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# New Frameworks for Constructing Special Curves via $f$-Weighted Direction Curves

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**Abstract**

In this paper, we investigate three classes of $f$-weighted direction curves, namely tangent, principal, and binormal direction curves, obtained by integrating positive smooth multiples of the corresponding Frenet vector fields of a generating Frenet curve in Euclidean three-space. We derive curvature and torsion relations for these associated curves and investigate their connections with general helices, slant helices, clad helices, and g-clad helices. In particular, the weighted tangent direction construction preserves certain helix invariants of the generating curve, while suitable choices of the weight function modify its curvature and torsion. We establish conditions on the weight functions under which the resulting curves satisfy the characterizations of rectifying, Mannheim, and Bertrand curves. We also investigate conditions for constructing spherical weighted direction curves and introduce $g$-weighted normal, rectifying, and osculating direction curves associated with them. Explicit examples illustrate successive weighted constructions, including the generation of a rectifying clad helix and a Salkowski curve. The main contribution is a common construction framework in which the choice of a Frenet direction field and a positive weight function allows several classes of special space curves to be investigated through their geometric characterizations, subject to the required regularity and nonvanishing conditions.

**Keywords:** $f$-weighted direction curves; general helix; slant helix; clad helix; g-clad helix; Bertrand curve; Mannheim curve; rectifying curve; spherical curve

**MSC:** 53A04; 53A55

## 1. Introduction

The characterization of special space curves through their curvature, torsion, and associated curves remains an important topic in this field.

In the Euclidean 3-space $\mathbb{R}^3$, a Frenet curve is called a general helix if its tangent vector makes a constant angle with a fixed straight line (the axis of the general helix). According to the well-known result stated by M. A. Lancret in 1802 and first proved by B. de Saint-Venant in 1845 (for details, see [1,2]), a Frenet curve is a general helix if and only if the ratio of its torsion to its curvature is constant.

The notions of slant, clad, and g-clad helices have been studied in [3–5] as successive extensions of the study of special helices. On this basis, some authors have systematically defined an $N_k$-slant helix (a slant helix of order $k$) as a generalization of these helix classes and have obtained a number of important results in [6,7]. The constancy of certain functions involving the curvature and torsion provides characterizations of these helices. For recent characterizations of the aforementioned helices in different spaces, we refer to [8]. Slant

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helices and their spherical indicatrices, as well as further characterizations of slant helices in Euclidean three-space, have also been studied in [9,10].

A curve is said to be associated with a given curve if the two curves satisfy a specified geometric relationship. Such relationships provide a useful framework for constructing new curves from a given curve and studying their geometric properties. In turn, associated curves can characterize an original curve and explain its behavior [11]. The concept of associated curves has long been studied; early examples include Bertrand mate curves and Mannheim partner curves. Mannheim curves, introduced by Mannheim in 1878, are space curves whose principal normal lines correspond to the binormal lines of their Mannheim partner curves [12,13]. A Bertrand curve is a space curve whose principal normal line coincides with the principal normal line of another curve, called its Bertrand mate, at corresponding points [14]. Bertrand curves have been applied in computer-aided geometric design [15]. Camci et al. [16] proposed a new approach to Bertrand curves in Euclidean 3-space and obtained necessary and sufficient conditions for a curve to be a Bertrand curve. The notion of a rectifying curve was introduced by Chen [17] for a curve whose position vector always lies in its rectifying plane. Kim et al. [18] considered space curves for which the ratio of torsion to curvature is a nonconstant linear function of the arc-length parameter. They proved that this property holds if and only if there exists a point through which all rectifying planes of the curve pass.

Several approaches to constructing special space curves have been developed using geometric relationships between associated curves. Chen [17] also provided a construction of rectifying curves based on spherical curves. Subsequent studies have further investigated rectifying curves and their constructions [19,20], including helical transformations of spherical curves. Önder [21] constructed Mannheim and Bertrand partner curves by integrating suitable combinations of the Frenet vector fields of a generating curve. Other investigations have addressed curves with constant curvature or constant torsion [22,23], as well as special curves generated from curves with constant torsion [24]. These studies illustrate the diversity of existing construction methods. Unlike constructions specifically designed to produce Mannheim or Bertrand partner pairs, the present approach selects weight functions so that the resulting curves themselves satisfy the corresponding curvature–torsion conditions. More generally, the same weighted construction can be used to investigate several classes of special curves through different choices of the weight function.

In this paper, we develop a unified framework for constructing special space curves by integrating suitably weighted Frenet vector fields of a generating curve. By selecting appropriate positive weight functions, we derive conditions under which the resulting curves satisfy prescribed curvature–torsion relations, including those characterizing various helix classes, rectifying curves, Mannheim curves, and Bertrand curves. In particular, the $f$-weighted tangent direction construction preserves certain helix invariants of the generating curve, allowing the curvature and torsion to be modified while retaining the corresponding helix properties. We also examine successive weighted constructions and their geometric consequences.

The results of this paper have three distinct roles. First, the curvature and torsion formulas, together with the invariant relations and helix-class conclusions in Theorems 1, 3, and 5, follow directly from the classical Frenet equations; these are not claimed as independent new geometric principles. Second, the positive-weight formulation extends the classical principal-direction and binormal-direction constructions of Choi and Kim [11] and treats the three Frenet direction fields within a common construction scheme. For a unit-speed generating curve, the choice $f = 1$ recovers their principal-direction and binormal-direction constructions. In the tangent case, the same choice simply yields a translate of the generating curve. The contribution in this case therefore concerns

the determination of admissible weights for prescribed geometric conditions, rather than the unweighted integration of the tangent field. Unlike the partner-pair constructions of Önder [21], the characterizations considered here impose geometric conditions on the generated curve itself.

Third, the specific characterization results of the present study determine the admissible weights that produce Mannheim, Bertrand, and rectifying curves in Theorems 2, 4, and 6, and spherical direction curves in Theorem 7. The rectifying characterizations are understood up to a suitable translation. The spherical curves obtained in the first stage are subsequently used, through their position vectors, to define $g$-weighted normal, rectifying, and osculating direction curves. These weight characterizations and the subsequent constructions constitute the specific contributions of the study within the common weighted framework.

The organization of this paper is as follows. In Section 2, we recall basic notions and properties of special space curves. In Sections 3–5, we introduce $f$-weighted tangent, principal, and binormal direction curves, respectively, and derive their curvature, torsion, and helix-invariant relations. We use these relations to investigate connections among general helices, slant helices, clad helices, and g-clad helices, and to establish conditions under which suitable weight functions yield rectifying, Mannheim, and Bertrand curves. In Section 6, we determine conditions for the construction of spherical $f$-weighted direction curves. We then introduce $g$-weighted normal, rectifying, and osculating direction curves and examine their relationships with spherical helices and the other helix classes considered in this study. Finally, in Section 7, we present explicit examples illustrating the main constructions and results.

## 2. Preliminaries

In this section, we recall some basic facts and definitions and introduce the notation used throughout the paper.

Let $\mathbb{R}^3$ denote the Euclidean 3-space equipped with the standard inner product

$$\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The norm of a vector $x \in \mathbb{R}^3$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. Also, the cross product of vectors $x$ and $y$ is given by

$$x \times y = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1).$$

Let $\boldsymbol{\beta} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular curve. We call $\boldsymbol{\beta}$ a Frenet curve if

$$\det(\boldsymbol{\beta}', \boldsymbol{\beta}'', \boldsymbol{\beta}''') \neq 0$$

for all $t \in I$. The unit tangent, principal normal, and binormal vectors of $\boldsymbol{\beta}$ are defined by

$$\boldsymbol{T}_\beta = \frac{\boldsymbol{\beta}'}{\|\boldsymbol{\beta}'\|}, \quad \boldsymbol{N}_\beta = \boldsymbol{B}_\beta \times \boldsymbol{T}_\beta, \quad \boldsymbol{B}_\beta = \frac{\boldsymbol{\beta}' \times \boldsymbol{\beta}''}{\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\|},$$

respectively. Then $\{\boldsymbol{T}_\beta, \boldsymbol{N}_\beta, \boldsymbol{B}_\beta\}$ is called the Frenet frame along $\boldsymbol{\beta}$. Also, the Frenet formulae are given by

$$\boldsymbol{T}_\beta{}' = \nu_\beta \kappa_\beta \boldsymbol{N}_\beta, \quad \boldsymbol{N}_\beta{}' = \nu_\beta(-\kappa_\beta \boldsymbol{T}_\beta + \tau_\beta \boldsymbol{B}_\beta), \quad \boldsymbol{B}_\beta{}' = -\nu_\beta \tau_\beta \boldsymbol{N}_\beta,$$

where $\nu_\beta = \|\beta'\|$ is the speed of $\beta$, while its curvature and torsion are given by

$$\kappa_\beta = \frac{\|\beta' \times \beta''\|}{\|\beta'\|^3}, \qquad \tau_\beta = \frac{\det(\beta', \beta'', \beta''')}{\|\beta' \times \beta''\|^2},$$

respectively. The Frenet frame, together with the curvature and torsion functions, is called the Frenet apparatus of the curve. If $\nu_\beta = 1$ throughout $I$, then $\beta$ is a unit-speed curve. It is well known that if a regular curve $\beta$ has a nonzero constant curvature $\kappa_\beta$ and a torsion $\tau_\beta = 0$, then the curve is a circle with the radius $\frac{1}{\kappa_\beta}$.

A Frenet curve $\beta$ is called a general helix if its unit tangent vector $T_\beta$ makes a constant angle $\theta$ with a fixed unit vector $u$ determining the axis of the helix. That is, $\langle T_\beta, u \rangle = \cos\theta$, where $\theta \in (0, \pi) \setminus \{\pi/2\}$.

For a Frenet curve $\beta$, the function

$$\sigma_0^\beta = \frac{\tau_\beta}{\kappa_\beta} \tag{1}$$

is a nonzero constant if and only if $\beta$ is a general helix. In particular, $\beta$ is a circular helix if and only if both $\kappa_\beta$ and $\tau_\beta$ are nonzero constants [1,25].

A Frenet curve $\beta$ is called a slant helix if its principal normal vector $N_\beta$ makes a constant angle $\theta$ with a fixed unit vector $u$. That is, $\langle N_\beta, u \rangle = \cos\theta$, where $\theta \in (0, \pi) \setminus \{\pi/2\}$. Moreover, the function

$$\sigma_1^\beta = \frac{(\sigma_0^\beta)'}{\nu_\beta \kappa_\beta \left(1 + (\sigma_0^\beta)^2\right)^{3/2}} \tag{2}$$

is a nonzero constant if and only if $\beta$ is a slant helix [3].

A Salkowski curve is a curve with constant curvature and nonconstant torsion, whereas an anti-Salkowski curve is a curve with constant torsion and nonconstant curvature; see [22,26].

Scofield [27] studied curves of constant precession and obtained an arc-length-parametrized closed-form solution of their natural equations through direct geometric analysis. For a curve of constant precession, the Darboux vector field

$$W_\beta = \tau_\beta T_\beta + \kappa_\beta B_\beta$$

revolves about a fixed axis with constant angle and constant speed. The curvature and torsion functions of the family considered in [27] are given by

$$\kappa_\beta = \omega_0 \sin(\mu_0 t), \qquad \tau_\beta = \omega_0 \cos(\mu_0 t),$$

where $t$ is the arc-length parameter, $\omega_0 > 0$, and $\mu_0$ is a constant. Salkowski curves, anti-Salkowski curves, and curves of constant precession are examples of proper slant helices.

A Frenet curve $\beta$ is called a clad helix if the spherical image of its principal normal vector $N_\beta$ is a spherical helix. The function

$$\sigma_2^\beta = \frac{(\sigma_1^\beta)'}{\nu_\beta \kappa_\beta \left(1 + (\sigma_0^\beta)^2\right)^{1/2} \left(1 + (\sigma_1^\beta)^2\right)^{3/2}} \tag{3}$$

is a nonzero constant if and only if $\beta$ is a clad helix [4,5].

A Frenet curve $\boldsymbol{\beta}$ is called a g-clad helix if the spherical image of its principal normal vector $\boldsymbol{N}_{\boldsymbol{\beta}}$ is a spherical slant helix. The function

$$\sigma_3^{\boldsymbol{\beta}} = \frac{(\sigma_2^{\boldsymbol{\beta}})'}{\nu_{\boldsymbol{\beta}}\kappa_{\boldsymbol{\beta}}\left(1+(\sigma_0^{\boldsymbol{\beta}})^2\right)^{1/2}\left(1+(\sigma_1^{\boldsymbol{\beta}})^2\right)^{1/2}\left(1+(\sigma_2^{\boldsymbol{\beta}})^2\right)^{3/2}} \tag{4}$$

is a nonzero constant if and only if $\boldsymbol{\beta}$ is a g-clad helix [4,5].

A Frenet curve $\boldsymbol{\beta}$ is called a rectifying curve if $\langle \boldsymbol{\beta}, \boldsymbol{N}_{\boldsymbol{\beta}} \rangle = 0$, that is, if its position vector lies in its rectifying plane. Moreover, $\boldsymbol{\beta}$ is congruent to a rectifying curve if and only if the ratio $\tau_{\boldsymbol{\beta}}/\kappa_{\boldsymbol{\beta}}$ is a linear function of its arc-length parameter [17,18]. For an arbitrary regular parametrization, this condition is equivalent to

$$\left(\frac{\tau_{\boldsymbol{\beta}}}{\kappa_{\boldsymbol{\beta}}}\right)' = c_0 \nu_{\boldsymbol{\beta}}, \tag{5}$$

where $c_0$ is a nonzero constant [28]. In this case, a suitable translation of $\boldsymbol{\beta}$ has the position vector

$$\frac{1}{c_0}\left(\frac{\tau_{\boldsymbol{\beta}}}{\kappa_{\boldsymbol{\beta}}}\boldsymbol{T}_{\boldsymbol{\beta}} + \boldsymbol{B}_{\boldsymbol{\beta}}\right).$$

This representation is also discussed in [29].

For a Mannheim curve $\boldsymbol{\beta}$ in $\mathbb{R}^3$, its curvature and torsion satisfy

$$\kappa_{\boldsymbol{\beta}} = c_1\left(\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2\right), \tag{6}$$

where $c_1$ is a positive constant. Further characterizations of Mannheim curves in $\mathbb{R}^3$ can be found in [14,30,31].

For a Bertrand curve $\boldsymbol{\beta}$ in $\mathbb{R}^3$, its curvature and torsion satisfy

$$c_2\kappa_{\boldsymbol{\beta}} + c_2\cot(\omega)\tau_{\boldsymbol{\beta}} = 1, \qquad c_2\cot(\omega)\kappa_{\boldsymbol{\beta}} - c_2\tau_{\boldsymbol{\beta}} \neq 0, \tag{7}$$

where $c_2$ is a nonzero constant and $\omega$ is the constant angle between the tangent vectors of the Bertrand curve and its Bertrand mate at corresponding points. Among general helices in $\mathbb{R}^3$, only circular helices are Bertrand curves [32,33].

Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a unit-speed Frenet curve. An integral curve of $\boldsymbol{N}_{\boldsymbol{\beta}}$ (resp. $\boldsymbol{B}_{\boldsymbol{\beta}}$) is called the principal-direction curve (resp. binormal-direction curve) of $\boldsymbol{\beta}$ [11].

Motivated by these classical direction-curve constructions, we introduce their $f$-weighted counterparts, together with the corresponding $f$-weighted tangent direction construction.

**Definition 1.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $f : I \to (0,\infty)$ be a smooth function. An integral curve of $f\boldsymbol{T}_{\boldsymbol{\beta}}$ (resp. $f\boldsymbol{N}_{\boldsymbol{\beta}}$, $f\boldsymbol{B}_{\boldsymbol{\beta}}$) is called an $f$-weighted tangent direction (resp. principal direction, binormal direction) curve of $\boldsymbol{\beta}$. The function $f$ is called the weight function.*

In the constructions developed below, the weight functions are prescribed in terms of the speed, curvature, and torsion of the generating curve, possibly involving their derivatives or integrals with respect to the chosen parameter.

Each construction is considered on an open subinterval $J \subseteq I$ on which the prescribed weight function is well-defined, smooth, and strictly positive. In particular, all denominators appearing in its expression are assumed to be nonzero on $J$. Consequently, the speed of each weighted direction curve equals its weight function, and the generated curve is

regular on $J$. Unless otherwise stated, all subsequent constructions and conclusions are understood to hold on such admissible subintervals.

## 3. $f$-Weighted Tangent Direction Curves of a Frenet Curve

In this section, we study the $f$-weighted tangent direction curve of a Frenet curve. For a unit-speed generating curve, the choice $f = 1$ simply yields a translate of that curve. The Frenet-frame identities and the invariant relations in Theorem 1 are direct consequences of the classical Frenet equations and are independent of the choice of admissible positive weight.

Suitable weight choices produce curves with constant curvature or constant torsion while preserving the corresponding helix class. These include Salkowski and anti-Salkowski curves, as well as clad and $g$-clad helices with constant curvature or constant torsion.

The specific characterization result in Theorem 2 determines the admissible weights for the generated curve to satisfy the Mannheim or Bertrand condition, or to be rectifying after a suitable translation. The subsequent corollaries combine these characterizations with the preservation of the corresponding helix classes.

Let $\boldsymbol{\beta} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an open subinterval on which the prescribed weight function $f$ is smooth and strictly positive. The $f$-weighted tangent direction curve $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ generated by $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\gamma_1}(t) = \int f(t)\boldsymbol{T_\beta}(t)\,dt. \tag{8}$$

Since $f > 0$ on $J$, the speed of $\boldsymbol{\gamma_1}$ is $\nu_{\gamma_1} = f$. The Frenet apparatus of $\boldsymbol{\gamma_1}$ satisfies

$$\begin{aligned} &\boldsymbol{T}_{\gamma_1} = \boldsymbol{T}_\beta, \quad \boldsymbol{N}_{\gamma_1} = \boldsymbol{N}_\beta, \quad \boldsymbol{B}_{\gamma_1} = \boldsymbol{B}_\beta, \\ &\kappa_{\gamma_1} = \frac{\nu_\beta}{f}\kappa_\beta, \qquad \tau_{\gamma_1} = \frac{\nu_\beta}{f}\tau_\beta. \end{aligned} \tag{9}$$

These identities give the following immediate consequences.

**Proposition 1.** *Let $\boldsymbol{\beta} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a Frenet curve, and let $\boldsymbol{\gamma_1}$ be its $f$-weighted tangent direction curve on an admissible open subinterval $J \subseteq I$. Set $\varepsilon = \operatorname{sgn}(\tau_\beta)$, which is constant on $J$. Then the following statements hold:*

1. *The curves $\boldsymbol{\beta}$ and $\boldsymbol{\gamma_1}$ have the same Frenet frame.*
2. *The products of speed and curvature, and of speed and torsion, are preserved:*

$$\nu_{\gamma_1}\kappa_{\gamma_1} = \nu_\beta\kappa_\beta, \qquad \nu_{\gamma_1}\tau_{\gamma_1} = \nu_\beta\tau_\beta.$$

3. *If $f = 1$, then $\boldsymbol{\gamma_1}$ is a unit-speed curve.*
4. *If $f = \nu_\beta$, then $\boldsymbol{\gamma_1}$ is a translate of $\boldsymbol{\beta}$.*
5. *If $f = \dfrac{\nu_\beta\kappa_\beta}{|\tau_\beta|}$, then $\kappa_{\gamma_1} = |\tau_\beta|$.*
6. *If $f = \dfrac{\nu_\beta|\tau_\beta|}{\kappa_\beta}$, then $\tau_{\gamma_1} = \varepsilon\kappa_\beta$.*
7. *If $f = \nu_\beta\kappa_\beta$, then $\kappa_{\gamma_1} = 1$.*
8. *If $f = \nu_\beta|\tau_\beta|$, then $\tau_{\gamma_1} = \varepsilon$.*

The tangent direction construction also applies to regular plane curves with positive curvature. If $\boldsymbol{\beta}$ is such a curve, then $\boldsymbol{\gamma_1}$ is also a plane curve. For the choice $f = \lambda\nu_\beta\kappa_\beta$, where $\lambda > 0$ is a constant, the curve $\boldsymbol{\gamma_1}$ has constant curvature $1/\lambda$ and zero torsion. Hence, it is an arc of a circle of radius $\lambda$.

Moreover, if $f = \nu_\beta \kappa_\beta$, then

$$\kappa_{\gamma_1} = 1, \qquad \tau_{\gamma_1} = \frac{\tau_\beta}{\kappa_\beta} \neq 0.$$

Thus, for $c_2 = 1$ (hence $\cot(\omega) = 0$),

$$c_2 \kappa_{\gamma_1} + c_2 \cot(\omega) \tau_{\gamma_1} = 1, \qquad c_2 \cot(\omega) \kappa_{\gamma_1} - c_2 \tau_{\gamma_1} = -\tau_{\gamma_1} \neq 0.$$

Hence, both conditions in Equation (7) are satisfied, and $\boldsymbol{\gamma_1}$ is a Bertrand curve.

We next use these relations to obtain characterizations of special curves.

**Theorem 1.** *Let $\boldsymbol{\beta} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a Frenet curve, and let $\boldsymbol{\gamma_1}$ be its $f$-weighted tangent direction curve on an admissible open subinterval $J \subseteq I$. Then the following properties are preserved on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_1}$ is a general helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_1}$ is a slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_1}$ is a clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_1}$ is a g-clad helix.*

**Proof.** By Equation (9),

$$\sigma_0^{\gamma_1} = \frac{\tau_{\gamma_1}}{\kappa_{\gamma_1}} = \frac{\tau_\beta}{\kappa_\beta} = \sigma_0^\beta, \qquad \nu_{\gamma_1} \kappa_{\gamma_1} = \nu_\beta \kappa_\beta.$$

Using these identities in the recursive definitions of the invariants given by Equations (2)–(4), we obtain

$$\sigma_i^{\gamma_1} = \sigma_i^\beta, \qquad 0 \leq i \leq 3,$$

wherever the corresponding invariants are defined on $J$. Hence, the stated helix properties of $\boldsymbol{\beta}$ are preserved by $\boldsymbol{\gamma_1}$ on $J$. □

The following two corollaries are immediate consequences of Proposition 1 and Theorem 1.

**Corollary 1.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be the $f = \nu_\beta \kappa_\beta$-weighted tangent direction curve generated by $\boldsymbol{\beta}$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_1}$ is a circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_1}$ is a Salkowski curve.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_1}$ is a clad helix with constant curvature* 1.
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_1}$ is a g-clad helix with constant curvature* 1.

**Corollary 2.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be the $f = \nu_\beta |\tau_\beta|$-weighted tangent direction curve generated by $\boldsymbol{\beta}$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_1}$ is a circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_1}$ is an anti-Salkowski curve.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_1}$ is a clad helix with constant torsion* $\operatorname{sgn}(\tau_\beta)$.
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_1}$ is a g-clad helix with constant torsion* $\operatorname{sgn}(\tau_\beta)$.

The conclusions of Corollaries 1 and 2 follow from Theorem 1 and the identities

$$f = \nu_\beta \kappa_\beta \quad \Longrightarrow \quad \kappa_{\gamma_1} = 1, \qquad \tau_{\gamma_1} = \sigma_0^\beta,$$
$$f = \nu_\beta |\tau_\beta| \quad \Longrightarrow \quad \kappa_{\gamma_1} = \frac{1}{|\sigma_0^\beta|}, \qquad \tau_{\gamma_1} = \operatorname{sgn}(\tau_\beta).$$

In particular, the nonzero constant condition in the characterization of slant helices implies that $\sigma_0^\beta$ is nonconstant, as required for the Salkowski and anti-Salkowski conclusions.

**Theorem 2.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be its $f$-weighted tangent direction curve on an admissible open subinterval $J \subseteq I$. In each item, the prescribed weight function is assumed to be well-defined, smooth, and strictly positive on $J$. Then the following statements hold:*

1. $f = \dfrac{c_1 \nu_\beta (\kappa_\beta^2 + \tau_\beta^2)}{\kappa_\beta}$ *for some positive constant $c_1$ if and only if $\boldsymbol{\gamma_1}$ is a Mannheim curve on $J$.*
2. $f = c_2 \nu_\beta \big(\kappa_\beta + \cot(\omega)\tau_\beta\big)$ *and $\cot(\omega)\kappa_\beta - \tau_\beta \neq 0$ on $J$, for some nonzero constant $c_2$ and some constant $\omega$, if and only if $\boldsymbol{\gamma_1}$ is a Bertrand curve on $J$.*
3. $f = \dfrac{1}{c_0}\left(\sigma_0^\beta\right)'$ *for some nonzero constant $c_0$ if and only if a suitable translation of $\boldsymbol{\gamma_1}$ is a rectifying curve on $J$.*

**Proof.** First, we prove (1). By Equation (6), $\boldsymbol{\gamma_1}$ is a Mannheim curve if and only if there exists a positive constant $c_1$ such that

$$\kappa_{\gamma_1} = c_1\left(\kappa_{\gamma_1}^2 + \tau_{\gamma_1}^2\right).$$

Substituting the curvature and torsion formulas from Equation (9), we obtain

$$\frac{\nu_\beta}{f}\kappa_\beta = c_1 \frac{\nu_\beta^2}{f^2}\left(\kappa_\beta^2 + \tau_\beta^2\right).$$

Since $f > 0$, $\nu_\beta > 0$, and $\kappa_\beta > 0$ on $J$, this equality is equivalent to

$$f = \frac{c_1 \nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)}{\kappa_\beta}. \tag{10}$$

Thus, the Mannheim condition for $\boldsymbol{\gamma_1}$ is equivalent to the prescribed weight function on $J$.

Next, we prove (2). By Equation (7), $\boldsymbol{\gamma_1}$ is a Bertrand curve if and only if there exist a nonzero constant $c_2$ and a constant $\omega$ such that

$$c_2 \kappa_{\gamma_1} + c_2 \cot(\omega)\tau_{\gamma_1} = 1$$

and

$$c_2 \cot(\omega)\kappa_{\gamma_1} - c_2 \tau_{\gamma_1} \neq 0.$$

Substituting the curvature and torsion formulas from Equation (9), the first equality is equivalent to

$$f = c_2 \nu_\beta \big(\kappa_\beta + \cot(\omega)\tau_\beta\big).$$

Since $c_2 \neq 0$, $\nu_\beta > 0$, and $f > 0$ on $J$, the second condition is equivalent to

$$\cot(\omega)\kappa_\beta - \tau_\beta \neq 0.$$

Thus, the two conditions in (2) are equivalent to the Bertrand conditions for $\boldsymbol{\gamma_1}$ on $J$.

Finally, we prove (3). By Equation (5), a suitable translation of $\boldsymbol{\gamma_1}$ is a rectifying curve if and only if there exists a nonzero constant $c_0$ such that

$$\left(\frac{\tau_{\gamma_1}}{\kappa_{\gamma_1}}\right)' = c_0 \nu_{\gamma_1}.$$

By Equation (9), we have

$$\frac{\tau_{\gamma_1}}{\kappa_{\gamma_1}} = \frac{\tau_{\boldsymbol{\beta}}}{\kappa_{\boldsymbol{\beta}}} = \sigma_0^{\boldsymbol{\beta}}, \qquad \nu_{\gamma_1} = f.$$

Therefore, the rectifying condition is equivalent to

$$\left(\sigma_0^{\boldsymbol{\beta}}\right)' = c_0 f,$$

or, equivalently,

$$f = \frac{1}{c_0}\left(\sigma_0^{\boldsymbol{\beta}}\right)'.$$

Since $f > 0$ on $J$ and $c_0 \neq 0$, the function $\sigma_0^{\boldsymbol{\beta}}$ cannot be constant on $J$, and hence $\boldsymbol{\beta}$ is not a general helix on $J$. This proves (3). □

The following three corollaries are direct consequences of Theorems 1 and 2.

**Corollary 3.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be the $f$-weighted tangent direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \frac{c_1 \nu_{\boldsymbol{\beta}}\left(\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2\right)}{\kappa_{\boldsymbol{\beta}}}$$

*for some positive constant $c_1$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_1}$ is a Mannheim circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_1}$ is a Mannheim slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_1}$ is a Mannheim clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_1}$ is a Mannheim g-clad helix.*

**Corollary 4.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be the $f$-weighted tangent direction curve generated by $\boldsymbol{\beta}$, where*

$$f = c_2 \nu_{\boldsymbol{\beta}}\left(\kappa_{\boldsymbol{\beta}} + \cot(\omega)\tau_{\boldsymbol{\beta}}\right)$$

*for some nonzero constant $c_2$ and some constant $\omega$. Assume that*

$$\cot(\omega)\kappa_{\boldsymbol{\beta}} - \tau_{\boldsymbol{\beta}} \neq 0$$

*throughout $J$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_1}$ is a Bertrand circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_1}$ is a Bertrand slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_1}$ is a Bertrand clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_1}$ is a Bertrand g-clad helix.*

**Corollary 5.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval on which $\boldsymbol{\beta}$ is not a general helix. Let $\boldsymbol{\gamma_1} : J \to \mathbb{R}^3$ be the $f$-weighted tangent direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \frac{1}{c_0}\left(\sigma_0^{\boldsymbol{\beta}}\right)'$$

*for some nonzero constant $c_0$. Then there exists a translation of $\boldsymbol{\gamma_1}$ for which the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a slant helix, then the translated curve is a rectifying slant helix.*
2. *If $\boldsymbol{\beta}$ is a clad helix, then the translated curve is a rectifying clad helix.*
3. *If $\boldsymbol{\beta}$ is a g-clad helix, then the translated curve is a rectifying g-clad helix.*

**Comparison with existing direction-curve constructions.**

In [11], principal-direction and binormal-direction curves are constructed from a unit-speed Frenet curve, and their relationships with special curve classes are investigated. In the present weighted tangent direction construction, the generating curve need not be unit speed, and an admissible positive weight is included. By Proposition 1(4), the weighted tangent direction curve obtained with $f = \nu_{\boldsymbol{\beta}}$ is a translate of the generating curve. For a unit-speed generating curve, this choice reduces to $f = 1$. The construction of associated curves by integrating Frenet vector fields, together with the relationships between such curves and helix classes, has been investigated in the direction-curve literature [11]. In the tangent-direction setting considered here, Equation (9) shows that the generating curve and its positively weighted tangent direction curve have the same Frenet frame and the same invariants $\sigma_i$, wherever these invariants are defined. Thus, the preservation statements in Theorem 1 follow directly from the Frenet relations. The role of the prescribed weights in the subsequent results is more specific: Theorem 2 identifies the weight functions corresponding to the Mannheim, Bertrand, and rectifying conditions, while Corollaries 3–5 combine these conditions with the preservation of the respective helix classes. In particular, the rectifying conclusions are understood up to a suitable translation of the generated curve.

## 4. $f$-Weighted Principal Direction Curves of a Frenet Curve

In this section, we study the $f$-weighted principal direction curve of a Frenet curve. For a unit-speed generating curve, the choice $f = 1$ recovers the classical principal-direction construction studied by Choi and Kim [11]. The positive-weight formulation extends this construction to a family of associated curves. The invariant relations and helix-class transitions in Theorem 3 follow directly from the classical Frenet equations and are independent of the choice of admissible positive weight.

Suitable weight choices impose constant curvature or constant torsion on the generated curve. Corollaries 6 and 7 combine these choices with the helix-class transitions to obtain the corresponding special curves.

The specific characterization result in Theorem 4 determines the admissible weights for the generated curve to satisfy the Mannheim or Bertrand condition, or to be rectifying after a suitable translation. Corollaries 8–10 combine these characterizations with the corresponding helix-class transitions.

Let $\boldsymbol{\beta} : I \subset \mathbb{R} \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval on which the prescribed weight function $f$ is well-defined, smooth, and strictly positive. The $f$-weighted principal direction curve $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ generated by $\boldsymbol{\beta}$ is defined by

$$\boldsymbol{\gamma_2}(t) = \int f(t)\mathbf{N}_{\boldsymbol{\beta}}(t)\,dt. \tag{11}$$

Since $f(t) > 0$ on $J$, the speed of $\boldsymbol{\gamma_2}$ is $\nu_{\gamma_2} = f$. Its Frenet frame is given by

$$\begin{aligned} \boldsymbol{T}_{\gamma_2} &= \boldsymbol{N}_{\beta}, \\ \boldsymbol{N}_{\gamma_2} &= \frac{-\kappa_{\beta}\boldsymbol{T}_{\beta} + \tau_{\beta}\boldsymbol{B}_{\beta}}{\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}}, \\ \boldsymbol{B}_{\gamma_2} &= \frac{\tau_{\beta}\boldsymbol{T}_{\beta} + \kappa_{\beta}\boldsymbol{B}_{\beta}}{\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}}. \end{aligned} \tag{12}$$

The curvature and torsion of $\boldsymbol{\gamma_2}$ satisfy

$$\begin{aligned} \kappa_{\gamma_2} &= \frac{\nu_{\beta}}{f}\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}, \\ \tau_{\gamma_2} &= \frac{\kappa_{\beta}^2\left(\frac{\tau_{\beta}}{\kappa_{\beta}}\right)'}{f\left(\kappa_{\beta}^2 + \tau_{\beta}^2\right)}. \end{aligned} \tag{13}$$

The following proposition records some consequences of Equation (13) for specific choices of the weight function.

**Proposition 2.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where the weight function specified in each item is assumed to be well-defined, smooth, and strictly positive on $J$. Then the following statements hold on $J$:*

1. *If $f = \nu_{\beta} = 1$, then $\boldsymbol{\gamma_2}$ is a principal-direction curve with unit speed.*
2. *If $f = \dfrac{\nu_{\beta}\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}}{\kappa_{\beta}}$, then $\boldsymbol{\beta}$ and $\boldsymbol{\gamma_2}$ have the same curvature.*
3. *If $f = \dfrac{\nu_{\beta}\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}}{|\tau_{\beta}|}$, then $\kappa_{\gamma_2} = |\tau_{\beta}|$.*
4. *If $f = \dfrac{\kappa_{\beta}^2\left(\frac{\tau_{\beta}}{\kappa_{\beta}}\right)'}{\tau_{\beta}\left(\kappa_{\beta}^2 + \tau_{\beta}^2\right)}$, then $\tau_{\gamma_2} = \tau_{\beta}$.*
5. *If $f = \dfrac{\kappa_{\beta}\left(\frac{\tau_{\beta}}{\kappa_{\beta}}\right)'}{\kappa_{\beta}^2 + \tau_{\beta}^2}$, then $\tau_{\gamma_2} = \kappa_{\beta}$.*
6. *If $f = \nu_{\beta}\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}$, then $\boldsymbol{\gamma_2}$ has constant curvature* 1.
7. *If $f = \dfrac{\kappa_{\beta}^2\left(\frac{\tau_{\beta}}{\kappa_{\beta}}\right)'}{\kappa_{\beta}^2 + \tau_{\beta}^2}$, then $\boldsymbol{\gamma_2}$ has constant torsion* 1.

In particular, let $\boldsymbol{\gamma_2}$ be the $f$-weighted principal direction curve of $\boldsymbol{\beta}$ corresponding to

$$f = \nu_{\beta}\sqrt{\kappa_{\beta}^2 + \tau_{\beta}^2}.$$

Then, by Equation (13),

$$\kappa_{\gamma_2} = 1, \qquad \tau_{\gamma_2} = \sigma_1^{\beta}.$$

If $\sigma_1^{\beta} \neq 0$ throughout $J$, then $\tau_{\gamma_2} \neq 0$. Thus, for $c_2 = 1$ (hence $\cot(\omega) = 0$),

$$c_2\kappa_{\gamma_2} + c_2\cot(\omega)\tau_{\gamma_2} = 1, \qquad c_2\cot(\omega)\kappa_{\gamma_2} - c_2\tau_{\gamma_2} = -\tau_{\gamma_2} \neq 0.$$

Hence, both conditions in Equation (7) are satisfied, and $\boldsymbol{\gamma_2}$ is a Bertrand curve on $J$.

We next use these relations to obtain characterizations of special curves.

**Theorem 3.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_2}$ is a plane curve.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_2}$ is a general helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_2}$ is a slant helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_2}$ is a clad helix.*

**Proof.** Suppose first that $\boldsymbol{\beta}$ is a general helix on $J$. Then $\sigma_0^{\beta} = \tau_\beta/\kappa_\beta$ is constant. By Equation (13), $\tau_{\gamma_2} = 0$ on $J$. Since $\kappa_{\gamma_2} > 0$, the curve $\boldsymbol{\gamma_2}$ is planar.

Next, Equations (2) and (13) give

$$\begin{aligned}\sigma_0^{\gamma_2} &= \frac{\tau_{\gamma_2}}{\kappa_{\gamma_2}} \\ &= \frac{\kappa_\beta^2\left(\tau_\beta/\kappa_\beta\right)'}{\nu_\beta\left(\kappa_\beta^2+\tau_\beta^2\right)^{3/2}} = \sigma_1^{\beta}.\end{aligned}$$

Moreover,

$$\begin{aligned}\nu_{\gamma_2}\kappa_{\gamma_2} &= \nu_\beta\sqrt{\kappa_\beta^2+\tau_\beta^2} \\ &= \nu_\beta\kappa_\beta\sqrt{1+\left(\sigma_0^{\beta}\right)^2}.\end{aligned}$$

Substituting these identities into the recursive relations (2)–(4), we obtain

$$\sigma_{i-1}^{\gamma_2} = \sigma_i^{\beta}, \qquad 1 \leq i \leq 3,$$

wherever the corresponding invariants are defined on $J$.

Consequently, if $\boldsymbol{\beta}$ is a slant helix, a clad helix, or a g-clad helix, then $\boldsymbol{\gamma_2}$ is a general helix, a slant helix, or a clad helix, respectively. This proves the remaining statements. □

The following two corollaries are immediate consequences of Proposition 2 and Theorem 3.

**Corollary 6.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \nu_\beta\sqrt{\kappa_\beta^2+\tau_\beta^2}.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_2}$ is an arc of a circle of radius* 1.
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_2}$ is a circular helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_2}$ is a Salkowski curve.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_2}$ is a clad helix with constant curvature* 1.

In particular, in item (3), $\sigma_2^{\boldsymbol{\beta}}$ is a nonzero constant, which implies that $\sigma_1^{\boldsymbol{\beta}}$ is nonconstant. Since the prescribed weight gives $\kappa_{\boldsymbol{\gamma_2}} = 1$ and $\tau_{\boldsymbol{\gamma_2}} = \sigma_1^{\boldsymbol{\beta}}$, the curve $\boldsymbol{\gamma_2}$ is a slant helix with constant curvature and nonconstant torsion, and hence is a Salkowski curve.

**Corollary 7.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval on which*

$$\left(\frac{\tau_{\boldsymbol{\beta}}}{\kappa_{\boldsymbol{\beta}}}\right)' > 0.$$

*Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \frac{\kappa_{\boldsymbol{\beta}}^2 \left(\frac{\tau_{\boldsymbol{\beta}}}{\kappa_{\boldsymbol{\beta}}}\right)'}{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2}.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_2}$ is a circular helix.*
2. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_2}$ is an anti-Salkowski curve.*
3. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_2}$ is a clad helix with constant torsion* 1.

In particular, in item (2), $\sigma_2^{\boldsymbol{\beta}}$ is a nonzero constant, so $\sigma_1^{\boldsymbol{\beta}}$ is nonconstant. Since the prescribed weight gives $\tau_{\boldsymbol{\gamma_2}} = 1$ and $\sigma_0^{\boldsymbol{\gamma_2}} = \sigma_1^{\boldsymbol{\beta}}$, the curve $\boldsymbol{\gamma_2}$ is a slant helix with constant torsion and nonconstant curvature, and hence is an anti-Salkowski curve.

**Theorem 4.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$. Assume also that $\left(\sigma_0^{\boldsymbol{\beta}}\right)' \neq 0$ throughout $J$. Then the following statements hold on $J$:*

1. *$f = c_1 \nu_{\boldsymbol{\beta}} \sqrt{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2}\left(1 + (\sigma_1^{\boldsymbol{\beta}})^2\right)$ for some positive constant $c_1$ if and only if $\boldsymbol{\gamma_2}$ is a Mannheim curve.*
2. *$f = c_2 \nu_{\boldsymbol{\beta}} \sqrt{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2}\left(1 + \cot(\omega)\sigma_1^{\boldsymbol{\beta}}\right)$ and $\cot(\omega) - \sigma_1^{\boldsymbol{\beta}} \neq 0$ on $J$, for some nonzero constant $c_2$ and some constant $\omega$, if and only if $\boldsymbol{\gamma_2}$ is a Bertrand curve on $J$.*
3. *$f = \frac{1}{c_0}\left(\sigma_1^{\boldsymbol{\beta}}\right)'$ for some nonzero constant $c_0$ if and only if a suitable translation of $\boldsymbol{\gamma_2}$ is a rectifying curve on $J$.*

**Proof.** By Equation (13), we have

$$\kappa_{\boldsymbol{\gamma_2}} = \frac{\nu_{\boldsymbol{\beta}}}{f}\sqrt{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2}, \qquad \frac{\tau_{\boldsymbol{\gamma_2}}}{\kappa_{\boldsymbol{\gamma_2}}} = \sigma_1^{\boldsymbol{\beta}}.$$

For (1), suppose that $\boldsymbol{\gamma_2}$ is a Mannheim curve. By Equation (6), there exists a constant $c_1 > 0$ such that

$$\frac{\kappa_{\boldsymbol{\gamma_2}}}{\kappa_{\boldsymbol{\gamma_2}}^2 + \tau_{\boldsymbol{\gamma_2}}^2} = c_1.$$

Using $\tau_{\boldsymbol{\gamma_2}} = \kappa_{\boldsymbol{\gamma_2}}\sigma_1^{\boldsymbol{\beta}}$ and solving for $f$, we obtain

$$f = c_1 \nu_{\boldsymbol{\beta}} \sqrt{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2}\left(1 + (\sigma_1^{\boldsymbol{\beta}})^2\right). \tag{14}$$

Conversely, substituting Equation (14) into Equation (13) yields

$$\frac{\kappa_{\gamma_2}}{\kappa_{\gamma_2}^2+\tau_{\gamma_2}^2}=c_1,$$

which proves (1).

For (2), suppose that $\boldsymbol{\gamma_2}$ is a Bertrand curve. By Equation (7), there exist constants $c_2 \neq 0$ and $\omega$ such that

$$c_2\kappa_{\gamma_2}+c_2\cot(\omega)\tau_{\gamma_2}=1,$$

and

$$\cot(\omega)\kappa_{\gamma_2}-\tau_{\gamma_2}\neq 0.$$

Substituting the curvature and torsion expressions from Equation (13) into the first relation gives

$$f=c_2\nu_{\boldsymbol{\beta}}\sqrt{\kappa_{\boldsymbol{\beta}}^2+\tau_{\boldsymbol{\beta}}^2}\Big(1+\cot(\omega)\sigma_1^{\boldsymbol{\beta}}\Big).$$

Since $\kappa_{\gamma_2}>0$, the second relation is equivalent to

$$\cot(\omega)-\sigma_1^{\boldsymbol{\beta}}\neq 0 \qquad \text{on } J.$$

Conversely, the stated expression for $f$ and the nonvanishing condition imply both relations in Equation (7). Hence $\boldsymbol{\gamma_2}$ is a Bertrand curve, proving (2).

For (3), by Equation (5), a suitable translation of $\boldsymbol{\gamma_2}$ is a rectifying curve if and only if there exists a nonzero constant $c_0$ such that

$$\left(\frac{\tau_{\gamma_2}}{\kappa_{\gamma_2}}\right)'=c_0\nu_{\gamma_2}.$$

By Equation (13), we have

$$\frac{\tau_{\gamma_2}}{\kappa_{\gamma_2}}=\sigma_1^{\boldsymbol{\beta}}, \qquad \nu_{\gamma_2}=f.$$

Therefore, the rectifying condition is equivalent to

$$\left(\sigma_1^{\boldsymbol{\beta}}\right)'=c_0 f,$$

or, equivalently,

$$f=\frac{1}{c_0}\left(\sigma_1^{\boldsymbol{\beta}}\right)'.$$

Since $f>0$ on $J$ and $c_0\neq 0$, the function $\sigma_1^{\boldsymbol{\beta}}$ cannot be constant on $J$. Consequently, $\boldsymbol{\beta}$ is not a slant helix on $J$. This proves (3). □

The following three corollaries combine Theorems 3 and 4.

**Corollary 8.** *Let $\boldsymbol{\beta}: I\to\mathbb{R}^3$ be a Frenet curve, and let $J\subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2}: J\to\mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where*

$$f=c_1\nu_{\boldsymbol{\beta}}\sqrt{\kappa_{\boldsymbol{\beta}}^2+\tau_{\boldsymbol{\beta}}^2}\Big(1+(\sigma_1^{\boldsymbol{\beta}})^2\Big), \qquad c_1>0.$$

*Assume also that $\left(\sigma_0^{\boldsymbol{\beta}}\right)'\neq 0$ throughout $J$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_2}$ is a Mannheim circular helix.*
2. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_2}$ is a Mannheim slant helix.*
3. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_2}$ is a Mannheim clad helix.*

**Corollary 9.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = c_2 \nu_{\boldsymbol{\beta}} \sqrt{\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2} \left(1 + \cot(\omega) \sigma_1^{\boldsymbol{\beta}}\right),$$

*for constants $c_2 \neq 0$ and $\omega$. Assume that $\left(\sigma_0^{\boldsymbol{\beta}}\right)' \neq 0$ throughout $J$, and that*

$$\cot(\omega) - \sigma_1^{\boldsymbol{\beta}} \neq 0 \qquad \text{on } J.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_2}$ is a Bertrand circular helix.*
2. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_2}$ is a Bertrand slant helix.*
3. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_2}$ is a Bertrand clad helix.*

**Corollary 10.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_2} : J \to \mathbb{R}^3$ be the $f$-weighted principal direction curve generated by $\boldsymbol{\beta}$, where the weight function*

$$f = \frac{1}{c_0} \left(\sigma_1^{\boldsymbol{\beta}}\right)', \qquad c_0 \neq 0,$$

*is assumed to be smooth and strictly positive on $J$. Assume also that $\left(\sigma_0^{\boldsymbol{\beta}}\right)' \neq 0$ throughout $J$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a clad helix, then a suitable translation of $\boldsymbol{\gamma_2}$ is a rectifying slant helix.*
2. *If $\boldsymbol{\beta}$ is a g-clad helix, then a suitable translation of $\boldsymbol{\gamma_2}$ is a rectifying clad helix.*

**Comparison with existing principal-direction curve constructions.** Principal-direction curves and their relationships with helix classes have been studied in the associated-curve literature [11]. For the construction considered here, the curvature and torsion formulas in Equation (13) reduce to the classical principal-direction formulas when the generating curve is parametrized by arc length and $f = 1$. Moreover, the invariant relations in Theorem 3 follow from these Frenet formulas and are independent of the choice of positive weight. Thus, the helix-class transitions in that theorem should be viewed in relation to the corresponding results for classical and iterated principal-direction curves.

The prescribed weights play a more specific role in the remaining results: Proposition 2 identifies weights that impose particular curvature or torsion relations, while Corollaries 6 and 7 combine constant curvature or constant torsion with the helix-class transitions. Theorem 4 determines the weights for which the generated curve satisfies the Mannheim or Bertrand condition, or has a suitable translation that is a rectifying curve, and Corollaries 8–10 combine these properties with the respective helix classes. As in the tangent-direction case, the rectifying conclusions hold up to a suitable translation of the generated curve.

## 5. $f$-Weighted Binormal Direction Curves of a Frenet Curve

In this section, we study the $f$-weighted binormal direction curve of a Frenet curve. For a unit-speed generating curve, the choice $f = 1$ recovers the classical binormal-direction construction studied by Choi and Kim [11]. The positive-weight formulation extends this construction to a family of associated curves. The invariant relations and helix-class preservation statements in Theorem 5 follow directly from the classical Frenet equations and are independent of the choice of admissible positive weight.

Suitable weight choices impose constant curvature or constant torsion on the generated curve. Corollaries 11 and 12 combine these choices with the preservation of the corresponding helix classes to obtain special curves.

The specific characterization result in Theorem 6 determines the admissible weights for the generated curve to satisfy the Mannheim or Bertrand condition, or to be rectifying after a suitable translation. Corollaries 13–15 combine these characterizations with the preservation of the corresponding helix classes.

Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve. On an admissible open subinterval $J \subseteq I$, define its $f$-weighted binormal direction curve $\boldsymbol{\gamma_3}$ by

$$\boldsymbol{\gamma_3}(t) = \int f(t)\boldsymbol{B_\beta}(t)\, dt. \tag{15}$$

Since $\boldsymbol{\gamma_3}'(t) = f(t)\boldsymbol{B_\beta}(t)$, the speed of $\boldsymbol{\gamma_3}$ is $\nu_{\gamma_3} = f > 0$. Set

$$\varepsilon = \operatorname{sgn}(\tau_\beta).$$

Since $\tau_\beta$ is continuous and nonzero on $J$, the value of $\varepsilon$ is constant on $J$. The Frenet frame, curvature, and torsion of $\boldsymbol{\gamma_3}$ are then given by

$$\begin{aligned} \boldsymbol{T}_{\gamma_3} &= \boldsymbol{B}_\beta, & \boldsymbol{N}_{\gamma_3} &= -\varepsilon \boldsymbol{N}_\beta, \\ \boldsymbol{B}_{\gamma_3} &= \varepsilon \boldsymbol{T}_\beta, & \kappa_{\gamma_3} &= \frac{\nu_\beta}{f}|\tau_\beta|, \\ \tau_{\gamma_3} &= \frac{\nu_\beta}{f}\kappa_\beta. \end{aligned} \tag{16}$$

Specific choices of the weight function in Equation (16) lead to the following consequences.

**Proposition 3.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$. Then the following statements hold on $J$:*

1. *If $f = \nu_\beta = 1$, then $\boldsymbol{\gamma_3}$ is a binormal-direction curve with unit speed.*
2. *If $f = \nu_\beta$, then $\kappa_{\gamma_3} = |\tau_\beta|$ and $\tau_{\gamma_3} = \kappa_\beta$.*
3. *If $f = \dfrac{\nu_\beta \kappa_\beta}{\tau_\beta}$, which requires $\tau_\beta > 0$ on $J$, then $\tau_{\gamma_3} = \tau_\beta$.*
4. *If $f = \dfrac{\nu_\beta |\tau_\beta|}{\kappa_\beta}$, then $\kappa_{\gamma_3} = \kappa_\beta$.*
5. *If $f = \nu_\beta \kappa_\beta$, then $\tau_{\gamma_3} = 1$.*
6. *If $f = \nu_\beta |\tau_\beta|$, then $\kappa_{\gamma_3} = 1$.*

For the particular choice

$$f = \nu_\beta |\tau_\beta|,$$

Equation (16) gives $\kappa_{\gamma_3} = 1$ and

$$\tau_{\gamma_3} = \frac{\kappa_\beta}{|\tau_\beta|} > 0 \qquad \text{on } J.$$

Thus, $\boldsymbol{\gamma_3}$ is a Bertrand curve on $J$, since it satisfies the linear relation

$$c_2 \kappa_{\gamma_3} + c_2 \cot\omega\, \tau_{\gamma_3} = 1$$

for $c_2 = 1$ (hence $\cot\omega = 0$). Moreover,

$$\cot\omega\,\kappa_{\gamma_3} - \tau_{\gamma_3} = -\tau_{\gamma_3} \neq 0,$$

so the second condition in Equation (7) is also satisfied.

We next use these relations to obtain characterizations of special curves.

**Theorem 5.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$. Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_3}$ is a general helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_3}$ is a slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_3}$ is a clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_3}$ is a g-clad helix.*

**Proof.** Since $\tau_\beta \neq 0$ on $J$, the quantity

$$\varepsilon = \operatorname{sgn}(\tau_\beta) \in \{-1, 1\}$$

is constant on $J$. By Equation (16),

$$\sigma_0^{\gamma_3} = \frac{\tau_{\gamma_3}}{\kappa_{\gamma_3}} = \frac{\kappa_\beta}{|\tau_\beta|} = \frac{\varepsilon}{\sigma_0^\beta}.$$

Consequently,

$$\left(\sigma_0^{\gamma_3}\right)' = -\frac{\varepsilon\left(\sigma_0^\beta\right)'}{\left(\sigma_0^\beta\right)^2}.$$

Moreover,

$$\nu_{\gamma_3}\kappa_{\gamma_3}\left(1 + \left(\sigma_0^{\gamma_3}\right)^2\right)^{3/2} = \frac{\nu_\beta\kappa_\beta\left(1 + \left(\sigma_0^\beta\right)^2\right)^{3/2}}{\left(\sigma_0^\beta\right)^2}.$$

Hence, the definition of the first successive invariant gives

$$\sigma_1^{\gamma_3} = -\varepsilon\sigma_1^\beta.$$

Applying the recursive relations (3) and (4), we similarly obtain

$$\sigma_i^{\gamma_3} = -\varepsilon\sigma_i^\beta, \qquad 2 \leq i \leq 3.$$

Since $\varepsilon$ is a nonzero constant, the relation for $\sigma_0$ preserves the property of being a nonzero constant, while the relations for $\sigma_i$, $1 \leq i \leq 3$, preserve the same property for each successive invariant. This proves the four assertions. □

The following two corollaries are immediate consequences of Proposition 3 and Theorem 5.

**Corollary 11.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \nu_\beta|\tau_\beta|.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_3}$ is a circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_3}$ is a Salkowski curve.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_3}$ is a clad helix with constant curvature* 1.
4. *If $\boldsymbol{\beta}$ is a $g$-clad helix, then $\boldsymbol{\gamma_3}$ is a $g$-clad helix with constant curvature* 1.

In particular, in item (2), $\sigma_1^{\boldsymbol{\beta}}$ is a nonzero constant, so $\sigma_0^{\boldsymbol{\beta}}$ is nonconstant. Since the prescribed weight gives $\kappa_{\gamma_3} = 1$ and

$$\tau_{\gamma_3} = \frac{\varepsilon}{\sigma_0^{\boldsymbol{\beta}}},$$

where $\varepsilon = \operatorname{sgn}(\tau_{\boldsymbol{\beta}})$ is a nonzero constant on $J$, the torsion of $\boldsymbol{\gamma_3}$ is nonconstant. Thus, $\boldsymbol{\gamma_3}$ is a Salkowski curve.

The constant-curvature constructions in Corollaries 1, 6, and 11 also provide further examples of Bertrand curves. In particular, the Salkowski curves, clad helices with constant curvature, and $g$-clad helices with constant curvature obtained in these corollaries satisfy the Bertrand conditions in Equation (7) on admissible subintervals where their torsion is nonzero.

**Corollary 12.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$, where $f = \nu_{\boldsymbol{\beta}}\kappa_{\boldsymbol{\beta}}$. Then the following statements hold on $J$:*

1. *$\boldsymbol{\beta}$ is a general helix if and only if $\boldsymbol{\gamma_3}$ is a circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_3}$ is an anti-Salkowski curve.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_3}$ is a clad helix with constant torsion* 1.
4. *If $\boldsymbol{\beta}$ is a $g$-clad helix, then $\boldsymbol{\gamma_3}$ is a $g$-clad helix with constant torsion* 1.

In particular, in item (2), $\sigma_1^{\boldsymbol{\beta}}$ is a nonzero constant, so $\sigma_0^{\boldsymbol{\beta}}$ is nonconstant. The prescribed weight gives $\tau_{\gamma_3} = 1$ and $\kappa_{\gamma_3} = |\sigma_0^{\boldsymbol{\beta}}|$. Since $\sigma_0^{\boldsymbol{\beta}}$ is nonconstant and has constant sign on $J$, the curvature of $\boldsymbol{\gamma_3}$ is nonconstant, as required for an anti-Salkowski curve.

The following theorem identifies weight functions for which the $f$-weighted binormal direction curve of a Frenet curve $\boldsymbol{\beta}$ is a Mannheim curve, a Bertrand curve, or has a suitable translation that is a rectifying curve.

**Theorem 6.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$. For each case below, the prescribed weight $f$ is assumed to be well-defined, smooth, and strictly positive on $J$. Then the following statements hold on $J$:*

1. *$f = \dfrac{c_1\nu_{\boldsymbol{\beta}}\left(\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2\right)}{|\tau_{\boldsymbol{\beta}}|}$ for some positive constant $c_1$ if and only if $\boldsymbol{\gamma_3}$ is a Mannheim curve on $J$.*
2. *$f = c_2\nu_{\boldsymbol{\beta}}\left(|\tau_{\boldsymbol{\beta}}| + \cot(\omega)\kappa_{\boldsymbol{\beta}}\right)$ and $\cot(\omega)|\tau_{\boldsymbol{\beta}}| - \kappa_{\boldsymbol{\beta}} \neq 0$ on $J$, for some nonzero constant $c_2$ and some constant $\omega$, if and only if $\boldsymbol{\gamma_3}$ is a Bertrand curve on $J$.*
3. *$f = \dfrac{1}{c_0}\left(\dfrac{1}{\sigma_0^{\boldsymbol{\beta}}}\right)'$ for some nonzero constant $c_0$ if and only if a suitable translation of $\boldsymbol{\gamma_3}$ is a rectifying curve on $J$.*

**Proof.** By Equation (16), we have

$$\kappa_{\gamma_3} = \frac{\nu_\beta}{f}|\tau_\beta|, \qquad \tau_{\gamma_3} = \frac{\nu_\beta}{f}\kappa_\beta.$$

For (1), suppose that $\boldsymbol{\gamma_3}$ is a Mannheim curve. By the Mannheim characterization in Equation (6), there exists a constant $c_1 > 0$ such that

$$\frac{\kappa_{\gamma_3}}{\kappa_{\gamma_3}^2 + \tau_{\gamma_3}^2} = c_1.$$

Substituting the expressions for the curvature and torsion of $\boldsymbol{\gamma_3}$ and solving for $f$, we obtain

$$f = \frac{c_1 \nu_\beta \left(\kappa_\beta^2 + \tau_\beta^2\right)}{|\tau_\beta|}. \tag{17}$$

Conversely, if $f$ is given by Equation (17), then

$$\frac{\kappa_{\gamma_3}}{\kappa_{\gamma_3}^2 + \tau_{\gamma_3}^2} = c_1,$$

so $\boldsymbol{\gamma_3}$ is a Mannheim curve.

For (2), by the Bertrand characterization in Equation (7), $\boldsymbol{\gamma_3}$ is a Bertrand curve if and only if there exist constants $c_2 \neq 0$ and $\omega$ such that

$$c_2 \kappa_{\gamma_3} + c_2 \cot(\omega) \tau_{\gamma_3} = 1$$

and

$$\cot(\omega)\kappa_{\gamma_3} - \tau_{\gamma_3} \neq 0 \qquad \text{on } J.$$

Substituting Equation (16) into the first relation gives

$$f = c_2 \nu_\beta \left(|\tau_\beta| + \cot(\omega)\kappa_\beta\right).$$

Since $\nu_\beta / f > 0$, the second relation is equivalent to

$$\cot(\omega)|\tau_\beta| - \kappa_\beta \neq 0 \qquad \text{on } J.$$

Thus, the conditions in (2) are both necessary and sufficient.

For (3), set

$$\varepsilon = \operatorname{sgn}(\tau_\beta).$$

Since $\boldsymbol{\beta}$ is a Frenet curve, $\varepsilon$ is constant on $J$. By Equation (5), a suitable translation of $\boldsymbol{\gamma_3}$ is a rectifying curve if and only if there exists a nonzero constant $a_0$ such that

$$\left(\frac{\tau_{\gamma_3}}{\kappa_{\gamma_3}}\right)' = a_0 \nu_{\gamma_3}.$$

By Equation (16), we have

$$\frac{\tau_{\gamma_3}}{\kappa_{\gamma_3}} = \frac{\kappa_\beta}{|\tau_\beta|} = \frac{\varepsilon}{\sigma_0^\beta}, \qquad \nu_{\gamma_3} = f.$$

Therefore, the rectifying condition is equivalent to

$$\varepsilon\left(\frac{1}{\sigma_0^{\boldsymbol{\beta}}}\right)' = a_0 f.$$

Since $\varepsilon \in \{-1, 1\}$ is constant on $J$, setting $c_0 = \varepsilon a_0 \neq 0$ gives

$$\left(\frac{1}{\sigma_0^{\boldsymbol{\beta}}}\right)' = c_0 f,$$

or, equivalently,

$$f = \frac{1}{c_0}\left(\frac{1}{\sigma_0^{\boldsymbol{\beta}}}\right)'.$$

Since $f > 0$ on $J$ and $c_0 \neq 0$, the function $\sigma_0^{\boldsymbol{\beta}}$ cannot be constant on $J$, and hence $\boldsymbol{\beta}$ is not a general helix on $J$. This proves (3). □

The following three corollaries are immediate consequences of Theorems 5 and 6.

**Corollary 13.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \frac{c_1 \nu_{\boldsymbol{\beta}}\left(\kappa_{\boldsymbol{\beta}}^2 + \tau_{\boldsymbol{\beta}}^2\right)}{|\tau_{\boldsymbol{\beta}}|}, \qquad c_1 > 0.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_3}$ is a Mannheim circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_3}$ is a Mannheim slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_3}$ is a Mannheim clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_3}$ is a Mannheim g-clad helix.*

**Corollary 14.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = c_2 \nu_{\boldsymbol{\beta}}\left(|\tau_{\boldsymbol{\beta}}| + \cot(\omega)\kappa_{\boldsymbol{\beta}}\right),$$

*for constants $c_2 \neq 0$ and $\omega$. Assume that*

$$\cot(\omega)|\tau_{\boldsymbol{\beta}}| - \kappa_{\boldsymbol{\beta}} \neq 0 \qquad \text{on } J.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\boldsymbol{\gamma_3}$ is a Bertrand circular helix.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\boldsymbol{\gamma_3}$ is a Bertrand slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\boldsymbol{\gamma_3}$ is a Bertrand clad helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\boldsymbol{\gamma_3}$ is a Bertrand g-clad helix.*

**Corollary 15.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Let $\boldsymbol{\gamma_3} : J \to \mathbb{R}^3$ be the $f$-weighted binormal direction curve generated by $\boldsymbol{\beta}$, where*

$$f = \frac{1}{c_0}\left(\frac{1}{\sigma_0^{\boldsymbol{\beta}}}\right)', \qquad c_0 \neq 0.$$

*Then the following statements hold on $J$:*

1. *If $\boldsymbol{\beta}$ is a slant helix, then a suitable translation of $\boldsymbol{\gamma_3}$ is a rectifying slant helix.*
2. *If $\boldsymbol{\beta}$ is a clad helix, then a suitable translation of $\boldsymbol{\gamma_3}$ is a rectifying clad helix.*
3. *If $\boldsymbol{\beta}$ is a g-clad helix, then a suitable translation of $\boldsymbol{\gamma_3}$ is a rectifying g-clad helix.*

**Comparison with existing binormal-direction curve constructions.** Binormal-direction curves generated from unit-speed Frenet curves and their relationships with helix classes have been studied in [11]. In the present construction, the Frenet relations in Equation (16) reduce to the classical binormal-direction relations when the generating curve is parametrized by arc length and $f = 1$. The invariant relations in Theorem 5 follow directly from these Frenet formulas and are independent of the choice of admissible positive weight. Accordingly, the weight does not alter the preservation of the respective helix classes described in that theorem.

The subsequent results use particular weights to obtain additional geometric properties. Proposition 3 identifies weights that impose particular curvature or torsion relations, while Corollaries 11 and 12 combine constant curvature or constant torsion with the preservation of the respective helix classes. Theorem 6 determines weights corresponding to the Mannheim, Bertrand, and rectifying conditions, and Corollaries 13–15 combine these conditions with the respective helix classes. The rectifying conclusions are understood up to a suitable translation of the generated curve.

## 6. Construction of Special Curves and Spherical Curves via $f$-Weighted Direction Curves of a Frenet Curve

This section distinguishes the classical spherical-curve characterization from its application to the weighted constructions. The spherical condition recalled below is due to Wong [34]; it is used here as an established criterion. The specific characterization result in Theorem 7 determines admissible weights for which the tangent, principal, and binormal direction constructions yield spherical curves.

A second stage uses the position vectors of these spherical curves to define $g$-weighted normal, rectifying, and osculating direction curves. This places successive associated-curve constructions within the common weighted framework. The subsequent results determine spherical conditions for these generated curves and establish relationships with the helix classes considered in this study. Thus, the specific contributions of this section concern the admissible weight choices and the resulting two-stage constructions, while the spherical criterion and the underlying Frenet equations remain the classical foundations.

We first recall a characterization of spherical curves and the corresponding expression for their position vectors.

Let $\bar{\gamma}$ be a Frenet curve with curvature $\kappa_{\bar{\gamma}}$, torsion $\tau_{\bar{\gamma}}$, and speed $\nu_{\bar{\gamma}}$. For convenience, set

$$\varphi(t) = \int \nu_{\bar{\gamma}}(t)\tau_{\bar{\gamma}}(t)\,dt.$$

The curve $\bar{\gamma}$ lies on a sphere if and only if

$$\frac{1}{\kappa_{\bar{\gamma}}} = A\cos\varphi(t) + B\sin\varphi(t), \tag{18}$$

where $A$ and $B$ are constants. The radius of the sphere is $\sqrt{A^2 + B^2}$ [34]. Henceforth, we restrict our attention to the unit sphere $S^2$ centered at the origin. Throughout this section, the integration constants defining the spherical direction curves are chosen so that their

position vectors lie on the unit sphere centered at the origin. If $\bar{\gamma}$ lies on $S^2$, its position vector can be expressed as follows [2]:

$$\bar{\gamma} = -\frac{1}{\kappa_{\bar{\gamma}}} N_{\bar{\gamma}} - \frac{1}{\nu_{\bar{\gamma}} \tau_{\bar{\gamma}}} \left( \frac{1}{\kappa_{\bar{\gamma}}} \right)' B_{\bar{\gamma}}.$$

Using the spherical curve characterization (18), the position vector of $\bar{\gamma}$ on the unit sphere $S^2$ can be written as

$$\bar{\gamma} = -\big(A \cos \varphi(t) + B \sin \varphi(t)\big) N_{\bar{\gamma}} + \big(A \sin \varphi(t) - B \cos \varphi(t)\big) B_{\bar{\gamma}}, \tag{19}$$

where $A^2 + B^2 = 1$. Denote by $\{T_{\bar{\gamma}}, N_{\bar{\gamma}}, B_{\bar{\gamma}}\}$ the Frenet frame along $\bar{\gamma}$.

In the constructions below, we use two representations of the spherical condition in Equation (18). For the $f$-weighted tangent and binormal direction curves, we use the normalized representation with $A = 1$ and $B = 0$. Thus,

$$\frac{1}{\kappa_{\bar{\gamma}_j}} = \cos\left( \int \nu_{\bar{\gamma}_j} \tau_{\bar{\gamma}_j} \, dt \right), \qquad j \in \{1, 3\}. \tag{20}$$

Accordingly, their position vectors on the unit sphere are written as

$$\bar{\gamma}_j = -\cos\left( \int \nu_{\bar{\gamma}_j} \tau_{\bar{\gamma}_j} \, dt \right) N_{\bar{\gamma}_j} + \sin\left( \int \nu_{\bar{\gamma}_j} \tau_{\bar{\gamma}_j} \, dt \right) B_{\bar{\gamma}_j}, \qquad j \in \{1, 3\}. \tag{21}$$

For the $f$-weighted principal direction curve, we retain the constants $A$ and $B$ in the representation (18). Although the normalized representation is also available in this case, retaining these constants allows us to display the phase dependence explicitly in the principal direction curve construction. In particular, the normalized choice $A = 1$ and $B = 0$ may lead to a special case in which the resulting integral is expressed in terms of a Frenet vector of the generating curve.

The following theorem identifies weight functions for which the corresponding $f$-weighted direction curves satisfy the spherical condition.

**Theorem 7.** *Let $\beta : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an open subinterval. For each of the following constructions, assume that the prescribed weight function $f$ is well-defined, smooth, and strictly positive on $J$. Then the following statements hold:*

1. *The $f$-weighted tangent direction curve $\bar{\gamma}_1$ generated by $\beta$ lies on $S^2$ if and only if*

$$f = \nu_\beta \kappa_\beta \cos\left( \int \nu_\beta \tau_\beta \, dt \right).$$

2. *Suppose that $\left(\sigma_0^\beta\right)' \neq 0$ throughout $J$.*
   *The $f$-weighted principal direction curve $\bar{\gamma}_2$ generated by $\beta$ lies on $S^2$ if and only if*

$$f = \nu_\beta \kappa_\beta \left(1 + (\sigma_0^\beta)^2\right)^{1/2} \left[ A \cos\left( \int \rho \, dt \right) + B \sin\left( \int \rho \, dt \right) \right],$$

   *where*

$$\rho = \nu_\beta \kappa_\beta \left(1 + (\sigma_0^\beta)^2\right)^{1/2} \sigma_1^\beta, \qquad A^2 + B^2 = 1.$$

3. *The $f$-weighted binormal direction curve $\bar{\gamma}_3$ generated by $\beta$ lies on $S^2$ if and only if*

$$f = \nu_\beta |\tau_\beta| \cos\left( \int \nu_\beta \kappa_\beta \, dt \right).$$

**Proof.** We first consider the $f$-weighted tangent direction curve $\bar{\gamma}_1$. Equation (20) gives the necessary and sufficient condition for $\bar{\gamma}_1$ to lie on $S^2$:

$$\frac{1}{\kappa_{\bar{\gamma}_1}} = \cos\left(\int \nu_{\bar{\gamma}_1}\tau_{\bar{\gamma}_1}\,dt\right). \tag{22}$$

Using Equation (9), this condition is equivalent to

$$f = \nu_\beta\kappa_\beta\cos\left(\int \nu_\beta\tau_\beta\,dt\right). \tag{23}$$

Furthermore, Equations (9) and (21) yield

$$\bar{\gamma}_1 = -\cos\left(\int \nu_\beta\tau_\beta\,dt\right)\boldsymbol{N}_\beta + \sin\left(\int \nu_\beta\tau_\beta\,dt\right)\boldsymbol{B}_\beta. \tag{24}$$

Thus, the position vector of $\bar{\gamma}_1$ lies in the normal plane of $\boldsymbol{\beta}$ at the corresponding point. In particular, the spherical curve $\bar{\gamma}_1$ is expressed in terms of the Frenet frame of $\boldsymbol{\beta}$.

Next, consider the $f$-weighted principal direction curve $\bar{\gamma}_2$. Using the general spherical condition in Equation (18), we obtain

$$\frac{1}{\kappa_{\bar{\gamma}_2}} = A\cos\left(\int \nu_{\bar{\gamma}_2}\tau_{\bar{\gamma}_2}\,dt\right) + B\sin\left(\int \nu_{\bar{\gamma}_2}\tau_{\bar{\gamma}_2}\,dt\right), \qquad A^2+B^2=1. \tag{25}$$

Set

$$\rho = \nu_\beta\kappa_\beta\left(1+\left(\sigma_0^\beta\right)^2\right)^{1/2}\sigma_1^\beta, \qquad m(t) = \int \rho\,dt.$$

By Equation (13), condition (25) is equivalent to

$$f = \nu_\beta\kappa_\beta\left(1+\left(\sigma_0^\beta\right)^2\right)^{1/2}(A\cos m(t) + B\sin m(t)). \tag{26}$$

For brevity, write

$$C(t) = A\cos m(t) + B\sin m(t), \qquad D(t) = A\sin m(t) - B\cos m(t).$$

Substituting the Frenet frame relations in Equation (12) into the position vector formula (19), we obtain

$$\bar{\gamma}_2 = \frac{C(t)+\sigma_0^\beta D(t)}{\sqrt{1+\left(\sigma_0^\beta\right)^2}}\boldsymbol{T}_\beta + \frac{D(t)-\sigma_0^\beta C(t)}{\sqrt{1+\left(\sigma_0^\beta\right)^2}}\boldsymbol{B}_\beta. \tag{27}$$

Hence, the position vector of $\bar{\gamma}_2$ lies in the rectifying plane of $\boldsymbol{\beta}$ at the corresponding point. Moreover, if

$$D(t) - \sigma_0^\beta C(t) = 0$$

throughout $J$, then $\bar{\gamma}_2 = \pm\boldsymbol{T}_\beta$ on $J$. Similarly, if

$$C(t) + \sigma_0^\beta D(t) = 0$$

throughout $J$, then $\bar{\gamma}_2 = \pm\boldsymbol{B}_\beta$ on $J$. In each case, the sign is constant on the interval $J$. Thus, with the chosen integration constants, these special cases coincide, up to sign, with the tangent and binormal indicatrices of $\boldsymbol{\beta}$, respectively.

Finally, consider the $f$-weighted binormal direction curve $\bar{\gamma}_3$. Equation (20) gives the necessary and sufficient condition for $\bar{\gamma}_3$ to lie on $S^2$:

$$\frac{1}{\kappa_{\bar{\gamma}_3}} = \cos\left(\int \nu_{\bar{\gamma}_3}\tau_{\bar{\gamma}_3}\,dt\right). \tag{28}$$

Using Equation (16), this condition is equivalent to

$$f = \nu_\beta|\tau_\beta|\cos\left(\int \nu_\beta\kappa_\beta\,dt\right). \tag{29}$$

Let

$$\varepsilon = \operatorname{sgn}(\tau_\beta).$$

Since $\boldsymbol{\beta}$ is a Frenet curve, $\varepsilon$ is constant on $J$. Equations (16) and (21) give

$$\bar{\gamma}_3 = \varepsilon\sin\left(\int \nu_\beta\kappa_\beta\,dt\right)\boldsymbol{T}_\beta + \varepsilon\cos\left(\int \nu_\beta\kappa_\beta\,dt\right)\boldsymbol{N}_\beta. \tag{30}$$

Therefore, the position vector of $\bar{\gamma}_3$ lies in the osculating plane of $\boldsymbol{\beta}$ at the corresponding point. □

The Frenet relations established in Theorems 1, 3, and 5 yield the following consequences for spherical weighted direction curves.

**Corollary 16.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Suppose that the spherical $f$-weighted direction curves considered below are defined on $J$. Then the following statements hold:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\bar{\gamma}_1$ and $\bar{\gamma}_3$ are spherical general helices.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\bar{\gamma}_1$ and $\bar{\gamma}_3$ are spherical slant helices, whereas $\bar{\gamma}_2$ is a spherical general helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\bar{\gamma}_1$ and $\bar{\gamma}_3$ are spherical clad helices, whereas $\bar{\gamma}_2$ is a spherical slant helix.*
4. *If $\boldsymbol{\beta}$ is a g-clad helix, then $\bar{\gamma}_1$ and $\bar{\gamma}_3$ are spherical g-clad helices, whereas $\bar{\gamma}_2$ is a spherical clad helix.*

The position vectors of the spherical $f$-weighted tangent, principal, and binormal direction curves obtained above lie in the normal, rectifying, and osculating planes of $\boldsymbol{\beta}$, respectively, at the corresponding parameter values. This observation motivates a second weighted construction in which these spherical position vectors are used as direction fields. To distinguish the weight in this second construction from the function $f$ used to generate the spherical direction curves, we denote the new positive weight function by $g$. This leads to the following definition.

**Definition 2.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. For $i \in \{1,2,3\}$, let $\bar{\gamma}_i : J \to S^2$ be a spherical $f$-weighted direction curve of $\boldsymbol{\beta}$, with its position vector chosen as in Theorem 7.*

*Let $g : J \to (0,\infty)$ be a smooth function, and define*

$$\bar{\boldsymbol{\alpha}}_i(t) = \int g(t)\bar{\gamma}_i(t)\,dt, \qquad i \in \{1,2,3\}.$$

*We call $\bar{\boldsymbol{\alpha}}_1$, $\bar{\boldsymbol{\alpha}}_2$, and $\bar{\boldsymbol{\alpha}}_3$ the g-weighted normal, rectifying, and osculating direction curves of $\boldsymbol{\beta}$, respectively.*

The notation $\bar{\boldsymbol{\alpha}}_{\boldsymbol{i}}$ will be used for a general $g$-weighted direction curve defined above, whereas $\tilde{\boldsymbol{\alpha}}_{\boldsymbol{i}}$ will denote the particular curve obtained when $g$ is chosen so that $\bar{\boldsymbol{\alpha}}_{\boldsymbol{i}}$ satisfies the spherical curve condition.

First, consider the spherical curve $\bar{\boldsymbol{\gamma}}_{\mathbf{1}}$ given by Equation (24):

$$\bar{\boldsymbol{\gamma}}_{\mathbf{1}} = -\cos\psi\, \boldsymbol{N}_{\boldsymbol{\beta}} + \sin\psi\, \boldsymbol{B}_{\boldsymbol{\beta}}, \qquad \psi(t) = \int \nu_{\beta}\tau_{\beta}\, dt.$$

The corresponding $g$-weighted normal direction curve is

$$\bar{\boldsymbol{\alpha}}_{\mathbf{1}}(t) = \int g(t)\bar{\boldsymbol{\gamma}}_{\mathbf{1}}(t)\, dt.$$

Since $\|\bar{\boldsymbol{\gamma}}_{\mathbf{1}}\| = 1$, the speed of $\bar{\boldsymbol{\alpha}}_{\mathbf{1}}$ is $\nu_{\bar{\alpha}_1} = g$. On an admissible subinterval $J$ where $\cos\psi > 0$, its Frenet frame is

$$\begin{aligned} \boldsymbol{T}_{\bar{\alpha}_1} &= -\cos\psi\, \boldsymbol{N}_{\boldsymbol{\beta}} + \sin\psi\, \boldsymbol{B}_{\boldsymbol{\beta}}, \\ \boldsymbol{N}_{\bar{\alpha}_1} &= \boldsymbol{T}_{\boldsymbol{\beta}}, \\ \boldsymbol{B}_{\bar{\alpha}_1} &= \sin\psi\, \boldsymbol{N}_{\boldsymbol{\beta}} + \cos\psi\, \boldsymbol{B}_{\boldsymbol{\beta}}. \end{aligned} \tag{31}$$

Consequently, its curvature and torsion are

$$\kappa_{\bar{\alpha}_1} = \frac{\nu_{\beta}\kappa_{\beta}\cos\psi}{g}, \qquad \tau_{\bar{\alpha}_1} = \frac{\nu_{\beta}\kappa_{\beta}\sin\psi}{g}. \tag{32}$$

Set

$$\Phi(t) = \int \nu_{\beta}\kappa_{\beta}\sin\psi\, dt.$$

On any admissible subinterval where $\tau_{\bar{\alpha}_1} \neq 0$, the normalized spherical curve condition in Equation (20) is equivalent to

$$g = \nu_{\beta}\kappa_{\beta}\cos\psi\cos\Phi. \tag{33}$$

The interval is further restricted so that $g > 0$. Let $\tilde{\boldsymbol{\alpha}}_{\mathbf{1}}$ denote the curve obtained with the weight in Equation (33). By Equations (21) and (31), its position vector is

$$\tilde{\boldsymbol{\alpha}}_{\mathbf{1}} = -\cos\Phi\, \boldsymbol{T}_{\boldsymbol{\beta}} + \sin\Phi\sin\psi\, \boldsymbol{N}_{\boldsymbol{\beta}} + \sin\Phi\cos\psi\, \boldsymbol{B}_{\boldsymbol{\beta}}. \tag{34}$$

Secondly, consider the spherical curve $\bar{\boldsymbol{\gamma}}_{\mathbf{2}}$ given by Equation (27). For brevity, set

$$r = \sqrt{1 + \left(\sigma_0^{\beta}\right)^2}.$$

Also, let

$$C(t) = A\cos m(t) + B\sin m(t), \qquad D(t) = A\sin m(t) - B\cos m(t),$$

where $A^2 + B^2 = 1$. Define

$$a(t) = \frac{C(t) + \sigma_0^{\beta}(t)D(t)}{r(t)}, \qquad b(t) = \frac{D(t) - \sigma_0^{\beta}(t)C(t)}{r(t)}.$$

Then Equation (27) becomes

$$\bar{\boldsymbol{\gamma}}_{\mathbf{2}} = a\, \boldsymbol{T}_{\boldsymbol{\beta}} + b\, \boldsymbol{B}_{\boldsymbol{\beta}}, \qquad a^2 + b^2 = 1.$$

In fact, the definitions of $m$, $C$, and $D$ imply

$$m' = \frac{\left(\sigma_0^{\beta}\right)'}{1+\left(\sigma_0^{\beta}\right)^2}, \qquad C' = -m'D, \qquad D' = m'C.$$

Consequently, $a' = b' = 0$, so $a$ and $b$ are constant on $J$.

Let $g : J \to (0,\infty)$ be a smooth function, and define the $g$-weighted rectifying direction curve by

$$\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}(t) = \int g(t)\bar{\boldsymbol{\gamma}}_{\mathbf{2}}(t)\,dt.$$

Since $\|\bar{\boldsymbol{\gamma}}_{\mathbf{2}}\| = 1$, the speed of $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$ is $\nu_{\tilde{\alpha}_2} = g$. On an admissible subinterval $J$ where $C > 0$, its Frenet frame is

$$\begin{aligned} \boldsymbol{T}_{\tilde{\alpha}_2} &= a\,\boldsymbol{T}_{\beta} + b\,\boldsymbol{B}_{\beta}, \\ \boldsymbol{N}_{\tilde{\alpha}_2} &= \boldsymbol{N}_{\beta}, \\ \boldsymbol{B}_{\tilde{\alpha}_2} &= -b\,\boldsymbol{T}_{\beta} + a\,\boldsymbol{B}_{\beta}. \end{aligned} \tag{35}$$

Consequently, its curvature and torsion are

$$\kappa_{\tilde{\alpha}_2} = \frac{\nu_{\beta}\kappa_{\beta} r C}{g}, \qquad \tau_{\tilde{\alpha}_2} = \frac{\nu_{\beta}\kappa_{\beta} r D}{g}. \tag{36}$$

Set

$$n(t) = \int \nu_{\beta}\kappa_{\beta} r D\,dt.$$

On any admissible subinterval where $\tau_{\tilde{\alpha}_2} \neq 0$, the spherical curve characterization in Equation (18) is equivalent to

$$g = \nu_{\beta}\kappa_{\beta} r C\big(P\cos n + Q\sin n\big), \qquad P^2 + Q^2 = 1. \tag{37}$$

The interval is restricted so that $g > 0$. Let $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$ denote the curve obtained with the weight in Equation (37). Set

$$E(t) = P\cos n(t) + Q\sin n(t), \qquad F(t) = P\sin n(t) - Q\cos n(t).$$

By Equations (19) and (35), its position vector is

$$\tilde{\boldsymbol{\alpha}}_{\mathbf{2}} = -bF\,\boldsymbol{T}_{\beta} - E\,\boldsymbol{N}_{\beta} + aF\,\boldsymbol{B}_{\beta}. \tag{38}$$

Finally, consider the spherical curve $\bar{\boldsymbol{\gamma}}_{\mathbf{3}}$ given by Equation (30). Set

$$\varepsilon = \operatorname{sgn}(\tau_{\beta}), \qquad \chi(t) = \int \nu_{\beta}\kappa_{\beta}\,dt.$$

Since $\boldsymbol{\beta}$ is a Frenet curve, $\varepsilon$ is constant on $J$. Thus,

$$\bar{\boldsymbol{\gamma}}_{\mathbf{3}} = \varepsilon\big(\sin\chi\,\boldsymbol{T}_{\beta} + \cos\chi\,\boldsymbol{N}_{\beta}\big).$$

Let $g : J \to (0,\infty)$ be a smooth function, and define the $g$-weighted osculating direction curve by

$$\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}(t) = \int g(t)\bar{\boldsymbol{\gamma}}_{\mathbf{3}}(t)\,dt.$$

Since $\|\bar{\gamma}_3\| = 1$, the speed of $\bar{\boldsymbol{\alpha}}_3$ is $\nu_{\bar{\alpha}_3} = g$. On an admissible subinterval $J$ where $\cos\chi > 0$, its Frenet frame is

$$\begin{aligned} \boldsymbol{T}_{\bar{\alpha}_3} &= \varepsilon(\sin\chi\,\boldsymbol{T}_\beta + \cos\chi\,\boldsymbol{N}_\beta), \\ \boldsymbol{N}_{\bar{\alpha}_3} &= \boldsymbol{B}_\beta, \\ \boldsymbol{B}_{\bar{\alpha}_3} &= \varepsilon(\cos\chi\,\boldsymbol{T}_\beta - \sin\chi\,\boldsymbol{N}_\beta). \end{aligned} \tag{39}$$

Consequently, its curvature and torsion are

$$\kappa_{\bar{\alpha}_3} = \frac{\nu_\beta|\tau_\beta|\cos\chi}{g}, \qquad \tau_{\bar{\alpha}_3} = \frac{\nu_\beta|\tau_\beta|\sin\chi}{g}. \tag{40}$$

Set

$$\Omega(t) = \int \nu_\beta|\tau_\beta|\sin\chi\,dt.$$

On any admissible subinterval where $\tau_{\bar{\alpha}_3} \neq 0$, the normalized spherical curve condition in Equation (20) is equivalent to

$$g = \nu_\beta|\tau_\beta|\cos\chi\cos\Omega. \tag{41}$$

The interval is further restricted so that $g > 0$.

Let $\tilde{\boldsymbol{\alpha}}_3$ denote the curve obtained with the weight in Equation (41). By Equations (21) and (39), its position vector is

$$\tilde{\boldsymbol{\alpha}}_3 = \varepsilon\sin\Omega\cos\chi\,\boldsymbol{T}_\beta - \varepsilon\sin\Omega\sin\chi\,\boldsymbol{N}_\beta - \cos\Omega\,\boldsymbol{B}_\beta. \tag{42}$$

The preceding curvature and torsion relations lead to the following result.

**Theorem 8.** *Let $\boldsymbol{\beta} : I \to \mathbb{R}^3$ be a Frenet curve, and let $J \subseteq I$ be an admissible open subinterval. Suppose that $\bar{\boldsymbol{\alpha}}_1$, $\bar{\boldsymbol{\alpha}}_2$, and $\bar{\boldsymbol{\alpha}}_3$ are the $g$-weighted normal, rectifying, and osculating direction curves of $\boldsymbol{\beta}$, respectively, where $g : J \to (0,\infty)$ is smooth. Assume that the curvature invariants in the following statements are well-defined on $J$. Then:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\bar{\boldsymbol{\alpha}}_1$ and $\bar{\boldsymbol{\alpha}}_3$ are slant helices.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\bar{\boldsymbol{\alpha}}_1$ and $\bar{\boldsymbol{\alpha}}_3$ are clad helices, whereas $\bar{\boldsymbol{\alpha}}_2$ is a slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\bar{\boldsymbol{\alpha}}_1$ and $\bar{\boldsymbol{\alpha}}_3$ are $g$-clad helices, whereas $\bar{\boldsymbol{\alpha}}_2$ is a clad helix.*

**Proof.** Let

$$\varepsilon = \operatorname{sgn}(\tau_\beta).$$

Since $\boldsymbol{\beta}$ is a Frenet curve, $\varepsilon$ is constant on $J$.

First, by Equation (32),

$$\frac{\tau_{\bar{\alpha}_1}}{\kappa_{\bar{\alpha}_1}} = \tan\psi, \qquad \psi' = \nu_\beta\tau_\beta.$$

Using the definition of the slant-helix invariant and the assumption $\cos\psi > 0$, we obtain

$$\sigma_1^{\bar{\alpha}_1} = \frac{(\tan\psi)'}{\nu_{\bar{\alpha}_1}\kappa_{\bar{\alpha}_1}(1+\tan^2\psi)^{3/2}} = \frac{\tau_\beta}{\kappa_\beta} = \sigma_0^\beta.$$

Applying the definitions of the higher-order helix invariants gives

$$\sigma_2^{\bar{\alpha}_1} = \sigma_1^\beta, \qquad \sigma_3^{\bar{\alpha}_1} = \sigma_2^\beta.$$

Next, for the cases in which $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$ is defined, Equation (36) and the definitions of $C$ and $D$ give

$$\frac{\tau_{\tilde{\alpha}_2}}{\kappa_{\tilde{\alpha}_2}} = \frac{D}{C}, \qquad C' = -m'D, \quad D' = m'C, \quad C^2 + D^2 = 1.$$

Consequently,

$$\begin{aligned}\sigma_1^{\tilde{\alpha}_2} &= \frac{(D/C)'}{\nu_{\tilde{\alpha}_2}\kappa_{\tilde{\alpha}_2}(1 + D^2/C^2)^{3/2}} \\ &= \frac{m'}{\nu_\beta \kappa_\beta \sqrt{1 + (\sigma_0^\beta)^2}} \\ &= \sigma_1^\beta.\end{aligned}$$

The higher-order invariants therefore satisfy

$$\sigma_2^{\tilde{\alpha}_2} = \sigma_2^\beta, \qquad \sigma_3^{\tilde{\alpha}_2} = \sigma_3^\beta.$$

Finally, Equation (40) yields

$$\frac{\tau_{\tilde{\alpha}_3}}{\kappa_{\tilde{\alpha}_3}} = \tan\chi, \qquad \chi' = \nu_\beta \kappa_\beta.$$

Hence,

$$\sigma_1^{\tilde{\alpha}_3} = \frac{(\tan\chi)'}{\nu_{\tilde{\alpha}_3}\kappa_{\tilde{\alpha}_3}(1 + \tan^2\chi)^{3/2}} = \frac{\kappa_\beta}{|\tau_\beta|} = \frac{\varepsilon}{\sigma_0^\beta}.$$

Applying the definitions of the higher-order invariants, we further obtain

$$\sigma_2^{\tilde{\alpha}_3} = -\varepsilon\,\sigma_1^\beta, \qquad \sigma_3^{\tilde{\alpha}_3} = -\varepsilon\,\sigma_2^\beta.$$

□

As a consequence of Theorem 8, the spherical curves given in Equations (34), (38), and (42) have the following properties.

**Corollary 17.** *Let $\tilde{\boldsymbol{\alpha}}_{\mathbf{1}}$, $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$, and $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ be the spherical curves given by Equations (34), (38), and (42), respectively, with the corresponding weights chosen as in Equations (33), (37), and (41). On any admissible open subinterval $J$ where these curves and the corresponding helix invariants are well-defined, the following assertions hold:*

1. *If $\boldsymbol{\beta}$ is a general helix, then $\tilde{\boldsymbol{\alpha}}_{\mathbf{1}}$ and $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ are spherical slant helices.*
2. *If $\boldsymbol{\beta}$ is a slant helix, then $\tilde{\boldsymbol{\alpha}}_{\mathbf{1}}$ and $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ are spherical clad helices, whereas $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$ is a spherical slant helix.*
3. *If $\boldsymbol{\beta}$ is a clad helix, then $\tilde{\boldsymbol{\alpha}}_{\mathbf{1}}$ and $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ are spherical $g$-clad helices, whereas $\tilde{\boldsymbol{\alpha}}_{\mathbf{2}}$ is a spherical clad helix.*

**Comparison with existing constructions.** Classical principal-direction curves and binormal-direction curves, together with their applications to helices and iterated associated-curve constructions, were investigated by Choi and Kim [11]. Related direction-curve methods for constructing associated curve pairs were studied by Önder [21], while Wong [34] established the spherical curve characterization employed in this section. Building on these geometric foundations, the present approach incorporates positive weight functions into the construction of tangent, principal, and binormal direction curves. In particular, Theorem 7 determines suitable weights for obtaining spherical curves from a prescribed generating curve. The position vectors of these spherical curves are then used to define three types of weighted associated curves, namely, $g$-weighted normal, rectifying,

and osculating direction curves. This second stage of the construction provides a geometric framework for establishing further relationships among different classes of helices. In this way, the classical geometric relations are brought together within a systematic weighted construction that connects spherical curves with the associated helix classes.

## 7. Examples

In this section, we illustrate the construction of special curves from a given Frenet curve. The corresponding curves are plotted using Mathematica.

Consider the circular helix $\boldsymbol{\beta} : I \to \mathbb{R}^3$ defined by

$$\boldsymbol{\beta}(t) = \big(3\cos t,\, 3\sin t,\, 4t\big),$$

where $I$ is an open interval containing $[-5\pi/8, 5\pi/8]$. Its speed is $\nu_{\boldsymbol{\beta}}(t) = 5$, and its Frenet apparatus is

$$\begin{aligned} \boldsymbol{T}_{\boldsymbol{\beta}}(t) &= \left(-\frac{3}{5}\sin t,\, \frac{3}{5}\cos t,\, \frac{4}{5}\right), \\ \boldsymbol{N}_{\boldsymbol{\beta}}(t) &= (-\cos t,\, -\sin t,\, 0), \\ \boldsymbol{B}_{\boldsymbol{\beta}}(t) &= \left(\frac{4}{5}\sin t,\, -\frac{4}{5}\cos t,\, \frac{3}{5}\right), \\ \kappa_{\boldsymbol{\beta}}(t) &= \frac{3}{25}, \qquad \tau_{\boldsymbol{\beta}}(t) = \frac{4}{25}. \end{aligned} \tag{43}$$

By item (1) of Theorem 7, the corresponding weight function is

$$\begin{aligned} f(t) &= \nu_{\boldsymbol{\beta}}(t)\kappa_{\boldsymbol{\beta}}(t)\cos\left(\int \nu_{\boldsymbol{\beta}}(t)\tau_{\boldsymbol{\beta}}(t)\,dt\right) \\ &= \frac{3}{5}\cos\left(\frac{4t}{5}\right). \end{aligned}$$

The weight function is strictly positive on

$$J = \left(-\frac{5\pi}{8}, \frac{5\pi}{8}\right),$$

so $J$ is an admissible open subinterval for this construction. The corresponding $f$-weighted tangent direction curve is therefore regular on $J$. We obtain

$$\bar{\boldsymbol{\gamma}}_{\mathbf{1}}(t) = \left(\frac{1}{10}\left(9\cos\frac{t}{5} + \cos\frac{9t}{5}\right), \frac{1}{10}\left(9\sin\frac{t}{5} + \sin\frac{9t}{5}\right), \frac{3}{5}\sin\frac{4t}{5}\right).$$

Indeed,

$$\|\bar{\boldsymbol{\gamma}}_{\mathbf{1}}(t)\| = 1, \qquad t \in J,$$

so $\bar{\boldsymbol{\gamma}}_{\mathbf{1}}$ lies on the unit sphere $S^2$. The speed, curvature, and torsion of $\bar{\boldsymbol{\gamma}}_{\mathbf{1}}$ are

$$\begin{aligned} \nu_{\bar{\boldsymbol{\gamma}}_{\mathbf{1}}}(t) &= \frac{3}{5}\cos\left(\frac{4t}{5}\right), \\ \kappa_{\bar{\boldsymbol{\gamma}}_{\mathbf{1}}}(t) &= \sec\left(\frac{4t}{5}\right), \\ \tau_{\bar{\boldsymbol{\gamma}}_{\mathbf{1}}}(t) &= \frac{4}{3}\sec\left(\frac{4t}{5}\right). \end{aligned}$$

Thus,

$$\frac{\tau_{\bar{\boldsymbol{\gamma}}_{\mathbf{1}}}(t)}{\kappa_{\bar{\boldsymbol{\gamma}}_{\mathbf{1}}}(t)} = \frac{4}{3},$$

and $\bar{\gamma}_1$ is a spherical general helix. Moreover, by Equation (9), the Frenet frame of $\bar{\gamma}_1$ coincides with that of $\beta$ at corresponding parameter values.

The circular helix $\beta$ and its spherical $f$-weighted tangent direction curve $\bar{\gamma}_1$ are illustrated in Figure 1.

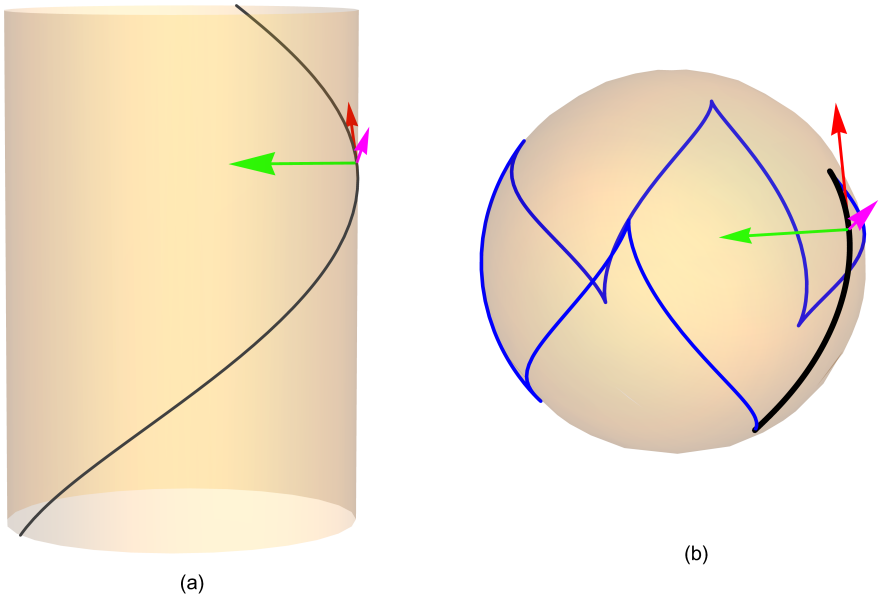



**Figure 1.** The circular helix $\beta$ on the cylinder (**a**) and its $f$-weighted tangent direction curve $\bar{\gamma}_1$, which is a spherical general helix on the unit sphere (**b**). In panel (**a**), the circular helix is shown in gray. In panel (**b**), the black portion corresponds to the admissible interval $J$, where the positive-weight construction is considered; the blue portions show the parametrized curve over a larger interval for geometric context. The tangent, principal normal, and binormal vectors are shown in red, green, and magenta, respectively, illustrating the preservation of the Frenet frame.

By item (1) of Corollary 17, we next construct a spherical slant helix from the spherical general helix $\bar{\gamma}_1$ obtained above. For the spherical $g$-weighted osculating direction curve construction, we have

$$\Omega_3(t) = -\frac{4}{3}\cos\left(\frac{3t}{5}\right),$$

and the corresponding weight function is

$$g(t) = \frac{4}{5}\cos\left(\frac{3t}{5}\right)\cos\Omega_3(t).$$

The weight function is strictly positive on

$$J = \left(-\frac{5\pi}{8}, \frac{5\pi}{8}\right).$$

Indeed, for every $t \in J$, we have

$$0 < \cos\left(\frac{3t}{5}\right) \leq 1, \qquad |\Omega_3(t)| \leq \frac{4}{3} < \frac{\pi}{2}.$$

Consequently, both cosine factors in $g(t)$ are positive on $J$. Thus, $J$ is an admissible open subinterval for this construction. The corresponding $g$-weighted osculating direction curve is

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}(t) = \Big( &-\frac{4}{5}\cos\Omega_3(t)\sin t + \frac{1}{5}\Big(4\sin\frac{2t}{5} - \sin\frac{8t}{5}\Big)\sin\Omega_3(t),\\ &\frac{4}{5}\cos\Omega_3(t)\cos t + \frac{1}{5}\Big(-4\cos\frac{2t}{5} + \cos\frac{8t}{5}\Big)\sin\Omega_3(t),\\ &-\frac{3}{5}\cos\Omega_3(t) - \frac{4}{5}\cos\frac{3t}{5}\sin\Omega_3(t)\Big).\end{aligned} \tag{44}$$

Since

$$\|\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}(t)\| = 1, \qquad t \in J,$$

the curve $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ lies on the unit sphere $S^2$. The speed of $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ is

$$\nu_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) = \frac{4}{5}\cos\Big(\frac{3t}{5}\Big)\cos\Omega_3(t) = g(t) > 0.$$

Its Frenet frame is given by

$$\begin{aligned}\boldsymbol{T}_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \Big(-\frac{1}{5}\Big(4\cos\frac{2t}{5} + \cos\frac{8t}{5}\Big), -\frac{1}{5}\Big(4\sin\frac{2t}{5} + \sin\frac{8t}{5}\Big), \frac{4}{5}\sin\frac{3t}{5}\Big),\\ \boldsymbol{N}_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \Big(\frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5}\Big),\\ \boldsymbol{B}_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \Big(\frac{1}{5}\Big(\sin\frac{8t}{5} - 4\sin\frac{2t}{5}\Big), \frac{1}{5}\Big(4\cos\frac{2t}{5} - \cos\frac{8t}{5}\Big), \frac{4}{5}\cos\frac{3t}{5}\Big).\end{aligned}$$

The curvature and torsion are

$$\begin{aligned}\kappa_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \sec\Omega_3(t),\\ \tau_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \tan\Big(\frac{3t}{5}\Big)\sec\Omega_3(t).\end{aligned}$$

Consequently,

$$\begin{aligned}\sigma_1^{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t) &= \frac{\dfrac{d}{dt}\Big(\dfrac{\tau_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)}{\kappa_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)}\Big)}{\nu_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)\kappa_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)\Big(1 + \Big(\dfrac{\tau_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)}{\kappa_{\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}}(t)}\Big)^2\Big)^{3/2}}\\ &= \frac{3}{4}.\end{aligned}$$

Thus, $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ is a spherical slant helix, in accordance with item (1) of Corollary 17.

The spherical $g$-weighted osculating direction curve $\tilde{\boldsymbol{\alpha}}_3$ is illustrated in Figure 2.

We next construct a Bertrand slant helix from the spherical slant helix $\tilde{\boldsymbol{\alpha}}_{\mathbf{3}}$ obtained above. By item (2) of Corollary 4, taking $c_2 = 1$ and $\omega = \pi/4$, the corresponding weight function becomes

$$f(t) = \frac{4}{5}\Big(\sin\frac{3t}{5} + \cos\frac{3t}{5}\Big).$$

The weight function is strictly positive on

$$J_1 = \Big(-\frac{5\pi}{12}, \frac{5\pi}{12}\Big),$$

so $J_1$ is an admissible open subinterval for this construction. The corresponding $f$-weighted tangent direction curve generated by $\tilde{\boldsymbol{\alpha}}_3$ is

$$\gamma_1(t) = \big(x_1(t), y_1(t), z_1(t)\big),$$

where

$$x_1(t) = \frac{2}{275}\left(-220\sin\frac{t}{5} - 5\sin\frac{11t}{5} - 55\sin t + 220\cos\frac{t}{5} + 5\cos\frac{11t}{5} + 33\cos t\right),$$
$$y_1(t) = \frac{2}{275}\left(-220\sin\frac{t}{5} + 5\sin\frac{11t}{5} + 33\sin t - 220\cos\frac{t}{5} + 5\cos\frac{11t}{5} + 55\cos t\right),$$
$$z_1(t) = -\frac{4}{75}\left(-6t + 5\sin\frac{6t}{5} + 5\cos\frac{6t}{5}\right).$$

The speed, curvature, and torsion of $\gamma_1$ are

$$\nu_{\gamma_1}(t) = \frac{4}{5}\left(\sin\frac{3t}{5} + \cos\frac{3t}{5}\right),$$
$$\kappa_{\gamma_1}(t) = \frac{\cos(3t/5)}{\sin(3t/5) + \cos(3t/5)},$$
$$\tau_{\gamma_1}(t) = \frac{\sin(3t/5)}{\sin(3t/5) + \cos(3t/5)}.$$

In particular,

$$\kappa_{\gamma_1}(t) + \tau_{\gamma_1}(t) = 1.$$

Thus, in the Bertrand relation (7), the corresponding constants are $c_2 = 1$ and $\cot(\omega) = 1$. Moreover,

$$\kappa_{\gamma_1}(t) - \tau_{\gamma_1}(t) = \frac{\cos(3t/5) - \sin(3t/5)}{\sin(3t/5) + \cos(3t/5)} \neq 0, \qquad t \in J_1.$$

Hence $\gamma_1$ is a Bertrand curve. Moreover, since $f > 0$ on $J_1$, the Frenet frame of $\gamma_1$ coincides with that of $\tilde{\boldsymbol{\alpha}}_3$ at corresponding parameter values. To verify the slant helix property directly, observe that

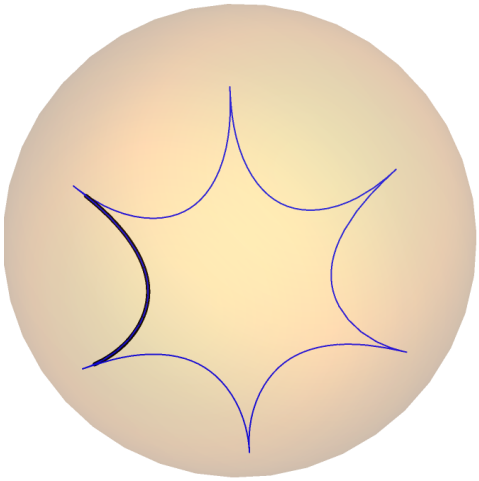


**Figure 2.** The spherical $g$-weighted osculating direction curve $\tilde{\boldsymbol{\alpha}}_3$ on the unit sphere. The black portion corresponds to the admissible interval $J$, where the $g$-weighted construction is considered; the blue portion shows the parametrized curve over a larger interval for geometric context.

$$\frac{\tau_{\gamma_1}(t)}{\kappa_{\gamma_1}(t)} = \tan\frac{3t}{5}.$$

Consequently,

$$\begin{aligned}\sigma_1^{\gamma_1}(t) &= \frac{\dfrac{d}{dt}\left(\dfrac{\tau_{\gamma_1}(t)}{\kappa_{\gamma_1}(t)}\right)}{\nu_{\gamma_1}(t)\kappa_{\gamma_1}(t)\left(1+\left(\dfrac{\tau_{\gamma_1}(t)}{\kappa_{\gamma_1}(t)}\right)^2\right)^{3/2}}\\ &= \frac{3}{4}.\end{aligned}$$

Thus, $\gamma_1$ is a Bertrand slant helix on $J_1$.

The Bertrand slant helix $\gamma_1$ is illustrated in Figure 3.

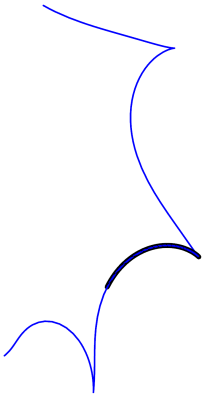


**Figure 3.** The Bertrand slant helix $\gamma_1$ generated by the spherical slant helix $\tilde{\boldsymbol{\alpha}}_3$. The black portion corresponds to the admissible interval $J_1$, where the weighted construction satisfies the required conditions; the blue portion shows the parametrized curve over a larger interval for geometric context.

By item (2) of Corollary 13, taking $c_1 = 1$, the corresponding weight function for the $f$-weighted binormal direction curve of the Bertrand slant helix $\gamma_1$ becomes

$$f(t) = \frac{4}{5}\csc\left(\frac{3t}{5}\right).$$

The weight function is strictly positive on

$$J_2 = \left(0, \frac{5\pi}{12}\right),$$

so $J_2$ is an admissible open subinterval for this construction. The corresponding curve is

$$\begin{aligned}\gamma_3(t) = \Bigg(&\frac{4}{25}\left(-\frac{5\sqrt{3}}{2}\ln\left(\frac{\sqrt{3}+2\sin\left(\frac{t}{5}\right)}{\sqrt{3}-2\sin\left(\frac{t}{5}\right)}\right)+2\sin t\right),\\ &-\frac{4}{25}\left(\frac{5}{2}\ln\left(\frac{1+2\cos\left(\frac{t}{5}\right)}{-1+2\cos\left(\frac{t}{5}\right)}\right)+\ln\left(\cot\left(\frac{t}{10}\right)\right)+2\cos t\right),\\ &\frac{16}{15}\ln\left(\sin\left(\frac{3t}{5}\right)\right)\Bigg).\end{aligned}$$

The Frenet frame of $\gamma_3$, whose speed is

$$\nu_{\gamma_3}(t) = \frac{4}{5}\csc\left(\frac{3t}{5}\right),$$

is given by

$$\boldsymbol{T}_{\gamma_3}(t) = \left(\frac{1}{5}\left(\sin\frac{8t}{5} - 4\sin\frac{2t}{5}\right), \frac{1}{5}\left(4\cos\frac{2t}{5} - \cos\frac{8t}{5}\right), \frac{4}{5}\cos\frac{3t}{5}\right),$$

$$\boldsymbol{N}_{\gamma_3}(t) = \left(-\frac{4}{5}\sin t, \frac{4}{5}\cos t, -\frac{3}{5}\right),$$

$$\boldsymbol{B}_{\gamma_3}(t) = \left(-\frac{1}{5}\left(4\cos\frac{2t}{5} + \cos\frac{8t}{5}\right), -\frac{1}{5}\left(4\sin\frac{2t}{5} + \sin\frac{8t}{5}\right), \frac{4}{5}\sin\frac{3t}{5}\right).$$

Its curvature and torsion are

$$\kappa_{\gamma_3}(t) = \sin^2\left(\frac{3t}{5}\right), \qquad \tau_{\gamma_3}(t) = \sin\left(\frac{3t}{5}\right)\cos\left(\frac{3t}{5}\right).$$

Consequently,

$$\kappa_{\gamma_3} = \kappa_{\gamma_3}^2 + \tau_{\gamma_3}^2,$$

so $\gamma_3$ satisfies the Mannheim condition with $c_1 = 1$. Moreover,

$$\frac{\tau_{\gamma_3}(t)}{\kappa_{\gamma_3}(t)} = \cot\left(\frac{3t}{5}\right).$$

Consequently,

$$\begin{aligned}\sigma_1^{\gamma_3}(t) &= \frac{\dfrac{d}{dt}\left(\dfrac{\tau_{\gamma_3}(t)}{\kappa_{\gamma_3}(t)}\right)}{\nu_{\gamma_3}(t)\kappa_{\gamma_3}(t)\left(1 + \left(\dfrac{\tau_{\gamma_3}(t)}{\kappa_{\gamma_3}(t)}\right)^2\right)^{3/2}} \\ &= -\frac{3}{4}.\end{aligned}$$

Thus, $\gamma_3$ is a Mannheim slant helix on $J_2$.

The Mannheim slant helix $\gamma_3$ is illustrated in Figure 4.

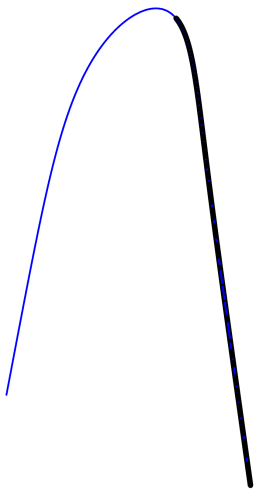


**Figure 4.** The Mannheim slant helix $\gamma_3$ generated by the Bertrand slant helix $\gamma_1$. The black portion shows the plotted part of the admissible interval $J_2$, where the Mannheim construction is valid; the blue portion shows the parametrized curve over a larger interval for geometric context.

Consider the Bertrand slant helix $\gamma_1$ constructed above. Since $\gamma_1$ is a slant helix, item (2) of Theorem 8 shows that its $g$-weighted normal direction curve $\bar{\boldsymbol{\alpha}}_1$ is a clad helix, where $g : J \to (0, \infty)$ is smooth. For $\gamma_1$, we have

$$\nu_{\gamma_1}\tau_{\gamma_1} = \frac{4}{5}\sin\frac{3t}{5}.$$

Thus,

$$\psi(t) = \int \nu_{\gamma_1}\tau_{\gamma_1}\,dt = -\frac{4}{3}\cos\frac{3t}{5}.$$

By (32),

$$\sigma_0^{\bar{\boldsymbol{\alpha}}_1} = \frac{\tau_{\bar{\boldsymbol{\alpha}}_1}}{\kappa_{\bar{\boldsymbol{\alpha}}_1}} = \tan\psi.$$

We now apply item (2) of Corollary 5 to $\bar{\boldsymbol{\alpha}}_1$. Taking $c_0 = 1$, the weight function prescribed by the corollary is

$$f(t) = \left(\sigma_0^{\bar{\boldsymbol{\alpha}}_1}\right)' = \frac{4}{5}\sin\frac{3t}{5}\sec^2\psi.$$

Let $\boldsymbol{\delta}_1$ be the $f$-weighted tangent direction curve generated by $\bar{\boldsymbol{\alpha}}_1$. By (31),

$$\boldsymbol{T}_{\bar{\boldsymbol{\alpha}}_1} = -\cos\psi\,\boldsymbol{N}_{\gamma_1} + \sin\psi\,\boldsymbol{B}_{\gamma_1}.$$

Since

$$\boldsymbol{B}'_{\gamma_1} = -\psi'\boldsymbol{N}_{\gamma_1},$$

we obtain

$$\boldsymbol{\delta}_1(t) = \sec\psi(t)\,\boldsymbol{B}_{\gamma_1}(t).$$

Hence,

$$\boldsymbol{\delta}_1(t) = \sec\psi(t)\left(\frac{1}{5}\left(\sin\frac{8t}{5} - 4\sin\frac{2t}{5}\right), \frac{1}{5}\left(4\cos\frac{2t}{5} - \cos\frac{8t}{5}\right), \frac{4}{5}\cos\frac{3t}{5}\right).$$

On the admissible open subinterval

$$J_2 = \left(0, \frac{5\pi}{12}\right),$$

the weight function $f$ is strictly positive. The curvature and torsion of $\boldsymbol{\delta}_1$ are

$$\kappa_{\boldsymbol{\delta}_1} = \cos^3\psi\,\cot\frac{3t}{5}, \qquad \tau_{\boldsymbol{\delta}_1} = \cos^2\psi\sin\psi\,\cot\frac{3t}{5}.$$

Moreover,

$$\sigma_1^{\boldsymbol{\delta}_1} = \tan\frac{3t}{5}, \qquad \sigma_2^{\boldsymbol{\delta}_1} = \frac{3}{4},$$

so $\boldsymbol{\delta}_1$ is a clad helix. Finally, since

$$\boldsymbol{\delta}_1 = \sec\psi\,\boldsymbol{B}_{\gamma_1}, \qquad \boldsymbol{N}_{\boldsymbol{\delta}_1} = \boldsymbol{T}_{\gamma_1},$$

we have

$$\left\langle \boldsymbol{\delta}_1, \boldsymbol{N}_{\boldsymbol{\delta}_1} \right\rangle = 0.$$

Thus, $\boldsymbol{\delta}_1$ is a rectifying clad helix.

The rectifying clad helix $\boldsymbol{\delta}_1$ is illustrated in Figure 5.

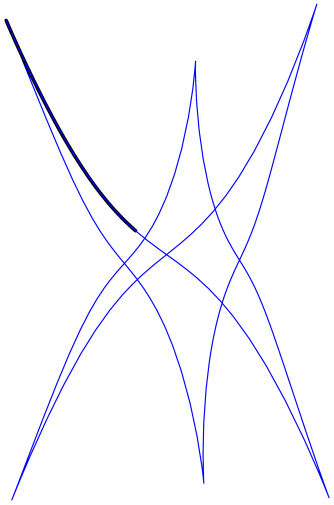


**Figure 5.** The rectifying clad helix $\delta_1$ obtained as the $f$-weighted tangent direction curve of $\bar{\boldsymbol{\kappa}}_{\mathbf{1}}$. The black portion corresponds to the admissible interval $J_2$, where the weighted construction satisfies the required conditions; the blue portion shows the parametrized curve over a larger interval for geometric context.

We next construct a Salkowski curve from the rectifying clad helix $\delta_1$ obtained above. By item (3) of Corollary 6, the weight function prescribed for its $f$-weighted principal direction curve is

$$f = \nu_{\delta_1}\sqrt{\kappa_{\delta_1}^2 + \tau_{\delta_1}^2} = \frac{4}{5}\cos\left(\frac{3t}{5}\right).$$

On

$$J_3 = \left(0, \frac{5\pi}{12}\right),$$

the weight function is strictly positive. Let $\delta_2$ denote the corresponding $f$-weighted principal direction curve generated by $\delta_1$. Then

$$\begin{aligned}\delta_2(t) = \Bigg( &-\frac{2}{55}\left(44\sin\frac{t}{5} + \sin\frac{11t}{5} + 11\sin t\right),\\ &\frac{2}{55}\left(-44\cos\frac{t}{5} + \cos\frac{11t}{5} + 11\cos t\right),\\ &-\frac{4}{15}\cos\frac{6t}{5}\Bigg).\end{aligned}$$

Indeed,

$$\delta_2'(t) = f(t)\,\boldsymbol{N}_{\delta_1}(t),$$

and hence

$$\nu_{\delta_2}(t) = \frac{4}{5}\cos\left(\frac{3t}{5}\right) > 0, \qquad t \in J_3.$$

Its Frenet frame is

$$\begin{aligned}\boldsymbol{T}_{\delta_2}(t) &= \left(-\frac{1}{5}\left(4\cos\frac{2t}{5} + \cos\frac{8t}{5}\right), -\frac{1}{5}\left(4\sin\frac{2t}{5} + \sin\frac{8t}{5}\right), \frac{4}{5}\sin\frac{3t}{5}\right),\\ \boldsymbol{N}_{\delta_2}(t) &= \left(\frac{4}{5}\sin t, -\frac{4}{5}\cos t, \frac{3}{5}\right),\\ \boldsymbol{B}_{\delta_2}(t) &= \left(\frac{1}{5}\left(\sin\frac{8t}{5} - 4\sin\frac{2t}{5}\right), \frac{1}{5}\left(4\cos\frac{2t}{5} - \cos\frac{8t}{5}\right), \frac{4}{5}\cos\frac{3t}{5}\right).\end{aligned}$$

The curvature and torsion are

$$\kappa_{\delta_2}(t) = 1, \qquad \tau_{\delta_2}(t) = \tan\left(\frac{3t}{5}\right).$$

Thus, $\delta_2$ has constant curvature and nonconstant torsion, and hence it is a Salkowski curve. Moreover,

$$\sigma_1^{\delta_2} = \frac{3}{4},$$

so $\delta_2$ is also a slant helix, in agreement with item (3) of Theorem 3.

The Salkowski curve $\delta_2$ is illustrated in Figure 6.

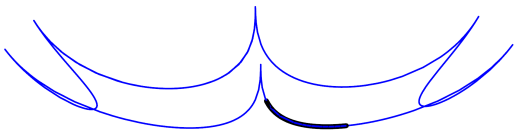


**Figure 6.** The Salkowski curve $\delta_2$ obtained as the $f$-weighted principal direction curve of the rectifying clad helix $\delta_1$. The black portion corresponds to the admissible interval $J_3$, where the weighted construction is valid; the blue portion shows the parametrized curve over a larger interval for geometric context.

## 8. Conclusions

In this paper, we have studied $f$-weighted direction curves associated with a Frenet curve in Euclidean 3-space. By integrating positive smooth multiples of the tangent, principal, and binormal vector fields, we have obtained curvature and torsion formulas for the resulting curves and examined their relationships with several classes of special space curves.

The weighted tangent direction curves share the Frenet frame of the generating curve, while the weighted principal and binormal direction curves satisfy corresponding relations between their Frenet frames and helix invariants. These formulas provide conditions under which the constructions yield general helices, slant helices, clad helices, and g-clad helices. Suitable choices of the weight function also yield Bertrand and Mannheim curves, as well as curves that are rectifying after a suitable translation. These conclusions hold on admissible intervals under the stated smoothness, positivity, regularity, and nonvanishing assumptions.

We have further examined weight functions that produce spherical direction curves. Using the position vectors of these spherical curves, we have defined $g$-weighted normal, rectifying, and osculating direction curves and derived spherical conditions for the resulting associated curves. The corresponding curvature and torsion relations establish further connections between spherical curves and the considered helix classes.

The examples illustrate how successive weighted constructions can be applied to an explicit generating curve. In particular, they exhibit spherical helices, a Bertrand slant helix, a Mannheim slant helix, a rectifying clad helix, and a Salkowski curve.

The contribution of the weighted formulation is to treat the choice of direction field and the choice of positive weight function within a common construction scheme. The classical unweighted direction curves are recovered by taking $f = 1$ when the generating curve is parametrized by arc length, whereas other admissible weights provide additional constructions subject to the geometric conditions established in the preceding sections.

The proposed approach is not limited to the classes explicitly considered here; its flexibility suggests possible applications to the construction and investigation of other special curve families. In particular, the spherical constructions presented here suggest a

further direction of research: the use of suitable weight functions to construct associated curves lying on prescribed surfaces.

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## References

1. Lancret, M.A. Mémoire sur les courbes à double courbure. *Mem. Present. Inst.* **1806**, *1*, 416–454.
2. Struik, D.J. *Lectures on Classical Differential Geometry*; Dover Publications: New York, NY, USA, 2012.
3. Izumiya, S.; Takeuchi, N. New special curves and developable surfaces. *Turk. J. Math.* **2004**, *28*, 153–163.
4. Takahashi, T. The generalization of helices. *Casp. J. Math. Sci.* **2019**, *8*, 178–195. [CrossRef]
5. Takahashi, T.; Takeuchi, N. Clad helices and developable surfaces. *Bull. Tokyo Gakugei Univ. Div. Nat. Sci.* **2014**, *66*, 1–9.
6. Li, Y.; Wang, Z.; Zhao, T. Slant helix of order $n$ and sequence of Darboux developables of principal-directional curves. *Math. Methods Appl. Sci.* **2020**, *43*, 9888–9903. [CrossRef]
7. Ramis, Ç.; Yilmaz, B.; Yayli, Y. New associated curves $k$-principle direction curves and $N_k$-slant helix. *Hagia Sophia J. Geom.* **2022**, *4*, 19–27.
8. Mak, M. Framed clad helices in Euclidean 3-space. *Filomat* **2023**, *37*, 9627–9640. [CrossRef]
9. Kula, L.; Yayli, Y. On slant helix and its spherical indicatrix. *Appl. Math. Comput.* **2005**, *169*, 600–607. [CrossRef]
10. Kula, L.; Ekmekci, N.; Yayli, Y.; Ilarslan, K. Characterizations of slant helices in Euclidean 3-space. *Turk. J. Math.* **2010**, *34*, 261–274. [CrossRef]
11. Choi, J.H.; Kim, Y.H. Associated curves of a Frenet curve and their applications. *Appl. Math. Comput.* **2012**, *218*, 9116–9124. [CrossRef]
12. Mannheim, A. Sur les surfaces trajectories des points d'une ligne de courbure d'une surface qui se d'eplace en restant tangente à une surface fixe. *C. R. Math. Acad. Sci. Paris* **1878**, *86*, 1254–1256.
13. Blum, R. A remarkable class of Mannheim-curves. *Can. Math. Bull.* **1966**, *9*, 223–228. [CrossRef]
14. Kuhnel, W. *Differential Geometry: Curves-Surfaces-Manifolds*; Student Mathematical Library; American Mathematical Society: Providence, RI, USA, 2002; Volume 16.
15. Papaioannou, S.G.; Kiritsis, D. An application of Bertrand curves and surfaces to CADCAM. *Comput. Aided Des.* **1985**, *17*, 348–352. [CrossRef]
16. Camci, Ç.; Uçum, A.; İlarslan, K. A new approach to Bertrand curves in Euclidean 3-space. *J. Geom.* **2020**, *111*, 49. [CrossRef]
17. Chen, B.Y. When does the position vector of a space curve always lie in its rectifying plane? *Am. Math. Mon.* **2003**, *110*, 147–152. [CrossRef]
18. Kim, D.S.; Chung, H.S.; Cho, K.H. Space curves satisfying $\tau/\kappa = as + b$. *Honam Math. J.* **1993**, *15*, 5–9.
19. Deshmukh, S.; Chen, B.Y.; Alshammari, S.H. On rectifying curves in Euclidean 3-space. *Turk. J. Math.* **2018**, *42*, 609–620. [CrossRef]
20. Altunkaya, B. Helical transformations from spherical to rectifying curves. *Konuralp J. Math.* **2025**, *13*, 292–299.
21. Önder, M. Construction of curve pairs and their applications. *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.* **2021**, *91*, 21–28. [CrossRef]
22. Monterde, J. Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion. *Comput. Aided Geom. Des.* **2009**, *26*, 271–278. [CrossRef]
23. Bates, L.M.; Melko, O.M. On curves of constant torsion I. *J. Geom.* **2013**, *104*, 213–227. [CrossRef]
24. Yüksel, N.; Saltık, B.; Karacan, M.K. The characterizations of the curve generated by a curve with constant torsion. *Konuralp J. Math.* **2023**, *11*, 1–7.
25. O'Neill, B. *Elementary Differential Geometry*; Academic Press: New York, NY, USA, 1966.
26. Salkowski, E. Zur Transformation von Raumkurven. *Math. Ann.* **1909**, *66*, 517–557. [CrossRef]
27. Scofield, P.D. Curves of constant precession. *Am. Math. Mon.* **1995**, *102*, 531–537. [CrossRef]
28. Mak, M.; Altınbas, H.; Kula, L. Ac-slant curves in Euclidean 3-space. *Asian-Eur. J. Math.* **2025**, *18*, 2450096. [CrossRef]
29. Altunkaya, B. The generalization of rectifying helices. *Turk. J. Math.* **2025**, *49*, 695–713. [CrossRef]

30. Liu, H.; Wang, F. Mannheim partner curves in 3-space. *J. Geom.* **2008**, *88*, 120–126. [CrossRef]
31. Ucum, A.; Camci, C.; Ilarslan, K. A new approach to Mannheim curve in Euclidean 3-Space. *Tamkang J. Math.* **2023**, *54*, 93–106.
32. Bertrand, J. Mémoire sur la théorie des courbes á double courbure. *J. Math. Pures Appl.* **1850**, *15*, 332–350.
33. Izumiya, S.; Takeuchi, N. Generic properties of helices and Bertrand curves. *J. Geom.* **2002**, *74*, 97–109. [CrossRef]
34. Wong, Y.C. On an explicit characterization of spherical curves. *Proc. Am. Math. Soc.* **1972**, *34*, 239–242. Erratum in *Proc. Am. Math. Soc.* **1973**, *38*, 668. [CrossRef]

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