the definition and essential curvatures characteristics of $(RCS)_m$-manifolds. In Section 3, we derive expressions for the Ricci curvature and scalar curvature of $(RCS)_m$-manifolds and show that the rate of change of HRS is decreasing (shrinking). In Sections 4 and 5, we compute the characteristic function $\mathcal{L}_{\mathfrak{S}}$ of the pseudo-symmetry condition in terms of HRS settings on Ricci pseudo-symmetric $(RCS)_m$-manifolds and concircular pseudo-symmetric $(RCS)_m$-manifolds. Sections 6 and 7 explore HRS with a torse forming vector field on projective pseudo-symmetric $(RCS)_m$-manifolds and conharmonic pseudo-symmetric $(RCS)_m$-manifolds. The hyperbolic Yamabe solitons on Riemannian concircular structure manifolds are covered in the final Section 8.

2. Preliminaries

Let $\mathbb{M}$ be an m-dimensional Riemannian manifold. A vector field ζ on Riemannian manifold $\mathbb{M}$ is considered to be a torse-forming type [26] if

$$\widetilde{\nabla}\zeta = \sigma I + \pi \otimes \zeta, \tag{9}$$

where I denotes unit transformation, σ is a smooth function on $\mathbb{M}$, π is a 1-form, and ζ is a vector field connected to the 1-form π, such that

$$\pi(\mathsf{X}_1) = \mathfrak{g}(\mathsf{X}_1, \zeta). \tag{10}$$

Yano [27] states that if 1-form π is closed on $\mathbb{M}$, then ζ is referred to as a concircular vector field.

Let ζ now represent a unit torse-forming vector field on $\mathbb{M}$.

$$\mathfrak{g}(\zeta, \zeta) = 1. \tag{11}$$

The above torse-forming vector field takes the following shape since

$$\nabla_{\mathsf{X}_1}\zeta = \sigma[\mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta], \tag{12}$$

where the smooth function σ that is not zero satisfies

$$\mathsf{X}_1(\sigma) = \mu\pi(\mathsf{X}_1), \quad \mathsf{X}_1(\mu) = \sigma\pi(\mathsf{X}_1) \tag{13}$$

for some smooth functions μ and σ on Riemannian manifold ζ.

Let us take

$$\nabla_{\mathsf{X}_1}\sigma = \sigma\varphi\mathsf{X}_1, \qquad \sigma \neq 0, \tag{14}$$

where φ is a $(1,1)$-tensor field on Riemannian manifold $\mathbb{M}$. Then we have

$$\varphi^2\mathsf{X}_1 = \mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta, \qquad \pi(\zeta) = 1, \tag{15}$$

$$\mathfrak{g}(\varphi\mathsf{X}_1, \varphi\mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2). \tag{16}$$

The manifold $\mathbb{M}$ equipped with structure $(\varphi, \zeta, \pi, \mathfrak{g})$ is referred to as an m-dimensional *Riemannian concircular structure manifold* and it is abbreviated as $((RCS)_m)$-manifold.

Now, from (15) and (16), easily we have

$$\varphi(\zeta) = 0, \qquad \pi \circ \varphi = 0. \tag{17}$$

Also from (12) we have [24]

$$(\nabla_{\mathsf{X}_1}\pi)(\mathsf{X}_2) = \sigma[\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)]. \tag{18}$$

$$(\nabla_{\mathsf{X}_1}\varphi)(\mathsf{X}_2) = \sigma[2\pi(\mathsf{X}_1)\pi(\mathsf{X}_2) - \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2)\zeta - \pi(\mathsf{X}_2)\mathsf{X}_1], \tag{19}$$

In addition, we also have some curvature identities on ζ [24]

$$\mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\zeta = (\sigma^2 + \mu)[\pi(\mathsf{X}_1)\mathsf{X}_2 - \pi(\mathsf{X}_2)\mathsf{X}_1], \tag{20}$$

$$\mathfrak{R}(\zeta, \mathsf{X}_1)\mathsf{X}_2 = (\sigma^2 + \mu)[\pi(\mathsf{X}_2)\mathsf{X}_1 - \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2)\zeta], \tag{21}$$

$$\pi(\mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3) = (\sigma^2 + \mu)[\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_3) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)], \tag{22}$$

$$\mathfrak{S}(\mathsf{X}_1, \zeta) = -(m-1)(\sigma^2 + \mu)\pi(\mathsf{X}_1), \tag{23}$$

$$\mathfrak{Q}\zeta = -(m-1)(\sigma^2 + \mu)\zeta, \tag{24}$$

$$\mathcal{R} = -m(m-1)(\sigma^2 + \mu), \tag{25}$$

for all vector fields $\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3 \in \chi(\mathbb{M})$, where $\mathfrak{R}$, $\mathfrak{S}$ and $\mathcal{R}$ stand for the Riemannian curvature, Ricci curvature, and scalar curvature, respectively.
Moreover, we have

Definition 1 ([28]). *A Riemannian manifold $(\mathbb{M}^n, \mathfrak{g})$ is said to be π-Einstein manifold or just π-Einstein if there are constants α, β and if the Ricci tensor $\mathfrak{S}$ is given by*

$$\mathfrak{S} = \alpha\mathfrak{g} + \beta\pi \otimes \pi. \tag{26}$$

3. Hyperbolic Ricci Solitons on $(RCS)_m$-Manifold

Let $(\zeta, \gtrdot, \Lambda, \mathfrak{g})$ be *HRS* on a m-dimensional $(RCS)_m$-manifold $\mathbb{M}$. Next, from (2), we have

$$\frac{1}{2}(\mathcal{L}_\zeta\mathcal{L}_\zeta\mathfrak{g})(\mathsf{X}_1, \mathsf{X}_2) + \gtrdot(\mathcal{L}_\zeta\mathfrak{g})(\mathsf{X}_1, \mathsf{X}_2) + \mathfrak{S}(\mathsf{X}_1, \mathsf{X}_2) = \Lambda\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2). \tag{27}$$

From (12) and (14), and using the property of Lie derivative, we obtain

$$(\mathcal{L}_\zeta\mathfrak{g})(\mathsf{X}_1, \mathsf{X}_2) = \mathfrak{g}(\nabla_{\mathsf{X}_1}\zeta, \mathsf{X}_2) + \mathfrak{g}(\mathsf{X}_1, \nabla_{\mathsf{X}_2}\zeta) = 2\sigma[\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)]. \tag{28}$$

Taking the Lie derivative along ζ of (28), we get

$$(\mathcal{L}_\zeta(\mathcal{L}_\zeta \mathfrak{g}))(\mathsf{X}_1,\mathsf{X}_2) = \mathcal{L}_\zeta((\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathsf{X}_2)) - (\mathcal{L}_\zeta \mathfrak{g})(\mathcal{L}_\zeta \mathsf{X}_1,\mathsf{X}_2) \tag{29}$$
$$-(\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathcal{L}_\zeta \mathsf{X}_2).$$

$$\mathcal{L}_\zeta((\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathsf{X}_2)) = 4\sigma^2[\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)] + 2\sigma(\mathfrak{g}(\mathcal{L}_\zeta \mathsf{X}_1,\mathsf{X}_2) \tag{30}$$
$$-\mathcal{L}_\zeta(\pi(\mathsf{X}_1)\pi(\mathsf{X}_2)) + \mathfrak{g}(\mathsf{X}_1,\mathcal{L}_\zeta \mathsf{X}_2)).$$

Furthermore,

$$(\mathcal{L}_\zeta \mathfrak{g})(\mathcal{L}_\zeta \mathsf{X}_1,\mathsf{X}_2) = 2\sigma[\mathfrak{g}(\mathcal{L}_\zeta \mathsf{X}_1,\mathsf{X}_2) - \pi(\mathcal{L}_\zeta \mathsf{X}_1)\pi(\mathsf{X}_2)]. \tag{31}$$

Likewise,

$$(\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathcal{L}_\zeta \mathsf{X}_2) = 2\sigma[\mathfrak{g}(\mathsf{X}_1,\mathcal{L}_\zeta \mathsf{X}_2) - \pi(\mathcal{L}_\zeta \mathsf{X}_1)\pi(\mathsf{X}_1)]. \tag{32}$$

Again, we find

$$\pi(\mathcal{L}_\zeta \mathsf{X}_1) = \mathfrak{g}(\nabla_\zeta \mathsf{X}_1,\zeta) - \mathfrak{g}(\nabla_{\mathsf{X}_1}\zeta,\zeta). \tag{33}$$

So,

$$\pi(\mathcal{L}_\zeta \mathsf{X}_1)\pi(\mathsf{X}_2) - \pi(\mathcal{L}_\zeta \mathsf{X}_2)\pi(\mathsf{X}_1) = \mathcal{L}_\zeta(\pi(\mathsf{X}_1)\pi(\mathsf{X}_2)). \tag{34}$$

Finally, if we apply (30), (31), (32) and (34) to (29), we obtain

$$\mathcal{L}_\zeta((\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathsf{X}_2)) = 4\sigma^2[\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)]. \tag{35}$$

We now utilize the *HRS* Equation (2) to obtain

$$2\sigma^2[g(\mathsf{X}_1,\mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)] + 2\diamond\sigma[\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)] \tag{36}$$
$$+S(\mathsf{X}_1,\mathsf{X}_2) = \Lambda\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2),$$

which implies

$$\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_2) = (\omega - 2\sigma^2 - 2\diamond\sigma)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) + 2\sigma(\sigma+\diamond)\pi(\mathsf{X}_1)\pi(\mathsf{X}_2). \tag{37}$$

We can therefore declare the following outcome:

Theorem 1. *Let $(\zeta,\diamond,\Lambda,\mathfrak{g})$ be HRS on a m-dimensional $(RCS)_m$-manifold $\mathbb{M}$, then, we have*

$$(\mathcal{L}_\zeta\mathcal{L}_\zeta \mathfrak{g})(\mathsf{X}_1,\mathsf{X}_2) = 4\sigma^2[\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)], \tag{38}$$

$$\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_2) = (\Lambda - 2\sigma^2 - 2\diamond\sigma)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_2) + 2\sigma(\sigma+\diamond)\pi(\mathsf{X}_1)\pi(\mathsf{X}_2) \tag{39}$$

$$\mathfrak{S}(\mathsf{X}_1,\zeta) = \Lambda\pi(\mathsf{X}_1), \tag{40}$$

$$\mathfrak{Q}\mathsf{X}_1 = (\omega - 2\sigma^2 - 2\diamond\sigma)\mathsf{X}_1 + 2\sigma(\sigma+\diamond)\pi(\mathsf{X}_1)\zeta, \tag{41}$$

$$\mathcal{R} = m\Lambda - 2\sigma(m-1)(\sigma-\diamond), \tag{42}$$

where $\mathfrak{Q}$ is the Ricci operator, expressed as $\mathfrak{g}(\mathfrak{Q}\mathsf{X}_1,\mathsf{X}_2) = \mathfrak{S}(\mathsf{X}_1,\mathsf{X}_2)$ for all $\mathsf{X}_1,\mathsf{X}_2 \in \chi(\mathbb{M})$.

Theorem 2. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits HRS $(\zeta,\diamond,\Lambda,\mathfrak{g})$, then the $(RCS)_m$-manifold $\mathbb{M}$ is an π-Einstein manifolds.*

Next, using (23) and (40),we get

$$\Lambda = -(m-1)(\sigma^2+\mu). \tag{43}$$

Consequently, we gain the next result.

Theorem 3. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits HRS $(\zeta, \gtrdot, \Lambda, \mathfrak{g})$, then the rate of change of HRS in $(RCS)_m$-manifolds $\mathbb{M}$ is decreasing.*

Example 1. *Let a 3-dimensional manifold $\mathbb{M} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0\}$, where (x_1, x_2, x_3) are the Cartesian coordinates in $\mathbb{R}^3$, and let the linearly independent vector fields are*

$$\mathsf{X}_1 = x_3^3 \frac{\partial}{\partial x_1}, \quad \mathsf{X}_2 = x_3^3 \frac{\partial}{\partial x_2}, \quad \mathsf{X}_3 = \frac{\partial}{\partial x_3},$$

where at every point in $\mathbb{M}$, X_1, X_2, and X_3 are linearly independent. The Riemannian metric, denoted by $\mathfrak{g}$, is described as:

$\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_1) = \mathfrak{g}(\mathsf{X}_2, \mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_3, \mathsf{X}_3) = 1$, $\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3) = \mathfrak{g}(\mathsf{X}_3, \mathsf{X}_1) = 0$,

Let φ be the (1,1) tensor field such that $\varphi(\mathsf{X}_1) = \mathsf{X}_1$, $\varphi(\mathsf{X}_2) = \mathsf{X}_2$, $\varphi(\mathsf{X}_3) = 0$, and let π be the 1-form defined by $\pi(\mathsf{X}_1) = \mathfrak{g}(\zeta, \mathsf{X}_3)$ for any vector field ζ on $\mathbb{M}$.
Then, utilizing the linearity of φ and $\mathfrak{g}$, we obtain $\varphi^2\mathsf{X}_1 = \mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta$, where $\mathsf{X}_3 = \zeta$. Moreover, for vector fields $\mathsf{X}_1, \mathsf{X}_2 \in \chi(\mathbb{M})$ with $\mathfrak{g}(\zeta, \zeta) = 1$, and

$$\mathfrak{g}(\phi\mathsf{X}_1, \phi\mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2).$$

Therefore, the structure defines an $(RCS)_3$-structure in $\mathbb{R}^3$.

If ∇ is the Levi-Civita connection with regard to the metric $\mathfrak{g}$, we have

$$2\mathfrak{g}(\nabla_{\mathsf{X}_1}\mathsf{X}_2, \mathsf{X}_3) = \mathsf{X}_1\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3) + \mathsf{X}_2\mathfrak{g}(\mathsf{X}_3, \mathsf{X}_1) - \mathsf{X}_3\mathfrak{g}\mathsf{X}_1, \mathsf{X}_2) - \mathfrak{g}(\mathsf{X}_1, [\mathsf{X}_2, \mathsf{X}_3])$$

$$-g(\mathsf{X}_2, [\mathsf{X}_1, \mathsf{X}_3]) + \mathfrak{g}(\mathsf{X}_3, [\mathsf{X}_1, \mathsf{X}_2]),$$

which is referred to as Koszul's formula.

$$\nabla_{\mathsf{X}_1}\mathsf{X}_1 = -\frac{3}{x_3}\mathsf{X}_3, \quad \nabla_{\mathsf{X}_1}\mathsf{X}_2 = 0, \quad \nabla_{\mathsf{X}_1}\mathsf{X}_3 = -\frac{3}{x_3}\mathsf{X}_1,$$
$$\nabla_{\mathsf{X}_2}\mathsf{X}_2 = -\frac{3}{x_3}\mathsf{X}_3, \quad \nabla_{\mathsf{X}_2}\mathsf{X}_1 = 0, \quad \nabla_{\mathsf{X}_2}\mathsf{X}_3 = -\frac{3}{x_3}\mathsf{X}_2,$$
$$\nabla_{\mathsf{X}_3}\mathsf{X}_1 = 0, \quad \nabla_{\mathsf{X}_3}\mathsf{X}_2 = 0, \quad \nabla_{\mathsf{X}_3}\mathsf{X}_3 = 0.$$

For each vector X_1 on $\mathbb{M}$, using the aforementioned expression, we have

$$\nabla_{\mathsf{X}_1}\zeta = \sigma[\mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta],$$

where $\sigma = -\frac{3}{x_3}$. Hence the $(\varphi, \zeta, \pi, \mathfrak{g}, \sigma)$ structure defines the $(RCS)_3$-structure in $\mathbb{R}^3$.

Indeed,the Riemann and the Ricci curvature tensor are computed as follows:

$$\mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_2 = \frac{6}{x_3^2}\mathsf{X}_1, \quad \mathfrak{R}(\mathsf{X}_1, \mathsf{X}_3)\mathsf{X}_3 = -\frac{9}{z^2}\mathsf{X}_1, \quad \mathfrak{R}(\mathsf{X}_2, \mathsf{X}_1)\mathsf{X}_1 = \frac{6}{x_3^2}\mathsf{X}_2, \tag{44}$$

$$\mathfrak{R}(\mathsf{X}_2, \mathsf{X}_3)\mathsf{X}_3 = -\frac{9}{x_3^2}\mathsf{X}_2, \quad \mathfrak{R}(\mathsf{X}_3, \mathsf{X}_1)\mathsf{X}_1 = \frac{6}{x_3^2}\mathsf{X}_3, \quad \mathfrak{R}(\mathsf{X}_3, \mathsf{X}_2)\mathsf{X}_2 = \frac{6}{x_3^2}\mathsf{X}_3.$$

Hence, we obtain the Ricci tensor

$$\mathfrak{S}(\mathsf{X}_1, \mathsf{X}_1) = \mathfrak{S}(\mathsf{X}_2, \mathsf{X}_2) = -\frac{3}{x_3^2}, \quad \mathfrak{S}(\mathsf{X}_3, \mathsf{X}_3) = -\frac{18}{{x_3}^2}. \tag{45}$$

Also, $\sigma = -\frac{3}{x_3}$, $\mu = -\frac{3}{x_3^2}$.

By the definition of hyperbolic Ricci solitons and using (2), (28) and (35), we have $2\sigma[g - \pi \otimes \pi]$ *and* $4\sigma^2[g - \pi \otimes \pi]$*, then, we have* $\backslash = -\frac{3}{x_3}$ *and* $\Lambda = -\frac{12}{x_3^2}$*, which satisfies (43). Thus Theorem 3 is verified.*

4. Hyperbolic Ricci Solitons on Ricci Pseudo-Symmetric $(RCS)_m$-Manifold

Let an $(RCS)_m$-manifold $(\mathbb{M}^m, \mathfrak{g})$ be a Ricci-pseudo-symmetric if its Riemannian curvature tensor $\mathfrak{R}$ obeys

$$(\mathfrak{R}(X_1, X_2)\mathfrak{S})(X_3, X_4) = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g}, \mathfrak{S})(X_3, X_4; X_1, X_2) \tag{46}$$

holds in a set $\mathbb{U}_{\mathfrak{S}} = \left\{ p \in \mathbb{M} \,\middle|\, \mathfrak{S} \neq \frac{\mathcal{R}}{m} \text{ at } p \right\}$, where $\mathcal{L}_{\mathfrak{S}}$ is a function on set $\mathbb{U}_{\mathfrak{S}}$.

Then from (5), (6), (7) and (46), we obtain

$$\begin{aligned} -\mathfrak{S}(\mathfrak{R}(X_1, X_2)X_3, X_4) - \mathfrak{S}(X_3, \mathfrak{R}(X_1, X_2)X_4) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(X_2, X_3)\mathfrak{S}(X_1, X_4) \\ -\mathfrak{g}(X_1, X_3)\mathfrak{S}(X_2, X_4) + \mathfrak{g}(X_2, X_4)\mathfrak{S}(X_1, X_3) - \mathfrak{g}(X_1, X_4)\mathfrak{S}(X_2, X_3)]. \end{aligned} \tag{47}$$

Putting $X_3 = \zeta$ in (47), then using (20), (39) and (40), we get

$$[2\sigma(\sigma + \backslash)\mathcal{L}_{\mathfrak{S}} - \sigma(\sigma^2 + \mu)][\pi(X_2)\mathfrak{g}(X_1, X_4) - \pi(X_2)\mathfrak{g}(X_2, X_4)] = 0. \tag{48}$$

Once again, putting $X_2 = \zeta$ in (48), we gain

$$[2\sigma(\sigma + \backslash)\mathcal{L}_{\mathfrak{S}} - \sigma(\sigma^2 + \mu)][\mathfrak{g}(X_1, X_4) - \pi(X_1)\pi(X_4)] = 0 \tag{49}$$

for all vector fields X_1 and X_2, we have

$$\mathcal{L}_{\mathfrak{S}} = \frac{\sigma(\sigma^2 + \mu)}{2\sigma(\sigma + \backslash)}, \qquad \sigma \neq 0, \quad \sigma + \backslash \neq 0. \tag{50}$$

Clearly, we can state the facts.

Theorem 4. *Let* $(\zeta, \backslash, \Lambda, \mathfrak{g})$ *be HRS on* $(RCS)_m$*-manifold* $\mathbb{M}$*, then* $\mathbb{M}$ *be a Ricci pseudo-symmetric, if* $\mathcal{L}_{\mathfrak{S}} = \frac{\sigma(\sigma^2 + \mu)}{2\sigma(\sigma + \backslash)}$*, provided,* $\sigma \neq 0$ *and* $\sigma + \backslash \neq 0$*.*

From (50), we also have

$$\backslash = \frac{\sigma(\sigma^2 + \mu)}{\mathcal{L}_{\mathfrak{S}}} - \sigma. \tag{51}$$

Theorem 5. *If a Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$ *admits the HRS, then the HRS are growing, stable or decreasing, referred to as*

1. $\frac{\sigma(\sigma^2 + \mu)}{\mathcal{L}_{\mathfrak{S}}} > \sigma$,
2. $\frac{\sigma(\sigma^2 + \mu)}{\mathcal{L}_{\mathfrak{S}}} = \sigma$, *and*
3. $\frac{\sigma(\sigma^2 + \mu)}{\mathcal{L}_{\mathfrak{S}}} < \sigma$, *respectively.*

Remark 1. *In view of (50), if* $\sigma + \backslash = 0$*, then* $\mathcal{L}_{\mathfrak{S}} \to \infty$*, the relationship can only mathematically hold in a projective sense, which means the* $(RCS)_m$*-manifold cannot be modeled or analyzed under standard Ricci-pseudo symmetry conditions at those points. For the condition to remain well-defined, the Tachibana tensor* $\mathfrak{Q}(\mathfrak{g}, \mathfrak{S}) = 0$*, this condition forces the left side of the equation (8) to vanish as well.* $\mathfrak{R}.\mathfrak{S} = 0$*, implying the* $(RCS)_m$*-manifold is Ricci-semi-symmetric.*

Remark 2. *If the point* $\lambda = -\sigma$ *on a Ricci pseudo-symmetric manifold* M *is a singular point associated with the function* $\mathcal{L}_{\mathfrak{S}}$*, then the manifold behaves exactly as a locally Ricci-symmetric manifold such that* $\nabla\mathfrak{S} = 0$ *(more details see [29]). Moreover, when* $\sigma + \lambda = 0$*, then* $\mathcal{L}_{\mathfrak{S}} \to \infty$ *refers to the function* $\mathcal{L}_{\mathfrak{S}}$ *diverging asymptotically.*

Thus, remark 2 entails the following outcome.

Corollary 1. *If* $(\zeta, \lambda, \Lambda, \mathfrak{g})$ *is a HRS on Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$*, and the function* $\mathcal{L}_{\mathfrak{S}}$ *is associated with the singular point* $\lambda = -\sigma$*, then* $\mathbb{M}$ *is locally Ricci-symmetric manifold.*

Proof. A limiting argument showing that as $\mathcal{L}_{\mathfrak{S}} \to -\sigma$, the condition $\mathfrak{R}.\mathfrak{S} = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g}, \mathfrak{S})$ degenerates to $\mathfrak{R}.\mathfrak{S} = 0$, which implies that $\mathbb{M}$ Ricci semi-symmetry. □

Corollary 2. *Let* $(\zeta, \lambda, \Lambda, \mathfrak{g})$ *be HRS on Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$*, and if the function* $\mathcal{L}_{\mathfrak{S}}$ *diverges ponitwise asymptotically, then the HRS is decreasing.*

Proof. After using (50) in (51), with condition $\lambda + \sigma = 0$, we obtain the required outcome. □

5. Hyperbolic Ricci Solitons on Concircular Pseudo-Symmetric $(RCS)_m$-Manifold

This section discusses the study of *HRS* on concircular Ricci-pseudo-symmetric $(RCS)_m$-manifolds.

An intriguing invariant of a concircular transformation is the concircular curvature tensor. A circular transformation is a transformation of an $(RCS)_m$-manifolds $\mathbb{M}$ that converts each geodesic circle of $\mathbb{M}$ into a geodesic circle [27].

The concircular curvature tensor $\mathcal{C}^c$ is the curious invariant of a circular transformation, and it is defined by [27]

$$\mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3 = \mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3 - \frac{\mathcal{R}}{m(m-1)}[\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)\mathsf{X}_1 - \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_3)\mathsf{X}_2], \tag{52}$$

where $\mathcal{R}$ is the scalar curvature of the manifold $\mathbb{M}$ and $\mathfrak{R}$ is the curvature tensor.

By using (10), (20), and (22), we obtain

$$\mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\zeta = \left[(\sigma^2 + \mu) - \frac{\mathcal{R}}{m(m-1)}\right][\pi(\mathsf{X}_2)\mathsf{X}_1 - \pi(\mathsf{X}_1)\mathsf{X}_2], \tag{53}$$

$$\pi(\mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3) = \left[(\sigma^2 + \mu) - \frac{\mathcal{R}}{m(m-1)}\right][\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_3) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)], \tag{54}$$

An $(RCS)_m$-manifold $\mathbb{M}$ is considered concircular Ricci pseudo-symmetric if its concircular curvature tensor $\mathcal{C}^c$ meets

$$(\mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\mathfrak{S})(\mathsf{X}_3, \mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g}, \mathfrak{S})(\mathsf{X}_3, \mathsf{X}_4; \mathsf{X}_1, \mathsf{X}_2) \tag{55}$$

holds in a set $\mathbb{U}_{\mathfrak{S}} = \left\{ p \in \mathbb{M} \,\middle|\, \mathfrak{S} \neq \frac{\mathcal{R}}{m} \text{ at } p \right\}$ with a function $\mathcal{L}_{\mathfrak{S}}$ on set $\mathbb{U}_{\mathfrak{S}}$.

Consider a concircular Ricci pseudo-symmetric $(RCS)_m$-manifold with *HRS* as its metric. Then, according to (55),

$$\mathfrak{S}(\mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3, \mathsf{X}_4) + \mathfrak{S}(\mathsf{X}_3, \mathcal{C}^c(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)\mathfrak{S}(\mathsf{X}_1, \mathsf{X}_4) \tag{56}$$

$$-\mathfrak{g}(X_1,X_3)\mathfrak{S}(X_2,X_4)+\mathfrak{g}(X_2,X_4)\mathfrak{S}(X_1,X_3)-\mathfrak{g}(X_1,X_4)\mathfrak{S}(X_2,X_3)].$$

It follows from (56) by virtue of (39) that

$$\pi(\mathcal{C}^c(X_1,X_2)X_3)\pi(X_4)+\pi(X_3)(\mathcal{C}^c(X_1,X_2)X_4)=\mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(X_2,X_3)\pi(X_1)\pi(X_4) \tag{57}$$

$$-\mathfrak{g}(X_1,X_3)\pi(X_2)\pi(X_4)+\mathfrak{g}(X_2,X_4)\pi(X_1)\pi(X_3)-\mathfrak{g}(X_1,X_4)\pi(X_2)\pi(X_3)].$$

By utilizing (53) and (54) and setting $X_3=\zeta$ in (57), we obtain

$$\left[\mathcal{L}_{\mathfrak{S}}-[(\sigma^2+\mu)-\frac{\mathcal{R}}{m(m-1)}]\right][\pi(X_2)\mathfrak{g}(X_1,X_4)-\pi(X_1)\mathfrak{g}(X_2,X_4)]=0. \tag{58}$$

Using (15), (16) and (42) and substituting $X_2=\zeta$ in (58), we obtain

$$\left[\mathcal{L}_{\mathfrak{S}}-[(\sigma^2+\mu)-\frac{m\Lambda}{(m-1)}+\frac{2\sigma(\sigma-\searrow)}{m}]\right][\mathfrak{g}(X_1,X_4)-\pi(X_1)\pi(X_4)]=0 \tag{59}$$

for every vector field X_1 and X_4, so that

$$\mathcal{L}_{\mathfrak{S}}=(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\frac{2\sigma(\sigma-\searrow)}{m}. \tag{60}$$

The following results from this:

Theorem 6. *If $(\zeta,\searrow,\Lambda,\mathfrak{g})$ is a HRS on a concircular Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$, then $\mathcal{L}_{\mathfrak{S}}=(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\frac{2\sigma(\sigma-\searrow)}{m}$.*

Again from (60), we get

$$\searrow=\sigma-\frac{m}{2\sigma}\left[(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\mathcal{L}_{\mathfrak{S}}\right]. \tag{61}$$

This leads to the following

Theorem 7. *If a concircular Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$ admits the HRS, then the HRS is growing, stable or decreasing, referred to as*

1. $\sigma>\frac{m}{2\sigma}\left[(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\mathcal{L}_{\mathfrak{S}}\right]$,
2. $\sigma=\frac{m}{2\sigma}\left[(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\mathcal{L}_{\mathfrak{S}}\right]$, *and*
3. $\sigma<\frac{m}{2\sigma}\left[(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}-\mathcal{L}_{\mathfrak{S}}\right]$, *respectively.*

Corollary 3. *If a concircular Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$ admits the HRS, then the critical value for $\mathcal{L}_{\mathfrak{S}}$ is $(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}$.*

Next, in view of Remark 2 and (61), we have

Corollary 4. *If a concircular Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$ admits the HRS, then the critical value for*

$$\mathcal{L}_{\mathfrak{S}}=(\sigma^2+\mu)+\frac{m\Lambda}{(m-1)}. \tag{62}$$

Example 2. *Let a 3-dimensional manifold* $\mathbb{M} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0\}$, *where* (x_1, x_2, x_3) *are the Cartesian coordinates in* $\mathbb{R}^3$, *and let the linearly independent vector fields are*

$$\mathsf{X}_1 = x_2^3 \frac{\partial}{\partial x_1}, \quad \mathsf{X}_2 = x_2^3 \frac{\partial}{\partial x_2}, \quad \mathsf{X}_3 = \frac{\partial}{\partial x_3},$$

where at every point in $\mathbb{M}$, X_1, X_2, *and* X_3 *are linearly independent. The Lorentzian metric, denoted by* $\mathfrak{g}$, *is described as:*

$$\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_1) = \mathfrak{g}(\mathsf{X}_2, \mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_3, \mathsf{X}_3) = 1, \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3) = \mathfrak{g}(\mathsf{X}_3, \mathsf{X}_1) = 0,$$

Let φ *be the (1,1) tensor field such that* $\varphi(\mathsf{X}_1) = \mathsf{X}_1, \quad \varphi(\mathsf{X}_2) = \mathsf{X}_2, \quad \varphi(\mathsf{X}_3) = 0$, *and* π *are the* 1*-form defined by* $\pi(\mathsf{X}_1) = g(\zeta, \mathsf{X}_3)$ *for any vector field* ζ *on* $\mathbb{M}$.
Then, utilizing the linearity of φ *and* $\mathfrak{g}$, *we obtain*

$$\varphi^2 \mathsf{X}_1 = \mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta,$$

where $\mathsf{X}_3 = \zeta$. *Moreover, for vector fields* $\mathsf{X}_1, \mathsf{X}_2 \in \chi(\mathbb{M})$ *with*

$$\mathfrak{g}(\phi \mathsf{X}_1, \phi \mathsf{X}_2) = \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2).$$

Therefore, the structure defines an $(RCS)_3$*-structure in* $\mathbb{R}^3$.

If ∇ *is the Levi-Civita connection with regard to the metric* $\mathfrak{g}$, *we have*

$$2\mathfrak{g}(\nabla_{\mathsf{X}_1} \mathsf{X}_2, \mathsf{X}_3) = \mathsf{X}_1 \mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3) + \mathsf{X}_2 \mathfrak{g}(\mathsf{X}_3, \mathsf{X}_1) - \mathsf{X}_3 \mathfrak{g} \mathsf{X}_1, \mathsf{X}_2) - \mathfrak{g}(\mathsf{X}_1, [\mathsf{X}_2, \mathsf{X}_3])$$

$$-g(\mathsf{X}_2, [\mathsf{X}_1, \mathsf{X}_3]) + \mathfrak{g}(\mathsf{X}_3, [\mathsf{X}_1, \mathsf{X}_2]),$$

which is referred to as Koszul's formula.

$$\nabla_{\mathsf{X}_1} \mathsf{X}_1 = -\frac{2}{x_3} \mathsf{X}_3, \quad \nabla_{\mathsf{X}_1} \mathsf{X}_2 = 0, \quad \nabla_{\mathsf{X}_1} \mathsf{X}_3 = -\frac{2}{x_3} \mathsf{X}_1,$$
$$\nabla_{\mathsf{X}_2} \mathsf{X}_2 = -\frac{2}{x_3} \mathsf{X}_2, \quad \nabla_{\mathsf{X}_2} \mathsf{X}_1 = 0, \quad \nabla_{\mathsf{X}_2} \mathsf{X}_3 = -\frac{2}{x_3} \mathsf{X}_2,$$
$$\nabla_{\mathsf{X}_3} \mathsf{X}_1 = 0, \quad \nabla_{\mathsf{X}_3} \mathsf{X}_2 = 0, \quad \nabla_{\mathsf{X}_3} \mathsf{X}_3 = 0.$$

For each vector X_1 *on* $\mathbb{M}$, *using the aforementioned expression, we have*

$$\nabla_{\mathsf{X}_1} \zeta = \sigma[\mathsf{X}_1 - \pi(\mathsf{X}_1)\zeta],$$

where $\sigma = -\frac{2}{x_3}$ *and such that* $\mathsf{X}_1(\sigma) = \mu\pi(\mathsf{X}_1)$, *implies* $\mu = -\frac{2}{x_3^2}$ *Hence the* $(\varphi, \zeta, \pi, \mathfrak{g}, \sigma)$ *structure defines the* $(RCS)_3$*-structure in* $\mathbb{R}^3$.
Indeed,the Riemann curvature $\mathfrak{R}$ *and the Ricci curvature tensor* $\mathfrak{S}$ *are computed as follows:*

$$\mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_2 = \frac{4}{x_3^2} \mathsf{X}_1, \quad \mathfrak{R}(\mathsf{X}_1, \mathsf{X}_3)\mathsf{X}_3 = -\frac{6}{z^2} \mathsf{X}_1, \quad \mathfrak{R}(\mathsf{X}_2, \mathsf{X}_1)\mathsf{X}_1 = \frac{4}{x_3^2} \mathsf{X}_2, \tag{63}$$

$$\mathfrak{R}(\mathsf{X}_2, \mathsf{X}_3)\mathsf{X}_3 = -\frac{6}{x_3^2} \mathsf{X}_2, \quad \mathfrak{R}(\mathsf{X}_3, \mathsf{X}_1)\mathsf{X}_1 = \frac{4}{x_3^2} \mathsf{X}_3, \quad \mathfrak{R}(\mathsf{X}_3, \mathsf{X}_2)\mathsf{X}_2 = \frac{4}{x_3^2} \mathsf{X}_3.$$

Hence, we obtain the Ricci tensor

$$\mathfrak{S}(\mathsf{X}_1, \mathsf{X}_1) = \mathfrak{S}(\mathsf{X}_2, \mathsf{X}_2) = -\frac{2}{x_3^2}, \quad \mathfrak{S}(\mathsf{X}_3, \mathsf{X}_3) = \frac{12}{{x_3}^2}. \tag{64}$$

Therefore, in view of (64), the scalar curvature $\mathcal{R}$ is computed as

$$\mathcal{R} = \sum_{i=1}^{3} \mathfrak{g}(\mathsf{X}_i, \mathsf{X}_i)\mathfrak{S}(\mathsf{X}_i, \mathsf{X}_i) = \frac{8}{x_3^2}. \tag{65}$$

Let $\{\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3\}$ be the basis of the $(RCS)_3$-manifold $\mathbb{M}$, every vector field $\mathsf{U}, \mathsf{V}, \mathsf{W}, \mathsf{Z} \in \chi(\mathbb{M})$ can be expressed as

$$\mathsf{U} = \alpha_1\mathsf{X}_1 + \beta_1\mathsf{X}_2 + \delta_1\mathsf{X}_3, \quad \mathsf{V} = \alpha_2\mathsf{X}_1 + \beta_2\mathsf{X}_2 + \delta_2\mathsf{X}_3, \tag{66}$$

$$\mathsf{W} = \alpha_3\mathsf{X}_1 + \beta_3\mathsf{X}_2 + \delta_3\mathsf{X}_3, \quad \mathsf{Z} = \alpha_4\mathsf{X}_1 + \beta_4\mathsf{X}_2 + \delta_4\mathsf{X}_3, \tag{67}$$

where $\alpha_i\beta_i, \delta_i \in \mathbb{R}$ for all $i = 1, 2, 3$ such that all $\alpha_i\beta_i, \delta_i$ are not proportional. Then

$$\begin{aligned}\mathfrak{R}(\mathsf{U}, \mathsf{V})\mathsf{W} = {} & \frac{2}{x_3^2}[3\beta_3(\alpha_1\beta_2 - \alpha_2\beta_1) - 3\delta_3(\alpha_1\delta_2 - \alpha_2\delta_1)]\mathsf{X}_1 \\ & -\frac{2}{x_3^2}[3\alpha_3(\alpha_1\beta_2 - \alpha_2\beta_1) + 3\delta_3(\beta_1\delta_2 - \beta_2\delta_1)]\mathsf{X}_2 \\ & -\frac{6}{x_3^2}[\beta_4(\beta_1\delta_2 - \beta_2\delta_1) + \alpha_4(\alpha_1\delta_2 - \alpha_2\delta_1)]\mathsf{X}_3.\end{aligned} \tag{68}$$

$$\begin{aligned}\mathfrak{R}(\mathsf{U}, \mathsf{V})\mathsf{Z} = {} & \frac{2}{x_3^2}[3\beta_4(\alpha_1\beta_2 - \alpha_2\beta_1) - 3\delta_4(\alpha_1\delta_2 - \alpha_2\delta_1)]\mathsf{X}_1 \\ & -\frac{2}{x_3^2}[3\alpha_4(\alpha_1\beta_2 - \alpha_2\beta_1) + 3\delta_4(\beta_1\delta_2 - \beta_2\delta_1)]\mathsf{X}_2 \\ & -\frac{6}{x_3^2}[\beta_4(\beta_1\delta_2 - \beta_2\delta_1) + \alpha_4(\alpha_1\delta_2 - \alpha_2\delta_1)]\mathsf{X}_3.\end{aligned} \tag{69}$$

Using (68) and (52), we get

$$\mathcal{C}^c(\mathsf{U}, \mathsf{V})\mathsf{W} = \mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3 - \frac{\mathcal{R}}{6}[\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)\mathsf{X}_1 - \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_3)\mathsf{X}_2] \tag{70}$$

$$\begin{aligned}= {} & \frac{2}{x_3^2}[3\beta_4(\alpha_1\beta_2 - \alpha_2\beta_1) - 3\delta_4(\alpha_1\delta_2 - \alpha_2\delta_1) \\ & -\frac{2}{3}[\alpha_1(\beta_2\beta_3 - \delta_2\delta 3) - \alpha_2(\beta_1\beta_3 - \delta_3\delta_1)]\mathsf{X}_1 \\ & -\frac{2}{x_3^2}[3\alpha_4(\alpha_1\beta_2 - \alpha_2\beta_1) + 3\delta_4(\beta_1\delta_2 - \beta_2\delta_1) \\ & +\frac{2}{3}[\beta_1(\alpha_2\alpha_3 - \delta_2\delta_3) - \beta_2(\alpha_1\alpha_3 - \delta_3\delta_1)]]\mathsf{X}_2 \\ & -\frac{2}{x_3^2}[\beta_4(\beta_1\delta_2 - \beta_2\delta_1) + \alpha_4(\alpha_1\delta_2 - \alpha_2\delta_1) \\ & +\frac{2}{3}\delta_1(\alpha_2\alpha_3 + \beta_2\beta_3) - \delta_2(\alpha_1\alpha_3 + \beta_1\beta_{3)}]\mathsf{X}_3.\end{aligned} \tag{71}$$

Then, we find

$$\begin{aligned}\mathfrak{S}(\mathcal{C}^c(\mathsf{U}, \mathsf{V})\mathsf{W}, \mathsf{Z}) = {} & -\frac{4\alpha_4}{x_3^2}[3\beta_4(\alpha_1\beta_2 - \alpha_2\beta_1) - 3\delta_4(\alpha_1\delta_2 - \alpha_2\delta_1) \\ & -\frac{2}{3}[\alpha_1(\beta_2\beta_3 - \delta_2\delta 3) - \alpha_2(\beta_1\beta_3 - \delta_3\delta_1)]\end{aligned} \tag{72}$$

$$+\frac{4\beta_4}{x_3^2}[3\alpha_4(\alpha_1\beta_2-\alpha_2\beta_1)+3\delta_4(\beta_1\delta_2-\beta_2\delta_1)$$
$$+\frac{2}{3}[\beta_1(\alpha_2\alpha_3-\delta_2\delta_3)-\beta_2(\alpha_1\alpha_3-\delta_3\delta_1)]$$
$$+\frac{24\delta_4}{x_3^2}[\beta_4(\beta_1\delta_2-\beta_2\delta_1)+\alpha_4(\alpha_1\delta_2-\alpha_2\delta_1)$$
$$+\frac{2}{3}\delta_1(\alpha_2\alpha_3+\beta_2\beta_3)-\delta_2(\alpha_1\alpha_3+\beta_1\beta_{3)}.$$

Similarly we obtain,

$$\mathfrak{S}(W,\mathcal{C}^c(U,V)Z)=-\frac{4\alpha_3}{x_3^2}[3\beta_3(\alpha_1\beta_2-\alpha_2\beta_1)-3\delta_3(\alpha_1\delta_2-\alpha_2\delta_1) \tag{73}$$
$$-\frac{2}{3}[\alpha_1(\beta_2\beta_4-\delta_2\delta 4)-\alpha_2(\beta_1\beta_4-\delta_1\delta_4)]$$
$$+\frac{4\beta_3}{x_3^2}[3\alpha_4(\alpha_1\beta_2-\alpha_2\beta_1)+3\delta_3(\beta_1\delta_2-\beta_2\delta_1)$$
$$+\frac{2}{3}[\beta_1(\alpha_2\alpha_4-\delta_2\delta_4)-\beta_2(\alpha_1\alpha_4-\delta_1\delta_4)]$$
$$+\frac{24\delta_3}{x_3^2}[\beta_4(\beta_1\delta_2-\beta_2\delta_1)+\alpha_4(\alpha_1\delta_2-\alpha_2\delta_1)$$
$$+\frac{2}{3}[\delta_1(\alpha_2\alpha_4+\beta_2\beta_4)-\delta_2(\alpha_1\alpha_4+\beta_1\beta_{4)}].$$

Now, we have

$$\mathfrak{g}(V,W)=\alpha_2\alpha_3+\beta_2\beta_3-\delta_2\delta_3,\quad \mathfrak{g}(U,W)=\alpha_1\alpha_3+\beta_1\beta_3-\delta_1\delta_3, \tag{74}$$

$$\mathfrak{g}(V,Z)=\alpha_2\alpha_4+\beta_2\beta_4-\delta_2\delta_4,\quad \mathfrak{g}(U,Z)=\alpha_1\alpha_4+\beta_1\beta_4-\delta_1\delta_4. \tag{75}$$

Also we have

$$\mathfrak{S}(V,W)=-\frac{2}{x_3^2}[\alpha_2\alpha_3+\beta_2\beta_3-\delta_2\delta_3],\quad \mathfrak{S}(U,W)=-\frac{2}{x_3^2}[\alpha_1\alpha_3+\beta_1\beta_3-\delta_1\delta_3], \tag{76}$$

$$\mathfrak{S}(V,Z)=-\frac{2}{x_3^2}[\alpha_2\alpha_4+\beta_2\beta_4-\delta_2\delta_4],\quad \mathfrak{S}(U,Z)=-\frac{2}{x_3^2}[\alpha_1\alpha_4+\beta_1\beta_4-\delta_1\delta_4]. \tag{77}$$

Therefore, from (74), (75), (76) and (77) we turn up

$$\mathfrak{g}(V,W)\mathfrak{S}(U,Z)-\mathfrak{g}(U,W)\mathfrak{S}(V,Z) \tag{78}$$
$$+\mathfrak{g}(V,Z)\mathfrak{S}(U,W)-\mathfrak{g}(U,Z)\mathfrak{S}(V,W)=$$
$$=\frac{14}{x_3^2}[(\alpha_1\delta_2-\alpha_2\delta_1)(\alpha_3\delta_4+\alpha_4\delta_3)+(\beta_1\delta_2-\beta_2\delta_1)(\beta_3\delta_4+\beta_4\delta_3)]\neq 0. \tag{79}$$

Since, $\alpha_i\beta_i,\delta_i$ are not proportional, then

$$(\alpha_1\delta_2-\alpha_2\delta_1)(\alpha_3\delta_4+\alpha_4\delta_3)+(\beta_1\delta_2-\beta_2\delta_1)(\beta_3\delta_4+\beta_4\delta_3)\neq 0.$$

Using (72) and (73), we have

$$\mathfrak{S}(\mathcal{C}^c(U,V)W,Z)+\mathfrak{S}(W,\mathcal{C}^c(U,V)Z)=\frac{196}{2x_3^4}\mathcal{K}\neq 0, \tag{80}$$

where $\mathcal{K} = (\alpha_1\delta_2 - \alpha_2\delta_1)(\alpha_3\delta_4 + \alpha_4\delta_3) + (\beta_1\delta_2 - \beta_2\delta_1)(\beta_3\delta_4 + \beta_4\delta_3)$.

Now, let us choose the function

$$\mathcal{L}_{\mathfrak{S}} = \frac{14}{2x_3^2}. \tag{81}$$

In light of (81), we have from (80) and (81) that

$$\mathfrak{S}(\mathcal{C}^c(\mathsf{U},\mathsf{V})\mathsf{W},\mathsf{Z}) + \mathfrak{S}(\mathsf{W},\mathcal{C}^c(\mathsf{U},\mathsf{V})\mathsf{Z}) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{U},\mathsf{W})\mathfrak{S}(\mathsf{U},\mathsf{Z}) \tag{82}$$

$$-\mathfrak{g}(\mathsf{U},\mathsf{W})\mathfrak{S}(\mathsf{V},\mathsf{Z}) + \mathfrak{g}(\mathsf{V},\mathsf{Z})\mathfrak{S}(\mathsf{U},\mathsf{W}) - \mathfrak{g}(\mathsf{U},\mathsf{Z})\mathfrak{S}(\mathsf{V},\mathsf{W})].$$

Thus, the $(RCS)_3$*-manifold under consideration is a concircular Ricci pseudo-symmetric manifold, which illustrates Equation (60) and Theorem 6.*

After using the definition of hyperbolic Ricci soliton, we have $\nwarrow = -\frac{2}{x_3}$ *and* $\Lambda = \frac{4}{x_3{}^2}$*. Hence, in view of (60), we have*

$$\mathcal{L}_{\mathfrak{S}} = (\sigma^2 + \mu) + \frac{3\frac{4}{x_3{}^2}}{(2)} - \frac{2\sigma(\sigma + \frac{2}{x_3})}{3} = \frac{14}{2x_3^2}, \quad as \quad \sigma = -\frac{2}{x_3}, \quad \mu = -\frac{2}{x_3^2}, \tag{83}$$

which satisfies (60). Thus Theorem 6 is also verified.

6. Hyperbolic Ricci Solitons on Projective Pseudo-Symmetric $(RCS)_m$-Manifold

HRS on projective Ricci pseudosymmetric $(RCS)_n$-manifolds are the subject of our current investigation. An essential idea in Riemannian geometry is the projective curvature tensor, which is used to compute the fundamental geometric measures on a manifold, such as angle, distance, and different invariants [9].

The projective curvature tensor is given by [30]

$$\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3 = \mathfrak{R}(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3 - \frac{1}{m-1}[\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_3)\mathsf{X}_1 - \mathfrak{S}(\mathsf{X}_1,\mathsf{X}_3)\mathsf{X}_2], \tag{84}$$

Adopting (10), (20), (22), and (40), we obtain

$$\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\zeta = \left[(\sigma^2 + \mu) - \frac{\Lambda}{(m-1)}\right][\pi(\mathsf{X}_1)\mathsf{X}_2 - \pi(\mathsf{X}_2)\mathsf{X}_1], \tag{85}$$

$$\pi(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3) = \left[(\sigma^2 + \mu) - \frac{\sigma + \Lambda}{(m-1)}\right][\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)], \tag{86}$$

Let an $(RCS)_m$-manifold $\mathbb{M}$ be a projective Ricci pseudo-symmetric if its projective curvature tensor $\mathcal{C}^c$ meets

$$(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathfrak{S})(\mathsf{X}_3,\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g},\mathfrak{S})(\mathsf{X}_3,\mathsf{X}_4;\mathsf{X}_1,\mathsf{X}_2) \tag{87}$$

holds in a set $\mathbb{U}_{\mathfrak{S}} = \left\{p \in \mathbb{M} \,\middle|\, \mathfrak{S} \neq \frac{\mathcal{R}}{m} \ at \ p \right\}$, with a function $\mathcal{L}_{\mathfrak{S}}$ on set $\mathbb{U}_{\mathfrak{S}}$.

Let us assume a projective Ricci pseudo-symmetric $(RCS)_m$-manifold admitting hyperbolic Ricci solitons. Then, according to (87),

$$\mathfrak{S}(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3,\mathsf{X}_4) + \mathfrak{S}(\mathsf{X}_3,\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_4) \tag{88}$$

$$-\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3)\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_4) + \mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_3) - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4)\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_3)].$$

After applying (39) in (88) we get

$$(\Lambda - 2\sigma^2 - 2\lambda\sigma)[\mathfrak{g}(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3,\mathsf{X}_4) + \mathfrak{g}(\mathsf{X}_3,\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_4)] \tag{89}$$

$$+2\sigma(\sigma+\lambda)[\pi(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3)\pi(\mathsf{X}_4) + \pi(\mathsf{X}_3)\pi(\mathcal{P}^r(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_4)$$

$$= 2\sigma(\sigma+\lambda)\mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)\pi(\mathsf{X}_1)\pi(\mathsf{X}_4) - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3)\pi(\mathsf{X}_2)\pi(\mathsf{X}_4)$$

$$+\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)\pi(\mathsf{X}_1)\pi(\mathsf{X}_3) - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4)\pi(\mathsf{X}_2)\pi(\mathsf{X}_3)].$$

Substituting $\mathsf{X}_3 = \zeta$ in (89), we find

$$[2\sigma(\sigma+\lambda)\mathcal{L}_{\mathfrak{S}} - (2\Lambda - 2\sigma^2 - 2\lambda\sigma)(\sigma^2+\mu)][\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)] = 0. \tag{90}$$

Again, putting $\mathsf{X}_2 = \zeta$ in (90), we get

$$[2\sigma(\sigma+\lambda)\mathcal{L}_{\mathfrak{S}} - (2\Lambda - 2\sigma^2 - 2\lambda\sigma)(\sigma^2+\mu)][\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_4)] = 0 \tag{91}$$

For every vector field X_1 and X_4, it may be inferred that

$$\mathcal{L}_{\mathfrak{S}} = \left[\frac{2\Lambda}{2\sigma(\sigma+\lambda)} - 1\right](\sigma^2+\mu), \quad \sigma+\lambda \neq 0. \tag{92}$$

Thus, we can state the following results.

Theorem 8. *If $(\zeta, \lambda, \Lambda, \mathfrak{g})$ is a HRS on a projective Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$, then $\mathcal{L}_{\mathfrak{S}} = \left[\frac{2\Lambda}{2\sigma(\sigma+\lambda)} - 1\right](\sigma^2+\mu)$, provides $\sigma+\lambda \neq 0$.*

Also, (91) provides

$$\lambda = \frac{\Lambda}{[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu)]} - \sigma^2. \tag{93}$$

Hence, (93) entail the next outcome.

Theorem 9. *If a projective Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$ admits the HRS, then the HRS is growing, stable or decreasing, referred to as*

1. $\frac{\Lambda}{[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu)]} > \sigma^2$,
2. $\frac{\Lambda}{[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu)]} = \sigma^2$, *and*
3. $\frac{\Lambda}{[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu)]} < \sigma^2$, *respectively.*

Corollary 5. *If a projective Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$ admits the HRS, then the critical value for $\mathcal{L}_{\mathfrak{S}}$ is $\Lambda - (\sigma^2+\mu)$.*

Using the fact $\sigma + \lambda = 0$ with Equation (92), we get $\mathcal{L}_{\mathfrak{S}} \to \infty$. Thus, we can state the following:

Corollary 6. *If $(\zeta, \lambda, \Lambda, \mathfrak{g})$ is a HRS on projective Ricci pseudo-symmetric $(RCS)_m$-manifold $\mathbb{M}$, and the function $\mathcal{L}_{\mathfrak{S}}$ associated with the singular point $\lambda = -\sigma$, then $\mathbb{M}$ is locally Ricci-symmetric manifold.*

Proof. A limiting argument showing that as $\mathcal{L}_{\mathfrak{S}} \to -\sigma$, the condition $\mathfrak{R}.\mathfrak{S} = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g},\mathfrak{S})$ degenerates to $\mathfrak{R}.\mathfrak{S} = 0$, which implies $(RCS)_m$-manifold $\mathbb{M}$ is Ricci semi-symmetric. □

7. Hyperbolic Ricci Solitons on Conharmonic Pseudo-Symmetric $(RCS)_m$-Manifold

Hyperbolic Ricci solitons on conharmonic Ricci pseudo-symmetric $(RCS)_m$-manifold are the subject of our current investigation. Conharmonic transformations, a unique kind of conformal transformation that preserves the harmonicity quality of smooth functions, are of great interest. Ishii [31] first described this kind of change in 1957, and it is currently being researched from a number of angles. The so-called conharmonic curvature tensor is a well-known tensor invariant of these transformations. This tensor has the conventional symmetry characteristics of the Riemannian curvature tensor, making it simple to confirm that it is an algebraic curvature tensor.

The conharmonic curvature tensor $\mathcal{H}^c$ of type (1,3) can be expressed as [31]

$$\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3 = \mathfrak{R}(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3 - \frac{1}{(m-2)}[\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_3)\mathsf{X}_1 - \mathfrak{S}(\mathsf{X}_1,\mathsf{X}_3)\mathsf{X}_2 \tag{94}$$

$$+\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)\mathfrak{Q}\mathsf{X}_1 - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3)\mathfrak{Q}\mathsf{X}_1].$$

Using (10), (20), (22), (40), and (41), we gain

$$\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\zeta = \left[(\sigma^2+\mu) - \frac{2\Lambda - 2\sigma(\sigma+\searrow)}{(m-1)}\right][\pi(\mathsf{X}_2)\mathsf{X}_1 - \pi(\mathsf{X}_1)\mathsf{X}_2], \tag{95}$$

$$\pi(\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3) = \left[(\sigma^2+\mu) - \frac{2\Lambda - 2\sigma(\sigma+\searrow)}{(m-1)}\right][\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)], \tag{96}$$

Let an $(RCS)_m$-manifold $\mathbb{M}$ be a conharmonic Ricci pseudo-symmetric if its projective curvature tensor $\mathcal{C}^c$ meets

$$(\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathfrak{S})(\mathsf{X}_3,\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g},\mathfrak{S})(\mathsf{X}_3,\mathsf{X}_4;\mathsf{X}_1,\mathsf{X}_2) \tag{97}$$

holds in a set $\mathbb{U}_{\mathfrak{S}} = \left\{p \in \mathbb{M} \;\middle|\; \mathfrak{S} \neq \frac{\mathcal{R}}{m} \;\; at \;\; p \;\right\}$, where $\mathcal{L}_{\mathfrak{S}}$ is a function on set $\mathbb{U}_{\mathfrak{S}}$.

Let us assume a conharmonic Ricci pseudo-symmetric $(RCS)_m$-manifold admitting hyperbolic Ricci solitons. Then, in the light of (97), we have

$$\mathfrak{S}(\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3,\mathsf{X}_4) + \mathfrak{S}(\mathsf{X}_3,\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_4) \tag{98}$$

$$-\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3)\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_4) + \mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)\mathfrak{S}(\mathsf{X}_1,\mathsf{X}_3) - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4)\mathfrak{S}(\mathsf{X}_2,\mathsf{X}_3)].$$

By virtue of (39), it follows from (98) that

$$\pi(\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_3)\pi(\mathsf{X}_4) + \pi(\mathsf{X}_3)(\mathcal{H}^c(\mathsf{X}_1,\mathsf{X}_2)\mathsf{X}_4) = \mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_3)\pi(\mathsf{X}_1)\pi(\mathsf{X}_4) \tag{99}$$

$$-\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_3)\pi(\mathsf{X}_2)\pi(\mathsf{X}_4) + \mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)\pi(\mathsf{X}_1)\pi(\mathsf{X}_3) - \mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4)\pi(\mathsf{X}_2)\pi(\mathsf{X}_3)].$$

By employing (95) and (96) and placing $\mathsf{X}_3 = \zeta$ in (99), we derive

$$\left[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu) + \frac{2\Lambda - 2\sigma(\sigma+\searrow)}{(m-1)}\right][\pi(\mathsf{X}_2)\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4) - \pi(\mathsf{X}_1)\mathfrak{g}(\mathsf{X}_2,\mathsf{X}_4)] = 0. \tag{100}$$

Adopting (15), (16) and (42) and substituting $\mathsf{X}_2 = \zeta$ in (100), we gain

$$\left[\mathcal{L}_{\mathfrak{S}} + (\sigma^2+\mu) + \frac{2\Lambda - 2\sigma(\sigma+\searrow)}{(m-1)}\right][\mathfrak{g}(\mathsf{X}_1,\mathsf{X}_4) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_4)] = 0 \tag{101}$$

for every vector field X_1 and X_4, so that

$$\mathcal{L}_{\mathfrak{S}} = \frac{2\Lambda - 2\sigma(\sigma + \lambda)}{(m-1)} - (\sigma^2 + \mu), \quad \sigma + \lambda \neq 0, \tag{102}$$

also, we have

$$\lambda = \frac{(m-1)[\mathcal{L}_{\mathfrak{S}} + (\sigma^2 + \mu)]}{2\sigma} - \frac{\Lambda + \sigma^2}{\sigma}. \tag{103}$$

Therefore, in view of (102) and (103), one can articulate the following results.

Theorem 10. *If* $(\zeta, \lambda, \Lambda, \mathfrak{g})$ *is a HRS on a conharmonic Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$*, then* $\mathcal{L}_{\mathfrak{S}} = \frac{2\Lambda - 2\sigma(\sigma + \lambda)}{(m-1)} - (\sigma^2 + \mu)$.

Theorem 11. *If a conharmonic Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$ *admits the HRS, then the HRS is growing, stable or decreasing, referred to as*

1. $\frac{(m-1)[\mathcal{L}_{\mathfrak{S}} + (\sigma^2 + \mu)]}{2\sigma} > \frac{\Lambda + \sigma^2}{\sigma}$,
2. $\frac{(m-1)[\mathcal{L}_{\mathfrak{S}} + (\sigma^2 + \mu)]}{2\sigma} = \frac{\Lambda + \sigma^2}{\sigma}$, *and*
3. $\frac{(m-1)[\mathcal{L}_{\mathfrak{S}} + (\sigma^2 + \mu)]}{2\sigma} < \frac{\Lambda + \sigma^2}{\sigma}$, *respectively.*

Corollary 7. *If a conharmonic Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$ *admits the HRS, then the critical value for* $\mathcal{L}_{\mathfrak{S}}$ *is* $\frac{\Lambda + \sigma^2}{(m-1)\sigma} - (\sigma^2 + \mu)$.

Again, employing (102) with the case, $\sigma + \lambda = 0$, we obtain

Corollary 8. *If a conharmonic Ricci pseudo-symmetric* $(RCS)_m$*-manifold* $\mathbb{M}$ *admits the HRS, then the critical value for* $\sigma + \lambda = 0$ *is*

$$\mathcal{L}_{\mathfrak{S}} = \frac{2\Lambda}{(m-1)} - (\sigma^2 + \mu). \tag{104}$$

Proof. A limiting argument showing that as $\mathcal{L}_{\mathfrak{S}} \to -\sigma$, the condition $\mathfrak{R}.\mathfrak{S} = \mathcal{L}_{\mathfrak{S}}\mathfrak{Q}(\mathfrak{g}, \mathfrak{S})$ degenerates to $\mathfrak{R}.\mathfrak{S} = 0$, which is Ricci semi-symmetry. □

We are now summarizing the four pseudo-symmetry settings, the growing/stable/shrinking $(\gtreqless)$ circumstances for λ, and the related value of $\mathcal{L}_{\mathfrak{S}}$ in the following table.

Solitonic Pseudo-Symmetry Type $(RCS)_m$	Critical Value $\mathcal{L}_{\mathfrak{S}}$	Value of λ Growing/Stable/Shrinking
Ricci pseudo-symmetric	$\mathcal{L}_{\mathfrak{S}} = \frac{\sigma(\sigma^2 + \mu)}{2\sigma(\sigma + \lambda)}$	$\frac{\sigma(\sigma^2 + \mu)}{\mathcal{L}_{\mathfrak{S}}} \gtreqless \sigma$
Concircular pseudo-symmetric	$\mathcal{L}_{\mathfrak{S}} = (\sigma^2 + \mu) + \frac{m\Lambda}{(m-1)} - \frac{2\sigma(\sigma - \lambda)}{m}$	$\sigma \gtreqless \frac{m}{2\sigma}\left[(\sigma^2 + \mu) + \frac{m\Lambda}{(m-1)} - \mathcal{L}_{\mathfrak{S}}\right]$
Projective pseudo-symmetric	$\mathcal{L}_{\mathfrak{S}} = \left[\frac{2\Lambda}{2\sigma(\sigma + \lambda)} - 1\right](\sigma^2 + \mu)$	$\frac{\Lambda}{\sigma[\frac{\mathcal{L}_{\mathfrak{S}}}{(\sigma^2 + \mu)} + 1]} \gtreqless \sigma$
Conharmonic pseudo-symmetric	$\mathcal{L}_{\mathfrak{S}} = \frac{2\Lambda - 2\sigma(\sigma + \lambda)}{(m-1)}$	$\frac{(m-1)[\mathcal{L}_{\mathfrak{S}} + (\sigma^2 + \mu)]}{2\sigma} \gtreqless \frac{\Lambda + \sigma^2}{\sigma}$

Example 3. *Using the Example 2, for conharmonic Ricci pseudo-symmetric* $(RCS)_3$*-manifold. Then from (94), we have*

$$\mathcal{H}^c(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3 = \mathfrak{R}(\mathsf{X}_1, \mathsf{X}_2)\mathsf{X}_3 - [\mathfrak{S}(\mathsf{X}_2, \mathsf{X}_3)\mathsf{X}_1 - \mathfrak{S}(\mathsf{X}_1, \mathsf{X}_3)\mathsf{X}_2 \tag{105}$$

$$+\mathfrak{g}(\mathsf{X}_2, \mathsf{X}_3)\mathfrak{Q}\mathsf{X}_1 - \mathfrak{g}(\mathsf{X}_1, \mathsf{X}_3)\mathfrak{Q}\mathsf{X}_1].$$

In view of (98), we have

$$\mathfrak{S}(\mathcal{H}^c(U,V)W,Z)+\mathfrak{S}(W,\mathcal{H}^c(U,V)Z)=\mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(V,W)\mathfrak{S}(U,Z) \tag{106}$$

$$-\mathfrak{g}(U,W)\mathfrak{S}(V,Z)+\mathfrak{g}(VZ)\mathfrak{S}(U,W)-\mathfrak{g}(U,Z)\mathfrak{S}(V,W)].$$

By virtue of (74), (75), (76) and (77), we have

$$\mathfrak{g}(V,W)\mathfrak{S}(U,Z)-\mathfrak{g}(U,W)\mathfrak{S}(V,Z) \tag{107}$$

$$+\mathfrak{g}(V,Z)\mathfrak{S}(U,W)-\mathfrak{g}(U,Z)\mathfrak{S}(V,W)=$$

$$=-\frac{14}{x_3^2}[(\alpha_1\delta_2-\alpha_2\delta_1)(\alpha_3\delta_4+\alpha_4\delta_3)+(\beta_1\delta_2-\beta_2\delta_1)(\beta_3\delta_4+\beta_4\delta_3)]\neq 0. \tag{108}$$

Applying (106) and (107), we find

$$\begin{aligned}&\mathfrak{S}(\mathcal{H}^c(U,V)W,Z)+\mathfrak{S}(W,\mathcal{H}^c(U,V)Z)\\&=\tfrac{196}{x_3^4}(\alpha_1\delta_2-\alpha_2\delta_1)(\alpha_3\delta_4+\alpha_4\delta_3)+(\beta_1\delta_2-\beta_2\delta_1)(\beta_3\delta_4+\beta_4\delta_3)\neq 0.\end{aligned} \tag{109}$$

Let us choose the function

$$\mathcal{L}_{\mathfrak{S}}=-\frac{14}{x_3^2}. \tag{110}$$

Thus, in view of (81) we have from (109) and (110) that

$$\mathfrak{S}(\mathcal{H}^c(U,V)W,Z)+\mathfrak{S}(W,\mathcal{H}^c(U,V)Z)=\mathcal{L}_{\mathfrak{S}}[\mathfrak{g}(U,W)\mathfrak{S}(U,Z) \tag{111}$$

$$-\mathfrak{g}(U,W)\mathfrak{S}(V,Z)+\mathfrak{g}(V,Z)\mathfrak{S}(U,W)-\mathfrak{g}(U,Z)\mathfrak{S}(V,W)].$$

Hence the $(RCS)_3$-manifold is conharmonic Ricci pseudo-symmetric manifold, which obeys Equation (98) and Theorem 10.
After, using $\sigma=-\frac{2}{x_3}$, $\mu=-\frac{2}{x_3^2}$, $\lambda=-\frac{2}{x_3}$ and $\Lambda=\frac{4}{x_3{}^2}$. Hence, in view of (102), we gain

$$\mathcal{L}_{\mathfrak{S}}=\frac{2\Lambda-2\sigma(\sigma+\lambda)}{2}-(\sigma^2+\mu)=-\frac{14}{x_3^2}, \tag{112}$$

which holds (102). Thus Theorem 10 is also verified.

8. Hyperbolic Yamabe Soliton on $(RCS)_m$-Manifold

In this section, we discuss the hyperbolic Yamabe soliton on $(RCS)_m$-manifolds.
Let $HYS\ (\zeta,\lambda,\Lambda,\mathfrak{g})$ on a m-dimensional $(RCS)_m$-manifold $\mathbb{M}$. Next, from (4), we have

$$(\mathcal{L}_\zeta\mathcal{L}_\zeta\mathfrak{g})(X_1,X_2)+\lambda(\mathcal{L}_\zeta\mathfrak{g})(X_1,X_2)=(\Lambda-\mathcal{R})\mathfrak{g}(X_1,X_2). \tag{113}$$

Now, using (28) and (35) in (113), we obtain

$$4\sigma^2[\mathfrak{g}(X_1,X_2)-\pi(X_1)\pi(X_2)]+2\sigma\lambda[\mathfrak{g}(X_1,X_2)-\pi(X_1)\pi(X_2)]=(\Lambda-\mathcal{R})\mathfrak{g}(X_1,X_2). \tag{114}$$

$$2\sigma[\mathfrak{g}(X_1,X_2)-\pi(X_1)\pi(X_2)](2\sigma+\lambda)=(\Lambda-\mathcal{R})\mathfrak{g}(X_1X_2). \tag{115}$$

$$[2\sigma(2\sigma+\lambda)+\mathcal{R}-\Lambda]\mathfrak{g}(X_1X_2)+2\sigma(2\sigma+\lambda)\pi(X_1)\pi(X_2)=0. \tag{116}$$

Putting $X_2=\zeta$ in (116), we get

$$[4\sigma(2\sigma+\lambda)+\mathcal{R}-\Lambda]\pi(X_1)=0, \tag{117}$$

which implies

$$\mathcal{R} = \Lambda - 4\sigma(2\sigma + \lambda), \tag{118}$$

$$\lambda = \frac{1}{4\sigma}[\Lambda - (\mathcal{R} + 8\sigma)]. \tag{119}$$

Theorem 12. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits HYS $(\zeta, \lambda, \Lambda, \mathfrak{g})$, then the scalar curvature of $(RCS)_m$-manifold $\mathbb{M}$ is $\Lambda - 4\sigma(2\sigma + \lambda)$.*

Theorem 13. *If $(RCS)_m$-manifold $\mathbb{M}$ admits HYS, then HYS is growing, stable or decreasing, referred to as*

1. $\Lambda > (\mathcal{R} + 8\sigma)$,
2. $\Lambda = (\mathcal{R} + 8\sigma)$, *and*
3. $\Lambda < (\mathcal{R} + 8\sigma)$, *respectively.*

9. Some Applications

For the hyperbolic flow (1) and (3), if there exists a function $\mathfrak{f}(t)$ and 1-parametric flow $\psi(t) : \mathbb{M} \longrightarrow \mathbb{M}$ such that the solution of (1) and (3) is

$$\mathfrak{g}(t) = \mathfrak{f}(t)\Psi(t)^*\mathfrak{g}(0). \tag{120}$$

We start by considering the initial conditions that give rise to hyperbolic soliton. Differentiating (120) twice yields

$$-2\mathfrak{S}(\mathfrak{g}(t)) = \mathfrak{f}''(t)\Psi(t)^*\mathfrak{g}(0) + 2\mathfrak{f}(t)'\Psi(t)^*(\mathfrak{L}_F\mathfrak{g}(0)) + \mathfrak{f}(t)\Psi(t)^*(\mathfrak{L}_F(\mathfrak{L}_F\mathfrak{g}(0))), \tag{121}$$

$$-\mathcal{R}(\mathfrak{g}(t)) = \mathfrak{f}''(t)\Psi(t)^*g(0) + 2\mathfrak{f}(t)'\Psi(t)^*(\mathfrak{L}_F\mathfrak{g}(0)) + \mathfrak{f}(t)\Psi(t)^*(\mathfrak{L}_F(\mathfrak{L}_F\mathfrak{g}(0))), \tag{122}$$

wherein $\mathfrak{g}(0) = \mathfrak{g}_0$. F is the time-dependent soliton vector field, and $\mathfrak{L}$ is the Lie derivative. For any $p \in \mathbb{M}$, $F(\Psi(t)(p)) = \frac{d}{dt}(\Psi(t))$; $\mathfrak{f}' = \frac{d\mathfrak{f}}{dt}$.
Since, we know that

$$\mathfrak{R}ic(\mathfrak{g}(t)) = \Psi(t)^*\mathfrak{S}(\mathfrak{g}(0)),$$

and

$$\mathcal{R}(\mathfrak{g}(t)) = \Psi(t)^*\mathcal{R}(g(0)),$$

we can skip the pullbacks in (121) and (122), we obtain

$$-2\mathfrak{S}(\mathfrak{g}(0)) = \mathfrak{f}''(t)\mathfrak{g}(0) + 2\mathfrak{f}(t)'(\mathfrak{L}_F\mathfrak{g}(0)) + \mathfrak{f}(t)\mathfrak{L}_F(\mathfrak{L}_F\mathfrak{g}(0)), \tag{123}$$

$$-\mathcal{R}(\mathfrak{g}(0)) = \mathfrak{f}''(t)\mathfrak{g}(0) + 2\mathfrak{f}(t)'(\mathfrak{L}_F\mathfrak{g}(0)) + \mathfrak{f}(t)\mathfrak{L}_F(\mathfrak{L}_F\mathfrak{g}(0)), \tag{124}$$

$\mathfrak{g}(t)$ is increasing, constant, or contracting at t_0 if $\mathfrak{f}'(t_0)$ is positive, zero, or negative. Therefore, we have the above correct definition of the hyperbolic Ricci soliton (2) and hyperbolic Yamabe soliton (4) after setting $\mathfrak{f}''(t_0) = \mathfrak{f}''(0) = \Lambda$ and $\mathfrak{f}'(t_0) = \mathfrak{f}'(0) = \lambda$.

It is important to focus on this situation since it is clear that we are getting close to a critical point when $\mathfrak{f}'(t_0) = \mathfrak{f}'(0) = \lambda = 0$ in (123) and (124). Inspired by this fact, we present the formulation of a new concept called modified second hyperbolic Ricci soliton and modified second hyperbolic Yamabe soliton [32,33], such that

$$\frac{1}{2}\mathfrak{L}_\zeta\mathfrak{L}_\zeta\mathfrak{g} + \mathfrak{S} = \Lambda\mathfrak{g}, \tag{125}$$

$$\mathfrak{L}_\zeta\mathfrak{L}_\zeta\mathfrak{g} = (\Lambda - \mathcal{R})\mathfrak{g}, \tag{126}$$

Now, by adopting the definition of modified second hyperbolic Ricci solitons (35) and (125), we find

$$\mathfrak{S}(\mathsf{X}_1, \mathsf{X}_2) = (\Lambda - 2\sigma^2)\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - 2\sigma^2\pi(\mathsf{X}_1)\pi(\mathsf{X}_2). \tag{127}$$

Setting $\mathsf{X}_2 = \zeta$ in (127) and using (23), we get

$$\Lambda = -(m-1)(\sigma^2 + \mu). \tag{128}$$

Next, using modified second hyperbolic Yamabe solitons from equation (126) with (35), we get

$$4\sigma^2[\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2) - \pi(\mathsf{X}_1)\pi(\mathsf{X}_2)] = (\Lambda - \mathcal{R})\mathfrak{g}(\mathsf{X}_1, \mathsf{X}_2). \tag{129}$$

Substituting $\mathsf{X}_2 = \zeta$, we obtain

$$\Lambda = \mathcal{R}. \tag{130}$$

Consequently, in the light of (127), (128), and (130) we gain some applications.

Theorem 14. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits modified second HRS $(\zeta, \Lambda, \mathfrak{g})$, then the $(RCS)_m$-manifolds $\mathbb{M}$ is a π-Einstein manifold.*

Theorem 15. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits modified second HRS $(\zeta, \Lambda, \mathfrak{g})$, then the rate of change of modified second HRS in $(RCS)_m$-manifolds is decreasing.*

Theorem 16. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits modified second HYS $(\zeta, \Lambda, \mathfrak{g})$, then the rate of change of modified second HYS in $(RCS)_m$-manifolds is equal to the scalar curvature.*

Corollary 9. *If an m-dimensional $(RCS)_m$-manifold $\mathbb{M}$ admits modified second HYS $(\zeta, \Lambda, \mathfrak{g})$, then the rate of change of modified second HYS in $(RCS)_m$-manifolds is decreasing.*

10. Conclusions

The current research centers around *HRS* and *HYS* on $(RCS)_m$-manifolds. We establish the scalar curvature and Ricci curvature expressions of $(RCS)_m$-manifolds that admit *HRS*. Furthermore, we demonstrate that on $(RCS)_m$-manifolds, the rate of change of *HRS* is decreasing. In this case, $(RCS)_m$-manifolds with *HRS* is π-Einstein manifold. Additionally, on Ricci pseudo-symmetric $(RCS)_m$-manifolds, concircular pseudo-symmetric $(RCS)_m$-manifolds, projective pseudo-symmetric $(RCS)_m$-manifolds, and conharmonic pseudo-symmetric $(RCS)_m$-manifolds, we explore the nature of *HRS* and critical value.

Contribution to Pre-Existing Literature: The results that followed were more convincing and part of an ongoing investigation of differential geometry's soliton-type equations and their rate of change. They included an application to a particular geometric problem, a geometric interpretation of the calculated values of $\mathcal{L}_{\mathfrak{S}}$, and a comparison with known results for related structures, such as Sasakian manifolds, Kenmotsu manifolds, or trans-Sasakian manifolds.

The following are the distinct new contributions.

1. Theorem 1: The expressions for the rate of *HRS* and *HYS* $(\mathfrak{L}_\zeta \mathfrak{L}_\zeta \mathfrak{g})$, the Ricci tensor, the Ricci operator and the scalar curvature on an $(RCS)_m$-manifold admitting *HRS*.
2. Theorem 3 and equation (42) showing that the rate of change is negative (shrinking soliton).
3. Theorems 4–11 and Corollaries 1–3: explicit formulas for $\mathcal{L}_{\mathfrak{S}}$ and for λ in the four pseudo-symmetric settings—Ricci pseudo-symmetric, concircular pseudo-symmetric, projective pseudo-symmetric, and conharmonic pseudo-symmetric $(RCS)_m$-manifolds—together with growing, stable or shrinking trichotomies.

4. Theorems 12–13: The scalar curvature formula and growing, stable or shrinking trichotomy for HYS and HYS on $(RCS)_m$-manifolds.

The study integrates two current frameworks: the $(RCS)_m$-manifolds structure of Chaubey and Suh [24,25] and hyperbolic solitons on semi-Riemannian manifolds. Determining the algebraic implications of applying the HRS and HYS equation on a manifold that already possesses the $(RCS)_m$-manifold structure and one of the four pseudo-symmetry criteria is the primary novelty.

Author Contributions: Conceptualization, N.M.A.-A., R.B. and M.D.S.; formal analysis, N.M.A.-A., R.B. and M.D.S.; investigation, N.M.A.-A., R.B. and M.D.S.; methodology, M.D.S.; N.M.A.-A. and R.B.; project administration and funding N.M.A.-A. and R.B.; validation, N.M.A.-A., R.B. and M.D.S.; and writing original draft, M.D.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Graduate studies and Scientific Research at Jazan University for funding this research work through the project number "JU-RSP2026".

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors extend their appreciation to the Deanship of Graduate studies and Scientific Research at Jazan University for funding this research work through the project number "JU-RSP2026". Additionally, the authors express their gratitude to the anonymous reviewers for their meticulous scrutiny of our work and their numerous perceptive remarks and recommendations.

Conflicts of Interest: The authors declare no conflicts of interest.

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