

Article

Analytic Continuation of the Hurwitz Transform

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Abstract

An integral over the unit interval of the product of a given function and the Hurwitz zeta function is known as the Hurwitz transform of the function. We give sufficient conditions on the function for the Hurwitz transform to continue meromorphically to a larger region and to the complex plane. Integral representations for the meromorphic extension of the Hurwitz transform are given, and some new integral identities involving the Hurwitz zeta function are derived.

Keywords: Hurwitz zeta function; Hurwitz transform; Weierstrass zeta function

MSC: 11M35; 30B50; 33E05

1. Introduction

The Hurwitz zeta function $\zeta(s, u)$ is given by the series

$$\zeta(s, u) = \sum_{n=0}^{\infty} \frac{1}{(n+u)^s}, \quad (1)$$

where $\operatorname{Re} s > 1$, $u > 0$, and the special case $\zeta(s, 1)$ gives the Riemann zeta function $\zeta(s)$. It has an integral representation

$$\zeta(s, u) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1} e^{-uz}}{1-e^{-z}} dz, \quad (2)$$

where C is a Hankel contour which goes around the positive real axis in the counterclockwise direction [1]. The integral in (2) without the gamma function factor is analytic in $(s, u) \in \mathbb{C} \times \mathfrak{R}$, where $\mathfrak{R} = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ is the right half plane, and it follows from (2) that $\zeta(s, u)$ for each $u \in \mathfrak{R}$ is meromorphic in s with a simple pole at $s = 1$. From (1), we have the relation

$$\zeta(s, u) = u^{-s} + \zeta(s, u+1), \quad (3)$$

and from (2), one can derive the Hurwitz Formula [1]:

$$\zeta(1-s, u) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n u}}{n^s} + e^{\frac{\pi i s}{2}} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n u}}{n^s} \right) \quad (4)$$

for $\operatorname{Re} s > 1$ and $0 < u \leq 1$. This formula can also be obtained from the Lipschitz summation Formula [2]. The Hurwitz Formula (4) also holds for $\operatorname{Re} s > 0$ and $0 < u < 1$, as the right-hand side of (4) converges and is analytic in this region if $0 < u < 1$.

Many integrals of the form

$$\int_0^1 f(u) \zeta(s, u) du \quad (5)$$



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have been studied in [3,4], and such an integral is named the Hurwitz transform of f . We will give the definition of the Hurwitz transform in Section 2 (Definition 1), with s in (5) replaced by $1 - s$.

The main tools in [3] of evaluating the integral (5) are the Hurwitz Formula (4) and the Fourier series of f , while in [4], it is demonstrated that many of the formulas can be derived from the results of Mikolás [5] on the Hurwitz zeta function.

There has not been a general theory of the Hurwitz transform, and some questions of extending the domain of the Hurwitz transform have been raised in [3,4]. In this article, we give sufficient conditions on the function f for the Hurwitz transform to extend meromorphically to a larger region (Theorem 1) and to the complex plane (Corollaries 1 and 2).

We also give integral representations for the Hurwitz transform valid in the corresponding regions. As an example, we derive the identity of Mikolás and its generalization (Theorem 3). Finally, we give a representation of a Lambert series studied previously in [6] as an integral involving the Hurwitz and the Weierstrass zeta functions (Theorem 5).

Notations

$L^p \Omega$	The complex L^p -space on Ω with the norm $\ \cdot\ _{L^p}$.
$L^1_{\text{loc}} \Omega$	The set of the equivalence classes of all locally integrable functions $f : \Omega \rightarrow \mathbb{C}$, identified if they agree almost everywhere.
$\mathcal{H}(f; s)$	The Hurwitz transform of f (Definition 1).
$H(f; s)$	The H -series attached to f (Definition 2).
$\mathbb{C}\langle x \rangle$	See Definition 3.
$\mathbb{C}\langle\langle x \rangle\rangle$	See Definition 4.
$\Gamma(s)$	The Euler gamma function.
$\zeta(s)$	The Riemann zeta function.
$\zeta(s, u)$	The Hurwitz zeta function.
$\zeta_{\mathbb{R}}(u, \tau)$	The Weierstrass zeta function (Section 7).
$G_{2k}(\tau)$	The Eisenstein series of weight $2k$ (Section 7).
$A(\tau, s)$	The Lambert series defined in Theorem 5 (Section 7).
$\Re$	The right half plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$.
R	The right half plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > r\}$ (Theorem 1).

2. Hurwitz Transform

Lemma 1. For an open subset $S \subseteq \mathbb{C}$, let $F : S \times (0, 1) \rightarrow \mathbb{C}$ be a function such that $F(s, u)$ is analytic in $s \in S$ for each $u \in (0, 1)$ and integrable in $u \in (0, 1)$ for each $s \in S$. Suppose for every compact subset $K \subseteq S$, there exists an integrable function $f_K : (0, 1) \rightarrow \mathbb{C}$ such that

$$|F(s, u)| \leq |f_K(u)| \quad (6)$$

for $(s, u) \in K \times (0, 1)$. Then, the integral

$$\int_0^1 F(s, u) du$$

is analytic in $s \in S$.

Proof. By the dominated convergence theorem with (6), the integral is continuous in $s \in S$. Let K be a triangle contractible in S . By Morera's theorem, it suffices to show

$$\int_K \left(\int_0^1 F(s, u) du \right) ds = 0.$$

By the bound (6), the order of integrals can be interchanged by Fubini's theorem, and the lemma follows. $\square$

Definition 1. For $f \in L^1(0,1)$, the function defined by the integral

$$\mathcal{H}(f;s) = \int_0^1 f(u)\zeta(1-s,u)du \quad (7)$$

for $\operatorname{Re} s > 1$ is called the Hurwitz transform of f .

The above definition differs from (5) as s is replaced by $1-s$, in view of the relation to the Mellin transform discussed in Section 4. It is a matter of convention for the notation $\mathcal{H}(f;s)$, as we have $\mathcal{H}(f;1-s) = \int_0^1 f(u)\zeta(s,u)du$. A statement on either $\mathcal{H}(f;s)$ or $\mathcal{H}(f;1-s)$ can easily be converted to the other by a change of variables.

The integral (7) converges for $\operatorname{Re} s > 1$ since $\zeta(1-s,u)$ is bounded in $u \in (0,1)$ for each s in the region $\operatorname{Re} s > 1$, as $\zeta(1-s,u) = u^{s-1} + \zeta(1-s,u+1)$ by (3).

Let us use the same notation for both $f \in L^1(0,1)$ and its representative function. For any compact subset K of the region $\operatorname{Re} s > 1$, with $B_K = 1 + \max_{(s,u) \in K \times [0,1]} |\zeta(1-s,u+1)|$, we have $|\zeta(1-s,u)| \leq B_K$ for $(s,u) \in K \times (0,1)$. By Lemma 1 with $F(s,u) = f(u)\zeta(1-s,u)$ and $f_K(u) = B_K f(u)$, it follows that $\mathcal{H}(f;s)$ is analytic in the region $\operatorname{Re} s > 1$.

One method of evaluating the Hurwitz transform is to express it as a series, using the Fourier series of the function f . We recall, for $f \in L^p(0,1)$ for $1 \leq p \leq \infty$, the Fourier series of f is the formal series

$$\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u},$$

where for $n \in \mathbb{Z}$,

$$a_n = \int_0^1 f(u) e^{-2\pi i n u} du, \quad (8)$$

and the Fourier series converges to f in the L^p -norm if $\sum_{|n| \leq k} a_n e^{2\pi i n u}$ converges to f in the L^p -norm as $k \rightarrow \infty$.

It is well known that if $f \in L^p(0,1)$ for any $1 < p < \infty$, then its Fourier series converges to f in the L^p -norm [7]. We are interested in the elements in $L^1(0,1)$ whose Fourier series converges to itself in the L^1 -norm. Since the inclusion $L^p(0,1) \subset L^q(0,1)$ is continuous for $1 \leq q < p \leq \infty$, we have the following sufficient condition for the convergence of the Fourier series of f in the L^1 -norm.

Lemma 2. If $f \in L^p(0,1)$ for any $1 < p \leq \infty$, then its Fourier series converges to f in the L^1 -norm.

Definition 2. Let $f \in L^1(0,1)$. Given the Fourier series $\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$ of f and $s \in \mathbb{C}$, let us call the series

$$H(f;s) = \sum_{n=1}^{\infty} \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right)$$

the H -series attached to f .

This series has been considered in [6] as a series attached to a Laurent series, and here the coefficients a_n of the series come from the Fourier coefficients of $f \in L^1(0,1)$. We discuss some of its properties in Section 6.

By definition, $H(f;s)$ is independent of the constant term a_0 of the Fourier series of f . In particular, $H(f;s) = 0$ if f is a constant.

Lemma 3. Suppose we have $f \in L^1(0, 1)$ such that the Fourier series

$$\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$$

converges to f in the L^1 -norm. Then, the Hurwitz transform $\mathcal{H}(f; s)$ of f is represented as the series

$$\begin{aligned} \mathcal{H}(f; s) &= \frac{\Gamma(s)}{(2\pi)^s} H(f; s) \\ &= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right), \end{aligned}$$

which converges for $\operatorname{Re} s > 1$.

Proof. Let $\sigma = \operatorname{Re} s > 1$. By (4), for $n \in \mathbb{Z}$,

$$\int_0^1 e^{2\pi i n u} \zeta(1-s, u) du = \frac{\Gamma(s)}{(2\pi)^s} \int_0^1 e^{2\pi i n u} \left(e^{-\frac{\pi i s}{2}} \sum_{k=1}^{\infty} \frac{e^{2\pi i k u}}{k^s} + e^{\frac{\pi i s}{2}} \sum_{k=1}^{\infty} \frac{e^{-2\pi i k u}}{k^s} \right) du.$$

We can interchange the integral and the summation by Tannery's theorem, since $|e^{\pm 2\pi i k u} k^{-s}| = k^{-\sigma}$ and $\sum_{k=1}^{\infty} k^{-\sigma} < \infty$. It follows from the orthonormality of the exponential functions that

$$\int_0^1 e^{2\pi i n u} \zeta(1-s, u) du = \begin{cases} \frac{\Gamma(s)}{(2\pi)^s} e^{\frac{\pi i s}{2}} n^{-s} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \frac{\Gamma(s)}{(2\pi)^s} e^{-\frac{\pi i s}{2}} |n|^{-s} & \text{if } n < 0. \end{cases} \quad (9)$$

Let

$$f_k(u) = \sum_{n=-k}^k a_n e^{2\pi i n u}.$$

By our assumption, we have $f(u) = f_k(u) + r_k(u)$, where $\|r_k(u)\|_{L^1} \rightarrow 0$ as $k \rightarrow \infty$, and

$$\begin{aligned} \int_0^1 f(u) \zeta(1-s, u) du &= \int_0^1 (f_k(u) + r_k(u)) \zeta(1-s, u) du \\ &= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^k \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right) + \int_0^1 r_k(u) \zeta(1-s, u) du. \end{aligned}$$

Since $\zeta(1-s, u)$ is bounded in $u \in (0, 1)$ for each s in the region $\operatorname{Re} s > 1$, we have $\|r_k(u) \zeta(1-s, u)\|_{L^1} \rightarrow 0$ as $k \rightarrow \infty$. $\square$

3. Meromorphic Continuation of the Hurwitz Transform

We now give sufficient conditions on f under which the Hurwitz transform $\mathcal{H}(f; s)$ can be analytically continued to a meromorphic function of s .

Lemma 4.

(i) We can consider $s\zeta(1-s, u)$ as an analytic function in $(s, u) \in \mathbb{C} \times \mathfrak{R}$, and

$$s\zeta(1-s, u)|_{s=0} = -1 \quad (10)$$

for all $u \in \mathfrak{R}$, where $\mathfrak{R} = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$.

(ii) We have

$$\int_0^1 s\zeta(1-s, u+1)du = -1 \quad (11)$$

for all $s \in \mathbb{C}$.

Proof. From (2), we have $s\zeta(1-s, u) = -s\Gamma(s)I(s, u)$ with

$$I(s, u) = \frac{1}{2\pi i} \int_C \frac{(-z)^{-s} e^{-uz}}{1 - e^{-z}} dz,$$

where C is the Hankel contour in (2). Clearly, $I(s, u)$ is analytic in $(s, u) \in \mathbb{C} \times \Re$. For $k \in \mathbb{Z}$, by closing C to a circle, we have

$$I(k, u) = \operatorname{Res}_{z=0} \frac{(-z)^{-k} e^{-uz}}{1 - e^{-z}}.$$

Thus, $I(k, u)$ vanishes for $k \in \mathbb{Z}_{<0}$, cancelling the poles of $-s\Gamma(s)$. Since $-s\Gamma(s)|_{s=0} = -1$ and $I(0, u) = 1$, it follows that $s\zeta(1-s, u)|_{s=0} = (-s\Gamma(s)|_{s=0})I(0, u) = -1$ for all $u \in \Re$. This shows (i).

From (3), we have the relation

$$s\zeta(1-s, u) = su^{s-1} + s\zeta(1-s, u+1)$$

for all $(s, u) \in \mathbb{C} \times \Re$. From $n = 0$ in (9), for $\operatorname{Re} s > 1$,

$$\begin{aligned} 0 &= \int_0^1 s\zeta(1-s, u)du = \int_0^1 su^{s-1}du + \int_0^1 s\zeta(1-s, u+1)du \\ &= 1 + \int_0^1 s\zeta(1-s, u+1)du. \end{aligned} \quad (12)$$

Since both sides of (12) are entire in s , (ii) follows. $\square$

Lemma 5. Let $f(u) \in L^1_{\text{loc}}(0, 1)$. If there is $r \in \mathbb{R}$ such that $f(u)u^{\delta-1} \in L^1(0, 1)$ for every $\delta > r$, the integral

$$\int_0^1 f(u)u^{s-1}du$$

is analytic in the region $\operatorname{Re} s > r$.

Proof. For $\operatorname{Re} s \geq \delta > r$, we have

$$|f(u)u^{s-1}| = |f(u)u^{\delta-1}u^{s-\delta}| \leq |f(u)u^{\delta-1}| \quad (13)$$

for $u \in (0, 1)$, for any representative function of f . From the bound (13) in any subset K of $\operatorname{Re} s \geq \delta$, the result follows by Lemma 1 with $F(s, u) = f(u)u^{s-1}$ and $f_K(u) = f(u)u^{\delta-1}$. $\square$

Proposition 1. Let $f(u) \in L^1(0, 1)$ and let $-\infty \leq r_0 \leq 1$ be the infimum of all $r \in \mathbb{R}$ such that

$$f(u)u^{r-1} \in L^1(0, 1).$$

Then, the integral

$$\mathcal{H}(f; s) = \int_0^1 f(u)\zeta(1-s, u)du \quad (14)$$

is well defined for any s in the region $\operatorname{Re} s > r_0$, $s \neq 0$, and the resulting function $\mathcal{H}(f; s)$ is meromorphic in the region $\operatorname{Re} s > r_0$. If $r_0 < 0$, it may have a simple pole at $s = 0$ with residue $-\int_0^1 f(u)du$.

Proof. Consider the integral $s\mathcal{H}(f; s) = \int_0^1 f(u)s\zeta(1-s, u)du$ for any s in the region $\operatorname{Re} s > r_0$. The integrand is

$$f(u)s\zeta(1-s, u) = f(u)su^{s-1} + f(u)s\zeta(1-s, u+1).$$

As both terms on the right-hand side are in $L^1(0, 1)$ by our assumption, so is the left-hand side. We have

$$s\mathcal{H}(f; s) = s \int_0^1 f(u)u^{s-1}du + \int_0^1 f(u)s\zeta(1-s, u+1)du. \quad (15)$$

The first integral is analytic in the region $\operatorname{Re} s > r_0$ by Lemma 5. Since the function $s\zeta(1-s, u+1)$ is entire in s for every $u \in (0, 1)$ and bounded on $(s, u) \in K \times (0, 1)$ for any compact subset $K \subseteq \mathbb{C}$, the second integral in (15) is an entire function of s by Lemma 1. Hence,

$$\mathcal{H}(f; s) = \int_0^1 f(u)u^{s-1}du + \frac{1}{s} \int_0^1 f(u)s\zeta(1-s, u+1)du$$

as a meromorphic function in the region $\operatorname{Re} s > r_0$. By (10), the value of the entire function $\int_0^1 f(u)s\zeta(1-s, u+1)du$ at $s = 0$ is $-\int_0^1 f(u)du$, and the proposition follows. $\square$

Lemma 6. For $n \in \mathbb{Z}$ and $\operatorname{Re} s > 0$,

$$\int_0^1 e^{2\pi i n u} \zeta(1-s, u)du = \begin{cases} \frac{\Gamma(s)}{(2\pi)^s} e^{\frac{\pi i s}{2}} n^{-s} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \frac{\Gamma(s)}{(2\pi)^s} e^{-\frac{\pi i s}{2}} |n|^{-s} & \text{if } n < 0. \end{cases}$$

Proof. We can view this equality as the Hurwitz transform of $f(u) = e^{2\pi i n u}$, $u \in (0, 1)$, which holds for $\operatorname{Re} s > 1$ by (9). Since $f(u)u^{\delta-1} \in L^1(0, 1)$ for all $\delta > 0$, it is valid for $\operatorname{Re} s > 0$ by Proposition 1. $\square$

In order to generalize Proposition 1, we introduce the following notation.

Definition 3. Let

$$\mathbb{C}\langle x \rangle = \bigoplus_{c \in \mathbb{C}} \mathbb{C}[x]e^{cx}.$$

An element $P(x) \in \mathbb{C}\langle x \rangle$ can be written as

$$P(x) = \sum_c P_c(x)e^{cx},$$

where $P_c(x) \in \mathbb{C}[x]$ is a polynomial for each $c \in \mathbb{C}$, which is zero for all but finitely many c 's. For $P(x) \in \mathbb{C}\langle x \rangle$,

$$P(\log u) = \sum_c P_c(\log u)u^c,$$

and we can regard $P(\log u)$ as a complex valued function of $u \in (0, 1)$. We also define, for a given function $f(s)$, the expression $P(\partial_s)f(s)$ to mean the finite sum

$$P(\partial_s)f(s) = \sum_c P_c(\partial_s)f(s+c). \quad (16)$$

We remark that the expression (16) in Definition 3 is not entirely arbitrary, as the identity $e^{c\partial_s} f(s) = f(s+c)$ is valid for an analytic function in a neighborhood of s . It is however used just as a notation for the sum. For example,

$$P(\partial_s)\zeta(1-s, u) = \sum_c P_c(\partial_s)\zeta(1-s-c, u),$$

$$P(\partial_s)\left(\frac{1}{s}\right) = \sum_c P_c(\partial_s)\left(\frac{1}{s+c}\right), \text{ etc.}$$

Theorem 1. Let $f(u) \in L^1(0, 1)$. Suppose there exists a real number $r < 1$ with $P(x) \in \mathbb{C}\langle x \rangle$ such that

$$(f(u) - P(\log u))u^{\delta-1} \in L^1(0, 1) \quad (17)$$

for all $\delta > r$, where we use the notations in Definition 3. Let $R = \{s \in \mathbb{C} \mid \operatorname{Re} s > r\}$. Then, the following hold.

- (i) The Hurwitz transform $\mathcal{H}(f; s)$ has the meromorphic continuation to the region R . We again denote by $\mathcal{H}(f; s)$ the meromorphic continuation on R .
- (ii) The principal part of $\mathcal{H}(f; s)$ at a nonzero $s = -c \in R$ is given by $P_c(\partial_s)\left(\frac{1}{s+c}\right)$. If $0 \in R$, the principal part at $s = 0$ is $P_0(\partial_s)\left(\frac{1}{s}\right) - \frac{1}{s} \int_0^1 f(u) du$.
- (iii) We have the equality

$$\mathcal{H}(f; s) = \int_0^1 (f(u)\zeta(1-s, u) - P(\partial_s)\zeta(1-s, u)) du \quad (18)$$

for all $s \in R$ which is not at a pole of the integrand (that is, for all $s \in R$ such that $s \neq -c$ for any nonzero c with $P_c(x) \neq 0$, and $s \neq 0$ unless $f(u) = P_0(\log u) = c_0$ for some constant $c_0 \in \mathbb{C}$).

Proof. The condition that both $f(u) \in L^1(0, 1)$ and $f(u) - \sum_c P_c(\log u)u^c \in L^1(0, 1)$ from letting $\delta = 1$ in (17) shows $\operatorname{Re} c > -1$ for all c for which $P_c(x) \neq 0$. Assuming $\operatorname{Re} s > 1$, the equality

$$\int_0^1 P_c(\log u)u^c u^{s-1} du = P_c(\partial_s)\frac{1}{s+c} \quad (19)$$

holds for all $c \in \mathbb{C}$, since either $\operatorname{Re} c > -1$ or $P_c(x) = 0$, and summing over c gives

$$\int_0^1 P(\log u)u^{s-1} du = P(\partial_s)\left(\frac{1}{s}\right).$$

We have $s\mathcal{H}(f; s) = \int_0^1 f(u)s\zeta(1-s, u) du$ for $\operatorname{Re} s > 1$, and

$$\begin{aligned} s\mathcal{H}(f; s) &= \int_0^1 f(u)su^{s-1} du + \int_0^1 f(u)s\zeta(1-s, u+1) du \\ &= s \int_0^1 (f(u) - P(\log u))u^{s-1} du + sP(\partial_s)\left(\frac{1}{s}\right) \\ &\quad + \int_0^1 f(u)s\zeta(1-s, u+1) du. \end{aligned} \quad (20)$$

We may now drop the assumption that $\operatorname{Re} s > 1$. The last integral of the right-hand side of (20) is an entire function of s , and the first integral is analytic in the region R by (17) and

Lemma 5. Dividing (20) by s shows that $\mathcal{H}(f; s)$ is meromorphic in the region R , showing (i), and it is given by

$$\begin{aligned}\mathcal{H}(f; s) &= \int_0^1 (f(u) - P(\log u)) u^{s-1} du \\ &\quad + \sum_c P_c(\partial_s) \frac{1}{s+c} + \frac{1}{s} \int_0^1 f(u) s \zeta(1-s, u+1) du.\end{aligned}\quad (21)$$

Since $\int_0^1 f(u) s \zeta(1-s, u+1) du|_{s=0} = -\int_0^1 f(u) du$, the principal parts of $\mathcal{H}(f; s)$ are as described in (ii).

We now prove (iii). First assume $s \in R$ such that $s \neq -c$ for all c such that $P_c(x) \neq 0$ and also $s \neq 0$. The integrand in (18) is

$$\begin{aligned}& f(u) \zeta(1-s, u) - \sum_c P_c(\partial_s) \zeta(1-s-c, u) \\ &= f(u) (u^{s-1} + \zeta(1-s, u+1)) - \sum_c P_c(\partial_s) (u^{s+c-1} + \zeta(1-s-c, u+1)) \\ &= (f(u) - \sum_c P_c(\log u) u^c) u^{s-1} + f(u) \zeta(1-s, u+1) - \sum_c P_c(\partial_s) \zeta(1-s-c, u+1),\end{aligned}\quad (22)$$

which shows that it is integrable. Replacing s with $s+c$ in (11) and dividing by it, we have

$$\int_0^1 \zeta(1-s-c, u+1) du = -\frac{1}{s+c} \quad (23)$$

for $s \neq -c$, and therefore,

$$\int_0^1 \left(-\sum_c P_c(\partial_s) \zeta(1-s-c, u+1) \right) du = \sum_c P_c(\partial_s) \frac{1}{s+c}.$$

Thus, the integral of (22) equals (21).

In the special case $f(u) = P_0(\log u) = c_0$ for $c_0 \in \mathbb{C}$, the integrand (22) becomes

$$\left(-\sum_{c \neq 0} P_c(\log u) u^c \right) u^{s-1} - \sum_{c \neq 0} P_c(\partial_s) \zeta(1-s-c, u+1).$$

The first term is integrable for $s \in R$ by our assumption, and we may also let $s = 0$ in the integrand if $0 \in R$. By (19) and (23), the integral evaluates to 0 which equals $\mathcal{H}(f; s)$. $\square$

Example 1. We present a simple example to illustrate the notations. Let

$$f(u) = \frac{\log u}{\sqrt{u}} + \sin^2\left(\frac{1}{u}\right), \quad u \in (0, 1).$$

Let $P(x) = xe^{-x/2} \in \mathbb{C}\langle x \rangle$. We have $P(\log u) = \frac{\log u}{\sqrt{u}}$, and

$$f(u) - P(\log u) = \sin^2\left(\frac{1}{u}\right).$$

Since $\int_0^1 \sin^2\left(\frac{1}{u}\right) u^{\delta-1} du = \int_1^\infty \sin^2(t) t^{-\delta-1} dt$, we see that $\sin^2\left(\frac{1}{u}\right) u^{\delta-1} \in L^1(0, 1)$ for all $\delta > 0$. Thus,

$$(f(u) - P(\log u)) u^{\delta-1} \in L^1(0, 1) \quad (24)$$

for all $\delta > 0$.

By Theorem 1 with $r = 0$, $\mathcal{H}(f; s)$ is meromorphic in the region $\operatorname{Re} s > 0$, with the principal part at $s = \frac{1}{2}$ given by

$$P_{-\frac{1}{2}}(\partial_s)\left(\frac{1}{s}\right) = \partial_s \frac{1}{s - \frac{1}{2}} = -\frac{1}{(s - \frac{1}{2})^2}. \quad (25)$$

Since $P(\partial_s)\zeta(1-s, u) = \partial_s(\zeta(1-s + \frac{1}{2}, u))$, we have the equality

$$\mathcal{H}(f; s) = \int_0^1 \left(\left(\frac{\log u}{\sqrt{u}} + \sin^2\left(\frac{1}{u}\right) \right) \zeta(1-s, u) - \partial_s \left(\zeta\left(\frac{3}{2} - s, u\right) \right) \right) du \quad (26)$$

for $\operatorname{Re} s > 0$, $s \neq \frac{1}{2}$.

For $r = 0$, the choice of $P(x) \in \mathbb{C}\langle x \rangle$ satisfying (24) is not unique, as we may add terms of the form $P_c(x)e^{cx}$ for $\operatorname{Re} c \geq 0$. This does not affect the conclusion that $\mathcal{H}(f; s)$ is meromorphic in the region $\operatorname{Re} s > 0$ nor its principal parts in this region. For example, consider $\tilde{P}(x) = xe^{-x/2} + 2e^{2x}$ with $\tilde{P}(\log u) = \frac{\log u}{\sqrt{u}} + 2u^2$. Then, $f(u) - \tilde{P}(\log u) = \sin^2\left(\frac{1}{u}\right) - 2u^2$ and also

$$(f(u) - \tilde{P}(\log u))u^{\delta-1} \in L^1(0, 1)$$

for all $\delta > 0$. Applying Theorem 1 with $\tilde{P}(x)$ also shows that $\mathcal{H}(f; s)$ is meromorphic on $\operatorname{Re} s > 0$, with the same principal part at $s = \frac{1}{2}$ as (25) since $\tilde{P}_{-\frac{1}{2}}(x) = P_{-\frac{1}{2}}(x) = x$.

Since $\tilde{P}(\partial_s)\zeta(1-s, u) = \partial_s(\zeta(\frac{3}{2} - s, u)) + 2\zeta(-1-s, u)$, we obtain

$$\mathcal{H}(f; s) = \int_0^1 \left(\left(\frac{\log u}{\sqrt{u}} + \sin^2\left(\frac{1}{u}\right) \right) \zeta(1-s, u) - \partial_s \left(\zeta\left(\frac{3}{2} - s, u\right) \right) - 2\zeta(-1-s, u) \right) du$$

for $\operatorname{Re} s > 0$, $s \neq \frac{1}{2}$. This integral formula is consistent with (26), as the additional term $\int_0^1 (-2\zeta(-1-s, u))du$ is zero in this region by Lemma 6.

We now consider further assumptions on the function $f(u) \in L^1(0, 1)$.

Definition 4. For $t \in \mathbb{R}$, let

$$\mathbb{C}\langle x \rangle_t = \bigoplus_{\operatorname{Re} c < t} \mathbb{C}[x]e^{cx}.$$

For $t_1 < t_2$, we have the map $\mathbb{C}\langle x \rangle_{t_2} \rightarrow \mathbb{C}\langle x \rangle_{t_1}$ which is the identity on the summands $\mathbb{C}[x]e^{cx}$ for $\operatorname{Re} c < t_1$ and sends all summands $\mathbb{C}[x]e^{cx}$ for $\operatorname{Re} c \geq t_1$ to zero. Let

$$\mathbb{C}\langle\langle x \rangle\rangle = \varprojlim \mathbb{C}\langle x \rangle_t$$

be the inverse limit of this system, with the projection maps $\pi_t : \mathbb{C}\langle\langle x \rangle\rangle \rightarrow \mathbb{C}\langle x \rangle_t$ for each $t \in \mathbb{R}$. An element $P(u) \in \mathbb{C}\langle\langle x \rangle\rangle$ can be written as a formal sum

$$P(x) = \sum_c P_c(x)e^{cx}$$

with $P_c(x) \in \mathbb{C}[x]$ for $c \in \mathbb{C}$, such that for every $t \in \mathbb{R}$,

$$\pi_t(P)(x) = \sum_{\operatorname{Re} c < t} P_c(x)e^{cx}$$

has a finite number of nonzero terms. We regard

$$\pi_t(P)(\log u) = \sum_{\operatorname{Re} c < t} P_c(\log u)u^c$$

as a complex valued function of $u \in (0, 1)$. A complex valued function $f(u)$ on $u \in (0, 1)$ has the asymptotic expansion

$$f(u) \sim P(\log u) \quad \text{as } u \rightarrow 0, \quad (27)$$

for some $P(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ if for every $t \in \mathbb{R}$,

$$f(u) - \pi_t(P)(\log u) = O(u^t) \quad \text{as } u \rightarrow 0. \quad (28)$$

An element $f(u) \in L^1(0, 1)$ has the asymptotic expansion (27) if there exists a representative function that has the asymptotic expansion.

Corollary 1. Suppose $f \in L^1(0, 1)$ has the asymptotic expansion

$$f(u) \sim P(\log u) \quad \text{as } u \rightarrow 0,$$

for some $P(x) \in \mathbb{C}\langle\langle x \rangle\rangle$, using the notations in Definition 4. Then, the following hold.

- (i) The Hurwitz transform $\mathcal{H}(f; s)$ continues to a meromorphic function of s in the complex plane, again denoted by $\mathcal{H}(f; s)$. The principal part at $s = -c \neq 0$ is given by $P_c(\partial_s)(\frac{1}{s+c})$ and at $s = 0$ is $P_0(\partial_s)(\frac{1}{s}) - \frac{1}{s} \int_0^1 f(u) du$.
- (ii) For any $r \in \mathbb{R}$, the identity

$$\mathcal{H}(f; s) = \int_0^1 (f(u)\zeta(1-s, u) - \pi_{-r}(P)(\partial_s)\zeta(1-s, u)) du \quad (29)$$

holds for any s in the region $\operatorname{Re} s > r$, where s is not at a pole of the integrand.

Proof. From (28), as $f \in L^1(0, 1)$, we have $\pi_t(P)(\log u) \in L^1(0, 1)$ for any $t > -1$. Hence, $\operatorname{Re} c > -1$ for any c with $P_c(x) \neq 0$. If $r \geq 1$, the identity (29) holds since $\pi_{-r}(P)(x) = 0$.

Let $r < 1$. By (28) with $t = -r$, we have

$$f(u) - \pi_{-r}(P)(\log u) = O(u^{-r}) \quad \text{as } u \rightarrow 0,$$

and therefore

$$(f(u) - \pi_{-r}(P)(\log u))u^{\delta-1} \in L^1(0, 1)$$

for every $\delta > r$. The result follows from Theorem 1. $\square$

A further simplification is possible by considering the special case of Corollary 1 when $P(x) \in \mathbb{C}\langle\langle x \rangle\rangle$ is of the form $P(x) = \sum_c b_c e^{cx}$ for $b_c \in \mathbb{C}$.

Corollary 2. Suppose $f \in L^1(0, 1)$ has the asymptotic expansion

$$f(u) \sim \sum_c b_c u^c \quad \text{as } u \rightarrow 0,$$

where b_c is a complex number for each $c \in \mathbb{C}$ such that the number of c 's satisfying $b_c \neq 0$ in the region $\operatorname{Re} c < t$ is finite for each $t \in \mathbb{R}$. Then, the Hurwitz transform $\mathcal{H}(f; s)$ continues to a meromorphic function of s in the complex plane, again denoted by $\mathcal{H}(f; s)$, which may have at most simple poles with

$$\operatorname{Res}_{s=-c} \mathcal{H}(f; s) = \begin{cases} b_c & \text{if } c \neq 0, \\ b_0 - \int_0^1 f(u) du & \text{if } c = 0. \end{cases} \quad (30)$$

We also have the identity

$$\mathcal{H}(f; s) = \int_0^1 \left(f(u) \zeta(1-s, u) - \sum_{\operatorname{Re} c < -r} b_c \zeta(1-s-c, u) \right) du$$

in the region $\operatorname{Re} s > r$ for any $r \in \mathbb{R}$, where s is not at a pole of the integrand.

Example 2. For a simple example, let

$$f(u) = a + e^{2\pi i n u}, \quad u \in (0, 1),$$

for an integer $n > 0$ and a constant $a \in \mathbb{C}$. By Lemma 6, we have

$$\mathcal{H}(f; s) = \int_0^1 (a + e^{2\pi i n u}) \zeta(1-s, u) du = \frac{\Gamma(s)}{(2\pi)^s} e^{\frac{\pi i s}{2}} n^{-s}$$

for $\operatorname{Re} s > 0$. Since $\Gamma(s)$ is meromorphic with simple poles at $s = -k \in \mathbb{Z}_{\leq 0}$ with residue $\frac{(-1)^k}{k!}$, it is evident that $\mathcal{H}(f; s)$ is meromorphic in the complex plane with simple poles at $s = -k$, and

$$\operatorname{Res}_{s=-k} \frac{\Gamma(s)}{(2\pi)^s} e^{\frac{\pi i s}{2}} n^{-s} = \frac{(2\pi i n)^k}{k!}$$

for $k \geq 0$.

We can verify that the residues are given by (30). From the Taylor expansion

$$f(u) = a + \sum_{k=0}^{\infty} \frac{(2\pi i n)^k}{k!} u^k,$$

the coefficients of the asymptotic expansion are given by $b_k = \frac{(2\pi i n)^k}{k!}$ for $k > 0$ and $b_0 = a + 1$. The residue of $\mathcal{H}(f, s)$ at $s = -k$ is indeed b_k for $k > 0$, and it is $b_0 - \int_0^1 f(u) du = (a + 1) - a = 1$ at $s = 0$.

Taking $r = 0, -1, -2, \dots$ in Corollary 2, we obtain

$$\frac{\Gamma(s)}{(2\pi)^s} e^{\frac{\pi i s}{2}} n^{-s} = \begin{cases} \int_0^1 (a + e^{2\pi i n u}) \zeta(1-s, u) du & \text{if } \operatorname{Re} s > 0, \\ \int_0^1 ((a + e^{2\pi i n u}) \zeta(1-s, u) - (a+1) \zeta(1-s, u)) du & \\ = \int_0^1 (e^{2\pi i n u} - 1) \zeta(1-s, u) du & \text{if } \operatorname{Re} s > -1, \\ \int_0^1 ((e^{2\pi i n u} - 1) \zeta(1-s, u) - 2\pi i n \zeta(-s, u)) du & \text{if } \operatorname{Re} s > -2, \\ \vdots & \vdots \end{cases}$$

where $s \notin \mathbb{Z}_{\leq 0}$.

Example 3. Theorem 1 is more general than Corollaries 1 and 2, as the Hurwitz transform may continue meromorphically to the complex plane without the function having an asymptotic expansion. For example, let $\chi_{(a,b)}(u)$ denote the characteristic function on the interval (a, b) , and let

$$g(u) = a + e^{2\pi i n u} + \sum_{k=2}^{\infty} \chi_{(\frac{1}{k}, \frac{1}{k} + 2^{-k})}(u), \quad u \in (0, 1),$$

for an integer $n > 0$ and a constant $a \in \mathbb{C}$. We can write $g(u) = f(u) + h(u)$, where $f(u)$ is as in Example 2, and

$$h(u) = \sum_{k=2}^{\infty} \chi_{(\frac{1}{k}, \frac{1}{k} + 2^{-k})}(u).$$

Let

$$P(x) = a + \sum_{k=0}^{\infty} \frac{(2\pi i n)^k}{k!} e^{kx} \in \mathbb{C}\langle x \rangle,$$

and let $r \in \mathbb{R}$. Then, $g(u) - \pi_{-r}(P)(\log u) = h(u) + O(u^{-r})$ as $u \rightarrow 0$. The function $h(u)$ does not have an asymptotic expansion, but

$$h(u)u^{\delta-1} \in L^1(0, 1)$$

for all $\delta \in \mathbb{R}$. Thus,

$$(g(u) - \pi_{-r}(P)(\log u))u^{\delta-1} \in L^1(0, 1)$$

for all $\delta > r$. By Theorem 1, as the argument holds for all $r \in \mathbb{R}$, we conclude that $\mathcal{H}(g; s)$ is meromorphic in the complex plane with simple poles at $s = -k \in \mathbb{Z}_{\leq 0}$, with the same residues as $\mathcal{H}(f; s)$ in Example 2, except at $s = 0$. The residue at $s = 0$ is now $(a + 1) - \int_0^1 g(u) du = 1 - \int_0^1 h(u) du = 1 - \frac{1}{2} = \frac{1}{2}$.

We can verify this conclusion directly, as we have

$$\mathcal{H}(g; s) = \mathcal{H}(f; s) + \int_0^1 h(u) \zeta(1 - s, u) du. \quad (31)$$

The integral in (31) can be written as $\int_0^1 h(u) u^{s-1} du + \frac{1}{s} \int_0^1 h(u) s \zeta(1 - s, u + 1) du$, and we see that it is a meromorphic function in the complex plane with only a simple pole at $s = 0$, with residue $-\frac{1}{2}$.

4. Relation to the Mellin Transform

Definition 5. For $f \in L^1(0, 1)$, let us call the unique element

$$\tilde{f} \in L^1_{\text{loc}}(0, \infty)$$

the periodic extension of f if there are representatives satisfying

$$\tilde{f}(x) = f(\{x\})$$

for all $x \in (0, \infty) - \mathbb{N}$, where $\{x\}$ is the fractional part of x .

Theorem 2. Suppose $f \in L^1(0, 1)$ has a representative such that $f(u) = O(u^\alpha)$ as $u \rightarrow 0$ for some $\alpha > 0$. Let $\tilde{f}$ be the periodic extension of f given in Definition 5. Then, the following identity

$$\int_0^1 f(u) \zeta(1 - s, u) du = \int_0^\infty \tilde{f}(u) u^{s-1} du \quad (32)$$

holds for $-\alpha < \operatorname{Re} s < 0$.

Proof. By Proposition 1, the integral on the left is well defined for all $\operatorname{Re} s > -\alpha$ and $s \neq 0$. If $-\alpha < \operatorname{Re} s < 0$, then by Tannery's theorem,

$$\begin{aligned}
\int_0^1 f(u) \zeta(1-s, u) du &= \int_0^1 f(u) u^{s-1} du + \int_0^1 f(u) \sum_{n=1}^{\infty} (n+u)^{s-1} du \\
&= \sum_{n=0}^{\infty} \int_0^1 f(u) (n+u)^{s-1} du \\
&= \sum_{n=0}^{\infty} \int_0^1 \tilde{f}(n+u) (n+u)^{s-1} du = \int_0^{\infty} \tilde{f}(u) u^{s-1} du,
\end{aligned}$$

as claimed. $\square$

5. Application to Mikolás' Identity

As an example of the previous discussion, we consider the following identity derived by Mikolás in [5], from which other identities are also derived in [4]:

$$\int_0^1 \zeta(s, u) \zeta(z, u) du = \frac{\Gamma(1-s)\Gamma(1-z)}{(2\pi)^{2-s-z}} 2 \cos\left(\frac{\pi}{2}(s-z)\right) \zeta(2-s-z) \quad (33)$$

in the region $\max\{\sigma, 0\} + \max\{x, 0\} < 1$, where $\sigma = \operatorname{Re} s$ and $x = \operatorname{Re} z$. By a change of variables, (33) is equivalent to

$$\int_0^1 \zeta(1-s, u) \zeta(1-z, u) du = \frac{\Gamma(s)\Gamma(z)}{(2\pi)^{s+z}} 2 \cos\left(\frac{\pi}{2}(s-z)\right) \zeta(s+z) \quad (34)$$

in the region $\min\{\sigma, 1\} + \min\{x, 1\} > 1$.

We can derive the identity (34) and its generalization by considering the Hurwitz transform of $f(u) = \zeta(1-z, u)$.

Proposition 2. Let $\operatorname{Re} z > 0$ and $f(u) = \zeta(1-z, u)$ for $u \in (0, 1)$. Then, $f \in L^1(0, 1)$ and the Fourier series $\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$ converges to f in the L^1 -norm, where

$$a_n = \begin{cases} \frac{\Gamma(z)}{(2\pi)^z} e^{-\frac{\pi i z}{2}} n^{-z} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \frac{\Gamma(z)}{(2\pi)^z} e^{\frac{\pi i z}{2}} |n|^{-z} & \text{if } n < 0. \end{cases}$$

We also have the equality

$$f(u) = u^{z-1} + \sum_{n=0}^{\infty} \binom{z-1}{n} \zeta(n+1-z) u^n, \quad (35)$$

valid for $u \in (0, 1)$, where $\binom{z-1}{n} = \frac{(z-1)(z-2)\cdots(z-n)}{n!}$. The expression $\binom{z-1}{n} \zeta(n+1-z)$ is well defined for all z for $n \geq 1$, as the pole of $\zeta(n+1-z)$ at $z = n$ cancels with the zero from the factor $(z-n)$ in $\binom{z-1}{n}$.

Proof. The identity

$$f(u) = u^{z-1} + \zeta(1-z, u+1) \quad (36)$$

from (3) shows $f(u) = O(1+u^{x-1})$ where $x = \operatorname{Re} z > 0$. Hence $f(u) \in L^1(0, 1)$ and in fact $f(u) \in L^{1+\varepsilon}(0, 1)$ for some $\varepsilon > 0$, and the Fourier series converges to f in the L^1 norm by Lemma 2. The Fourier coefficients a_n can be read off from the Hurwitz Formula (4) or obtained by replacing n with $-n$ in Lemma 6.

The second term $\zeta(1-z, u+1)$ in (36) is analytic in $|u| < 1$, and we have the Taylor series

$$\zeta(1-z, u+1) = \sum_{n=0}^{\infty} \frac{\partial_u^n \zeta(1-z, u+1)|_{u=0}}{n!} u^n, \quad (37)$$

valid in the unit disk. Since $\partial_u^n \zeta(1-z, u+1)|_{u=0}$ for each $n \geq 0$ is meromorphic in z , the relation

$$\frac{\partial_u^n \zeta(1-z, u+1)|_{u=0}}{n!} = \binom{z-1}{n} \zeta(n+1-z) \quad (38)$$

that holds for $\operatorname{Re} z < 0$ from (1) can be continued meromorphically to the complex plane. We also see that the right-hand side of (38) is entire for $n \geq 1$. The formula (35) follows by adding u^{z-1} to (37) for $u \in (0, 1)$. $\square$

Theorem 3. Let s and z be complex numbers with $\sigma = \operatorname{Re} s$, $x = \operatorname{Re} z$. Let

$$F(s, z) = \frac{\Gamma(s)\Gamma(z)}{(2\pi)^{s+z}} 2 \cos\left(\frac{\pi}{2}(s-z)\right) \zeta(s+z).$$

For a given z , assume that s is not at the poles of the integrand in each of the following. Then, we have the following identities.

(i) For $\min\{x, 1\} + \min\{\sigma, 1\} > 1$,

$$F(s, z) = \int_0^1 \zeta(1-s, u) \zeta(1-z, u) du,$$

and more generally, for any integer $m \geq 0$ and for $\min\{x, 1+m\} + \min\{\sigma, 1-m\} > 1$,

$$F(s, z) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \sum_{0 \leq n < m} \binom{z-1}{n} \zeta(n+1-z) \zeta(1-s-n, u)) du.$$

(ii) For $\sigma > 0$ and $0 < x < 1$,

$$F(s, z) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \zeta(2-s-z, u)) du,$$

and more generally for any integer $m \geq 0$ and for $\sigma > -m$ and $0 < x < m+1$,

$$F(s, z) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \zeta(2-s-z, u) - \sum_{0 \leq n < m} \binom{z-1}{n} \zeta(n+1-z) \zeta(1-s-n, u)) du.$$

Proof. Fix z such that $x > 0$, and let $f(u) = \zeta(1-z, u)$ for $u \in (0, 1)$ as in Proposition 2. By Lemma 3, the Hurwitz transform of f is given by

$$\begin{aligned} \mathcal{H}(f; s) &= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right) \\ &= \frac{\Gamma(s)}{(2\pi)^s} \frac{\Gamma(z)}{(2\pi)^z} \left(e^{\frac{\pi i s}{2}} e^{-\frac{\pi i z}{2}} \sum_{n=1}^{\infty} \frac{1}{n^z n^s} + e^{-\frac{\pi i s}{2}} e^{\frac{\pi i z}{2}} \sum_{n=1}^{\infty} \frac{1}{n^z n^s} \right), \end{aligned}$$

which, since $x > 0$, converges at least for $\sigma > 1$ to $F(s, z) = \frac{\Gamma(s)\Gamma(z)}{(2\pi)^{s+z}} 2 \cos\left(\frac{\pi}{2}(s-z)\right) \zeta(s+z)$.

From the equality (35), we have the asymptotic expansion

$$f(u) \sim \sum_c b_c u^c,$$

with the following coefficients. If $z - 1 \notin \mathbb{Z}_{\geq 0}$, then, $b_{z-1} = 1$ and $b_n = \binom{z-1}{n} \zeta(n+1-z)$ for $n \in \mathbb{Z}_{\geq 0}$. If $z - 1 = k$ for some $k \in \mathbb{Z}_{\geq 0}$, then $b_k = 1 + \zeta(0) = \frac{1}{2}$, $b_n = \binom{k}{n} \zeta(n-k)$ for $n \neq k$. In this case, $b_{k+1} = -\frac{1}{k+1}$, and $b_n = 0$ for all $n > k+1$.

We obtain integral representations of $F(s, z)$ by considering the following cases, depending on the choice of r in Corollary 2.

1. Let $r = 1 - x$. Since $x > 0$, we have $m - 1 < x - 1 \leq m$ for some $m \in \mathbb{Z}_{\geq 0}$, and

$$\mathcal{H}(f; s) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \sum_{0 \leq n < m} \binom{z-1}{n} \zeta(n+1-z) \zeta(1-s-n, u)) du \quad (39)$$

in the region $\sigma > 1 - x$.

2. Let $r = -m$ for some $m \in \mathbb{Z}_{\geq 0}$ and suppose $x - 1 \geq m$. We have

$$\mathcal{H}(f; s) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \sum_{0 \leq n < m} \binom{z-1}{n} \zeta(n+1-z) \zeta(1-s-n, u)) du \quad (40)$$

in the region $\sigma > -m$.

3. Let $r = -m$ for some $m \in \mathbb{Z}_{\geq 0}$ and suppose $x - 1 < m$. Then,

$$\mathcal{H}(f; s) = \int_0^1 (\zeta(1-s, u) \zeta(1-z, u) - \zeta(2-s-z, u) - \sum_{0 \leq n < m} \binom{z-1}{n} \zeta(n+1-z) \zeta(1-s-n, u)) du \quad (41)$$

in the region $\sigma > -m$.

We thus have the following region of validity for each of the above equations, for each $m \in \mathbb{Z}_{\geq 0}$.

Equation	Region of validity
(39)	$\sigma + x > 1$ and $m < x \leq m + 1$
(40)	$\sigma > -m$ and $x \geq m + 1$
(41)	$\sigma > -m$ and $0 < x < m + 1$

The first two Equations (39) and (40) are identical. Hence, this identity is valid for $\sigma + x > 1$ and $m < x \leq m + 1$, or $\sigma > -m$ and $x \geq m + 1$. This condition on σ and x can be written as

$$\min\{x, 1 + m\} + \min\{\sigma, 1 - m\} > 1,$$

and we obtain (i). The Equation (41) shows (ii). $\square$

6. Relation to the H -Series

Suppose $f \in L^1(0, 1)$ and the Fourier series converges to f in the L^1 -norm. By Lemma 3, the H -series $H(f; s)$ attached to f is related to the Hurwitz transform $\mathcal{H}(f; s)$ of f by

$$H(f; s) = \frac{(2\pi)^s}{\Gamma(s)} \mathcal{H}(f; s)$$

for $\operatorname{Re} s > 1$. We obtain the following sufficient condition for $H(f; s)$ to continue to the complex plane as an entire function, under a weaker condition on f than the condition

considered in [6]. We also obtain its representations in terms of an integral involving the Hurwitz zeta function.

Theorem 4. Suppose we have $f \in L^1(0, 1)$ such that the Fourier series

$$\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$$

converges to f in the L^1 -norm, and f has an asymptotic expansion

$$f(u) \sim \sum_{n=0}^{\infty} b_n u^n \quad \text{as } u \rightarrow 0, \quad b_n \in \mathbb{C}, \quad n \in \mathbb{Z}_{\geq 0}.$$

Then, the following hold.

(i) The H -series attached to f given by

$$H(f; s) = \sum_{n=1}^{\infty} \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right)$$

for $\operatorname{Re} s > 1$ continues to an entire function of s , which we again denote by $H(f; s)$.

(ii) For any integer $m \geq 0$, we have the identity

$$H(f; s) = \int_0^1 \left(f(u) \frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s, u) - \sum_{0 \leq n < m} b_n \frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s-n, u) \right) du \quad (42)$$

for any s in the region $\operatorname{Re} s > -m$. Here, the terms $\frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s-n, u)$ are regarded as entire functions of s and are thus defined for all $s \in \mathbb{C}$.

Proof. By Corollary 2, the Hurwitz transform $\mathcal{H}(f; s)$ of f is a meromorphic function with at most simple poles at the set $\mathbb{Z}_{\leq 0}$, and $\Gamma(s)$ has simple poles with residue $\frac{(-1)^n}{n!}$ at $s = -n \in \mathbb{Z}_{\leq 0}$. Hence $H(f; s) = \frac{(2\pi)^s}{\Gamma(s)} \mathcal{H}(f; s)$ is an entire function.

The residue of $\mathcal{H}(f; s)$ at $s = -n < 0$ is b_n while it is $b_0 - \int_0^1 f(u) du = b_0 - a_0$ at $s = 0$. Hence, for $n \in \mathbb{Z}_{\geq 0}$,

$$H(f; -n) = \begin{cases} b_0 - a_0 & \text{if } n = 0, \\ (-2\pi)^{-n} n! b_n & \text{if } n > 0. \end{cases}$$

For $s \notin \mathbb{Z}_{\leq 0}$, the identity (42) follows from Corollary 2. We can verify that (42) also holds for $s = -n \in \mathbb{Z}_{\leq 0}$, using $\frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s-n, u)|_{s=-n} = -(-2\pi)^{-n} n!$ for $n \in \mathbb{Z}_{\geq 0}$. Indeed, for $s = 0$,

$$\int_0^1 (-f(u) + b_0) du = b_0 - a_0 = H(f; 0),$$

and at $s = -n$ for $n > 0$,

$$\int_0^1 b_n (-2\pi)^{-n} n! du = (-2\pi)^{-n} n! b_n = H(f; -n),$$

which completes the proof. $\square$

7. Application to a Lambert Series

As an example, we can apply Theorem 4 to find another integral representation of a Lambert series considered in [6]. We first define the classical Eisenstein series and the Weierstrass zeta function [8–10].

For $\text{Im } \tau > 0$, the Eisenstein series of weight $2k$, for any integer $k \geq 2$, is defined by

$$G_{2k}(\tau) = \sum_{\omega \in (\mathbb{Z} + \mathbb{Z}\tau)^\times} \omega^{-2k},$$

where $(\mathbb{Z} + \mathbb{Z}\tau)^\times = \{m + n\tau \mid m, n \in \mathbb{Z}\} \setminus \{0\}$. The Eisenstein series of weight 2 is defined, with the notation $\omega_{m,n} = m + n\tau$, by

$$G_2(\tau) = \sum_{m \neq 0} \omega_{m,0}^{-2} + \sum_{n \neq 0} \left(\sum_{m \in \mathbb{Z}} \omega_{m,n}^{-2} \right).$$

The Weierstrass zeta function is given, for $u \in \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau)$, by

$$\zeta_{\mathbb{R}}(u, \tau) = \frac{1}{u} + \sum_{\omega \in (\mathbb{Z} + \mathbb{Z}\tau)^\times} \left(\frac{1}{u - \omega} + \frac{1}{\omega} + \frac{u}{\omega^2} \right),$$

where the notation “ $\zeta_{\mathbb{R}}$ ” is used to distinguish it from the Hurwitz zeta function.

Lemma 7. Let $\text{Im } \tau > 0$. We have the equality

$$\zeta_{\mathbb{R}}(u, \tau) - G_2(\tau)u - \pi \cot(\pi u) = -2\pi i \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} (e^{2\pi i n u} - e^{-2\pi i n u})$$

with $q = e^{2\pi i \tau}$, valid for $|\text{Im } u| < \text{Im } \tau$, and

$$\zeta_{\mathbb{R}}(u, \tau) - G_2(\tau)u - \pi \cot(\pi u) = \sum_{j=1}^{\infty} (2\zeta(2j) - G_{2j}(\tau)) u^{2j-1},$$

valid for $|u| < \min\{|m + \tau| \mid m \in \mathbb{Z}\}$.

Lemma 7 follows from the definitions, and a proof can be found in ([6], Proposition 2, Corollary 1). The Lambert series $A(\tau, s)$ in the following theorem was introduced in [11] and has been studied in several works, for which some references can also be found in [6].

Theorem 5. Let $\text{Im } \tau > 0$ and $s \in \mathbb{C}$. Let

$$A(\tau, s) = \sum_{n=1}^{\infty} n^{s-1} \frac{q^n}{1 - q^n},$$

where $q = e^{2\pi i \tau}$. Then, we have the identity

$$4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s) = \int_0^1 (\zeta_{\mathbb{R}}(u, \tau) - G_2(\tau)u - \pi \cot(\pi u)) \frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s, u) du \quad (43)$$

for $\text{Re } s > -1$, and for any integer $k \geq 0$,

$$\begin{aligned} 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1-s) &= \int_0^1 \left((\zeta_{\mathbb{R}}(u, \tau) - G_2(\tau)u - \pi \cot(\pi u)) \frac{(2\pi)^s}{\Gamma(s)} \zeta(1-s, u) \right. \\ &\quad \left. - \sum_{j=1}^k (2\zeta(2j) - G_{2j}(\tau)) \frac{(2\pi)^s}{\Gamma(s)} \zeta(2-s-2j, u) \right) du \end{aligned} \quad (44)$$

for $\text{Re } s > -(2k+1)$.

Proof. Let $f(u) = \zeta_{\mathbb{R}}(u, \tau) - G_2(\tau)u - \pi \cot(\pi u)$. The Fourier coefficients a_n and the asymptotic expansion coefficients b_n are given in Lemma 7, and the Fourier series converges to f uniformly in $u \in (0, 1)$. The H -series attached to f is

$$\begin{aligned} H(f; s) &= \sum_{n=1}^{\infty} \left(e^{\frac{\pi i s}{2}} \frac{a_n}{n^s} + e^{-\frac{\pi i s}{2}} \frac{a_{-n}}{n^s} \right) \\ &= -2\pi i \left(e^{\frac{\pi i s}{2}} - e^{-\frac{\pi i s}{2}} \right) \sum_{n=1}^{\infty} n^{-s} \frac{q^n}{1 - q^n} \\ &= 4\pi \sin\left(\frac{\pi s}{2}\right) A(\tau, 1 - s). \end{aligned}$$

By Theorem 4, $H(f; s)$ is an entire function, as can also be seen directly. The integral representations (43) and (44) also follow from Theorem 4 by letting $m = 2k + 1$ with $b_{2j-1} = 2\zeta(2j) - G_{2j}(\tau)$ for $j \geq 1$. $\square$

8. Conclusions

We have studied some properties of the Hurwitz transform, especially the conditions under which it extends meromorphically to the complex plane. We have given two applications, first Mikolás' identity (Theorem 3) as an example of Corollary 2, and second an integral representation of a Lambert series (Theorem 5) as an example of Theorem 4.

In view of Theorem 2, the results in this article give a representation of the Mellin transform of a periodic function in terms of an integral involving the Hurwitz zeta function. We observe the left-hand side of (32) is valid on a right half plane, while the right-hand side is valid only on a strip inside the half plane. Conceptually, for periodic functions, one may say the integral defining the Hurwitz transform gives the zeta-regularized version of the Mellin transform, giving the analytic continuation of the latter to a right half plane.

The new integral representations (43) and (44) of the Lambert series are valid in half planes instead of strips. Another advantage is that the integrands do not have poles to be avoided. Interestingly, the Formulas (43) and (44) involve the product of two rather different zeta functions, Hurwitz and Weierstrass.

The Lambert series is closely related to the sum of positive divisors function. It is hoped that further study finds some applications of these formulas to this arithmetic function.

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