

Article

Generalized Derivations in Rings and Their Applications to Banach Algebra

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Abstract

Let $\mathfrak{R}$ be a prime ring, and let F denote a generalized derivation associated with a derivation d of $\mathfrak{R}$. Consider I as a nonzero ideal of $\mathfrak{R}$, and let m, n, k, l be fixed positive integers. In this study, we explore the behavior of the generalized derivation F within the structures of both prime and semiprime rings that satisfy the functional identity $[F(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_l \eta^k$, $\forall \eta, \tau \in I$. Furthermore, we extend this investigation to the framework of Banach algebras, analyzing how generalized derivations operate in such algebras. A comparative discussion is also presented to highlight the distinctions and similarities in the behavior of generalized derivations within Banach algebraic settings under the above structural condition.

Keywords: prime ring; semiprime ring; ideal; derivation; generalized derivation; Banach algebras

MSC: 16W20; 16W25; 16U70; 16U80; 16N60

1. Introduction

Let $\mathfrak{R}$ be an associative ring, and denote its center by $Z(\mathfrak{R})$. The Martindale quotient ring of $\mathfrak{R}$ will be written as Q , while U will represent the Utumi quotient ring associated with $\mathfrak{R}$. The center of the Utumi quotient ring, usually written as $C = Z(U)$, is known in ring theory as the extended centroid of $\mathfrak{R}$. This construction plays an important role in the structural analysis of noncommutative rings, especially when extending scalars or studying derivations and identities.

A ring $\mathfrak{R}$ is called prime if it does not allow zero products in a nontrivial way: whenever elements $\eta, \tau \in \mathfrak{R}$ satisfy $\eta\mathfrak{R}\tau = 0$, then at least one of the elements must be zero (i.e., $\eta = 0$ or $\tau = 0$). This condition ensures that the ring behaves analogously to an integral domain from the viewpoint of its multiplicative structure, though without requiring commutativity.

Likewise, $\mathfrak{R}$ is said to be semiprime if it has no nonzero nilpotent ideals. Equivalently, if an element $\eta \in \mathfrak{R}$ satisfies $\eta\mathfrak{R}\eta = 0$, then this forces $\eta = 0$. Thus, a semiprime ring is one in which self-annihilating elements cannot exist unless they are already zero, making it a central object when studying radicals and decomposition theorems.



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Let A be a Banach algebra, and denote by $\text{rad}(A)$ the Jacobson radical of A , defined as the intersection of all primitive ideals of A . If $\text{rad}(A) = 0$, the algebra A is said to be semisimple. A Banach algebra A that lacks a unity element can be embedded into a unital Banach algebra $\tilde{A} = A \oplus \mathbb{C}$ as an ideal of codimension one. Under the isometric isomorphism

$$\eta \mapsto (\eta, 0),$$

we may identify A with the ideal $(\eta, 0) \mid \eta \in A \subset \tilde{A}$.

An additive mapping $d : \mathfrak{R} \rightarrow \mathfrak{R}$ is called a derivation if it satisfies

$$d(\eta\tau) = d(\eta)\tau + \eta d(\tau), \quad \forall \eta, \tau \in \mathfrak{R}.$$

Following the definition introduced by Brešar [1], an additive mapping $F : \mathfrak{R} \rightarrow \mathfrak{R}$ is termed a generalized derivation if there exists a derivation $d : \mathfrak{R} \rightarrow \mathfrak{R}$ such that

$$F(\eta\tau) = F(\eta)\tau + \eta d(\tau), \quad \forall \eta, \tau \in \mathfrak{R}.$$

Generalized derivations have been extensively investigated, particularly in the context of operator algebras. Building upon these notions, the concept of generalized derivation can thus be formalized as an extension of the classical derivation, preserving a similar functional identity but incorporating the influence of an associated derivation. Additionally, derivations, generalized derivations, and commuting maps are studied in the field of alternative rings and alternative algebras (see [2,3]).

Example 1. Let $\mathbb{Z}$ denote the ring of integers, and consider

$$\mathfrak{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

For any nonzero element $b \in \mathbb{Z}$, we have

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mathfrak{R} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0),$$

showing that $\mathfrak{R}$ is not semiprime.

Now define

$$I = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\},$$

which is easily verified to be a two-sided ideal of $\mathfrak{R}$.

Define maps $F, d : \mathfrak{R} \rightarrow \mathfrak{R}$ by

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a - c \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that I is indeed an ideal of $\mathfrak{R}$, and that $\mathfrak{R}$ is a generalized derivation on $\mathfrak{R}$ associated with the derivation d .

In [4], Lee et al. established that any generalized derivation on a prime ring $\mathfrak{R}$ can be uniquely extended to a generalized derivation on its Utumi quotient ring U . Consequently, all generalized derivations of $\mathfrak{R}$ are henceforth assumed to act on the entirety of U . Specifically, Lee demonstrated that every generalized derivation F defined on a dense right ideal of $\mathfrak{R}$ can be uniquely extended to U in the form

$$F(\eta) = a\eta + d(\eta),$$

where $a \in U$ and d is a derivation on U (see Theorem 3 of [5]).

In [6], Singer and Wermer proved that for a commutative Banach algebra, every continuous derivation maps into the Jacobson radical of the algebra. They also conjectured that the assumption of continuity might be unnecessary, a conjecture later confirmed by Thomas in [7]. However, this result does not hold for noncommutative Banach algebras, prompting efforts to extend the theorem to that setting. In this direction, Sinclair [8] showed that any continuous derivation of a Banach algebra leaves its primitive ideals invariant, inspiring numerous subsequent studies that removed or generalized the commutativity restriction.

Let S be a nonempty subset of $\mathfrak{R}$. A mapping $F : \mathfrak{R} \rightarrow \mathfrak{R}$ is said to be commutativity preserving on S if

$$[\eta, \tau] = 0 \implies [F(\eta), F(\tau)] = 0, \quad \forall \eta, \tau \in S.$$

Moreover, F is said to be strongly commutativity preserving (SCP) on S if

$$[F(\eta), F(\tau)] = [\eta, \tau], \quad \forall \eta, \tau \in S.$$

Bell and Daif [9] were among the first to study SCP derivations on ideals of semiprime rings. Brešar later extended this work to Lie ideals in [10]. Further developments were made by Ma and Xu [11], who examined generalized derivations satisfying SCP conditions, while Koç and Gölbaşı [12] generalized these ideas to multiplicative generalized derivations on semiprime rings. In [13], Ali et al. demonstrated that if $\mathfrak{R}$ is a semiprime ring and f is an endomorphism that is SCP on a nonzero ideal U of $\mathfrak{R}$, then f must be commuting on U . Similarly, Samman [14] proved that an epimorphism of a semiprime ring is SCP if and only if it is centralizing.

Research on derivations and SCP mappings has since been actively pursued across operator algebras, prime rings, and semiprime rings. Motivated by these studies, Huang [15] showed that if $\mathfrak{R}$ is a prime ring, I a nonzero ideal, and d a nontrivial derivation satisfying

$$[d(\eta), d(\tau)]_m = [\eta, \tau]^n, \quad \forall \eta, \tau \in I,$$

for some fixed positive integers m, n , then $\mathfrak{R}$ must be commutative. Extending these results further, Dhara et al. [16] proved that if $\mathfrak{R}$ is a 2-torsion-free semiprime ring, I is a nonzero ideal of $\mathfrak{R}$, and F is a generalized derivation corresponding to a derivation d satisfying $d(I) = 0$, then the following conclusion holds:

$$[d(\eta), F(\tau)] = \pm[\eta, \tau], \quad \forall \eta, \tau \in I,$$

implies that $\mathfrak{R}$ contains a nonzero central ideal. Subsequently, Raza and Rehman [17] further analyzed this condition in the framework of both prime and semiprime rings, extending and refining these foundational results.

For elements $\eta, \tau \in \mathfrak{R}$, the notation

$$[\eta, \tau] = \eta\tau - \tau\eta$$

denotes their commutator, which measures the extent to which η and τ fail to commute. This definition extends recursively to define higher-order commutators. For any non-negative integer k , set

$$[\eta, \tau]_0 = \eta, \quad [\eta, \tau]_1 = \eta\tau - \tau\eta,$$

and for all $k > 1$,

$$[\eta, \tau]_k = [[\eta, \tau]_{k-1}, \tau].$$

A ring $\mathfrak{R}$ is said to satisfy an *Engel condition* if there exists a positive integer k such that

$$[\eta, \tau]_k = 0 \quad \text{for all } \eta, \tau \in \mathfrak{R}.$$

In such a case, repeated commutation with τ eventually yields zero at the k -th step.

The expression $[\eta, \tau]_k$ can also be written explicitly as a noncommutative polynomial:

$$[\eta, \tau]_k = \sum_{m=0}^k (-1)^m \binom{k}{m} \tau^m \eta \tau^{k-m}.$$

Furthermore, the Engel commutator is additive in its first argument:

$$[\eta + z, \tau]_k = [\eta, \tau]_k + [z, \tau]_k \quad \text{for all } \eta, \tau, z \in \mathfrak{R}.$$

This paper aims to establish results corresponding to the previously stated conditions for generalized derivations acting on ideals in semiprime and prime rings. Furthermore, the obtained results are applied to the study of continuous generalized derivations within non-commutative Banach algebras. A comparative discussion is also presented to analyze the behavior of Banach algebras under these algebraic conditions.

2. The Results in Prime Rings

For a comprehensive treatment of generalized polynomial identities involving derivations, we refer the reader to Chapter 7 of [18].

Let $\text{der}(U)$ denote the collection of all derivations of U . A derivation word is an additive map of the form $\Delta = d_1 d_2 \cdots d_m$, where each $d_i \in \text{der}(U)$. A differential polynomial is then a generalized polynomial with coefficients in U , written as $\phi(\Delta_j \eta_i)$, where the η_i are noncommuting variables, and the derivation words Δ_j act on them as unary operators. We say that $\phi(\Delta_j \eta_i)$ is a differential identity (DI) on a subset $T \subseteq U$ if the evaluation of the polynomial is zero whenever elements of T are substituted for the variables η_i .

Let D_{int} denote the C -subspace of $\text{der}(U)$ that consists of all inner derivations of U , and let d be a nonzero derivation of $\mathfrak{R}$. According to Theorem 2 in [19] (see also Theorem 1 in the same reference), the following holds:

If $(\eta_1, \dots, \eta_n, d_{\eta_1}, \dots, d_{\eta_n})$ forms a differential identity on $\mathfrak{R}$, then one of the following conditions is satisfied:

- (i) $d \in D_{\text{int}}$
- (ii) $\mathfrak{R}$ satisfies the GPI $(\eta_1, \dots, \eta_n, \tau_1, \dots, \tau_n)$.

Theorem 1. *Let $\mathfrak{R}$ be a prime ring of characteristic not equal to 2, and let I be a nonzero ideal of $\mathfrak{R}$. Take fixed positive integers m, n, k, l . Assume that $\mathfrak{R}$ possesses a generalized derivation F corresponding to a nonzero derivation d , and that for all $\eta, \tau \in I$, the identity*

$$[F(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_l \eta^k$$

is satisfied. Under these conditions, the ring $\mathfrak{R}$ must be commutative.

Proof. Since $\mathfrak{R}$ is a prime ring and F is a generalized derivation on $\mathfrak{R}$, [[5] Theorem 3] ensures that there exists an element $a \in \mathfrak{R}$ and a derivation d on the Utumi quotient ring U such that the desired equation holds.

$$F(\eta) = a\eta + d(\eta).$$

This relation represents a differential identity, and hence the ideal I satisfies

$$[a\eta + d(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_I \eta^k, \quad \forall \eta, \tau \in I. \quad (1)$$

That is,

$$[a\eta, d(\tau)]_m + [d(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_I \eta^k.$$

In accordance with Kharchenko's theorem [[19] Theorem 2], the proof divides into two cases.

Case 1. Suppose the derivation d is not inner. Then the ideal I satisfies the polynomial identity

$$[a\eta, t]_m + [s, t]_m = \eta^n [\eta, \tau]_I \eta^k, \quad \forall \eta, \tau, s, t \in I.$$

For $\eta = 0$, this reduces to $[s, t]_m = 0$ for all $s, t \in I$. That is, $[s, t]_m = 0 = [I_s(t), t]_{m-1}$ for all $s, t \in I$. By [[20] Theorem 1], it follows that either $\mathfrak{R}$ is commutative or $I_s = 0$, implying $I \subseteq Z(\mathfrak{R})$. Consequently, by [[21] Lemma 3], $\mathfrak{R}$ itself is commutative.

Case 2. Suppose d is an inner derivation, i.e., there exists $q \in Q$ such that $d(\eta) = [q, \eta]$ for all $\eta \in \mathfrak{R}$. Substituting into Equation (1) gives, for all $\eta, \tau \in I$,

$$[a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m = \eta^n [\eta, \tau]_I \eta^k.$$

By [[22] Theorem 2], this generalized polynomial identity (GPI) also holds in Q , i.e.,

$$[a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m = \eta^n [\eta, \tau]_I \eta^k, \quad \forall \eta, \tau \in Q.$$

If the center C of Q is infinite, then

$$[a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m = \eta^n [\eta, \tau]_I \eta^k, \quad \forall \eta, \tau \in Q \otimes {}^*C\overline{C},$$

Let $\overline{C}$ denote the algebraic closure of the centroid C . Since both Q and the scalar extension $Q \otimes_C \overline{C}$ are prime and centrally closed (see [23], Theorems 2.5 and 3.5), we may replace $\mathfrak{R}$ with either Q or $Q \otimes_C \overline{C}$, according to whether C is finite or infinite. Hence, without loss of generality, we can assume that $\mathfrak{R}$ is centrally closed over C (that is, $\mathfrak{R}C = \mathfrak{R}$), where C is either finite or algebraically closed, and that the identity

$$[a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m = \eta^n [\eta, \tau]_I \eta^k, \quad \forall \eta, \tau \in \mathfrak{R},$$

holds.

By Martindale's theorem [24], the ring $\mathfrak{R}C$ (and hence $\mathfrak{R}$) is primitive and contains a nonzero socle H , whose corresponding division ring will be denoted by $\mathbb{D}$. According to Jacobson's characterization of primitive rings [[25] p. 75], $\mathfrak{R}$ is isomorphic to a dense ring of linear operators on a vector space V over $\mathbb{D}$, with H identified with the finite-rank operators contained in $\mathfrak{R}$.

If V is finite-dimensional over $\mathbb{D}$, the density of the action implies that $\mathfrak{A}$ is isomorphic to the full matrix algebra $M_k(\mathbb{D})$, where $k = \dim_{\mathbb{D}} V$. Since $\mathfrak{A}$ is noncommutative, it follows that $k \geq 2$.

Assume that $\dim_{\mathbb{D}} V \geq 2$; otherwise, the argument is already complete.

We first aim to prove that s and qs are $\mathbb{D}$ -linearly dependent for every $s \in V$. If $qs = 0$, then the pair $\{s, qs\}$ is clearly $\mathbb{D}$ -dependent. Thus, it suffices to consider the case where $qs \neq 0$.

Assume, toward a contradiction, that the set $\{s, qs\}$ is $\mathbb{D}$ -linearly independent for some $s \in V$. We examine two possibilities:

If $q^2s \notin \text{span}_{\mathbb{D}}\{s, qs\}$, then the three vectors $\{s, qs, q^2s\}$ are $\mathbb{D}$ -linearly independent. By the density of $\mathfrak{A}$, we may choose elements $\eta, \tau \in R$ such that the following conditions hold:

$$\begin{aligned}\eta s &= 0, \eta qs = qs, \eta q^2s = 0 \\ \tau s &= 0, \tau qs = s, \tau q^2s = 3qs.\end{aligned}\tag{2}$$

Multiplying (2) by s from the right, we have

$$0 = ([a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m - \eta^n [\eta, \tau]_l \eta^k) s$$

which can be rewritten as

$$\begin{aligned}0 &= \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k a\eta (q\tau - \tau q)^{m-k} s \\ &\quad + \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k (q\eta - \eta q) (q\tau - \tau q)^{m-k} s \\ &\quad + \eta^n \left(\sum_{t=0}^l (-1)^t \binom{l}{k} \tau^t \eta \tau^{l-t} \right) \eta^k s\end{aligned}$$

expanding this equation and using (2), we get

$$\begin{aligned}0 &= \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k a\eta (q\tau - \tau q)^{m-k-1} (q\tau s - \tau qs) \\ &\quad + \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k (q\eta - \eta q) (q\tau - \tau q)^{m-k-1} (q\tau s - \tau qs) \\ &\quad + \eta^n \left(\sum_{t=0}^l (-1)^t \binom{l}{k} \tau^t \eta \tau^{l-t} \eta^{k-1} (\eta s) \right) \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k a\eta (q\tau - \tau q)^{m-k-1} (-s) \\ &\quad + \sum_{k=0}^m (-1)^k \binom{m}{k} (q\tau - \tau q)^k (q\eta - \eta q) (q\tau - \tau q)^{m-k-1} (-s) \\ &\quad + 0\end{aligned}$$

and so, we obtain that

$$\begin{aligned}
 0 &= \sum_{k=0}^m (-1)^m \binom{m}{k} (q\tau - \tau q)^k a(\eta s) + \sum_{k=0}^m (-1)^m \binom{m}{k} (q\tau - \tau q)^k (q(\eta s) - \eta qs) \\
 &= 0 + \sum_{k=0}^m (-1)^m \binom{m}{k} (q\tau - \tau q)^k qs \\
 &= \sum_{k=0}^m (-1)^m \binom{m}{k} (q\tau - \tau q)^{k-1} (q\tau qs - \tau q^2 s) \\
 &= \sum_{k=0}^m (-1)^m \binom{m}{k} (q\tau - \tau q)^{k-2} (q\tau - \tau q) (-2qs) \\
 &= 2 \sum_{k=0}^m (-1)^{m+1} \binom{m}{k} (q\tau - \tau q)^{k-2} (q\tau qs - \tau q^2 s) \\
 &= 2 \sum_{k=0}^m (-1)^{m+2} \binom{m}{k} (q\tau - \tau q)^{k-2} (-2qs) \\
 &= \dots = 2^m (-1)^{m+1} qs.
 \end{aligned}$$

This yields that $2^m (-1)^{m+1} qs = 0$; this is a contradiction.

If $q^2 s \in \text{span}_{\mathcal{D}}\{s, qs\}$, then $q^2 s = \alpha s + qs\beta$, for some $\alpha, \beta \in \mathcal{D}$. In view of the density of $\mathfrak{A}$, there exists $\eta, \tau \in \mathfrak{A}$ such that

$$\begin{aligned}
 \eta s &= 0, \eta qs = qs \\
 \tau s &= 0, \tau qs = s.
 \end{aligned}$$

By applying a method similar to the above, we can easily see that

$$([a\eta, [q, \tau]]_m + [[q, \eta], [q, \tau]]_m - \eta^n [\eta, \tau]_l \eta^k) s = (-1)^{m+1} 2^m qs + (-1)^m 2^{m-1} s\beta = 0$$

for some $\beta \in \mathcal{D}$. Since $\mathfrak{A}$ is a prime ring with characteristic different from 2, $\beta = 0$. In this case, $\{s, qs\}$ are linearly $\mathcal{D}$ -dependent. This contradiction shows that, for every $s \in V$, the element qs must be of the form sb for some $b \in \mathbb{D}$.

Our next goal is to show that $qs = s\gamma(s)$, where $\gamma(s) \in \mathcal{D}$. Let's first show that $\gamma(s)$ is independent of the choice of $s \in V$.

If $\{s, t\}$ are $\mathcal{D}$ -independent for any $s, t \in V$, there exist $\gamma(s), \gamma(t), \gamma(s+t) \in \mathcal{D}$ such that

$$qs = s\gamma(s), qt = t\gamma(t) \text{ and } q(s+t) = (s+t)\gamma(s+t).$$

Moreover,

$$s\gamma(s) + t\gamma(t) = q(s+t) = (s+t)\gamma(s+t).$$

Hence,

$$s(\gamma(s) - \gamma(s+t)) + t(\gamma(t) - \gamma(s+t)) = 0.$$

Since $\{s, t\}$ are linearly $\mathcal{D}$ -independent, we obtain that $\gamma(s) = \gamma(s+t) = \gamma(t)$.

If $\{s, t\}$ are $\mathcal{D}$ -dependent for any $s, t \in V$, say $s = t\zeta$ where $\zeta \in \mathcal{D}$, then

$$s\gamma(s) = qs = qt\zeta = t\gamma(t)\zeta = s\gamma(t)$$

and so $\gamma(s) = \gamma(t)$ as claimed. So, there exists $\lambda \in \mathcal{D}$ such that $qs = s\lambda$ for all $s \in V$.

For any $r \in \mathfrak{A}$ and $s \in V$, we have $qs = s\lambda$, and therefore $r(qs) = r(s\lambda)$, while $q(rs) = (rs)\lambda$. Hence,

$$[q, r]s = 0 \quad \text{for all } r \in \mathfrak{A}, s \in V.$$

This shows that $[q, r]V = (0)$ for every $r \in \mathfrak{R}$. Since V is a faithful and irreducible left $\mathfrak{R}$ -module, it follows that $[q, r] = 0$ for all $r \in \mathfrak{R}$, meaning that $q \in Z(\mathfrak{R})$.

Consequently, for every $\eta \in \mathfrak{R}$ we obtain $d(\eta) = [q, \eta] = 0$, so d must be the zero derivation, which contradicts the assumption that d is nonzero. This completes the proof. $\square$

3. The Results in Semiprime Rings

Throughout this section, let $\mathfrak{R}$ denote a semiprime ring and U its left Utumi quotient ring. We now proceed to establish the main result of this section as follows.

Theorem 2. *Let $\mathfrak{R}$ be a semiprime ring of characteristic different from 2, and let m, n, k be fixed positive integers. Assume that $\mathfrak{R}$ admits a generalized derivation F associated with a nonzero derivation d such that*

$$[F(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_l \eta^k, \quad \text{for all } \eta, \tau \in \mathfrak{R}.$$

Then, there exists a central idempotent e in the Utumi quotient ring U of $\mathfrak{R}$ for which the decomposition

$$U = eU \oplus (1 - e)U$$

holds. Moreover, the derivation d vanishes identically on the component eU , while the complementary component $(1 - e)U$ is a commutative ring.

Proof. Since $\mathfrak{R}$ is semiprime, and F acts as a generalized derivation on $\mathfrak{R}$, by using [[5] Theorem 3], we have $Z(U) = C$, where C denotes the extended centroid of $\mathfrak{R}$. Moreover, again by Theorem 3 of [5], the derivation d extends uniquely to the Utumi quotient ring U . According to [[5] Theorem 3], the generalized derivation F can be expressed as

$$F(\eta) = a\eta + d(\eta),$$

for some $a \in U$ and a derivation d on U .

Given that

$$[a\eta + d(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_l \eta^k, \quad \forall \eta, \tau \in \mathfrak{R},$$

and using Theorem 3 of [26], which asserts that $\mathfrak{R}$ and U satisfy the same DI, we obtain

$$[a\eta + d(\eta), d(\tau)]_m = \eta^n [\eta, \tau]_l \eta^k, \quad \forall \eta, \tau \in U.$$

Let B be the complete Boolean algebra consisting of all idempotents in the extended centroid C , and let M be any maximal ideal of B . Since the Utumi quotient ring U is an orthogonally complete B -algebra (see p. 42 of [27]), [[21] Lemma 3] implies that MU is a d -invariant prime ideal of U . Define a derivation $\bar{d}$ on the factor ring $\bar{U} = U/MU$ by

$$\bar{d}(\bar{u}) = \overline{d(u)} \quad \text{for all } u \in U.$$

For the identity

$$[a\bar{\eta} + \bar{d}(\bar{\eta}), \bar{d}(\bar{\tau})]_m = \bar{\eta}^n [\bar{\eta}, \bar{\tau}]_l \bar{\eta}^k,$$

it is clear that $\bar{U}$ is a prime ring. Thus, by Theorem 1, either $\bar{U}$ is commutative, or the induced derivation $\bar{d}$ is zero. Equivalently, we obtain either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$. In both cases, we conclude that

$$d(U) [U, U] \subseteq MU.$$

Since the intersection of all such ideals is trivial, i.e.,

$$\bigcap_M MU = 0,$$

it follows that

$$d(U) [U, U] = 0.$$

Applying the theory of orthogonal completion for semiprime rings (see Chapter 3 of [18]), one can deduce the existence of a central idempotent element e in U such that

$$U = eU \oplus (1 - e)U.$$

With this decomposition, the derivation d is identically zero on eU , while the component ring $(1 - e)U$ is commutative. This establishes the desired result and completes the proof of the theorem. $\square$

4. The Results in Banach Algebras

In this section, we investigate conditions under which every continuous antiderivation on a Banach algebra takes values in the Jacobson radical, thereby extending the purely algebraic results obtained in the previous section to the topological setting. As discussed earlier, Thomas generalized the Singer–Wermer theorem by showing that any derivation on a commutative Banach algebra maps the entire algebra into its Jacobson radical. Motivated by this result, it is natural to ask whether a similar conclusion can be drawn in the absence of the commutativity assumption.

Throughout this section, A denotes a Banach algebra with Jacobson radical $\text{rad}(A)$. Moreover, $F = La + d$ will represent a spectrally bounded generalized derivation on A .

Theorem 3. *Let A be a noncommutative Banach algebra, and let $\text{rad}(A)$ denote its Jacobson radical. Fix positive integers m, n, k, l , and consider a continuous generalized derivation F on A of the form $F(\eta) = a\eta + d(\eta)$, where $a \in A$ and d is a derivation of A . Assume that for every $\eta, \tau \in A$ the expression $[F(\eta), d(\tau)]_m - \eta^n [\eta, \tau]_l \eta^k$ lies in $\text{rad}(A)$. Under these assumptions, it follows that $d(A) \subseteq \text{rad}(A)$, that is, the image of the derivation d is contained entirely in the Jacobson radical.*

Proof. Let P be a primitive ideal of A . By the hypothesis, we have $\mathfrak{R}$ is continuous. Using left multiplication map is continuous, we obtain that d is continuous. Using d is continuous derivation on the Banach algebra, d of a Banach algebra leaves the primitive ideals invariant by [[8] Theorem 2.2]. We obtain that $d(P) \subseteq P$. Therefore, we get $F(P) \subseteq aP + d(P) \subseteq P$. It means that the continuous generalized derivation $\mathfrak{R}$ leaves the primitive ideals invariant. Moreover, because the Jacobson radical is equal to the intersection of all primitive ideals, it follows that

$$d(\text{rad}(A)) \subseteq \text{rad}(A).$$

Denote A/P and a nonzero derivation such that

$$\begin{aligned} F_P : A/P &\rightarrow A/P \\ F_P(\bar{\eta}) &= F_P(\eta + P) = F(\eta) + P = a\eta + d(\eta) + P, \end{aligned}$$

for all $\eta \in A$ and $\bar{\eta} = \eta + P$, where A/P is a factor Banach algebra. By the hypothesis, we obtain that

$$[F(\eta), d(\tau)]_m - \eta^n [\eta, \tau]_l \eta^k \in \text{rad}(A) \text{ for all } \eta, \tau \in A.$$

That is,

$$[F_p(\bar{\eta}), d_p(\bar{\tau})]_m - \bar{\eta}^n [\bar{\eta}, \bar{\tau}]_l \bar{\eta}^k = \bar{0} \text{ for all } \bar{\eta}, \bar{\tau} \in A/\mathfrak{P}.$$

Since P is a primitive ideal, the factor algebra A/P is primitive, and so it is prime. There is no loss of generality in assuming that A is semisimple. That is, A/P is semisimple. By Theorem 1, we get A/P is commutative or $d_P = 0$.

If A/P is commutative, then by [[28] Remark 4.3], every derivation on a semisimple Banach algebra is continuous. Furthermore, from [[6] Singer-Wermer Theorem], it follows that $d_P = 0$, since A/P is semisimple. Hence, in both situations, we arrive at $d_P = 0$. Consequently, $d(A) \subseteq P$ for any primitive ideal P of A . Noting that the Jacobson radical $\text{rad}(A)$ is the intersection of all primitive ideals of A , we conclude that

$$d(A) \subseteq \text{rad}(A).$$

This completes the proof. $\square$

Theorem 4. Let A be a noncommutative Banach algebra with Jacobson radical $\text{rad}(A)$, and let m, n, k, l be fixed positive integers. Consider a spectrally bounded generalized derivation $F = L_a + d$, where L_a denotes left multiplication by some element $a \in A$, and d is a derivation on A .

If for all $\eta, \tau \in A$, the relation

$$[F(\eta), d(\tau)]_m - \eta^n [\eta, \tau]_l \eta^k \in \text{rad}(A)$$

holds, then it follows that

$$d(A) \subseteq \text{rad}(A).$$

Proof. Since A is spectrally bounded, [[29] Lemma 2.8] ensures that both L_a and d are spectrally bounded. Let P be a primitive ideal of A . By [[29] Lemma 2.7], we obtain $d(P) \subseteq P$. From this point on, following the line of argument used in the proof of Theorem 3 leads to the desired conclusion. $\square$

5. Open Problem

The techniques developed in this paper suggest possible extensions to nonassociative algebraic structures. In particular, it is an interesting open problem to determine whether the results established for prime and semiprime rings remain true when these classes are replaced by prime alternative rings and semiprime alternative rings, respectively. Such an extension may reveal new connections between derivation theory and the structure of alternative rings.

6. Conclusions

This study reexamines known results on generalized derivations acting on ideals of prime and semiprime rings under a new set of conditions, resulting in several strengthened and extended conclusions. These findings are further applied to continuous generalized derivations on noncommutative Banach algebras, enabling a systematic analysis of the interplay between algebraic properties and topological structures.

The results presented in this paper provide a foundation for future research, with potential applications in operator algebras, C^* -algebras, alternative algebras, and other mathematical areas in which ring-theoretic methods play a central role.

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