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Asymptotic Behavior of Distance near the Origin Between Two Elements of the Semigroup in a Banach Algebra

Mohamed Agti [1,2,*] and Abdelkader Mokhtari [3,4]

[1] Department of Mathematics, University of Biskra, BP 145 RP, Biskra 07000, Algeria
[2] Laboratory of Applied Mathematics, Ouargla 30000, Algeria
[3] Department of Mathematics, University of Laghouat, Laghouat 03000, Algeria; a.mokhtari@mail.lagh-univ.dz
[4] Pure and Applied Mathematical Laboratory, Laghouat 03000, Algeria
* Correspondence: agti.mohamed@univ-ouargla.dz

Abstract

This paper examines the local asymptotic behavior of the distance $\|T(t+\xi(t))-T(t)\|$ near the origin, where $(T(t))_{t>0}$ is a semigroup on a commutative Banach algebra and $\xi :]0,+\infty[\to]0,+\infty[$ is a continuous function satisfying $\liminf_{t\to 0^+} \frac{\xi(t)}{t} > 0$. Our first main result establishes that if the closed subalgebra A generated by the semigroup $(T(t))_{t>0}$ is non-unital, then $\limsup_{t\to 0^+} \|T(t+\xi(t))-T(t)\| > 0$. As an application, we derive an exact asymptotic formula for the disk algebra $A\left(\overline{\mathbb{D}}\right)$. Considering the semigroup $(T(t))_{t>0}$ on $A\left(\overline{\mathbb{D}}\right)$, given by $(T(t)f)(z) = f(e^{-t}z)$ for $f \in A\left(\overline{\mathbb{D}}\right)$, $z \in \mathbb{D}$, and assuming that $\lim_{t\to 0^+} \frac{\xi(t)}{t} = \gamma$ is finite, we prove that $\limsup_{t\to 0^+} \|T(t+\xi(t))-T(t)\| = \frac{\gamma}{(1+\gamma)^{1+\frac{1}{\gamma}}}$. Furthermore, the closed subalgebra generated by this semigroup is non-unital.

Keywords: semigroups; C_0-semigroups; Banach algebras; closed subalgebras; asymptotic behavior

MSC: 47D03; 46J40; 46H20

1. Introduction

This article examines the behavior of $\|T(t+\xi(t))-T(t)\|$ near the origin for a continuous semigroup $(T(t))_{t>0}$ on a commutative Banach algebra, where the parameter t is perturbed by a continuous function ξ.

We begin by recalling Esterle's foundational result [1], which states that if T is a bounded linear operator on a complex Banach space, with $\sigma(T) = \{1\}$ and $T \neq I$, then $\liminf_{n\to+\infty} n\left\|T^{n+1}-T^n\right\| \geq 1/12$.

The first application to semigroups was introduced by the second author [2]: for any nonzero continuous semigroup $(T(t))_{t>0}$ in a Banach algebra satisfying $\limsup_{t\to 0^+} \|T(t)-T(2t)\| < \frac{1}{4}$, the Banach algebra A generated by $(T(t))_{t>0}$ includes a unit e with $\lim_{t\to 0^+} \|T(t)-e\| = 0$ and, consequently, $\lim_{t\to 0^+} \|T(t)-T(2t)\| = 0$.

Esterle and Mokhtari [3] later extended this result to semigroups defined on $\mathbb{R}^+$, proving that, for some integer $n \geq 1$, $\limsup_{t\to 0^+} \|T(t)-T((n+1)t)\| < \frac{n}{(n+1)^{1+\frac{1}{n}}}$. Then, the semigroup is trivial $T(t) = 0$, $\forall t > 0$, or it generates a unital closed subalgebra A, with

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$T(t) = e^{tu}$ for some $u \in A$. The constant $\frac{n}{(n+1)^{1+\frac{1}{n}}}$ is optimal for this result; moreover, $\lim_{t\to 0^+} \|T(t) - e\| = 0$ and $\lim_{t\to 0^+} \|T(t) - T((n+1)t)\| = 0$.

In complementary work, Kalton et al. [4] developed methods for constructing broad generalizations of Esterle's results. If T is a bounded operator with $\sigma(T) = \{1\}$ and $T \neq I$, then $\liminf_{n\to+\infty} n\left\|T^{n+1} - T^n\right\| \geq \frac{1}{e}$. The constant $\frac{1}{e}$ is sharp.

Later, for the C_0-semigroups Esterle described in [5], if $(T(t))_{t>0}$ is nontrivial, strongly continuous quasinilpotent semigroup. If there exists a constant $\delta > 0$ such that $0 < t < s \leq \delta \ \|T(t) - T(s)\| > (s-t)\frac{s^{(\frac{s}{s-t})}}{t^{(\frac{t}{s-t})}} = \theta\left(\frac{s}{t}\right)$ and if $(T(t))_{t>0}$ is C_0-semigroup on a Banach space and there exists a constant $\eta > 0$ and a continuous function $t \to s(t)$ defined on $[0, \eta]$ with $s(0) = 0$ such that $0 < t < s(t)$ and $\|T(t) - T(s(t))\| < \theta\left(\frac{s}{t}\right)$ for $t \in [0, \eta]$, then $(T(t))_{t>0}$ is a uniformly continuous semigroup and its infinitesimal generator is bounded.

Bendaoud et al. [6] obtained analogous results for analytic semigroups $(T(t))_{t\in S_\alpha}$ defined on a sector $S_\alpha = \{z \in \mathbb{C}: Re(z) > 0 \text{ and } |arg(z)| < \alpha\}$. They found that if $\sup_{t\in S_\alpha,\ 0<|t|<\delta} \left\|t^n T^{(n)}(t)\right\| < \left(\frac{n}{e\cos(\alpha)}\right)^n$ holds for some $\delta > 0$, then the closed algebra generated by the semigroup $(T(t))_{t\in S_\alpha}$ is unital and the infinitesimal generator is bounded.

For related results, Chalendar et al. [7] showed that if $(T(t))_{t\in\mathbb{C}^+}$ is a quasinilpotent semigroup on the open half plane, and if there exist $\delta > 0$, $\mu > 0$ and $\lambda > 0$ such that $\sup_{n\in\mathbb{Z}} \lambda^{|n|}\|T(\delta + in\,\mu)\| < +\infty$, then for every $y > 0$ and every $h > 0$:

$$\max_{v=\pm 1} \lim_{\varepsilon\to 0^+} inf\|T(ivy + \varepsilon) - T(iv(y+h) + \varepsilon)\| \geq 2$$

The same authors [8] established estimates for the functional calculus $F(-uA)$ near the origin for the infinitesimal generator A of a C_0-semigroup $(T(t))_{t>0}$ in the Banach algebra of bounded linear operators. Here, F is an entire function given by a Laplace transform with $F(0) = 0$. They proved the existence of either an identity element or an exhaustive sequence of idempotents in the Banach algebra generated by the semigroup.

In a complementary direction, Arendt et al. [9] recently studied the asymptotic behavior of semigroups of composition operators $(C_t)_{t\geq 0}$ on the Hardy space $H^2(\mathbb{D})$, where $C_t f = f \circ \phi_t$ for a holomorphic semiflow ϕ_t. Their main result establishes a sharp characterization: C_t converges strongly as $t \to +\infty$ if and only if the Denjoy–Wolff point α of the semiflow lies in $\mathbb{D}$ and the generator G has a zero in $\mathbb{D}$ with $Re(\bar{z}G(z)) < 0$ on a positive-measure subset of $\mathbb{T}$. Under these conditions, the convergence is exponentially fast: $\|C_t - P\| \leq Me^{\varepsilon t}$, where $Pf = f(\alpha) \cdot 1_{\mathbb{D}}$. This provides a continuous-time analog of the discrete dichotomy established earlier in the same paper, where strong convergence of powers of a composition operator was shown to imply uniform convergence.

We present, in the following, a simple generalization of the above results. Our proofs employ the Gelfand theory for commutative Banach algebra to reformulate the problem into spectral terms. We apply Shilov's idempotent theorem to establish the existence of a unit element in the algebra generated by a continuous semigroup, and use the Feller technique to handle semigroups unbounded near the origin. The remainder of this paper is organized as follows. In Section 2, we state some necessary preliminaries. In Section 3, we list the main results. Section 4 provides detailed proofs of the main results. We conclude in Section 5 with an example of a contraction semigroup on the disk algebra with exact asymptotic estimates.

2. Preliminaries

Let $(T(t))_{t>0}$ be a semigroup of elements on a commutative Banach algebra, indexed by $\mathbb{R}^+$. The closed commutative subalgebra generated by this semigroup is defined as $A = \overline{span}\{T(t);\ t > 0\}$. If the Banach algebra A does not have a unit, we denote its uniti-

zation by A_e. The Gelfand spectrum of A, denoted by $\hat{A}$, is the set of all characters on the Banach algebra A, that is, nonzero multiplicative linear functionals $\chi : \mathrm{A} \to \mathbb{C}$, equipped with the weak-* topology. The radical of A, denoted by $rad(A)$, is defined as the intersection of the kernels of all the characters [10]:

$$rad(A) = \bigcap_{\chi \in \hat{A}} ker\chi.$$

We say that A is radical if $rad(A) = \mathrm{A}$.

For any element $a \in A$, its Gelfand transform is the function $\hat{a} : \hat{A} \to \mathbb{C}$ given by

$$\hat{a}(\chi) = \chi(a) \ (\chi \in \hat{A})$$

Let I be an ideal in the algebra A. For $a \in \mathrm{A}$, we denote its equivalence class in the quotient algebra A/I by

$$[a] = a + \mathrm{I} \in A/I.$$

We now prove several lemmas that will be used later.

Lemma 1. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, bounded near the origin. Let $A = \overline{span}\{T(t);\ t > 0\}$ be the closed subalgebra generated by the semigroup. Then, for all $x \in A$, $\lim_{t \to 0^+} T(t)x = x$, and $(T(t_n))_{t_n>0}$ is a sequential bounded approximate identity for A, with $t_n \to 0$.*

Proof. Since the application $t \to T(t)$ is continuous, we have $T(t+s) \overset{t \to 0^+}{\longrightarrow} T(s)$ for all $s > 0$. Hence, $T(t)\sum_{i=1}^{m} \lambda_i T(s_i) = \sum_{i=1}^{m} T(t)\lambda_i T(s_i) \overset{t \to 0^+}{\longrightarrow} \sum_{i=1}^{m} \lambda_i T(s_i)$ for $s_1, s_2, \ldots, s_m > 0$, $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$. Since $\sup_{t \in]0,\delta[} \|T(t)\| < +\infty$, for some $\delta > 0$, it follows that for all $x \in A$, $T(t)x \overset{t \to 0^+}{\to} x$. □

The construction that follows is related to a technique developed by Feller [11].

Lemma 2. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, and let the closed subalgebra $A = \overline{span}\{T(t);\ t > 0\}$. Consider the ideal $I = \left\{ x \in A;\ T(t)x \overset{t \to 0^+}{\longrightarrow} x \right\}$, and let $a = T(1)$ with spectral radius $r(a)$. Define $\lambda = log(1 + r(a))$ and set*

$$p(x) = \sup_{t>0} \left\| T(t)e^{-\lambda t}x \right\|, \text{ for all } x \in I$$

$$And q(x) = \sup_{\substack{y \in I \\ y \neq 0}} \frac{p(xy)}{p(y)}, \text{ for all } x \in A.$$

Then:

1. *q is a seminorm on A.*
2. *$H = \{x \in A: q(x) = 0\}$ is a closed two-sided ideal of A, and $B = A/H$ is a well-defined algebra.*
3. *The quotient norm*

$$\| [x] \|_{A/H} = \inf_{h \in H} q(x+h), \qquad [x] = x + H \in A/H,$$

makes $(B, \|\cdot\|_{A/H})$ a normed commutative algebra.

4. *If $\bar{B}$ is the completion of $(B, \|\cdot\|_{A/H})$, then $\theta(T(t))$ is bounded near the origin, where $\theta = i \circ \pi$, $\pi: A \to B$ is the canonical surjection and $i : B \to \bar{B}$ is the canonical injection.*

5. $(\theta(T(t_n)))_{t_n>0}$ *is a sequential bounded approximate identity for* $\bar{B}$ *for any sequence* $t_n \to 0$.
6. (I, p) *is a Banach algebra.*
7. I *is a Banach* $\bar{B}$*-module.*

Proof. We split the proof into seven parts, each corresponding to one assertion.

Preliminary: p is a norm on I. For $x, y \in I$,

$$p(x+y) = \sup_{t>0} \left\| T(t)e^{-\lambda t}(x+y) \right\| \leq p(x) + p(y).$$

Also,

$$p(xy) = \sup_{t>0} \left\| T(t)e^{-\lambda t}(xy) \right\|$$

$$p(xy) = \sup_{t>0} \left\| \left(T(t)e^{-\lambda t}x \right) y \right\|$$

$$p(xy) \leq \sup_{t>0} \left(\left\| T(t)e^{-\lambda t}x \right\| \|y\| \right)$$

$$p(xy) \leq \left(\sup_{t>0} \left\| T(t)e^{-\lambda t}x \right\| \right) \|y\|$$

$$p(xy) \leq p(x)\|y\|$$

Since

$$\|y\| = \lim_{t\to 0^+} \left\| T(t)e^{-\lambda t}y \right\| \leq \sup_{t>0} \left\| T(t)e^{-\lambda t}y \right\| = p(y),$$

we obtain

$$p(xy) \leq p(x)p(y).$$

Finally, If $p(x) = 0$, then

$$\|T(t)e^{-\lambda t}x\| = 0 \text{ for all } t > 0. \text{ Since } \|x\| = \lim_{t\to 0^+} \left\| T(t)e^{-\lambda t}x \right\| \leq p(x) = 0$$

Hence, $x = 0$. Thus, p is a norm on I.

Part 1: q is a seminorm on A. For x_1,$x_2 \in A$, and $y \in I$,

$$\begin{aligned} q(x_1 + x_2) &= \sup_{\substack{y \in I \\ y \neq 0}} \frac{p((x_1+x_2)y)}{p(y)} \\ &\leq \sup_{\substack{y \in I \\ y \neq 0}} \left(\frac{p(x_1 y)}{p(y)} + \frac{p(x_2 y)}{p(y)} \right) \\ &\leq \sup_{\substack{y \in I \\ y \neq 0}} \frac{p(x_1 y)}{p(y)} + \sup_{\substack{y \in I \\ y \neq 0}} \frac{p(x_2 y)}{p(y)} \end{aligned}$$

$$q(x_1 + x_2) \leq q(x_1) + q(x_1).$$

Moreover,

$$\begin{aligned} q(x_1x_2) &= \sup_{\substack{y \in I \\ y \neq 0}} \frac{p((x_1x_2)y)}{p(y)} \\ &= \sup_{\substack{y \in I \\ y \neq 0}} \frac{p(x_1(x_2y))}{p(y)} \\ &\leq \sup_{\substack{y \in I \\ y \neq 0}} \left(q(x_1) \frac{p(x_2y)}{p(y)} \right) \end{aligned}$$

$$q(x_1x_2) \leq q(x_1)q(x_2)$$

Hence, q is a seminorm on A.

Part 2: $H = \ker(q)$ is a closed two-sided ideal of A. We have,

$$H = \ker(q) = \{x \in A : q(x) = 0\}.$$

- H is a linear subspace: If $x, y \in H$ and $\alpha \in \mathbb{C}$, then

$$q(x + y) \leq q(x) + q(y) = 0, \qquad q(\alpha x) = |\alpha| q(x) = 0.$$

- H is a two-sided ideal: If $x \in H$ and $z \in A$, then, by submultiplicativity,

$$q(xz) \leq q(x)q(z) = 0, \qquad q(zx) \leq q(z)q(x) = 0.$$

Since A is commutative, left and right ideals coincide. Hence, H is a two-sided ideal.

Since

$$q(x)\|x\|, \qquad x \in A,$$

the seminorm q is continuous with respect to the norm of A. Therefore, $H = q^{-1}(\{0\})$ is closed in A.

Consequently, the quotient $B = A/H$ is a well-defined algebra.

Part 3: $(B, \|\cdot\|_{A/H})$ is a normed commutative algebra. For $[x] = x + H \in B$, define

$$\| [x] \|_{A/H} = \inf_{h \in H} q(x + h)$$

We verify that this is indeed a norm. If $\|[x]\|_{A/H} = 0$, then there exists a sequence $(h_n)_n \subset H$ such that $q(x + h_n) \to 0$. Since $q(h_n) = 0$,

$$|q(x + h_n) - q(x)| \leq q(h_n) = 0,$$

so $q(x) = q(x + h_n) \to 0$. Hence, $x \in H$ and $[x] = 0$.

For $[x], [z] \in B$,

$$\begin{aligned} \| [x][z] \|_{A/H} &= \inf_{h \in H} q(xz + h) \\ &\leq \inf_{h_1, h_2 \in H} q((x + h_1)(z + h_2)) \\ &\leq \inf_{h_1 \in H} q(x + h_1) \inf_{h_2 \in H} q(z + h_2) \\ &= \| [x] \|_{A/H} \| [z] \|_{A/H}. \end{aligned}$$

Thus, $(B, \|\cdot\|_{A/H})$ is a normed commutative algebra.

Part 4: Boundedness of $\theta(T(t))$ near the origin. Let $\bar{B}$ be the completion of $(B, \|\cdot\|_{A/H})$, and let $\theta = i \circ \pi$. Where $\pi : A \to B$ is the canonical surjection and $i : B \to \overline{B}$ is the canonical injection. we have the following:

$$\|\theta(T(t))\|_{\overline{B}} = \|i(\pi(T(t)))\|_{\overline{B}} = \|(\pi(T(t)))\|_{A/H} \leq q(T(t)),$$

since $\|\pi(x)\|_{A/H} \leq q(x)$ for all $x \in A$. For any $y \in I\backslash\{0\}$, we have

$$\begin{aligned} p(T(t)y) &= \sup_{s>0}\left\|T(s)e^{-\lambda s}T(t)y\right\| \\ &= \sup_{s>0}\left\|T(s+t)e^{-\lambda s}y\right\| \\ &= \sup_{s>0}\left\|T(s+t)e^{-\lambda(s+t)}e^{\lambda t}y\right\| \\ &= |e^{\lambda t}|\sup_{s>0}\left\|T(s+t)e^{-\lambda(s+t)}y\right\| \\ &\leq |e^{\lambda t}|p(y). \end{aligned}$$

Therefore,

$$q(T(t)) \leq \mid e^{\lambda t} \mid, \text{ for all } t > 0$$

It follows that $\theta(T(t))$ is bounded near the origin.

Part 5: $\theta(T(t_n))$ is a sequential bounded approximate identity for $\bar{B}$. Let $(t_n)_n$ be any sequence with $t_n \to 0^+$. By Lemma 1, $(T(t_n))_n$ is a sequential bounded approximate identity for A,

$$T(t_n)x \to x \qquad \text{for all } x \in A,$$

and $\sup_n\|T(t_n)\| < +\infty$. Since θ is a continuous algebra homomorphism, for every $x \in A$,

$$\theta(T(t_n))\theta(x) = \theta(T(t_n)x) \to \theta(x).$$

Moreover, by Part 4, the sequence $(\theta(T(t_n)))_n$ is bounded in $\bar{B}$. Since $\theta(A)$ is dense in $\bar{B}$, it follows that

$$\theta(T(t_n))y \to y \qquad \text{for all } y \in \bar{B}.$$

Thus, $(\theta(T(t_n)))_n$ is a sequential bounded approximate identity for $\bar{B}$.

Part 6: (I, p) is a Banach algebra. We prove the completeness of (I, p); let $(x_n)_n$ be a Cauchy sequence in (I, p). Since

$$\|x_n - x_m\| \leq p(x_n - x_m),$$

$(x_n)_n$ is Cauchy in A, Therefore, it converges to some $x \in A$. We now show that $x \in I$. We fix $\varepsilon > 0$. We choose N such that $p(x_n - x_m) < \varepsilon$ for all $n,m \geq N$, and fix $n \geq N$. For any $t > 0$,

$$\left\|T(t)e^{-\lambda t}(x_n - x)\right\| = \lim_{m\to+\infty}\left\|T(t)e^{-\lambda t}(x_n - x_m)\right\| \leq \limsup_{m\to+\infty} p(x_n - x_m) \leq \varepsilon.$$

Thus, $p(x_n - x) < \varepsilon$. Now,

$$\begin{aligned} \|T(t)(x - x)\| &\leq \|T(t)(x - x_n)\| + \|T(t)x_n - x_n\| + \|x_n - x\| \\ &\leq e^{\lambda t}p(x - x_n) + \|T(t)x_n - x_n\| + \|x_n - x\| \\ &\leq e^{\lambda t}\varepsilon + \|T(t)x_n - x_n\| + \|x_n - x\|. \end{aligned}$$

Since $x_n \in I$, $\lim_{t\to0^+}\|T(t)x_n - x_n\| = 0$, and since $\|x_n - x\| \leq p(x_n - x) \leq \varepsilon$, we have

$$\limsup_{t\to0^+}\|T(t)x - x\| \leq e^{|\lambda|}\varepsilon + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $\lim_{t\to0^+} T(t)x = x$; therefore, $x \in I$.

Thus, (I, p) is a Banach algebra.

Part 7: I is a Banach $\bar{B}$-module. Define the application $\overline{B} \times I \to I$ by

$$([x], u) \to [x].\ u = xu$$

where x is any representative of $[x]$.

To check that this definition is well-posed, suppose that $[x_1] = [x_2]$; thus, $[x_1] - [x_2] \in H$. Then, $q(x_1 - x_2) = 0$, which implies that

$$\sup_{\substack{y \in I \\ y \neq 0}} \frac{p((x_1 - x_2)y)}{p(y)} = 0.$$

Thus, $p((x_1 - x_2)y) = 0$ for all $y \in I$; thus, $(x_1 - x_2)y = 0$ for all $y \in I$, particularly for $y = u \in I$; therefore, $x_1 u = x_2 u$.

Thus the application is well-defined.

Boundedness. For $[x] \in B$ and $u \in I$,

$$p([x].\ u) = p(xu) \leq q(x)p(u) \leq \|[x]\|_{A/H}\, p(u).$$

For $[x] \in \overline{B}$, we choose a sequence $([x_n])_n \subset B$ with $\|[x_n] - [x]\|_{A/H} \to 0$. Then, for $u \in I$,

$$p(([x_n] - [x_m]).u) \leq \|[x_n] - [x_m]\|_{A/H} p(u) \to 0 \text{ as } n,\ m \to +\infty.$$

Hence, $([x_n].u)_n$ is a Cauchy sequence in (I, p), and converges to some $v \in I$. We define $[x].\ u = v$.

This definition is independent of the chosen approximating sequence, and

$$p([x].u) = \lim_{n \to +\infty} p([x_n].u) \leq \lim_{n \to +\infty} \|[x_n]\|_{A/H} p(u) = \|[x]\|_{\overline{B}} p(u).$$

Thus, I is a Banach $\overline{B}$-module. □

Lemma 3. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, and let the closed subalgebra $A = \overline{span}\{T(t);\ t > 0\}$. Then,*

(1) The map $\widetilde{\theta} :\ \hat{\overline{B}} \to \hat{A}$ is a homeomorphism.

(2) If $\overline{B}$ is unital, then the continuous semigroup $(T(t))_{t>0}$ is bounded near the origin, A is unital and the unit element $g \in A$ satisfies

$$g = \lim_{t \to 0^+} T(t).$$

where $\overline{B}$ and θ are defined as in Lemma 2.

Proof. We consider $\theta = i \circ \pi$: $A \to B$ and define the map $\widetilde{\theta} :\ \hat{\overline{B}} \to \hat{A}$ by $\widetilde{\theta}(\phi) = \phi \circ \theta$. Thus, $\widetilde{\theta}$ is well defined. Additionally, if $\widetilde{\theta}(\phi_1) = \widetilde{\theta}(\phi_2)$, then $\phi_1 \circ \theta = \phi_2 \circ \theta$, which implies that $\phi_1 = \phi_2$ on $\theta(A)$. Since $\overline{\theta(A)} = \overline{B}$, we have $\phi_1 = \phi_2$ on $\overline{B}$; thus, $\widetilde{\theta}$ is injective.

Surjectivity of $\widetilde{\theta}$. Let $\psi \in \hat{A}$; then, $\psi(a) \neq 0$. For $x \in A$ we set $\phi_0(\theta(x)) = \psi(x)$. If $\theta(x_1) = \theta(x_2)$, then $ax_1 = ax_2$ because $a \in I$; thus, $\psi(x_1) = \psi(x_2)$. Therefore, ϕ_0 is well defined on $\theta(A)$. Moreover, $|\psi(y)| \leq \|y\| \leq p(y)$ for all $y \in I$. We have

$$|\phi_0(\theta(x))| = |\psi(x)| = \frac{|\psi(ax)|}{|\psi(a)|} \leq \frac{p(ax)}{|\psi(a)|} \leq \frac{p(a)}{|\psi(a)|} q(x) = \frac{p(a)}{|\psi(a)|} \dot{q}(\dot{x}).$$

Thus, ϕ_0 is a continuous character on $\left(\theta(A), \dot{q}\right)$. We extend ϕ_0 to $\phi \in \hat{\overline{B}}$ by continuity, and $\widetilde{\theta}(\phi) = \phi \circ \theta = \psi$. Therefore, $\widetilde{\theta}$ is surjective.

Continuity of $\widetilde{\theta}$. Let $(\phi_\alpha)_\alpha$ be a net in $\hat{\overline{B}}$ converging to $\phi \in \hat{\overline{B}}$ in the weak-* topology. For any $x \in A$ we have

$$\widetilde{\theta}(\phi_\alpha)(x) = \phi_\alpha(\theta(x)) \to \phi(\theta(x)) = \widetilde{\theta}(\phi)(x).$$

Thus, $\widetilde{\theta}(\phi_\alpha) \to \widetilde{\theta}(\phi)$ pointwise on A, showing that $\widetilde{\theta}$ is continuous.

Continuity of $\widetilde{\theta}^{-1}$. Let $(\psi_\alpha)_\alpha$ be a net in $\hat{A}$ converging to $\psi \in \hat{A}$ in the weak-* topology. We set $\phi_\alpha = \widetilde{\theta}^{-1}(\psi_\alpha)$, $\phi = \widetilde{\theta}^{-1}(\psi)$. For any $y \in \overline{B}$ and $\varepsilon > 0$, we choose $z = \theta(x) \in \theta(A)$ with $\|y - z\|_{\overline{B}} < \varepsilon/3$. Then,

$$\begin{aligned} |\phi_\alpha(y) - \phi(y)| &\leq |\phi_\alpha(y) - \phi_\alpha(z)| + |\phi_\alpha(z) - \phi(z)| + |\phi(z) - \phi(y)|. \\ &\leq \|y - z\| + |\psi_\alpha(x) - \psi(x)| + \|z - y\|. \end{aligned}$$

Since $\psi_\alpha(x) \to \psi(x)$, there exists α_0 such that $|\psi_\alpha(x) - \psi(x)| < \varepsilon/3$ for $\alpha \geq \alpha_0$.

Thus, for $\alpha \geq \alpha_0$:

$$|\phi_\alpha(y) - \phi(y)| < \varepsilon.$$

Therefore, $\phi_\alpha(y) \to \phi(y)$ for all $y \in \overline{B}$, and $\widetilde{\theta}^{-1}$ is continuous.

Now, assume that $\overline{B}$ is unital with unit element $e \in \overline{B}$. We prove that $(T(t))_{t>0}$ is bounded near the origin.

From Lemma 2, we know that $\theta(T(t))$ is bounded near the origin and there exists a sequence $(t_n)_n$ with $t_n \to 0^+$ such that $\theta(T(t_n))_{t_n}$ forms a sequential bounded approximate identity for $\overline{B}$.

Since $\overline{B}$ is unital, the approximation identity must converge to e. Thus, there exists $\delta_1 > 0$ and $M_1 > 0$ such that:

$$\|\theta(T(t))\|_{\overline{B}} \leq M_1 \text{ for all } 0 < t < \delta_1$$

Consider the operator $S_t{:}A \to A$ defined by $S_t(x) = T(t)x$. For each fixed $x \in A$, by the continuity of the semigroup,

$$\lim_{t \to 0^+} S_t(x) = \lim_{t \to 0^+} T(t)x = x$$

Thus, for each $x \in A$, $\sup_{0<t<\delta_x} \|S_t(x)\| < +\infty$ for some $\delta_x > 0$.

By the uniform boundedness principle, there exists $\delta_2 > 0$ and $M_2 > 0$ such that:

$$\|S_t\| = \sup_{\|x\|<1} \|T(t)x\| \leq M_2 \text{ for all } 0 < t < \delta_2$$

Thus,

$$\|T(t)\| \leq M_2 \text{ for all } 0 < t < \delta_2$$

proving that the semigroup is bounded near the origin.

Existence of a unit. Let $\delta = min(\delta_1, \delta_2)$. For any sequence $t_n \to 0^+$ with $t_n < \delta$, the sequence $(T(t_n))_n$ is bounded by M_2. For any $x \in A$, we have:

$$\|(T(t_n) - T(t_m))x\| \leq \|T(t_n)(x - y)\| + \|(T(t_n) - T(t_m))y\| + \|T(t_m)(y - x)\|$$

where $y \in span\{T(s); s > 0\}$ approximates x.

Using boundedness and the continuity, we estimate:

$$\|T(t_n)(x-y)\| \leq M_2\|x-y\|$$

$$\|(T(t_n)-T(t_m))y\| \to 0 \text{ as } n, m \to \infty$$

$$\|T(t_m)(y-x)\| \leq M_2\|x-y\|$$

Thus, $(T(t_n)x)_n$ is a Cauchy sequence in A for each $x \in A$. By completeness of A, the limit:

$$g(x) = \lim_{n\to+\infty} T(t_n)x$$

exists. Moreover, for any $x \in A$:

$$g(x) = \lim_{t\to 0^+} T(t)x = x$$

by the continuity of the semigroup. Thus, g is the unit element of A, and $g = \lim_{t\to 0^+} T(t)$. □

Lemma 4. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, and let $A = \overline{span}\{T(t);\ t > 0\}$. Consider a continuous function k: $]0, +\infty \to [0, +\infty[$ satisfying the following:*

(1) $k(t) \geq 2$ *for all* $t \in]0, \delta_0[$ *(some* $\delta_0 > 0$*).*

(2) $k(t) = o\left(\frac{1}{t}\right)$ *as* $t \to 0^+ (\lim_{t\to 0^+} tk(t) = 0)$*.*

Then,

(1) *If A is unital, then* $\limsup_{t\to 0^+}\|T(tk(t)) - T(t)\| = 0$.

(2) *A is unital if and only if* $\limsup_{t\to 0^+}\|T(tk(t)) - T(t)\| < \frac{1}{4}$.

Proof. Assume that A is unital with identity e. Since the semigroup is continuous and A is unital, we have $\lim_{t\to 0^+} T(t) = e$; it follows that $\lim_{t\to 0^+} T(tk(t)) = e$, since $\lim_{t\to 0^+} tk(t) = 0$. Therefore,

$$\lim_{t\to 0^+}\|T(tk(t)) - T(t)\| = \lim_{t\to 0^+}\|T(tk(t)) - e + e - T(t)\| = 0,$$

which implies that

$$\limsup_{t\to 0^+}\|T(tk(t)) - T(t)\| = 0.$$

Now, assume that $\|T(tk(t)) - T(t)\| < \delta$, for $0 < t < t_0$ and $\delta \in]0, 1/4[$.

$$\begin{aligned}
&\dot{q}[\theta(T((2k(t)-2)t)) - \theta(T((k(t)-1)t))] \\
&= \dot{q}[\theta(T((k(t)-2)t))[\theta(T(tk(t))) - \theta(T(t))]] \\
&\leq \dot{q}[\theta(T((k(t)-2)t))].\dot{q}[\theta(T(tk(t))) - \theta(T(t))] \\
&= q[T((k(t)-2)t)].q[T(tk(t)) - T(t)] \\
&\leq |e^{\lambda(k(t)-2)t}|||T(tk(t)) - T(t)|| \\
&\leq \delta e^{\lambda(k(t)-2)t} \\
&\leq \delta e^{\lambda(k(t)-1)t}.
\end{aligned}$$

For all $s \in]0, (k(t_0)-1)t_0[$, there exists $t \in]0, t_0[$ such that $s = (k(t)-1)t$. Therefore,

$$\limsup_{t\to 0^+} \dot{q}[\theta(T(2s)) - \theta(T(s))] \leq \delta < \frac{1}{4}.$$

Therefore,

$$\limsup_{t\to 0^+} \dot{q}\Big[(\theta(T(s)))^2 - \theta(T(s))\Big] \leq \delta < \frac{1}{4}.$$

Now, define, for $s > 0$,

$$S(s) = \theta(T(s)) \in \bar{B}.$$

Then $S(s)_{s>0}$ is a continuous semigroup in $\bar{B}$, because

$$S(s)S(\mathrm{t}) = \theta(T(s))\theta(T(t)) = \theta(T(s)T(t)) = \theta(T(s+t)) = S(s+t).$$

The closed subalgebra generated by $S(s)_{s>0}$ is the closure of span$\{\theta(T(s)){:}s > 0\}$ in $\bar{B}$. Since $\theta(A)$ is dense in $\bar{B}$ and $A = \overline{\text{span}}\{T(t) : t > 0\}$, this closed subalgebra is all of $\bar{B}$.

For s in a neighborhood of 0^+. Since $S(s) = \theta(T(s))$, this is $\dot{q}\Big[(S(s))^2 - S(s)\Big] < \frac{1}{4}$.

Now observe that

$$(S(s))^2 - S(s) = \theta(T(s))^2 - \theta(T(s)) = \theta(T(2s)) - \theta(T(s)) = S(2s) - S(s).$$

Therefore,

$$\| S(2s) - S(s) \|_{\bar{B}} = \dot{q}[(S(s))^2 - S(s)] < \frac{1}{4}.$$

Taking the limit superior as $s \to 0^+$, we get

$$\limsup_{s\to 0^+} \| S(s) - S(2s) \|_{\bar{B}} \leq \delta < \frac{1}{4}.$$

We conclude that $\bar{B}$ is unital; this follows from [3] (p. 180). By Lemma 3, A is unital. □

3. Main Results

The main results of this paper are as follows:

Theorem 1. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, and let $\zeta :]0, +\infty[\to]0, +\infty[$ is a continuous function. If the algebra $A = \overline{span}\{T(t);\ t > 0\}$ is non-unital and if*

$$\limsup_{t\to 0^+} \zeta(t) > 0,$$

then

$$\limsup_{t\to 0^+} \|T(\zeta(t)) - T(t)\| > 0.$$

Theorem 2. *Let $(T(t))_{t>0}$ be a continuous semigroup on a commutative Banach algebra, and let $\xi :]0,+\infty[\to]0,+\infty[$ be a continuous function such that*

$$\liminf_{t\to 0^+} \frac{\xi(t)}{t} > 0$$

If the algebra $A = \overline{span}\{T(t);\ t > 0\}$ is non-unital, then

$$\limsup_{t\to 0^+} \|T(t + \xi(t)) - T(t)\| > 0$$

4. Main Results Proofs

In this section, we present detailed proofs related to the main results. We define the Banach algebra $\left(\overline{B}, \dot{q}\right)$ and the homomorphism θ as in Lemma 2.

Proof of Theorem 1. We proceed by contradiction. Assume that

$$\limsup_{t\to 0^+}\|T(\zeta(t))-T(t)\|=0,$$

which implies the following:

$$\lim_{t\to 0^+}\|T(\zeta(t))-T(t)\|=0. \tag{1}$$

Since $\limsup_{t\to 0^+}\zeta(t)>0$, there exists a sequence with $t_n\to 0^+$ and a constant $c>0$ such that $\zeta(t_n)\geq c$ for all n. By the Bolzano–Weierstrass property in extended real line, there exists a subsequence (still denoted $(t_n)_n$) such that $\zeta(t_n)$ converges to some $\acute{c}\in[c,\ +\infty]$. We consider two cases:

Case 1: $\acute{c}<+\infty$ ($\zeta(t_n)\to\ \acute{c}>0$).

By continuity of the semigroup for $t>0$:

$$\lim_{n\to+\infty}T(\zeta(t_n))=T(\acute{c}). \tag{2}$$

From (1), $\|T(\zeta(t_n))-T(t_n)\|\to 0$; therefore, by (2):

$$\lim_{n\to+\infty}T(t_n)=T(\acute{c}). \tag{3}$$

For any fixed $s>0$, we have the following:

$$T(s)T(t_n)=T(s+t_n)\overset{n\to+\infty}{\longrightarrow}T(s). \tag{4}$$

By (3) and the continuity of multiplication, we find that

$$T(s)T(t_n)\overset{n\to+\infty}{\longrightarrow}T(s)T(\acute{c}). \tag{5}$$

From (4) and (5), we conclude that $T(s)=T(s)T(\acute{c})$. By commutativity, $T(\acute{c})T(s)=T(s)$. Thus, $T(\acute{c})$ is an identity for $A=\overline{span}\{T(t);\ t>0\}$, contradicting the non-unitality of A.

Case 2: $\acute{c}=+\infty$ ($\zeta(t_n)\to+\infty$). We have two subcases:

(a) If A is radical.

Since A is radical, every element of A is quasinilpotent, By the Gelfand–Beurling formula, we conclude that: $\|T(\zeta(t_n))\|\to 0$ as $\zeta(t)\to+\infty$

Using the hypothesis

$$\|T(\zeta(t_n))-T(t_n)\|\to 0,$$

the triangle inequality gives

$$\|T(t_n)\|\leq\|T(t_n)-T(\zeta(t_n))\|+\|T(\zeta(t_n))\|\to 0.$$

Therefore,

$$\|T(t_n)\|\to 0.$$

In particular, $(T(t_n))$ is bounded.

By the Banach–Alaoglu theorem, there exists a subsequence (still denoted $(t_n)_n$) and an element $a\in A$ such that $T(t_n)\to a$ weakly. For fixed $s>0$:

$$T(s)T(t_n)=T(s+t_n)\overset{\text{norm}}{\longrightarrow}T(s),$$

$$T(s)T(t_n) \overset{weakly}{\longrightarrow} T(s)a.$$

Thus, $T(s)a = T(s)$. By density and continuity, a is identity for A, a contradiction.

(b) If A is not radical,

$\hat{A} \neq \varnothing$. For each $\phi \in \hat{A}$, $t \mapsto \phi(T(t))$ is a continuous semigroup; thus, $\phi(T(t)) = e^{t\lambda_\phi}$ for some $\lambda_\phi \in \mathbb{C}$ with $Re\lambda_\phi \leq 0$. From (1), we have

$$\left|e^{\zeta(t_n)\lambda_\phi} - e^{t_n\lambda_\phi}\right| \leq \|\phi\|.\|T(\zeta(t_n)) - T(t_n)\| \to 0.$$

Since $e^{t\lambda_\phi} \to 1$, we must have

$$e^{\zeta(t)\lambda_\phi} \to 1 \quad \text{as} t \to 0^+. \tag{6}$$

We now show this forces $\lambda_\phi = 0$.

If $\mathrm{Re}(\lambda_\phi) < 0$, then $e^{\zeta(t)\lambda_\phi} \to 0$, contradicting (6).

If $\lambda_\phi = i\beta$ with $\beta \neq 0$, then (6) becomes

$$e^{i\beta\zeta(t)} \to 1 \quad \text{as } t \to 0^+. \tag{7}$$

By continuity of ζ and the fact that $\zeta(t) \to +\infty$, the image of ζ on any interval $]0, \delta[$ contains some interval $[M, +\infty[$. Therefore, the function $t \mapsto e^{i\beta\zeta(t)}$ is a continuous function on $]0, \delta[$, whose range contains

$$\{e^{i\beta s}: s \in [M, +\infty[\ \} = \mathbb{T},$$

the unit circle. Such a function cannot converge to a single point 1 as $t \to 0^+$.

More explicitly, for each $n \in \mathbb{N}$, choose $s_n > M$ such that $e^{i\beta s_n} = -1$ (e.g., $s_n = \frac{(2k_n+1)\pi}{|\beta|}$ with k_n large enough so that $s_n > M$). Since $\zeta(t) \to +\infty$ and ζ is continuous, by the intermediate value theorem there exists $t_n \in]0, \delta_n[$ (with $\delta_n \to 0$) such that

$$\zeta(t_n) = s_n. \text{ Then } t_n \to 0^+ \text{ and } e^{i\beta\zeta(t_n)} = e^{i\beta s_n} = -1,$$

which contradicts (7), where the limit must be 1.

Thus, $\lambda_\phi = 0$ for all $\phi \in \hat{A}$; thus, $\phi(T(t)) = 1$ for all $t > 0$.

Fix $s > 0$; then, $\phi(T(s)) = 1$ for all $\phi \in \hat{A}$.

Now, consider the set

$$K = \left\{\phi \in \hat{A} : \phi(T(s)) = 1\right\}.$$

Since $\phi(T(s)) = 1$ for every $\phi \in \hat{A}$, we have $K = \hat{A}$.

We now verify that $K = \hat{A}$ is a compact open subset of $\hat{A}$.

1. Since $T(s) \in A$, its Gelfand transform $\widehat{T(s)}$ belongs to $C_0(\hat{A})$. But, by assumption,

$$\widehat{T(s)}(\phi) = \phi(T(s)) = 1 \qquad \text{for all } \phi \in \hat{A}.$$

Thus, the constant function 1 belongs to $C_0(\hat{A})$. We now recall that if the constant function 1 belongs to $C_0(\hat{A})$, then $\hat{A}$ must be compact. Indeed, by definition of $C_0(\hat{A})$, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \hat{A}$, such that

$$\left|\widehat{T(s)}(\phi)\right| < \varepsilon \qquad \text{for all } \phi \in \hat{A}\backslash K_\varepsilon.$$

Taking $\varepsilon = 1/2$ and using $\widehat{T(s)} = 1$, we obtain

$$1 < 1/2 \qquad \text{for all } \phi \in \hat{A} \backslash K_{1/2},$$

which is impossible unless $\hat{A} \backslash K_{1/2} = \varnothing$. Hence, $\hat{A} = K_{1/2}$ is compact. Therefore, $K = \hat{A}$ is compact.

2. The set $K = \hat{A}$ is trivially open in itself with respect to the relative Gelfand topology [12].

Thus, $K = \hat{A}$ is a compact open subset of $\hat{A}$.

By Shilov's idempotent theorem [13] (pp. 178–180), there exists an idempotent $q \in A$ such that $\phi(q) = 1$ for all $\phi \in \hat{A}$. For any $b \in A$, the Gelfand transform satisfies the following:

$$\widehat{bT(s) - b} = \hat{b} \cdot 1 - \hat{b} = 0,$$

Thus, $bT(s) - b \in rad(A)$. In particular, for $b = T(t)$ $(t > 0)$:

$$T(t)q - \mathrm{T}(t) \in rad(A).$$

Now, $T(t_n)$ is bounded. We define:

$$r_n = T(t)q - T(t) \in rad(A).$$

Suppose that $r_n \nrightarrow 0$ in the norm. Then, $\delta > 0$ and the subsequence $\left(r_{n_k}\right)_k$ with $\left\|r_{n_k}\right\| \geq \delta$ exist. By the Banach–Alaoglu theorem, we pass to subsequence $r_{n_j} \to r$ weakly for some $r \in rad(A)$. For fixed $s > 0$:

$$T(s)r_{n_j} = T(s)\left(T\left(t_{n_j}\right)q - T\left(t_{n_j}\right)\right) = T\left(s + t_{n_j}\right)q - T\left(s + t_{n_j}\right) \overset{weakly}{\longrightarrow} T(s)q - \ T(s)$$

$$T(s)r_{n_j} \overset{weakly}{\longrightarrow} T(s)r.$$

Thus, $T(s)r = T(s)q - T(s)$.

Consider the unitization $A_e = A \times \mathbb{C}$ with identity e. Let $v = r - (q - e) \in A_e$, for any $b \in A$:

$$bv = br - b(q - e) \in rad(A),$$

Both terms are in the radical. However,

$$T(s)v = T(s)r - T(s)(q - e) = (T(s)q - T(s)) - (T(s)q - T(s)) = 0.$$

By density, $bv = 0$ for all $b \in A$. Since A is essential in A_e, $v = 0$; thus, $r = q - e$. However, $q - e \notin A$ is a contradiction. Thus, $r_n \to 0$ in norm.

Finally, for a fixed $s > 0$ and a large n (so that $s > t_n$):

$$T(s)q - T(s) = T(s - t_n)(T(t_n)q - T(t_n)) = T(s - t_n)r_n.$$

As $n \to +\infty$. $\|T(s - t_n)\|$ remains bounded, and $\|r_n\| \to 0$; thus, $\|T(s)q - T(s)\| \to 0$.

Thus, $T(s)q = T(s)$ for all $s > 0$; thus, q is an identity for A, a contradiction.

All the cases lead to contradiction; therefore, $\limsup_{t \to 0^+} \|T(\zeta(t)) - T(t)\| > 0$. The proof is completed. □

Proof of Theorem 2. If $\limsup_{t \to 0^+} \xi(t) > 0$, then the result from Theorem 1 applies with $\zeta(t) = t + \xi(t)$.

Otherwise, we have $\limsup_{t\to 0^+}\zeta(t)=0$. Since $\zeta(t)=t+\xi(t)\geq 0$ and $t\to 0$, this implies $\limsup_{t\to 0^+}\xi(t)=0$. Let $\delta\in\left]0,\liminf_{t\to 0^+}\frac{\xi(t)}{t}\right[$, and let $n_0\in\mathbb{N}$ such that $\delta n_0\geq 1$. We have:

$$\begin{aligned}
&\dot{q}[\theta(T(t+n_0\xi(t)))-\theta(T(t))]=\\
&=\dot{q}[\theta(T(t+n_0\xi(t)))-\theta(T(t+(n_0-1)\xi(t)))+\theta(T(t+(n_0-1)\xi(t)))\\
&\quad-\theta(T(t+(n_0-2)\xi(t)))+\theta(T(t+(n_0-2)\xi(t)))+\cdots\\
&\quad+\theta(T(t+\xi(t)))-\theta(T(t))]\\
&\leq\dot{q}[\theta(T(t+n_0\xi(t)))-\theta(T(t+(n_0-1)\xi(t)))]\\
&\quad+\dot{q}[\theta(T(t+(n_0-1)\xi(t)))-\theta(T(t+(n_0-2)\xi(t)))]+\cdots\\
&\quad+\dot{q}[\theta(T(t+2\xi(t)))-\theta(T(t+\xi(t)))]\\
&\quad+\dot{q}[\theta(T(t+\xi(t)))-\theta(T(t))]\\
&\quad\leq\dot{q}[\theta(T(t+\xi(t)))-\theta(T(t))]\left[\dot{q}[\theta(T(n_0-1)\xi(t)))]\right.\\
&\quad\left.+\dot{q}[\theta(T(n_0-2)\xi(t))]+\cdots+1\right]\\
&=q[T(t+\xi(t))-T(t)]\{q[T(n_0-1)\xi(t)]+q[T(n_0-2)\xi(t)]+\cdots+1\}\\
&=\|T(t+\xi(t))-T(t)\|\{e^{\lambda(n_0-1)\xi(t)}+e^{\lambda(n_0-2)\xi(t)}+\cdots+1\}.
\end{aligned}$$

Let us define

$$k(t)=\frac{t+n_0\xi(t)}{t}=1+n_0\frac{\xi(t)}{t}\geq 1+n_0\delta.$$

We have $k(t)\geq 2$, since A is non-unital and $\overline{B}$ is also non-unital:

$$\limsup_{t\to 0^+}\dot{q}[\theta(T(tk(t)))-\theta(T(t))]\geq 1/4,$$

(according to Lemma 4 applied to $\overline{B}=\overline{span}\{\theta(T(t));\ t>0\}$).

Since $\lim_{t\to 0^+}\xi(t)=0$, we have $\limsup_{t\to 0^+}\|T(t+\xi(t))-T(t)\|\geq\frac{1}{4n_0}$. The proof is completed. □

5. Example: A Contraction Semigroup on the Disk Algebra $A\left(\overline{\mathbb{D}}\right)$

As an application, we provide the following example to determine the explicit form of $\limsup_{t\to 0^+}\|T(t+\xi(t))-T(t)\|$ within the disk algebra $A\left(\overline{\mathbb{D}}\right)$.

Example 1. *Consider the disk algebra* $A\left(\overline{\mathbb{D}}\right)=\left\{f\in C\left(\overline{\mathbb{D}}\right):f\text{ analytic on }\mathbb{D}\right\}$ *with the uniform norm* $\|f\|_\infty=\sup_{z\in\overline{\mathbb{D}}}|f(z)|$. *Let* $(T(t))_{t>0}$ *be a contraction semigroup defined as follows* [14,15]*:*

$$(T(t)f)(z)=f\left(e^{-t}z\right),\ f\in A\left(\overline{\mathbb{D}}\right),\ z\in\overline{\mathbb{D}}.$$

It is easily verified that the disk algebra is a commutative Banach algebra with identity [16]*. Given* $\xi(t)>0$ *with* $\lim_{t\to 0^+}\frac{\xi(t)}{t}=\gamma>0$ *finite, we have the following:*

$$\|T(t+\xi(t))-T(t)\|=\sup_{\|f\|_\infty\leq 1}\sup_{z\in\overline{\mathbb{D}}}\left|f\left(e^{-(t+\xi(t))}z\right)-f\left(e^{-t}z\right)\right|.$$

We substitute $w=e^{-t}z$*:*

$$\|T(t+\xi(t))-T(t)\|=\sup_{\|f\|_\infty\leq 1}\sup_{|w|<e^{-t}}\left|f\left(e^{-\xi(t)}w\right)-f(w)\right|.$$

The supremum is attained for monomials $f_m(w)=w^m$ $(m\in\mathbb{N})$, $\|f_m\|_\infty=1$*:*

$$|f_m(e^{-\xi(t)}w) - f_m(w)| = |w|^m |e^{-m\xi(t)} - 1|.$$

Thus,

$$\sup_{|w|<e^{-t}} |w|^m \left|e^{-m\xi(t)} - 1\right| = e^{-mt}\left(1 - e^{-m\xi(t)}\right),$$

We aim to determine

$$\sup_{m\geq 0} g(m)$$

for the function

$$g(m) = e^{-mt}(1 - e^{-m\xi(t)}).$$

We first find the derivative $g'(m)$. To simplify the computations, set

$$a = t > 0, \qquad b = \xi(t) > 0, \qquad h(t) = \frac{b}{a} = \frac{\xi(t)}{t} > 0.$$

Then,

$$g(m) = e^{-am}(1 - e^{-bm}), \qquad m \geq 0.$$

Hence,

$$g'(m) = -ae^{-am}(1 - e^{-bm}) + be^{-am}e^{-bm}.$$

Therefore,

$$\begin{aligned} g'(m) &= -ae^{-am} + ae^{-am}e^{-bm} + be^{-am}e^{-bm} \\ &= e^{-am}\left(-a + (a+b)e^{-bm}\right) \end{aligned}$$

Setting $g'(m) = 0$, we obtain the critical equation

$$-a + (a+b)e^{-bm} = 0,$$

Hence,

$$e^{-bm} = \frac{a}{a+b}.$$

Since $h = b/a$, we also have

$$\frac{a}{a+b} = \frac{1}{1+b/a} = \frac{1}{1+h(t)}.$$

Therefore,

$$e^{-bm_{max}} = \frac{1}{1+h(t)}.$$

Taking logarithms,

$$-bm_{max} = ln\left(\frac{1}{1+h(t)}\right) = -ln(1+h(t)),$$

so

$$m_{max} = \frac{ln(1+h(t))}{b} = \frac{ln(1+h(t))}{h(t)t}$$

Verify that m_{max} is a global maximum. for $m < m_{max}$,

$$e^{-bm} > e^{-bm_{max}} = \frac{a}{a+b},$$

hence

$$(a+b)e^{-bm} > a, \quad so \quad B(m) = -a + (a+b)e^{-bm} > 0,$$

which gives $g'(m) > 0$. *Thus* g *is strictly increasing on* $]0, m_{max}[$.

For $m > m_{max}$,

$$e^{-bm} < e^{-bm_{max}} = \frac{a}{a+b},$$

hence

$$(a+b)e^{-bm} < a, \quad so \quad B(m) = -a + (a+b)e^{-bm} < 0,$$

which gives $g'(m) < 0$. *Thus* g *is strictly decreasing on* $]m_{max}, +\infty[$.

Moreover,

$$g(0) = 0, \qquad \lim_{m\to+\infty} g(m) = \lim_{m\to+\infty}\left(e^{-am} - e^{-(a+b)m}\right) = 0.$$

Therefore, g *rises from* 0, *attains its maximum uniquely at* m_{max}, *and decreases back to* 0. *Hence,* m_{max} *is the unique global maximum.*

Evaluate $g(m_{max})$. *From the critical equation, we obtain:*

$$e^{-bm_{max}} = \frac{a}{a+b} = \frac{1}{1+h(t)}.$$

Therefore,

$$1 - e^{-bm_{max}} = 1 - \frac{1}{1+h} = \frac{h}{1+h}.$$

Also,

$$e^{-am_{max}} = e^{-\frac{a}{b}ln(1+h)} = e^{-\frac{1}{h}ln(1+h)} = (1+h)^{-1/h}.$$

Multiplying the two factors:

$$g(m_{max}) = e^{-am_{max}}\left(1 - e^{-bm_{max}}\right) = (1+h)^{-1/h} \cdot \frac{h}{1+h}.$$

Thus,

$$\sup_{m\geq 0} g(m) = h(t)\left(1+h(t)\right)^{-1-\frac{1}{h(t)}}.$$

As $t \to 0^+$, $h(t) \to \gamma$, *and by continuity, we have:*

$$\limsup_{t\to 0^+} \|T(t+\xi(t)) - T(t)\| = \frac{\gamma}{(1+\gamma)^{1+\frac{1}{\gamma}}}$$

Let $\mathcal{H} = \overline{span}\{T(t) : t > 0\}$ *be the closed subalgebra of* $A\left(\overline{\mathbb{D}}\right)$ *generated by the semigroup.*

The linear span of $\{T(t)$: $t > 0\}$ *is, by definition, the set of all finite linear combinations of semigroup elements:*

$$span\{T(t) : t > 0\} = \left\{\sum_{k=1}^{m} c_k\, T(t_k) \;\middle|\; m \in \mathbb{N},\ c_k \in \mathbb{C},\ t_k > 0\right\}$$

Since $\mathcal{H}$ *is the norm-closure of span*$\{T(t)$: $t > 0\}$, *an element* $x \in \mathcal{H}$ *if and only if there exists a sequence* $(x_n)_{n\geq 1}$, *such that*

$$x_n \in span\{T(t)\text{: } t > 0\} \qquad for\ every\ n,$$

and

$$\|x_n - x\| \to 0 \qquad (n \to \infty).$$

Now suppose, for contradiction, that $\mathcal{H}$ *is unital, and let* $e \in \mathcal{H}$ *be its unit.*

Since $e \in \mathcal{H}$ *and* $\mathcal{H}$ *is the closure of the span of the semigroup elements, there exists a sequence* $(S_n)_{n\geq 1}$ *of finite linear combinations of the form*

$$S_n = \sum_{k=1}^{m_n} c_{k,n} T(a_{k,n}), \qquad c_{k,n} \in \mathbb{C},\ a_{k,n} > 0,$$

such that

$$\|S_n - e\| \to 0 \qquad \text{as } n \to \infty.$$

The operator norm is:

$$\|S_n - e\| = \sup_{\substack{f \in A(\overline{\mathbb{D}}) \\ f \neq 0}} \frac{\| S_n f - ef \|_\infty}{\| f \|_\infty}.$$

Hence, for every $f \in A\left(\overline{\mathbb{D}}\right)$,

$$\| S_n f - ef \|_\infty \leq \|S_n - e\| \, \| f \|_\infty.$$

Since $\| S_n - e \| \to 0$ *and* $\|f\|_\infty < +\infty$, *we obtain*

$$\| S_n f - ef \|_\infty \to 0 \qquad (n \to \infty).$$

In particular, choosing:

$$f = f_m(z) = z^m \in A\left(\overline{\mathbb{D}}\right) \text{for each } m \in \mathbb{N}.$$

Then, $\|f_m\|_\infty = 1$ *and, by definition of the semigroup,*

$$T(a) f_m(z) = f_m(e^{-a} z) = e^{-ma} z^m.$$

Therefore,

$$S_n f_m(z) = \sum_{k=1}^{m_n} c_{k,n} e^{-m a_{k,n}} z^m = c_n(m) z^m,$$

where

$$c_n(m) := \sum_{k=1}^{m_n} c_{k,n} e^{-m a_{k,n}}.$$

Since e is the unit of $\mathcal{H}$, *we have* $ef_m = f_m$. *Hence,*

$$\|S_n f_m - f_m\|_\infty = \|(c_n(m) - 1) z^m\|_\infty = |\, c_n(m) - 1 \,|.$$

But $\|S_n - e\| \to 0$ *implies*

$$\|S_n f_m - e f_m\|_\infty \leq \|S_n - e\| \cdot \|f_m\|_\infty \to 0.$$

Therefore, for each fixed $m \in \mathbb{N}$,

$$c_n(m) \to 1 \qquad \text{as } n \to \infty. \tag{8}$$

On the other hand, for each fixed n, since $a_{k,n} > 0$ for all k, we have

$$e^{-ma_{k,n}} \to 0 \qquad \text{as } m \to \infty.$$

Thus,

$$\lim_{m\to\infty} c_n(m) = \sum_{k=1}^{m_n} c_{k,n} \lim_{m\to\infty} e^{-ma_{k,n}} = 0. \tag{9}$$

Combining (2) with (1), we see that for each fixed n, the sequence $(c_n(m))_{m\geq 1}$ converges to 0 as $m \to \infty$, while it must converge to 1 as $n \to \infty$ for each fixed m. In particular, for each n, there exists m such that

$$|c_n(m) - 1| \geq \frac{1}{2}.$$

Hence,

$$\| S_n - e \| \geq \sup_m \| S_n f_m - e f_m \|_\infty = \sup_m |c_n(m) - 1| \geq \frac{1}{2}.$$

This contradicts the assumption that $\|S_n - e\| \to 0$. Therefore, $\mathcal{H}$ cannot be unital.

6. Conclusions

This study establishes a rigorous bridge between the local asymptotic behavior of the distance between elements of a continuous semigroup near the origin and the global algebraic structure of the closed Banach algebra it generates. Our main result provides a sharp quantitative criterion for detecting unitality, relying exclusively on the semigroup's infinitesimal behavior under a general time perturbation $\xi(t)$. Through a continuous approach that is completely indifferent to the choice of parametrization, this criterion broadly generalizes the classical theorems on asymptotic behavior due to Esterle, Mokhtari, and Kalton. Moving beyond their confinement to fixed numerical ratios in selected Banach algebras, this new approach applies to all commutative algebras and accommodates a much wider spectrum of perturbations.

The proof of this result relies on a sophisticated synthesis of three powerful functional-analytic tools: Gelfand representation theory, which translates norm estimates into pointwise evaluations on the character space; Shilov's idempotent theorem, which decomposes the spectrum into components responsible for the asymptotic growth; and an original extension of Feller's regularization technique. The Feller construction is particularly vital, as it permits handling semigroups that are unbounded near the origin. We adapt Feller's regularization to continuous semigroups on commutative Banach algebras. Given a possibly unbounded semigroup on such an algebra, we build a new Banach algebra as the completion of a quotient. In this new algebra, the regularized semigroup is bounded near the origin. The quotient construction preserves the algebraic structure: the induced map on spectra is a homeomorphism, and if the new algebra is unital, then the original algebra is unital and the original semigroup is bounded near the origin. Thus, unitality is faithfully detected in the regularized quotient, which is exactly the purpose of Feller's technique in this setting.

Sharpness is demonstrated explicitly on the disk algebra $A(\overline{\mathbb{D}})$, where we obtain a sharp asymptotic constant. This constant matches the classical optimal constants established by Esterle and Mokhtari in the discrete setting, thereby confirming the optimality and coherence of our approach.

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