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Characterizations of Pseudo-Symmetric Space–Times in Gray's Subspaces and $f(R)$-Gravity Vacuum Solutions

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Abstract

This paper investigates pseudo-symmetric space–times within two interrelated frameworks: vacuum $f(R)$-gravity and Gray's seven canonical decomposition subspaces. First, it is established that any conformally flat pseudo-symmetric space–time satisfying the vacuum field equations of $f(R)$-gravity necessarily corresponds to a perfect fluid. Subsequently, a detailed analysis of Gray's subspaces reveals the following structural results: In the trivial and $\mathcal{A}$ subspaces, pseudo-symmetric space–times are Ricci-simple and Weyl-harmonic, and thus are necessarily generalized Robertson–Walker space–times. In the $\mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ subspaces, the associated time-like vector field ξ^l is shown to be an eigenvector of the Ricci tensor with the eigenvalue $-R/2$. Furthermore, for a perfect fluid pseudo-symmetric space–time obeying $f(R)$-gravity and belonging to the trivial, $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{A} \oplus \mathcal{B}$ subspaces, the isotropic pressure p and energy density σ are proven to be constants. Additionally, it is demonstrated that Gray's $\mathcal{I}$ subspace reduces to the $\mathcal{B}$ subspace in the pseudo-symmetric setting. Finally, under specific geometric conditions, pseudo-symmetric space–times in the $\mathcal{I} \oplus \mathcal{A}$ and $\mathcal{I} \oplus \mathcal{B}$ subspaces are also shown to admit perfect fluid representations. These results collectively clarify the geometric and physical constraints imposed by pseudo-symmetry within $f(R)$-gravity and Gray's classification scheme.

Keywords: pseudo-symmetric space-times; the vacuum solution of $f(R)$-gravity; Gray's decomposition subspaces

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1. Introduction

In physics, an n-dimensional space–time M^n is modeled as a Lorentzian manifold endowed with a Lorentzian metric g. The metric g has the signature $(-,+,\ldots,+)$ or $(+,-,\ldots,-)$, that is, one time-like direction and $(n-1)$ space-like directions. This structure ensures the existence of a globally defined time-orientation, allowing one to distinguish consistently between future- and past-directed time-like vectors throughout M^n.

A manifold M is called a pseudo-symmetric manifold $(\mathrm{PS})_n$ if its Riemann curvature tensor R_{ijkl} satisfies [1]

$$\nabla_h R_{ijkl} = 2\xi_h R_{ijkl} + \xi_i R_{hjkl} + \xi_j R_{ihkl} + \xi_k R_{ijhl} + \xi_l R_{ijkh}, \tag{1}$$

where ξ_i is a nonzero 1-form, referred to as the associated vector of $(\mathrm{PS})_n$. If ξ_i vanishes identically in a neighborhood of a point, then (1) reduces to $\nabla_h R_{ijkl} = 0$, and the manifold is locally symmetric in the sense of Cartan [2]. This identity expresses that the covariant derivative of the Riemann curvature tensor is entirely determined by a vector field ξ, in the sense that the rate of change of R_{ijkl} in any direction is given by a linear combination of curvature components weighted by the components of ξ. This condition imposes a strong geometric constraint on the manifold, indicating that the departure of the curvature tensor from parallelism is controlled by ξ. This condition also represents a natural extension of several classical symmetry requirements in Riemannian geometry, encompassing locally symmetric spaces as well as a variety of weaker curvature–symmetry structures. It is worth noting that the term pseudo-symmetry appears in the literature with different meanings. In the present work, we exclusively consider the notion of pseudo-symmetry introduced by M. C. Chaki. This concept is distinct from the notion of pseudo-symmetry defined by R. Deszcz [3]. Throughout this paper, the term pseudo-symmetric manifold is used solely in the sense of Chaki.

Pseudo-symmetric manifolds have been extensively studied from different geometric viewpoints. Numerous contributions—including [4–8] among many others—demonstrate the mathematical richness and structural significance of pseudo-symmetry within Riemannian and semi-Riemannian geometry.

A Lorentzian manifold is characterized as a pseudo-symmetric space–time whenever its Riemann curvature tensor meets the condition given in Equation (1). Throughout this study, the vector field associated with the pseudo-symmetric structure is assumed to be a unit time-like vector field, that is,

$$\xi_i \xi^i = -1.$$

A natural motivation for this choice is that taking the associated vector field to be unit time-like provides a canonical normalization compatible with the causal structure of Lorentzian geometry. In particular, such a vector field may be interpreted as the four-velocity of an observer or a physically meaningful flow in space–time, thereby removing arbitrary scaling freedom and facilitating the geometric and physical interpretation of the curvature condition.

Transvecting Equation (1) with g^{il} yields the covariant derivative of the Ricci tensor,

$$\nabla_h R_{jk} = 2\xi_h R_{jk} + \xi^l R_{hjkl} + \xi_j R_{hk} + \xi_k R_{jh} + \xi^i R_{ijkh}. \tag{2}$$

This relation provides a structural description of how the Ricci tensor varies along the space–time under the influence of a vector field ξ. It indicates that the covariant derivative of R_{jk} is governed not only by the Ricci tensor itself but also by specific contractions of the Riemann tensor with ξ, thereby linking first-order Ricci variations to the full curvature structure. This identity shows that the deviation of the Ricci tensor from parallelism is controlled by ξ, in a manner consistent with the broader pseudo-symmetric curvature condition. As such, it represents a curvature restriction that parallels and complements the defining equation of pseudo-symmetric manifolds, reflecting how the geometry is constrained by the interplay between ξ and the curvature tensors.

A further contraction with g^{jk} gives the covariant derivative of the scalar curvature,

$$\nabla_h R = 2\xi_h R + 4\xi^l R_{hl}. \tag{3}$$

A pseudo-symmetric space–time is said to represent a perfect fluid space–time (PFS) when its Ricci tensor takes the form

$$R_{ij} = a\, g_{ij} + b\, \xi_i \xi_j,$$

where a and b denote smooth scalar fields [9–11]. Under this condition, the corresponding energy–momentum tensor assumes the classical structure

$$T_{ij} = (p + \sigma)\, \xi_i \xi_j + p\, g_{ij},$$

with p representing the isotropic pressure and σ the energy density [12]. A functional relation between the pressure p and the energy density σ is prescribed by the equation of state (EoS) [13], which specifies a law of the form $p = p(\sigma)$. Within this framework, the perfect fluid is described as isentropic, meaning that its thermodynamic evolution is governed by a single state variable [14]. Different choices of this dependence correspond to distinct cosmological regimes: $\sigma = -3p$ describes the quintessence phase, $p = 0$ characterizes dust, $\sigma = 3p$ represents radiation, and $\sigma = -p$ corresponds to dark energy [14].

The Weyl conformal curvature tensor is given by [12]

$$C_{ijkh} = R_{ijkh} + \frac{g_{ih}R_{jk} - g_{jh}R_{ik} + g_{jk}R_{ih} - g_{ik}R_{jh}}{n-2} - R\,\frac{g_{ih}g_{jk} - g_{jh}g_{ik}}{(n-2)(n-1)}.$$

Its divergence is

$$\nabla^j C_{ijkh} = -\frac{n-3}{n-2}\left[\nabla_i R_{kh} - \nabla_k R_{ih} - \frac{g_{kh}\nabla_i R - g_{ih}\nabla_k R}{2(n-1)}\right]. \tag{4}$$

A space–time is conformally flat if $C_{ijkh} = 0$, and it is said to be Weyl-harmonic if $\nabla^j C_{ijkh} = 0$.

In 1978, Alfred Gray introduced and studied seven decomposition subspaces [15], which are classified according to the properties of the covariant derivative of the Ricci tensor (see also [16], Ch. 16). These decomposition subspaces are denoted as trivial, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \oplus \mathcal{B}$, $\mathcal{I}$, $\mathcal{I} \oplus \mathcal{A}$, and $\mathcal{I} \oplus \mathcal{B}$. A manifold M belongs to the trivial subspace if it is Ricci-symmetric, that is,

$$\nabla_i R_{hk} = 0.$$

The subspace $\mathcal{A}$ consists of manifolds with a Killing–Ricci tensor, satisfying

$$\nabla_h R_{jk} + \nabla_j R_{hk} + \nabla_k R_{jh} = 0.$$

Gray's subspace $\mathcal{B}$ contains manifolds whose Ricci tensor is of Codazzi type, that is,

$$\nabla_h R_{jk} = \nabla_k R_{jh}.$$

Manifolds with constant scalar curvature are classified within the $\mathcal{A} \oplus \mathcal{B}$ subspace, that is,

$$\nabla_i R = 0.$$

The subspace $\mathcal{I}$ consists of manifolds whose Ricci tensor satisfies

$$\begin{aligned}\nabla_h R_{jk} \;=\; & \frac{n\,\nabla_h R}{(n-1)(n-2)}\, g_{jk} + \frac{(n-2)\,\nabla_k R}{2(n-1)(n+2)}\, g_{jh} \\ & + \frac{(n-2)\,\nabla_j R}{2(n-1)(n+2)}\, g_{kh}.\end{aligned}$$

Manifolds belonging to Gray's $\mathcal{I}$ subspace are referred to as Sinyukov manifolds [17]. In $\mathcal{I} \oplus \mathcal{A}$ subspace, the Ricci tensor is conformal-Killing, that is,

$$\nabla_h R_{jk} + \nabla_k R_{hj} + \nabla_j R_{hk} = \frac{2\nabla_h R}{(n+2)} g_{jk} + \frac{2\nabla_k R}{(n+2)} g_{hj} + \frac{2\nabla_j R}{(n+2)} g_{kh}.$$

Finally, the $\mathcal{I} \oplus \mathcal{B}$ subspace contains Weyl-harmonic manifolds, that is

$$\begin{aligned} \nabla^j C_{ijkh} &= 0, \\ \nabla_i R_{kh} - \nabla_k R_{ih} - \frac{g_{kh}\nabla_i R - g_{ih}\nabla_k R}{2(n-1)} &= 0. \end{aligned}$$

In recent years, extensive work has been devoted to studying various classes of manifolds within Gray's seven decomposition subspaces, yielding several notable results; see, for example, [18–21], among many others.

The $f(R)$–gravity theory constitutes a significant extension of General Relativity, obtained by replacing the linear dependence on the scalar curvature in the Einstein–Hilbert action with a general differentiable function $f(R)$. This modification leads to altered gravitational field equations, which in turn produce new geometric and physical effects at both cosmological and local scales. Such theories can account for the accelerated expansion of the universe without invoking dark energy and provide a geometric framework for exploring phenomena such as black holes, wormholes, and possible alternatives to dark matter. Owing to its broad implications and its capacity to modify the underlying geometric structure of space–time, $f(R)$–gravity remains an active and influential area of research (see [22–24]).

The theory of $f(R)$–gravity admits two principal variational formulations: the metric formalism and the Palatini formalism. In general, these frameworks yield distinct sets of field equations, agreeing only in the special linear case that reproduces General Relativity. A dynamical equivalence between them may be established by introducing an auxiliary scalar field together with a divergence-free current; with an appropriate specification of this scalar field, one obtains an equivalent unified description, as demonstrated in [25].

In what follows, we adopt the classical metric formulation. In this setting, the field equations of $f(R)$–gravity take the form [26]

$$\begin{aligned} \kappa T_{ij} &= f'(R)\, R_{ij} - f'''(R)\, \nabla_i R \nabla_j R - f''(R)\, \nabla_i \nabla_j R \\ &\quad + g_{ij}\left[f'''(R)\, \nabla_k R \nabla^k R + f''(R)\, \nabla^2 R - \frac{1}{2} f(R) \right], \end{aligned} \tag{5}$$

where κ is the gravitational coupling constant, and a prime denotes differentiation with respect to the scalar curvature R.

In the setting of $f(R)$–gravity, the term vacuum solutions designates space–time configurations in which matter fields are absent, that is, $T_{ij} = 0$. To investigate the geometric properties inherent to $f(R)$–gravity, it is natural to analyze such vacuum configurations in an n-dimensional framework. In [27], Capozziello et al. examined vacuum solutions by prescribing particular forms of the function $f(R)$ or by imposing symmetry constraints on the metric. Subsequently, ref. [24] established that, in vacuum $f(R)$–gravity, the Ricci tensor must necessarily exhibit the structure of a perfect fluid. In [28], space–times possessing a Codazzi-type Z–tensor were analyzed within the context of vacuum $f(R)$–gravity, and among several conclusions it was demonstrated that pseudo Z-symmetric space–times form a distinguished class of perfect fluid geometries. Furthermore, ref. [29] investigated the non-linear differential equations satisfied by $f'(R)$ in spherically symmetric vacuum solutions and obtained explicit solutions for various choices of the function $f(R)$.

The pseudo-symmetric condition (1) is imposed in the framework of $f(R)$-gravity for both geometric and physical reasons. From a geometric viewpoint, Chaki's notion of pseudo-symmetry constitutes a controlled relaxation of local symmetry, whereby the covariant derivative of the Riemann curvature tensor is governed by a vector field rather than vanishing identically. This condition restricts the variation of curvature while allowing a broader class of non-symmetric space–times. From a physical perspective, in Lorentzian geometry the associated unit time-like vector field naturally represents the four-velocity of an observer or fluid flow, rendering the pseudo-symmetric condition compatible with relativistic space–time models. In the context of $f(R)$-gravity, where higher-order curvature effects play a fundamental role, pseudo-symmetry serves as an effective curvature constraint that yields analytically manageable models and leads naturally to physically relevant space–times, including perfect fluid and generalized Robertson–Walker geometries. The present work is distinct from existing studies on related curvature conditions. In particular, pseudo-Ricci symmetric space–times, as studied in [21], impose restrictions only on the covariant derivative of the Ricci tensor, whereas pseudo-symmetry involves the full Riemann curvature tensor and represents a stronger condition; moreover, ref. [21] is confined to General Relativity and does not address $f(R)$-gravity. Likewise, ref. [30] investigates pseudo-Ricci symmetric generalized Robertson–Walker space–times within Einstein gravity, while the present work neither assumes pseudo-Ricci symmetry nor restricts the geometry to the GRW class. Instead, GRW space–times arise here as a consequence of pseudo-symmetry and Gray's decomposition in vacuum $f(R)$-gravity.

The primary objective of this paper is to examine pseudo-symmetric space–times within the framework of vacuum solutions of $f(R)$-gravity, and to systematically explore their relationship with Gray's seven canonical decomposition subspaces. Our investigation proceeds in two principal stages. First, in Section 2, we analyze the structure of conformally flat pseudo-symmetric space–times that arise from the vacuum field equations of $f(R)$-gravity. We derive necessary geometric conditions on the associated vector field ξ_i and establish the conditions under which such space–times reduce to perfect fluid configurations. Second, in Section 3, we conduct a detailed study of pseudo-symmetric space–times lying within each of Gray's seven subspaces—namely the trivial, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \oplus \mathcal{B}$, $\mathcal{I}$, $\mathcal{I} \oplus \mathcal{A}$, and $\mathcal{I} \oplus \mathcal{B}$ subspaces. For each class, we derive explicit forms of the Ricci tensor, characterize the behavior of the associated vector field, and determine when these space–times correspond to perfect fluid models. This dual approach allows us to unify the geometric constraints imposed by pseudo-symmetry with the physical framework of $f(R)$-gravity, offering new insights into the mathematical structure and cosmological relevance of such space–times. In General Relativity, a spacetime is usually taken to be four-dimensional, consisting of three spatial coordinates and one temporal coordinate. However, in this work we consider an n-dimensional spacetime ($n \geq 4$), which arises naturally in several physical and geometrical contexts. Higher-dimensional spacetimes are commonly used, for example, in Kaluza–Klein theory, string theory, and other higher-dimensional gravitational models. Our results are therefore formulated in the general n-dimensional setting, with the four-dimensional case appearing as a particular instance.

2. Conformally Flat Pseudo-Symmetric Space–Times in the Vacuum Solution of $f(R)$-Gravity

In this section, we establish characterizations of conformally flat pseudo-symmetric space–times arising in the vacuum solutions of $f(R)$-gravity. In the absence of matter, i.e., when $T_{ij} = 0$, the field equations of $f(R)$-gravity given by Equation (5) reduce to

$$R_{ij} = \eta_1 \nabla_i R \, \nabla_j R + \eta_2 \nabla_i \nabla_j R - \eta_3 g_{ij}, \tag{6}$$

where the scalar functions η_1, η_2, and η_3 are defined by

$$\eta_1 = \frac{f'''(R)}{f'(R)}, \qquad \eta_2 = \frac{f''(R)}{f'(R)}, \tag{7}$$

$$\eta_3 = \eta_1 \nabla_k R \nabla^k R + \eta_2 \nabla^2 R - \frac{f(R)}{2f'(R)}. \tag{8}$$

The derivatives of η_1, η_2, and η_3 are given by

$$\eta_1' = \frac{f^{(4)} f' - f'' f'''}{(f')^2}, \qquad \eta_2' = \frac{f''' f' - (f'')^2}{(f')^2}, \qquad \eta_3' = -\frac{f' + f\,\eta_2}{2(n-1)f'}. \tag{9}$$

It follows immediately that

$$\eta_1 - \eta_2' = \eta_2^2. \tag{10}$$

Multiplying Equation (6) by g^{ij} and using Equation (8), we obtain

$$\eta_3 = \frac{f - 2f'R}{2(n-1)f'}. \tag{11}$$

If $C_{ijkh} = 0$, then

$$\begin{aligned} R_{ijkh} = {} & \frac{1}{n-2}\Big(g_{jh}R_{ik} + g_{ik}R_{jh} - g_{ih}R_{jk} - g_{jk}R_{ih}\Big) \\ & + \frac{R}{(n-1)(n-2)}\Big(g_{ih}g_{jk} - g_{jh}g_{ik}\Big). \end{aligned} \tag{12}$$

Contracting Equation (12) with ξ^i yields

$$\begin{aligned} \xi^i R_{ijkh} = {} & \frac{1}{n-2}\Big(g_{jh}\,\xi^i R_{ik} + \xi_k R_{jh} - \xi_h R_{jk} - g_{jk}\,\xi^i R_{ih}\Big) \\ & + \frac{R}{(n-1)(n-2)}\Big(\xi_h g_{jk} - \xi_k g_{jh}\Big). \end{aligned} \tag{13}$$

Contracting (13) with ξ^h, we obtain

$$\begin{aligned} \xi^h R_{ijkh} = {} & \frac{1}{n-2}\Big(\xi_j R_{ik} + g_{ik}\,\xi^h R_{jh} - \xi_i R_{jk} - g_{jk}\,\xi^h R_{ih}\Big) \\ & + \frac{R}{(n-1)(n-2)}\Big(\xi_i g_{jk} - \xi_j g_{ik}\Big). \end{aligned} \tag{14}$$

Using Equations (13) and (14) in Equation (2), we obtain

$$\begin{aligned} \nabla_h R_{jk} = {} & \frac{n-1}{n-2}\Big(\xi_j R_{hk} + \xi_k R_{jh}\Big) \\ & + \frac{1}{n-2}\Big(g_{hk}\,\xi^l R_{jl} + g_{jh}\,\xi^i R_{ik} - 2g_{jk}\,\xi^l R_{hl}\Big) \\ & + \frac{2n-6}{n-2}\,\xi_h R_{jk} + \frac{R}{(n-1)(n-2)}\Big(2\xi_h g_{jk} - \xi_j g_{hk} - \xi_k g_{jh}\Big). \end{aligned} \tag{15}$$

Contracting Equation (15) with g^{jk} yields

$$\nabla_h R = 2\xi_h R. \tag{16}$$

Substituting Equation (16) into Equation (3), we obtain

$$\xi^l R_{hl} = 0. \tag{17}$$

Theorem 1. *In a conformally flat pseudo-symmetric space–time, ξ^l is an eigenvector of the Ricci tensor with an eigenvalue of zero.*

Applying the operator ∇_k to Equation (16), we obtain

$$\nabla_k \nabla_h R = 2R\,\nabla_k \xi_h + 4\xi_h \xi_k R. \tag{18}$$

Interchanging the indices h and k yields

$$\nabla_h \nabla_k R = 2R\,\nabla_h \xi_k + 4\xi_k \xi_h R. \tag{19}$$

Subtracting Equation (19) from Equation (18), we conclude

$$(\nabla_k \nabla_h - \nabla_h \nabla_k)R = 2R(\nabla_k \xi_h - \nabla_h \xi_k).$$

It is well known that $(\nabla_k \nabla_h - \nabla_h \nabla_k)R = 0$, and therefore

$$2R(\nabla_k \xi_h - \nabla_h \xi_k) = 0.$$

Since $R \neq 0$, it follows that

$$\nabla_k \xi_h - \nabla_h \xi_k = 0, \tag{20}$$

which shows that ξ_h is irrotational. Contracting with ξ^h, we deduce

$$\dot{\xi}_k = \xi^h \nabla_h \xi_k = 0, \tag{21}$$

indicating that ξ_h is acceleration-free. The vorticity tensor μ_{kh} is defined by [31]:

$$\mu_{kh} = \frac{1}{2}(\nabla_k \xi_h - \nabla_h \xi_k) + \frac{1}{2}\dot{\xi}_h \xi_k + \xi_h \dot{\xi}_k. \tag{22}$$

Using Equations (20) and (21) in (22), we obtain

$$\mu_{kh} = 0,$$

which shows that ξ_h is vorticity-free. Since ξ_h is both acceleration-free and vorticity-free, its covariant derivative can be written as [24]

$$\nabla_k \xi_h = \psi(g_{kh} + \xi_k \xi_h) + \sigma_{kh}, \tag{23}$$

where ψ is a scalar field and σ_{kh} is the shear tensor, which is symmetric, traceless, and satisfies $\xi^h \sigma_{kh} = 0$. Accordingly, we have the following result:

Theorem 2. *The associated vector field ξ_h of a conformally flat pseudo-symmetric space–time is irrotational, acceleration-free, and vorticity-free, and its covariant derivative can be expressed as in Equation (23).*

Using Equation (16) in Equation (6), we obtain

$$R_{ij} = 4(\eta_1 R + \eta_2)\xi_i \xi_j R + 2\eta_2 R \nabla_i \xi_j - \eta_3 g_{ij}. \tag{24}$$

Employing Equation (23) in Equation (24), one infers

$$R_{ij} = 2[2\eta_1 R + \eta_2(2+\psi)]\xi_i\xi_j R + [2\eta_2\psi R - \eta_3]g_{ij} + 2\eta_2 R\sigma_{ij}. \tag{25}$$

Contracting the above equation with ξ^i, and using Equation (17), we find

$$2[2\eta_1 R + \eta_2(2+\psi)]R = 2\eta_2\psi R - \eta_3. \tag{26}$$

Transvecting Equation (25) with g^{ij}, we acquire

$$2[2\eta_1 R + \eta_2(2+\psi)]R = n(2\eta_2\psi R - \eta_3) - R. \tag{27}$$

Solving Equations (26) and (27), we obtain

$$2\eta_2\psi R - \eta_3 = \frac{R}{n-1}, \tag{28}$$

$$2\eta_1 R + \eta_2(2+\psi) = \frac{1}{2(n-1)}. \tag{29}$$

Inserting Equations (28) and (29) into Equation (25), we obtain

$$R_{ij} = \frac{R}{n-1}(g_{ij} + \xi_i\xi_j) + 2\eta_2 R\,\sigma_{ij}. \tag{30}$$

Theorem 3. *A conformally flat* $(\mathrm{PS})_n$ *space–time satisfying the vacuum solution of* $f(R)$*-gravity is perfect fluid, provided that* $\sigma_{ij} = 0$.

In the subsequent analysis, our aim is to demonstrate that $\sigma_{ij} = 0$. Differentiating Equation (6), one obtains

$$\begin{aligned}\nabla_h R_{ij} &= \eta_1'(\nabla_h R)(\nabla_i R)(\nabla_j R) + \eta_1(\nabla_j R)(\nabla_h\nabla_i R) + \eta_1(\nabla_i R)(\nabla_h\nabla_j R)\\ &\quad +\eta_2'(\nabla_h R)(\nabla_i\nabla_j R) + \eta_2(\nabla_h\nabla_i\nabla_j R) - \eta_3' g_{ij}\nabla_h R.\end{aligned} \tag{31}$$

Using Equation (16) in Equation (31), we obtain

$$\begin{aligned}\nabla_h R_{ij} &= 8\eta_1'\,\xi_h\xi_i\xi_j R^3 + 16\eta_1\,\xi_j\xi_i\xi_h R^2 + 8\eta_2'\,\xi_h\xi_j\xi_i R^2 + 8\eta_2\,\xi_j\xi_i\xi_h R\\ &\quad +4\eta_1\,\xi_j R^2\nabla_h\xi_i + 4\eta_1\,\xi_i R^2\nabla_h\xi_j + 4\eta_2'\,\xi_h R^2\nabla_i\xi_j + 4\eta_2\,\xi_h R\nabla_i\xi_j\\ &\quad +2\eta_2 R\nabla_h\nabla_i\xi_j + 4\eta_2\,\xi_j R\nabla_h\xi_i + 4\eta_2\,\xi_i R\nabla_h\xi_j - 2\eta_3' g_{ij}\xi_h R.\end{aligned}$$

By swapping the indices h and i, we find

$$\begin{aligned}\nabla_i R_{hj} &= 8\eta_1'\,\xi_i\xi_h\xi_j R^3 + 16\eta_1\,\xi_j\xi_h\xi_i R^2 + 8\eta_2'\,\xi_i\xi_j\xi_h R^2 + 8\eta_2\,\xi_j\xi_h\xi_i R\\ &\quad +4\eta_1\,\xi_j R^2\nabla_i\xi_h + 4\eta_1\,\xi_h R^2\nabla_i\xi_j + 4\eta_2'\,\xi_i R^2\nabla_h\xi_j + 4\eta_2\,\xi_i R\nabla_h\xi_j\\ &\quad +2\eta_2 R\nabla_i\nabla_h\xi_j + 4\eta_2\,\xi_j R\nabla_i\xi_h + 4\eta_2\,\xi_h R\nabla_i\xi_j - 2\eta_3' g_{hj}\xi_i R.\end{aligned}$$

The last two equations are subtracted to give

$$\begin{aligned}\nabla_h R_{ij} - \nabla_i R_{hj} &= 4\eta_1\,\xi_j R^2(\nabla_h\xi_i - \nabla_i\xi_h) + 4\eta_1 R^2(\xi_i\nabla_h\xi_j - \xi_h\nabla_i\xi_j)\\ &\quad +4\eta_2'R^2(\xi_h\nabla_i\xi_j - \xi_i\nabla_h\xi_j) + 2\eta_2 R(\nabla_h\nabla_i\xi_j - \nabla_i\nabla_h\xi_j)\\ &\quad +4\eta_2\,\xi_j R(\nabla_h\xi_i - \nabla_i\xi_h) + 2\eta_3' R\Big(g_{hj}\xi_i - g_{ij}\xi_h\Big).\end{aligned} \tag{32}$$

Utilizing Equation (20) in Equation (32), we obtain

$$\begin{aligned}\nabla_h R_{ij} - \nabla_i R_{hj} &= 4R^2(\eta_1 - \eta_2')(\xi_i \nabla_h \xi_j - \xi_h \nabla_i \xi_j) \\ &\quad + 2\eta_2 R(\nabla_h \nabla_i \xi_j - \nabla_i \nabla_h \xi_j) + 2\eta_3' R\Big(g_{hj}\xi_i - g_{ij}\xi_h\Big).\end{aligned}$$

Making use of Equation (10) and the identity

$$\nabla_h \nabla_i \xi_j - \nabla_i \nabla_h \xi_j = \xi^m R_{hijm},$$

we finally arrive at

$$\nabla_h R_{ij} - \nabla_i R_{hj} = 4R^2\eta_2^2(\xi_i \nabla_h \xi_j - \xi_h \nabla_i \xi_j) + 2\eta_2 R\,\xi^m R_{hijm} + 2\eta_3' R\Big(g_{hj}\xi_i - g_{ij}\xi_h\Big). \quad (33)$$

Employing Equations (14) and (15) in Equation (33), with the help of Equation (17), one uncovers

$$\begin{aligned}4R^2\eta_2^2(\xi_i \nabla_h \xi_j - \xi_h \nabla_i \xi_j) &= \frac{5 - n - 2\eta_2 R}{n-2}\Big(\xi_i R_{hj} - \xi_h R_{ij}\Big) \\ &\quad + \left[\frac{(3 - 2\eta_2 R)R}{(n-1)(n-2)} + 2\eta_3' R\right]\Big(\xi_h g_{ij} - \xi_i g_{hj}\Big).\end{aligned}$$

Transvecting with ξ^h and using Equations (17), (21) and (23), we obtain

$$\begin{aligned}R_{ij} &= \frac{R}{5 - n - 2\eta_2 R}\left[\frac{3 - 2\eta_2 R}{n-1} + 2(n-2)\left(\eta_3' + 2\psi R\eta_2^2\right)\right](g_{ij} + \xi_i \xi_j) \\ &\quad + \frac{4(n-2)}{5 - n - 2\eta_2 R} R^2 \eta_2^2 \sigma_{ij}. \end{aligned} \quad (34)$$

Using Equation (11) in Equation (28) yields

$$\eta_2 \psi R = \frac{f}{4(n-1)f'}. \quad (35)$$

Employing Equations (9) and (35) in Equation (34), we obtain the simplified form

$$R_{ij} = \frac{R}{n-1}(g_{ij} + \xi_i \xi_j) + \frac{4(n-2)R^2\eta_2^2}{5 - n - 2\eta_2 R}\sigma_{ij}. \quad (36)$$

Comparing Equations (30) and (36), it follows that

$$\left[2(n-1)R\eta_2 - 5 + n\right]\eta_2 R\,\sigma_{ij} = 0.$$

If

$$\eta_2 \neq \frac{n-5}{-2(n-1)R},$$

then necessarily

$$\sigma_{ij} = 0.$$

Consequently, Equation (36) reduces to

$$R_{ij} = \frac{R}{n-1} g_{ij} + \frac{R}{n-1}\xi_i \xi_j, \quad (37)$$

which shows that the space–time is a perfect fluid. Thus, we can state the following result:

Theorem 4. *A conformally flat $(PS)_n$ space–time satisfying the vacuum field equation of $f(R)$–gravity is necessarily a perfect fluid space–time.*

Remark 1. *Using Equation (7), namely $\eta_2 = \dfrac{f''(R)}{f'(R)}$, the equality case*

$$\eta_2 = -\frac{n-5}{2(n-1)R}$$

yields the differential equation

$$\frac{f''(R)}{f'(R)} = -\frac{n-5}{2(n-1)}\,\frac{1}{R}.$$

Equivalently,

$$\frac{d}{dR}\left(\ln f'(R)\right) = -\frac{n-5}{2(n-1)}\,\frac{1}{R}.$$

Integrating, we obtain

$$f'(R) = C_1\, R^{-\frac{n-5}{2(n-1)}},$$

where C_1 is a constant. A further integration gives

$$f(R) = C_2\, R^{\frac{n+3}{2(n-1)}} + C_3,$$

with C_2 and C_3 constants.

This equality therefore corresponds to a special power-law form of the function $f(R)$. In this case, the coefficient multiplying the shear tensor term in the Ricci decomposition vanishes, and the argument leading to $\sigma_{ij} = 0$ no longer applies. Consequently, this represents a degenerate branch of solutions for which the perfect fluid characterization cannot be ensured.

3. Pseudo-Symmetric Space–Times Under Gray's Decomposition Subspaces

In this section, our objective is to investigate pseudo-symmetric space–times within the seven subspaces of Gray's decomposition.

The trivial subspace: For a pseudo-symmetric space–time belonging to this subspace, the Ricci tensor satisfies

$$\nabla_h R_{jk} = 0, \tag{38}$$

which, after contraction with g^{jk}, immediately yields

$$\nabla_h R = 0. \tag{39}$$

Combining Equations (38) and (2), we obtain

$$2\xi_h R_{jk} + \xi^l R_{hjkl} + \xi_j R_{hk} + \xi_k R_{jh} + \xi^i R_{ijkh} = 0. \tag{40}$$

Contracting Equation (40) with g^{jk}, one uncovers

$$\xi^l R_{hl} = -\frac{1}{2}\xi_h R. \tag{41}$$

Transvecting Equation (40) with ξ^j, and using Equation (41), we obtain

$$R_{hk} = -\frac{3}{2}\,\xi_h \xi_k R + \xi^l \xi^j R_{hjkl}. \tag{42}$$

Similarly, contracting Equation (40) with ξ^h and employing Equation (41), we infer

$$R_{jk} = \xi^l \xi^h R_{hjkl} - \frac{1}{2}\xi_j \xi_k R. \tag{43}$$

In view of Equations (42) and (43), we conclude that

$$R_{jk} = -\xi_j \xi_k R. \tag{44}$$

This shows that the $(\mathrm{PS})_n$ space–time is Ricci-simple.

From Equations (38) and (39) substituted into Equation (4), we reveal that

$$\nabla^i C_{kijh} = 0,$$

which shows that the $(\mathrm{PS})_n$ space–time is Weyl-harmonic.

Gray's subspace $\mathcal{A}$: In this subspace, the Ricci tensor of a $(\mathrm{PS})_n$ space–time is Killing, that is,

$$\nabla_h R_{jk} + \nabla_j R_{hk} + \nabla_k R_{jh} = 0. \tag{45}$$

Contracting this identity leads to

$$\nabla_h R = 0. \tag{46}$$

Combining Equations (2) and (45), we obtain

$$4\xi_h R_{jk} + \xi^l R_{kjhl} + 4\xi_j R_{hk} + 4\xi_k R_{jh} + \xi^i R_{ihkj} = 0. \tag{47}$$

Transvecting Equation (47) with g^{jk}, we obtain

$$\xi^k R_{hk} = -\frac{1}{2}\,\xi_h R. \tag{48}$$

Next, contracting Equation (47) with ξ^j and using Equation (48), it follows that

$$R_{hk} = -\xi_h \xi_k R, \tag{49}$$

showing that the $(\mathrm{PS})_n$ space–time is Ricci-simple.

Finally, transvecting Equation (49) with ξ^k and applying Equation (48) yields

$$R = 0. \tag{50}$$

Applying Equation (46) in Equation (4) gives

$$\nabla^i C_{kijh} = -\frac{n-3}{n-2}\Big[\nabla_k R_{jh} - \nabla_j R_{kh}\Big].$$

Using Equation (49), this becomes

$$\nabla^i C_{kijh} = -\frac{n-3}{n-2}\big[\nabla_j(\xi_h \xi_k R) - \nabla_k(\xi_h \xi_j R)\big].$$

Finally, by employing Equation (50), we deduce

$$\nabla^i C_{kijh} = 0,$$

which shows that the $(\mathrm{PS})_n$ space–time is Weyl-harmonic.

Accordingly, we can state the following result:

Theorem 5. *A pseudo-symmetric space–time belonging to the trivial and $\mathcal{A}$ subspaces is Ricci-simple and Weyl-harmonic.*

A Ricci-simple space–time satisfying $\nabla^i C_{kijh} = 0$ necessarily admits a proper concircular vector field. As a consequence, such a space–time can be identified as a generalized Robertson–Walker (GRW) space–time [32].

Hence, we can state the following:

Corollary 1. *A pseudo-symmetric space–time in the trivial and $\mathcal{A}$ subspaces is a GRW space–time.*

The subspace $\mathcal{B}$: The Ricci tensor of a $(\mathrm{PS})_n$ space–time in this subspace satisfies

$$\nabla_h R_{jk} = \nabla_k R_{jh}. \tag{51}$$

A contraction yields

$$\nabla_h R = 0.$$

Consequently, from Equation (3), we obtain

$$\xi^l R_{hl} = -\frac{R}{2}\xi_h. \tag{52}$$

Employing Equation (2) in Equation (51), we deduce

$$\xi_h R_{jk} - \xi_k R_{jh} + \xi^l \left(R_{hjkl} - R_{kjhl} \right) + 2\xi^i R_{ijkh} = 0. \tag{53}$$

Now, contracting Equation (53) with ξ^h and using Equation (52), we obtain

$$R_{jk} = \frac{1}{2}\xi_k \xi_j R + 3\,\xi^i \xi^h R_{ijkh}. \tag{54}$$

Again, contracting Equation (53) with ξ^j and using Equation (52), one infers

$$\xi^l \xi^j \left(R_{hjkl} - R_{kjhl} \right) = 0. \tag{55}$$

Theorem 6. *Let M be a pseudo-symmetric space–time in Gray's subspace $\mathcal{B}$. Then:*

1. *ξ^l is an eigenvector of the Ricci tensor with eigenvalue $-\frac{R}{2}$.*
2. *The Ricci tensor is given by*

$$R_{jk} = \frac{1}{2}\,\xi_j \xi_k R + 3\,\xi^i \xi^h R_{ijkh}.$$

3. *The Riemann tensor satisfies*

$$\xi^l \xi^j \left(R_{hjkl} - R_{kjhl} \right) = 0.$$

The subspace $\mathcal{A} \oplus \mathcal{B}$: A $(\mathrm{PS})_n$ space–time in this subspace is characterized by having constant scalar curvature, i.e.,

$$\nabla_h R = 0.$$

Consequently, from Equation (3), we obtain

$$\xi^l R_{hl} = -\frac{R}{2}\,\xi_h.$$

Theorem 7. *For a pseudo-symmetric space–time in Gray's subspace $\mathcal{A} \oplus \mathcal{B}$, the vector field ξ^l is an eigenvector of the Ricci tensor with corresponding eigenvalue $-\frac{R}{2}$.*

Since the scalar curvature R is constant in the above four subspaces, the $f(R)$-gravity field equations reduce to

$$R_{ij} - \frac{f}{2f'}g_{ij} = \frac{\kappa}{f'}T_{ij}.$$

For a perfect fluid space–time, the above equation becomes

$$R_{ij} - \frac{f}{2f'}g_{ij} = \frac{\kappa}{f'}\Big[(p+\sigma)\xi_i\xi_j + pg_{ij}\Big].$$

Two contractions with g^{ij} and ξ^j yield

$$R - \frac{nf}{2f'} = \frac{\kappa}{f'}\big[(n-1)p - \sigma\big], \tag{56}$$

$$\sigma = \frac{Rf' + f}{2\kappa}. \tag{57}$$

Substituting Equation (57) into Equation (56) gives

$$p = \frac{3Rf' - (n-1)f}{2(n-1)\kappa}. \tag{58}$$

Hence, we can state:

Theorem 8. *The isotropic pressure p and the energy density σ for a perfect fluid pseudo-symmetric space–time obeying $f(R)$-gravity in the trivial, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \oplus \mathcal{B}$ subspaces are constants, given respectively by Equations (57) and (58).*

In virtue of Equations (57) and (58), we obtain

$$\frac{p}{\sigma} = \frac{3Rf' - (n-1)f}{(n-1)(Rf' + f)}.$$

For $n = 4$, this reduces to

$$\frac{p}{\sigma} = \frac{Rf' - f}{Rf' + f}.$$

Thus, we can state:

Remark 2. *Based on the varying stages of cosmic development in the universe, a pseudo-symmetric space–time obeying $f(R)$-gravity in the trivial, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \oplus \mathcal{B}$ subspaces exhibits different ratios of pressure to energy density, corresponding to different cosmological phases as in Table 1.*

Table 1. Equations of state in pseudo-symmetric space–times.

Space–Time	Equation of State (EoS)	$f(R)$
Quintessence era	$\sigma + 3p = 0$	$aR^{\frac{1}{2}}$ for some constant a
Dust matter era	$p = 0$	aR for some constant a
Radiation era	$\sigma - 3p = 0$	aR^2 for some constant a
Dark energy era	$\sigma + p = 0$	Constant

The subspace $\mathcal{I}$: The covariant derivative of the Ricci tensor of a pseudo-symmetric space–time in this subspace satisfies

$$\nabla_h R_{jk} = \frac{n\nabla_h R}{(n-1)(n-2)} g_{jk} + \frac{(n-2)\nabla_k R}{2(n-1)(n+2)} g_{jh} + \frac{(n-2)\nabla_j R}{2(n-1)(n+2)} g_{kh}. \tag{59}$$

This condition implies that

$$\nabla_k R_{jh} - \nabla_j R_{kh} - \frac{g_{jh}\nabla_k R - g_{kh}\nabla_j R}{2(n-1)} = 0. \tag{60}$$

Inserting Equation (2) into Equation (59), we have

$$2\xi_h R_{jk} + \xi^l R_{hjkl} + \xi_j R_{hk} + \xi_k R_{jh} + \xi^i R_{ijkh} = \frac{n\nabla_h R}{(n-1)(n-2)} g_{jk} + \frac{(n-2)\nabla_k R}{2(n-1)(n+2)} g_{jh} + \frac{(n-2)\nabla_j R}{2(n-1)(n+2)} g_{kh}.$$

Transvecting with g^{jk}, we obtain

$$2\xi_h R + 4\xi^l R_{hl} = \frac{n^2(n+2) + (n-2)^2}{(n-1)(n-2)(n+2)} \nabla_h R.$$

Using Equation (3), we find

$$\xi^l R_{hl} = -\frac{R}{2}\xi_h.$$

Substituting this result into Equation (3), it follows that

$$\nabla_h R = 0.$$

Finally, inserting this outcome into Equation (60) gives

$$\nabla_k R_{jh} = \nabla_j R_{kh}.$$

Hence, we can state

Theorem 9. *For a pseudo-symmetric space–time, the subspace $\mathcal{I}$ reduces to the subspace $\mathcal{B}$.*

The subspace $\mathcal{I} \oplus \mathcal{A}$: The Ricci tensor of a pseudo-symmetric space–time in this subspace is conformal-Killing, that is,

$$\nabla_h R_{jk} + \nabla_k R_{hj} + \nabla_j R_{hk} = \frac{2\nabla_h R}{n+2} g_{jk} + \frac{2\nabla_k R}{n+2} g_{hj} + \frac{2\nabla_j R}{n+2} g_{kh}. \tag{61}$$

Merging Equations (2) and (61), we obtain

$$4\xi_h R_{jk} + \xi^l R_{kjhl} + 4\xi_j R_{hk} + 4\xi_k R_{jh} + \xi^i R_{ihkj} = \frac{2\nabla_h R}{n+2} g_{jk} + \frac{2\nabla_k R}{n+2} g_{hj} + \frac{2\nabla_j R}{n+2} g_{kh}.$$

Multiplying by ξ^h and using Equation (3), we get

$$R_{jk} = \frac{R - 2\xi^h \xi^l R_{hl}}{n+2} g_{jk} - \frac{2R}{n+2}\xi_j \xi_k + \frac{n\,\xi^l \left(\xi_j R_{kl} + \xi_k R_{jl}\right)}{n+2}. \tag{62}$$

Thus, we have

Theorem 10. *The Ricci tensor of a pseudo-symmetric space–time under Gray's subspace $\mathcal{I} \oplus \mathcal{A}$ is given by Equation (62).*

If $\xi^l R_{hl} = 0$, then

$$R_{jk} = \frac{R}{n+2} g_{jk} - \frac{2R}{n+2} \xi_j \xi_k.$$

Hence, we get

Corollary 2. *A pseudo-symmetric space–time in Gray's subspace $\mathcal{I} \oplus \mathcal{A}$ is perfect fluid, provided $\xi^l R_{hl} = 0$.*

The subspace $\mathcal{I} \oplus \mathcal{B}$: A pseudo-symmetric space–time in this subspace is Weyl-harmonic, that is,

$$\nabla_h R_{jk} - \nabla_j R_{kh} = \frac{1}{2(n-1)} \Big[g_{jk} \nabla_h R - g_{kh} \nabla_j R \Big]. \tag{63}$$

Employing Equations (2) and (3) in Equation (63), one acquires

$$\xi_h R_{jk} - \xi_j R_{hk} + 2\xi^l R_{hjkl} + \xi^i \Big(R_{ijkh} - R_{ihkj} \Big) = \frac{1}{n-1} \Big[(\xi_h R + 2\xi^l R_{hl}) g_{jk} - (\xi_j R + 2\xi^l R_{jl}) g_{kh} \Big]. \tag{64}$$

Transvecting with ξ^k yields

$$\xi^k \xi^i \Big(R_{ijkh} - R_{ihkj} \Big) = \frac{n+1}{n-1} \xi^k \Big[\xi_j R_{hk} - \xi_h R_{jk} \Big]. \tag{65}$$

Contracting Equation (64) with ξ^j gives

$$R_{hk} = \frac{R - 2\xi^l \xi^j R_{jl}}{n-1} g_{kh} + \frac{R}{n-1} \xi_h \xi_k + \frac{\xi^l [2\xi_k R_{hl} - (n-1)\xi_h R_{lk}]}{n-1} - 3\, \xi^l \xi^j R_{hjkl}. \tag{66}$$

Thus, we can conclude

Theorem 11. *The Ricci tensor of a pseudo-symmetric space–time in Gray's subspace $\mathcal{I} \oplus \mathcal{B}$ is given by Equation (66). Furthermore, the Riemann tensor satisfies Equation (65).*

If

$$\xi^l \xi^j R_{hjkl} = \frac{\xi^l [2\xi_k R_{hl} - (n-1)\xi_h R_{lk}]}{3(n-1)}, \tag{67}$$

then

$$R_{hk} = \frac{R - 2\xi^l \xi^j R_{jl}}{n-1} g_{kh} + \frac{R}{n-1} \xi_h \xi_k.$$

Thus, we have

Corollary 3. *A pseudo-symmetric space–time in Gray's subspace $\mathcal{I} \oplus \mathcal{B}$ is perfect fluid if the Riemann tensor obeys Equation (67).*

The following table, see Table 2, summarizes the results of Gray's decomposition.

Table 2. Summary of results for pseudo-symmetric space–times in Gray's decomposition subspaces

Gray's Subspace	Defining Condition	Main Results for Pseudo-Symmetric Space–Times
Trivial	$\nabla_h R_{jk} = 0$	Ricci tensor is Ricci-simple; space–time is Weyl-harmonic and hence a generalized Robertson–Walker (GRW) space–time.
A	$\nabla_h R_{jk} + \nabla_j R_{hk} + \nabla_k R_{jh} = 0$	Ricci tensor is Ricci-simple with vanishing scalar curvature; space–time is Weyl-harmonic and GRW.
B	$\nabla_h R_{jk} = \nabla_k R_{jh}$	The associated vector field ξ^l is an eigenvector of the Ricci tensor with eigenvalue $-\frac{R}{2}$.
$A \oplus B$	$\nabla_h R = 0$	Scalar curvature is constant; ξ^l is an eigenvector of the Ricci tensor with eigenvalue $-\frac{R}{2}$.
I	Sinyukov condition	The subspace I reduces to the subspace B in the pseudo-symmetric setting.
$I \oplus A$	Conformal Killing–Ricci tensor	Ricci tensor admits a specific decomposition; space–time is perfect fluid under the condition $\xi^l R_{hl} = 0$.
$I \oplus B$	Weyl harmonic	Ricci tensor admits an explicit form; space–time becomes perfect fluid under an additional curvature constraint.

Remark 3. *From a physical viewpoint, the geometric restrictions derived in this work have direct implications for properties of space–times in extended gravity theories. The emergence of perfect fluid configurations constrains the energy density and isotropic pressure, thereby fixing the equation of state and determining the corresponding cosmological phase. The reduction to generalized Robertson–Walker space–times in certain Gray subspaces reflects large-scale homogeneity and isotropy, which are key features of cosmological models consistent with observational data. Moreover, conditions such as the vanishing of shear, vorticity, and acceleration of the associated time-like vector field are closely related to the kinematical behavior of cosmological flows and influence expansion dynamics. In the context of $f(R)$-gravity, these geometric properties restrict admissible curvature evolutions and affect quantities such as the effective gravitational coupling and cosmological expansion rate, thereby linking the underlying curvature structure to physically measurable phenomena.*

4. Conclusions

A systematic investigation has been conducted into the geometric and physical nature of pseudo-symmetric space–times within vacuum $f(R)$-gravity, particularly through the lens of Gray's seven decomposition subspaces. The analysis yields significant constraints that govern the structure and physical properties of these space–times.

First, we have demonstrated that conformally flat pseudo-symmetric space–times satisfying the vacuum field equations of $f(R)$-gravity are necessarily perfect fluid space–times (Theorem 4). Moreover, the associated time-like vector field ξ_i is shown to be irrotational, acceleration-free, and vorticity-free, with its covariant derivative admitting a shear-based decomposition (Theorem 2).

Within Gray's classification, we have established that pseudo-symmetric space–times belonging to the trivial and $\mathcal{A}$ subspaces are Ricci-simple and Weyl-harmonic (Theorem 5), and consequently, they are generalized Robertson–Walker space–times (Corollary 1). For space–times in the $\mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ subspaces, we derived explicit expressions for the Ricci tensor and established that ξ_i is an eigenvector of the Ricci tensor with eigenvalue $-R/2$ (Theorems 6 and 7).

Furthermore, we have shown that in the trivial, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \oplus \mathcal{B}$ subspaces, where the scalar curvature is constant, the isotropic pressure p and energy density σ of a per-

fect fluid pseudo-symmetric space–time in $f(R)$-gravity are constants, given explicitly by Equations (57) and (58). The ratio p/σ depends on the functional form of $f(R)$ and corresponds to different cosmological eras, as illustrated in Table 1.

We also proved that the $\mathcal{I}$ subspace reduces to the $\mathcal{B}$ subspace for pseudo-symmetric space–times (Theorem 9), while in the $\mathcal{I} \oplus \mathcal{A}$ and $\mathcal{I} \oplus \mathcal{B}$ subspaces, the Ricci tensor admits specific decompositions, with conditions under which the space–time becomes perfect fluid (Theorems 10 and 11, Corollaries 2 and 3).

In summary, this work provides a comprehensive characterization of pseudo-symmetric space–times in the context of $f(R)$-gravity vacuum solutions and Gray's decomposition, clarifying their geometric nature, physical interpretation, and relevance to cosmological models. The results underscore the rich interplay between pseudo-symmetry, $f(R)$-gravity, and the canonical subspaces of Gray, offering a unified perspective on the structure of space–times in extended theories of gravity.

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