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# Maintenance Modeling for a Multi-State System Under Competing Failures and Imperfect Repairs

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**Abstract**

A condition-based maintenance modeling approach is proposed for a multi-state system under competing failures and imperfect repairs. The system experiences three states (normal, defective and failed) over its lifecycle. Two competing failure processes, i.e., natural degradation and external shocks, cause these state changes. If the system becomes defective, an imperfect repair is adopted to restore it to a normal state. Imperfect repairs addressing defects are mathematically characterized. Based on this, two system renewal scenarios and their occurrence probabilities are simulated and derived. The cost of downtime caused by hidden failures is then deduced. A maintenance model of the expected cost rate is constructed, and the optimal inspection period that minimizes the expected cost rate is determined. Finally, a numerical example verifies the correctness and effectiveness of the maintenance model.

**Keywords:** multi-state system; maintenance model; competing failures; imperfect repair; inspection strategy

**MSC:** 65C20; 65E05

## 1. Introduction

With rapid scientific and technological advances, engineering systems today are being designed with increasingly powerful and complex functions. Consequently, growing economic losses and societal impacts from system risks and failures have prompted researchers to develop methods for reducing risks and improving system performance and reliability [1]. In classical reliability theory, systems are typically modeled as having two distinct states: perfect functioning and complete failure. However, this binary representation is often too simplistic to capture the behavior of complex real-world systems. In reality, a system and its components do not fail abruptly but go through intermediate states of partial performance degradation. In other words, the system can operate at finite distinct performance levels. Hence, the system is modeled as a multi-state system (MSS) [2]. MSSs are more suitable for the actual state of systems throughout their lifecycle since they can exhibit multiple performance levels between full functionality and total failure. For instance, permanent magnet synchronous motor in ship propulsion aerospace [3], electrical power system, production and manufacturing system, pipe transmission system [4–7], and the like can all be regarded as MSSs.

The evolutions of MSSs are shaped by the simultaneous influence of degradation and external shocks. Upon exposure to external shocks, these systems may undergo progressive

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degradation, transitioning from a stable state to one of heightened vulnerability, and ultimately to complete failure. For instance, the performance of critical systems in aerospace, rail transit, and power systems is influenced by both their natural degradation and random shocks, including high voltage, vibration, impact, turbulence, and extreme weather. Internal degradation generally results from the operational workload of the system. External shocks stem from random environmental damages, including both natural and human-induced factors. For instance, a battery's operation is fundamentally driven by internal chemical reactions, and its progressive degradation results from the gradual consumption of these active materials. Simultaneously, environmental shocks can cause an abrupt acceleration of degradation, such as overheating or over-voltage. Both internal degradation and external shocks can lead to system failure, though they manifest differently.

Failures due to internal degradation are often termed soft failures, characterized by a gradual decline in performance until the system can no longer meet operational requirements. These includes phenomena such as mechanical wear and tear or the aging of insulating materials. Failures that result from external shocks are called hard failures, which involve sudden losses of functionality, such as device breakdowns or short circuits [8]. A system is considered failed when either of the two competing failure modes occurs first. Maintenance is crucial for enhancing the reliability and safety of engineered systems. Among the various maintenance approaches, condition-based maintenance (CBM) is distinguished by its potential advantages through real-time condition monitoring since it can monitor and respond to the system's health status, thereby preventing failures [9,10]. At present, it has been widely applied to developing maintenance plans. Both academic research and engineering practice have demonstrated that CBM is more effective and offers distinct advantages in reducing or preventing failures [11]. In particular, with the rapid advancement of sensor technology and signal processing methods, the benefits of CBM in supporting maintenance decisions have become even more pronounced.

Numerous studies have explored CBM for MSSs subject to two competing failure modes, with a primary focus on developing and optimizing inspection and maintenance models. The first step in CBM involves assessing the system state through inspections, followed by appropriate maintenance actions based on the assessment. A key contribution of these works lies in their modeling approaches, where formulated models are largely influenced by inspection and maintenance policies. In practice, however, inspection quality is often compromised by factors such as measurement errors, sensor degradation, and other uncertainties, meaning inspections are typically imperfect. The issue of imperfect inspection in MSSs with competing failure modes has been addressed by several researchers [12,13]. In these studies, maintenance actions based on inspection outcomes predominantly involved replacement. Nevertheless, replacement is often more costly than repair, particularly for addressing partial system defects. In such cases, imperfect repair strategies may be more suitable, as they allow maintenance personnel to control the extent of reliability improvement, thus making repair levels a decision variable [14,15]. Despite its potential, imperfect repair has not received sufficient attention within CBM frameworks for MSSs subject to competing failure modes.

Motivated by both theoretical foundations and industrial practice, this study develops CBM models for MSSs with two competing failure modes to derive an optimal inspection policy. The system undergoes gradual degradation, leading to a decline in performance from a normal state to a defective state, and eventually to failure. Additionally, the system is exposed to random shocks from the external environment. A sudden failure occurs if the load of any shock exceeds a certain threshold.

Periodical inspections are conducted to monitor the system state. Based on the inspection results, the operator decides whether to perform maintenance. When a defect is

identified, the system undergoes an imperfect repair, modeled using a proportional age reduction approach. This repair restores the system from a defective state to a normal state but does not return it to an "as good as new" condition. If a failure is detected, whether due to gradual degradation or a fatal shock, a failure-based replacement is performed. Furthermore, if the system remains in a normal state after a predetermined number of inspections, an age-based preventive replacement is conducted. Both types of replacement restore the system to an "as good as new" state.

We first derive the probabilities associated with these two system renewal scenarios under imperfect repair. Based on these probabilities, we formulate a CBM model for the expected cost rate. To evaluate the performance of the proposed model, optimization and sensitivity analysis are conducted. The results demonstrate that the optimal inspection policy can be determined by minimizing the expected cost rate. Furthermore, improving the repair effectiveness significantly reduces inspection frequency and lowers maintenance costs. A thoroughly comparison with highly relevant previous studies is provided in Table 1, highlighting the key differences and novelties of our work.

**Table 1.** Comparison with highly relevant studies.

| Literature | Multiple States | Multiple Failure Modes | Hidden Failures | Imperfect Repairs |
|---|---|---|---|---|
| Peng et al. [16] | √ | √ | × | × |
| Zhang et al. [13] | √ | √ | √ | × |
| Tang et al. [17] | √ | × | √ | √ |
| Liang et al. [18] | √ | × | × | √ |
| Tekin et al. [19] | √ | × | √ | √ |
| This paper | √ | √ | √ | √ |

Note: "√" indicates that the feature of the corresponding column is considered. An "×" indicates that it was not taken into consideration.

In summary, the main contributions of this study are as follows:

- A novel CBM model is developed for MSSs subject to two competing failure modes, incorporating imperfect repairs. The proposed methodology accommodates multiple system states, diverse failure modes, and imperfect maintenance actions, offers enhanced realism and addresses greater practical complexity.
- The model explicitly incorporates imperfect repair to better represent the impact of maintenance on system condition. Results underscore the importance of considering imperfect repair, as it significantly affects both the optimal inspection policy and the minimal expected cost rate.
- A comprehensive sensitivity analysis is conducted to examine how the optimal inspection policy responds to changes in key parameters. The findings provide valuable managerial insights that can guide effective maintenance decision-making for real-world MSSs exposed to competing failure mechanisms.

The paper is organized as follows. Section 2 provides a review of the related literature. In Section 3, we present a detailed description of the system and the underlying assumptions. Section 4 develops the CBM model for the expected cost rate, incorporating the derived probabilities of imperfect repair, failure-based corrective replacement, and age-based preventive replacement, along with the computed costs associated with downtime. The solution approach for determining the optimal inspection interval and the minimum expected cost rate is introduced in Section 5. Section 6 offers a numerical example to demonstrate the application of the proposed model and presents a sensitivity analysis.

Finally, Section 7 summarizes the main conclusions and suggests potential directions for future research.

## 2. Literature Review

In this section, we first focus on studies about MSSs. Then, we discuss studies about maintenance policies for MSSs. Finally, we introduce papers on imperfect repair for MSSs.

### *2.1. Research Topics on MSSs*

A substantial body of research has been dedicated to MSSs, with major efforts concentrated in areas such as reliability analysis [20–22], resilience analysis [23–25], and maintenance policies [26–28]. For example, Janada et al. [29] developed a new design of Angular Control Chart to monitor the reliability of MSSs, where failure data can be modeled by any continuous probability distribution. The $L_z$ transformation has been widely used in the reliability modeling of MSSs. To address the problem of complex calculations caused by the dense random combination of multi-state performance parameters in the $L_z$ transformation, Zheng et al. [30] defined a screening function prior to the $L_z$ transformation and combined with the performance threshold to pre-screen the state performance parameters in advance. Mi et al. [31] introduced a reliability analysis method for complex MSSs with epistemic uncertainty, which was quantified by adding an uncertain state to the root nodes in the multi-state evidential network. Tan et al. [32] focused on MSSs exposed to disruptive events and developed a comprehensive resilience modeling and quantifying framework based on Markov processes. To enable a more comprehensive understanding of system characteristics, Dui et al. [33] investigated transient resilience evaluation for systems subjected to competing failures. They introduced the overall resilience of the system based on its multi-state division, along with resistibility, absorbability, and recoverability indices, and subsequently proposed resilience models based on both reliability and cost.

### *2.2. Maintenance of MSSs*

In the study of MSSs, maintenance strategies represent a prominent research focus. Timely and effective maintenance planning can restore aging systems to improved conditions, thereby preventing unexpected failures and mitigating undesirable consequences during operation [34]. Chen et al. [35] proposed a joint optimization model for fleet-level sequential selective maintenance and repairpersons assignment under flow dependency and uncertain maintenance duration. In the proposed optimization model, the reliability of multi-state manufacturing systems was incorporated. Pandey et al. [36] developed a selective maintenance strategy for a series-parallel MSS. In their study, imperfect maintenance was considered as a maintenance option alongside replacement and the "do-nothing" option. Ma et al. [37] developed a new selective maintenance model by integrating maintenance decisions and task assignment planning for multi-unit systems executing multiple missions, and a cooperative co-evolutionary genetic algorithm was tailored to solve the joint optimization problem. The results from a numerical example and an air defense system showed that the proposed method can efficiently improve the success probability of future missions by integrating selective maintenance and task assignment.

Advancements in sensing technologies have significantly enhanced the application of condition monitoring data in maintenance practices, leading to increased effectiveness and adoption of CBM strategies. As a result, research on CBM for multi-state systems has attracted growing interest [38–41]. For example, Cao et al. [42] designed a condition-based inspection policy to capture the states of a multi-state degradation system in a timely manner, and based on this, a new CBM policy for systems subject to multi-state deterioration and random shocks was proposed. Zhao et al. [43] focused on the effect of environmen-

tal factors on system failure behavior and maintenance decisions and investigated the optimal joint inspection interval, CBM, and loading policies for systems operating in a random shock environment. Zhang et al. [13] constructed a CBM model for a three-state system subject to competing and hidden failures, where periodic inspection was utilized to identify all states of the system. Wang and Zhu [44] considered the joint optimization of periodic condition-based replacement of components and inventory control for a k-out-of-n: F system, which comprised degrading components that have multiple statuses and are non-repairable.

In recent years, the increasing complexity of engineering systems has presented significant challenges in addressing maintenance optimization problems. Machine learning methods with advanced computational techniques demonstrate powerful capability in solving complicated maintenance problems under uncertainty, such as Markov theory [45], Bayesian method [46,47], Gaussian process [48], and reinforcement learning [49]. Wang et al. [50] systematically reviewed the applications of the Bayesian network in structural health monitoring and found that Bayesian network enhanced the accuracy and reliability of monitoring systems, providing a theoretical foundation for optimization of maintenance strategies. A step-by-step guideline on the utilization of reinforcement learning algorithms for resolving maintenance optimization problems was offered by Zhang et al. [51]. Zhou et al. [52] proposed a deep learning-based intelligent multilevel predictive maintenance framework for series-parallel MSS considering comprehensive cost. A comparative analysis demonstrated that their proposed framework yielded a more effective maintenance strategy.

### *2.3. Imperfect Repair in Maintenance Policies of MSSs*

Over the past few decades, significant attention has been devoted to the concept of imperfect repair. It refers to a maintenance intervention that falls between perfect repair and minimal repair. A perfect repair restores an item to an "as good as new" condition, while a minimal repair returns it to an "as bad as old" state. Researchers have increasingly incorporated imperfect repair into maintenance models for MSSs. Dong et al. [53] scheduled an imperfect maintenance policy for a single-unit MSS. They incorporated a CBM action and a corrective maintenance (CM) activity were incorporated in each operation stage, assuming that the CBM action constituted a minimal repair and the CM activity was imperfect effective. Finkelstein and Cha [54] investigated a new approach to modeling the imperfect repair and described the corresponding imperfect repair processes for items with an observable degradation process. They introduced the concept of random virtual age was introduced to define the state of an item after an imperfect repair, which reduced the degradation of an item upon failure to some intermediate level. In the research by Liang et al. [18], the state of the system before repair was categorized into three classes based on its internal degradation level: failure, major defect, and minor defect, with three corresponding thresholds. Corresponding repairs were carried out to reduce the system's degradation to different levels. If the internal degradation level of the system was recognized as a minor defect, an imperfect repair was implemented to scale down the degradation level below the minor defect threshold. If the degradation level was identified as a major defect, an upgraded imperfect repair was executed. Otherwise, if the degradation was beyond the failure threshold, replacement would be carried out. Ultimately, a novel hierarchical imperfect maintenance structure was introduced, and a multi-variable repair cost model was constructed. Tang et al. [17] considered imperfect inspection and imperfect repair for a system subject to a three-stage degradation process. The concepts of virtual age and the improvement factor were adopted to characterize the imperfect repair effect. The case of a steel converter plant was applied to verify the effectiveness of the proposed

model. Chen and Zhao [55] proposed a maintenance optimization method applicable to dependent two-component systems subject to degradation and imperfect repair. They established a random-effect imperfect repair model to describe the degradation process and maintainability of components. The study discovered that the characteristics of imperfect repair can considerably influence the optimal policies. Hu et al. [40] proposed a CBM model for multi-state systems that operate under time-varying environmental conditions, the evolution of which was described by a Markov process. Both imperfect maintenance and replacement actions were considered in the constructed CBM model. The inspection interval, preventive maintenance threshold and number of imperfect maintenance actions in a replacement cycle were jointly determined to minimize the long-run average cost.

Existing research has primarily focused on MSSs influenced by a single failure mode, whereas the imperfect repair of MSSs subject to two competing failure modes has received comparatively little attention.

## 3. System Description and Basic Assumptions

In this paper, an MSS subject to two competing failure modes is taken as a research object to explore its maintenance modeling methodology. Furthermore, the failures of the MSS are considered hidden. Such failures are not obvious to operators and can only be identified through inspections [13]. When there are no external shocks, it is assumed that the system goes through three states during the degradation process: normal, defective, and failed. As shown in Figure 1, initially, the system is in a normal state. As the system's operational time increases, its performance gradually degrades, entering a defective state. Eventually, the core functions are lost, resulting in system failure.

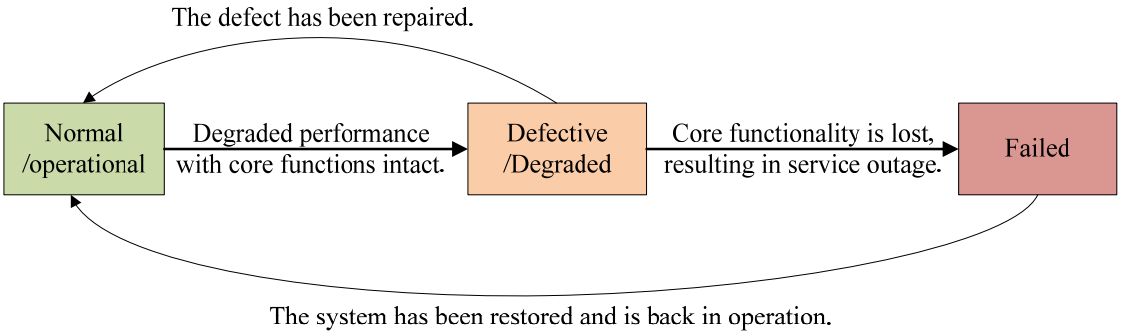



**Figure 1.** State transition diagram of the system without external shocks.

However, the system will inevitably be affected by external shocks. As shown in Figure 2, suppose that shocks reach the system at $t_1$, $t_2$, …, $t_j$ and their loads are $W_1$, $W_2$, $W_3$, …, $W_j$. If the magnitude $W_j$ of the $j$th shock (where $j = 1, 2, \ldots, \infty$) exceeds the sudden failure threshold $D$, the system is deemed to have experienced a sudden failure. The degradation failure and sudden failure compete with each other, and the failure mode that occurs first ultimately leads to system failure.

In this paper, the maintenance model of the MMS under competing failures is constructed based on the CBM strategy, which is based on the state of the system to formulate the appropriate maintenance action. The state of the MSS needs to be identified through inspection. Among various inspection technologies, periodic inspection is widely used due to its ease of implementation and effectiveness in improving system performance. Therefore, this article adopts a periodic inspection strategy to identify the state of the system and then arranges corresponding maintenance activities for the MSS based on the detected states. Maintenance behavior based on the system state consists of the following 3 main types: (1) If the system is identified as a defective state at the $k$th ($k = 1, 2, \ldots, n$)

inspection, an imperfect repair is performed on the system. (2) At the $k$th ($k$ = 1, 2, . . ., $n$) inspection, if the system is detected to have entered a failed state, a corrective replacement is implemented to restore the system as new. (3) If the system is still normal at the $n$th inspection, a preventive replacement is performed to restore the system as new to enhance the system's ability to withstand external shocks.

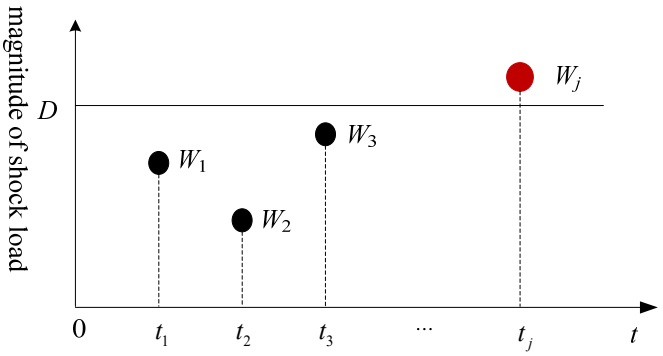



**Figure 2.** Schematic diagram of external shocks. Note: Black circles indicate non-fatal shocks, and red circles indicate fatal shocks.

The following research hypotheses are proposed to better construct a maintenance model for the MSS subject to competing failures and imperfect repairs.

**Assumption 1.** *The MSS is susceptible to hidden failures, which result in downtime and consequent economic loss.*

**Assumption 2.** *Repairs for system defects are considered imperfect repairs. The effect of repair in engineering practice is usually imperfect. A system can seldom be restored to an "as-new" state but is instead returned to a condition intermediate between "as new" and "as old".*

**Assumption 3.** *Degradation failure and sudden failure are independent, with system failure resulting from the competition between these two modes. The degradation process is modeled using a delay-time approach, while the sudden failure process is described by a homogeneous Poisson process.*

**Assumption 4.** *The system's duration in the normal and defective states are random variables, denoted by X and Y, respectively, with probability density functions $f_X(x)$ and $f_Y(y)$. Shocks arrive at the system with intensity $\lambda$, and their magnitudes are independent and identically distributed normal random variables,$W_j \sim N(\mu_W, \sigma_W^2), j = 1, 2, \ldots \infty$.*

**Assumption 5.** *Inspections are assumed to be perfect and all maintenance actions (repair, preventive, and corrective replacement) are assumed to be instantaneous.*

## 4. Maintenance Modeling for MSSs Subject to Competing Failure Processes and Imperfect Repairs

There is a critical trade-off in maintenance scheduling: overly frequent inspections waste resources, but excessively long intervals may cause missed opportunities for timely repairs. The objective of this study is to establish the optimal inspection interval for the MSS through the development and optimization of a maintenance model, with the goal of minimizing the total maintenance cost over the long term.

Two of the three maintenance behaviors mentioned in Section 3 will renew the system. The first system renewal scenario is triggered upon detecting a system failure at the $k$th ($k$ = 1, 2, . . ., $n$) inspection, leading to a corrective replacement. The second type of system renewal scenario is a preventive replacement when the system is inspected as normal until the $n$th inspection. Let the period inspection interval be $T$. If the time interval between two consecutive replacements is defined as a renewal cycle, the system renewal cycle length

for the first system renewal scenario is $kT$ ($k = 1, 2, \ldots, n$), while it equals $nT$ for the second system renewal scenario.

Let the occurrence probabilities of a corrective replacement and a preventive replacement be $P_{cor}(L = kT)$ ($k = 1, 2, \ldots, n$) and $P_{pre}(L = nT)$, respectively. Then, the expected length of the system renewal cycle can be expressed by the following equation.

$$E(L) = \sum_{k=1}^{n} kTP_{cor}(L = kT) + nTP_{pre}(L = nT). \tag{1}$$

Different system renewal scenarios correspond to different renewal cycle lengths and are associated with distinct maintenance costs. For the first renewal scenario, the total cost from system startup to failure comprises the expenses for $k$ ($k = 1, 2, \ldots, n$) inspections, any imperfect repairs conducted prior to failure, one corrective replacement and the downtime costs attributable to hidden failures. Similarly, the costs incurred during a renewal cycle initiated by a preventive replacement primarily include the expenses for $n$ inspections, those for any imperfect repairs performed before the $n$th inspection, and the cost of one preventive replacement.

Denote the unit inspection cost, imperfect repair cost, preventive replacement cost, and corrective replacement cost as $C_I$, $C_R$, $C_P$, and $C_C$, respectively. The cost of downtime due to hidden failures is denoted by $C_D$. Assume that the probability of identifying a system defect at the $i$th inspection (where $i = 0, 1, \ldots, k-1$) and subsequently addressing it with an imperfect repair is $P(T, i)$. Based on these cost parameters, the probability of an imperfect repair, and the probabilities associated with the two system renewal scenarios, the expected total cost per renewal cycle can be derived as follows.

$$\begin{aligned} E(C) = & \sum_{k=1}^{n} P_{cor}(L = kT)(kC_I + \sum_{i=0}^{k-1} C_R P(T, i) + C_C + C_D) + \\ & P_{pre}(L = nT)(nC_I + \sum_{i=0}^{n-1} C_R P(T, i) + C_P), \end{aligned} \tag{2}$$

where $\sum_{i=0}^{k-1} C_R P(T, i)$ represents the aggregate cost of imperfect repairs undertaken for defects prior to a corrective replacement, and $\sum_{i=0}^{n-1} C_R P(T, i)$ represents the total cost of all imperfect repairs performed prior to a preventive replacement.

Based on Equations (1) and (2), the expected cost rate model for the condition-based maintenance of the MSS is constructed as follows.

$$\text{minECR}(T) = \frac{E(C)}{E(L)}. \tag{3}$$

The primary objective of this study is to determine the optimal inspection interval $T^*$ that minimizes the expected cost rate function ECR($T$). As shown in Equations (1) and (2), the maintenance model involves four key unknown terms: the probability of imperfect repair $P(T, i)$, the probability of corrective replacement $P_{cor}(L = kT)$, the probability of preventive replacement $P_{pre}(L = nT)$, and the cost of downtime due to hidden failures $C_D$.

By examining system renewal scenarios, it is evident that the system may degrade into defective states and undergo imperfect repairs before either corrective or preventive replacement occurs. Consequently, the derivation of the corrective replacement probability $P_{cor}(L = kT)$ and the preventive replacement probability $P_{pre}(L = nT)$ depends on the probability $P(T, i)$ of imperfect repairs. Thus, the subsequent analysis proceeds as follow. First, the probability $P(T, i)$ of imperfect repairs for system defects is derived. Next, the probabilities $P_{cor}(L = kT)$ and $P_{pre}(L = nT)$ are computed based on $P(T, i)$. Subsequently,

the downtime costs $C_D$ due to hidden failures are analyzed and evaluated. Finally, the maintenance model of the system is formulated by integrating the explicit expressions of these four key terms with the relevant cost parameters.

### *4.1. Mathematical Modeling of Imperfect Repair for System Defects*

In this study, repair activities are scheduled to address system defects. However, these repairs can only restore the system from a defective state to a normal operating state, rather than making it "as good as new". Such interventions are therefore considered imperfect. To model this imperfection, the proportional age reduction model is employed. This model is based on the premise that the $(k+1)$th repair only influences the system's operating time between the $(k+1)$th and the $k$th repair.

Let $\tau_k$ be the time at which the $k$th repair is performed and $\omega$ the improvement factor of the repair. Then the $(k+1)$th repair reduces the virtual age of the system by $\omega(\tau_{k+1} - \tau_k)$. Therefore, the virtual age of the system at time $t$ is given by

$$V_t = t - \tau_{k+1} + \sum_{l=0}^{k} (1-\omega)(\tau_{l+1} - \tau_l) = t - \omega\tau_{k+1}. \tag{4}$$

The virtual age defined in Equation (4) corresponds to the "normal age", representing the actual duration during which the system remains in a normal state. This quantity exclusively influences the intensity function associated with the initiation of defects. Although an imperfect repair cannot restore the system to an "as good as new" condition, it entirely eliminates existing defects. As a result, the defective age of the system is reset to zero following each repair.

As illustrated in Figure 3, if a defect first occurs at time $x_1$ and is identified at $\tau_1$, the normal operating age of the system at that moment is $x_1$, and its defective age is $\tau_1 - x_1$. Since inspections in this study are assumed to be perfect, the defect will be detected and repaired during the subsequent inspection, i.e., at time $\tau_1$. The imperfect repair completely eliminates the defect and converts the accumulated defective age into normal operating age.

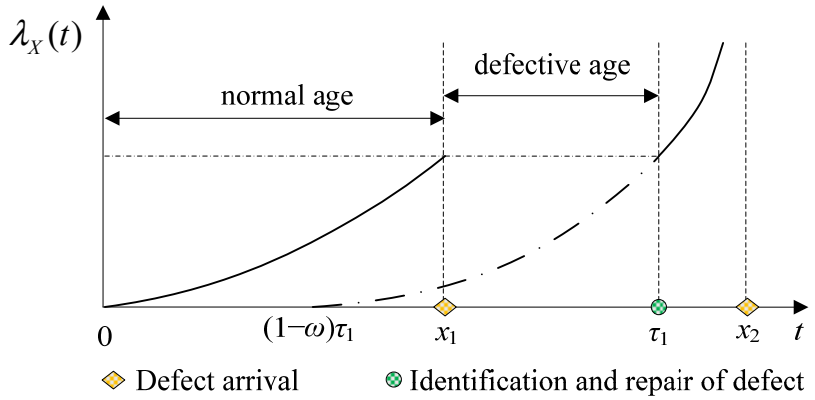



**Figure 3.** Impact of imperfect repairs on the intensity function $\lambda_X(t)$.

Figure 4 shows the difference in system age at $\tau_1$ under scenarios without and with imperfect repair. It is not difficult to find that by carrying out imperfect repair at $\tau_1$, the service life of the system is shortened by $\omega\tau_1$. Therefore, the service life of the system is shortened to $\tau_1 - \omega\tau_1$, which is equal to the virtual age of the system at $\tau_1$. In short, the imperfect repair reduces the system age from $\tau_1$ to $\tau_1 - \omega\tau_1$, and changes the system from a defective state to a normal state. Consequently, after the repair, the normal operating age of the system becomes $(1-\omega)x_1 + (1-\omega)(\tau_1 - x_1) = (1-\omega)\tau_1$, while the defective age is reset to zero.

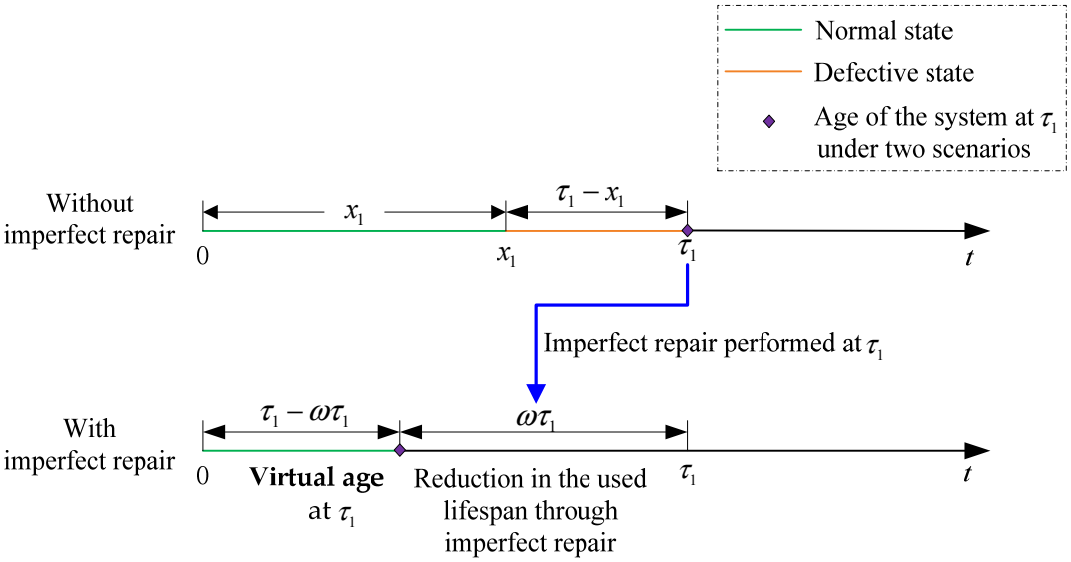



**Figure 4.** Comparison of system age at $\tau_1$ under scenarios with and without imperfect repairs.

If a system defect is detected at the $i$th inspection and is subsequently subjected to an imperfect repair, the density function of the defect initiation time $X$ following this imperfect repair can be expressed as follows:

$$f_X^i(u) = f_X(u - \omega T_i), \tag{5}$$

where $T_i = iT$.

Additionally, the probability of an imperfect repair following the $i$th inspection can be derived using the following expression.

$$P(T,i) = \sum_{m=0}^{i-1} \sum_{j=m}^{i-1} P(T,m) \int_{T_j}^{T_{j+1}} \frac{f_X^m(u) R_Y(T_i - u)}{R_X((1-\omega)T_m)} du, \tag{6}$$

where $P(T,m)$ $(0 \le m \le i-1)$ denotes the probability that the last imperfect repair was performed at time $mT$ and $P(T,0) = 1$.

### *4.2. Probability of Implementing a Corrective Replacement*

The system renewal process can be categorized into two scenarios. They are, respectively, corrective replacement following system failure after the $k$th inspection (where $k = 1, 2, \ldots, n$) and preventive replacement performed if the system remains in a normal state up to the $n$th inspection. This section first derives the probability of a corrective replacement. A corrective replacement is triggered when the system fails at the $k$th inspection ($k = 1, 2, \ldots, n$). System failure may arise from one of three causes: (i) degradation failure, (ii) sudden failure, or (iii) a combination of both degradation and sudden failure.

- *Case 1. System failure caused exclusively by degradation*

In Case 1, as illustrated in Figure 5, the system degradation evolves such that the most recent defect occurs within the interval $((k-1)T, kT)$ for $k = 1, 2, \ldots, n$), and the system failure is detected during the $k$th inspection. Furthermore, the shock process affecting the system is characterized by the absence of any fatal shocks throughout the period $(0, kT)$.

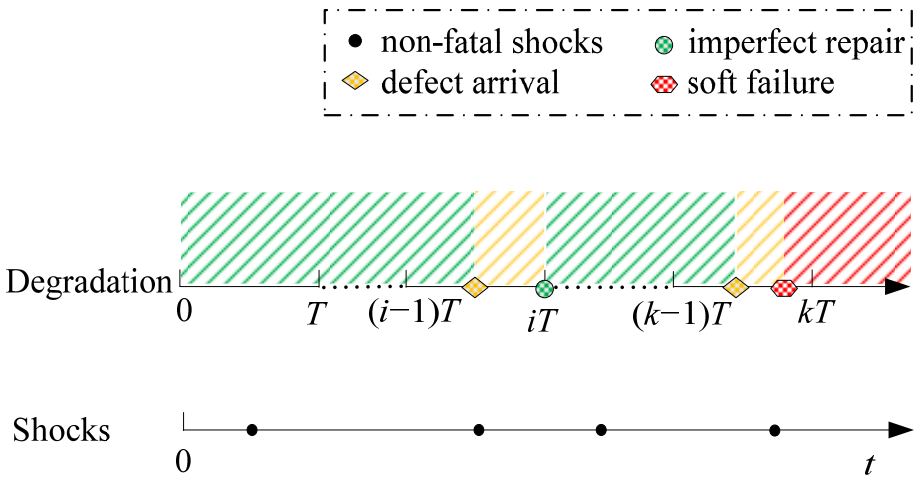



**Figure 5.** Evolution pattern of system state in Case 1.

Denoting the probability of this event by $P_{nok}$, it can be calculated as follows.

$$\begin{aligned} P_{nok} &= \sum_{j=0}^{\infty} P(W_1 < D, W_2 < D, \ldots, W_j < D | N(kT) = j) P(N(kT) = j) \\ &= \sum_{j=0}^{\infty} \Phi\left(\frac{D - \mu_W}{\sigma_W}\right)^j \frac{e^{-\lambda kT}(\lambda kT)^j}{j!}, k = 1, \ldots, n. \end{aligned} \tag{7}$$

where $\lambda$ represents the arrival rate of shocks, and $D$ is the failure threshold of the shock process.

The probability of a corrective replacement, considering degradation only, is denoted by $P_1$ and can be expressed as follows.

$$P_1 = \sum_{i=0}^{k-1} P(T, i) \int_{T_{k-1}}^{T_k} \frac{f_X^i(u) F_Y(T_k - u)}{R_X((1-\omega)T_i)} du. \tag{8}$$

Therefore, the probability of a corrective replacement triggered solely by degradation failure can be expressed as

$$P_{C1} = P_1 P_{nok}. \tag{9}$$

- *Case 2. System failure due solely to a sudden failure*

Next, calculate the probability of a system failure attributable to a sudden failure. As illustrated in Figure 6 (Case 2), a sudden failure is identified at an inspection time $kT$ ($k = 1, 2, \ldots, n$). Critically, at these same inspection intervals, the system's degradation process may either be in a normal state (see degradation path (a) in Figure 6) or in a defective state (see degradation path (b) in Figure 6).

The shock process in Case 2 is characterized by the first fatal shock arriving between $(k-1)T$ and $kT$, $k = 1, 2, \ldots, n$. This necessitates that no fatal shock occurs from time 0 to $(k-1)T$ (event with probability $P_{\mathrm{M}}$, and at least one fatal shock occurs in the subsequent interval $((k-1)T, kT)$ (event with probability $P_{\mathrm{N}}$). Consequently, the overall probability for this shock scenario is

$$P_h = P_{\mathrm{M}} P_{\mathrm{N}}. \tag{10}$$

Define $j_1$ as the total number of shocks reaching the system within the time period $(0, (k-1)T)$. Then, $P_{\mathrm{M}}$ can be computed as

$$\begin{aligned} P_{\mathrm{M}} &= \sum_{j_1=0}^{\infty} P(W_1 < D, W_2 < D, \ldots, W_{j_1} < D | N((k-1)T) = j_1) P(N((k-1)T) = j_1) \\ &= \sum_{j_1=0}^{\infty} \Phi\left(\frac{D-\mu_W}{\sigma_W}\right)^{j_1} \frac{e^{-\lambda(k-1)T}(\lambda(k-1)T)^{j_1}}{j_1!}, k = 1, \ldots, n. \end{aligned} \tag{11}$$

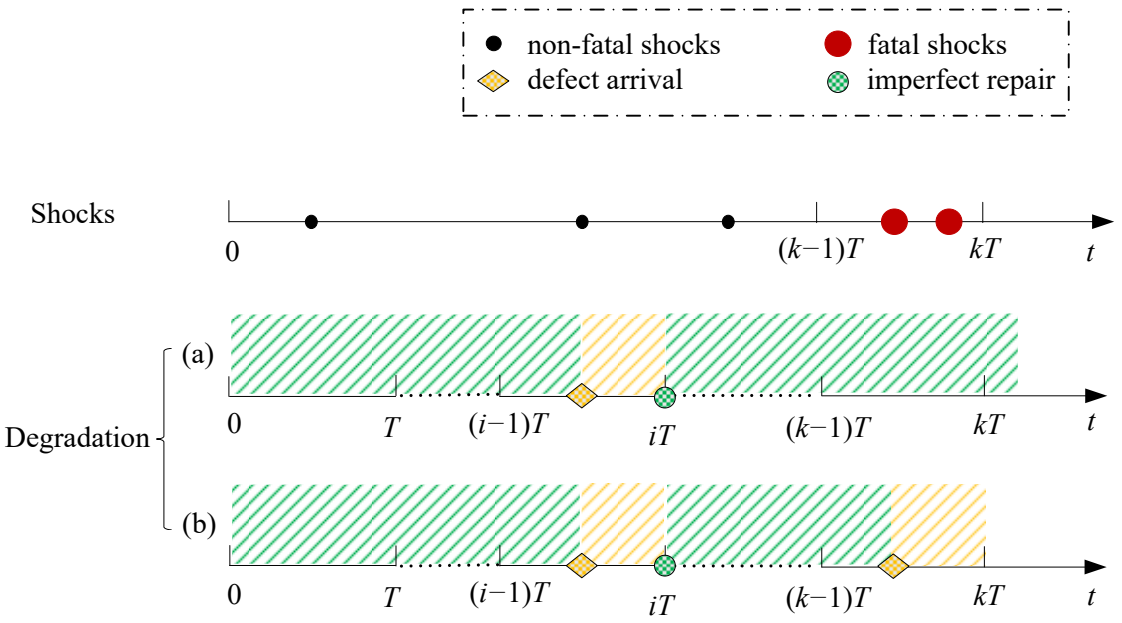



**Figure 6.** Evolution pattern of system state in Case 2.

The complement of the event of "at least one fatal shock reaches the system during $((k-1)T, kT)$" is that "all shocks arriving at the system during $((k-1)T, kT)$ are non-fatal shocks". Let $j_2$ be the number of shocks during $((k-1)T, kT)$. The probability $P_{\mathrm{N}}$ can then be calculated as

$$\begin{aligned} P_{\mathrm{N}} &= 1 - \sum_{j_2=0}^{\infty} P(W_1 < D, W_2 < D, \ldots, W_{j_2} < D | N(kT - (k-1)T) = j_2) P(N(T) = j_2) \\ &= 1 - \sum_{j_2=0}^{\infty} \Phi\left(\frac{D-\mu_W}{\sigma_W}\right)^{j_2} \frac{e^{-\lambda T}(\lambda T)^{j_2}}{j_2!}. \end{aligned} \tag{12}$$

The occurrence probabilities for the two evolutionary patterns of the degradation process can be calculated using the following two equations, respectively.

$$P_{2a} = \sum_{i=0}^{k-1} \frac{P(T,i) \int_{kT}^{\infty} f_X^i(u) du}{R_X((1-\omega)T_i)} = \sum_{i=0}^{k-1} \frac{P(T,i) R_X^i(kT)}{R_X((1-\omega)T_i)}, \tag{13}$$

$$P_{2b} = \sum_{i=0}^{k-1} \sum_{j=i}^{k-1} P(T,i) \int_{T_j}^{T_{j+1}} \frac{f_X^i(u) R_Y(T_k - u)}{R_X((1-\omega)T_i)} du, \tag{14}$$

where $R_X^i(u) = R_X(u - \omega T_i)$.

Therefore, the overall probability of system failure attributable only to a sudden failure is given by the following expression.

$$P_{\mathrm{C2}} = (P_{2a} + P_{2b}) P_h. \tag{15}$$

- *Case 3. System failure resulting from the combined effects of degradation and sudden failure*

Finally, the probability of system failure resulting from the combined effects of degradation and sudden failure is computed. As illustrated in Figure 7, the shock process follows the same evolutionary pattern as in Case 2; its probability, denoted as $P_h$, is therefore given by Equation (10). In Case 3, the degradation process evolves such that the system transitions to a defective state within the interval $((k-1)T, kT)$, and a degradation failure is identified at the $k$th inspection. The probability of this degradation evolutionary pattern can be calculated as

$$P_3 = \sum_{i=0}^{k-1} P(T,i) \int_{T_{k-1}}^{T_k} \frac{f_X^i(u) F_Y(T_k - u)}{R_X((1-\omega)T_i)} du. \tag{16}$$

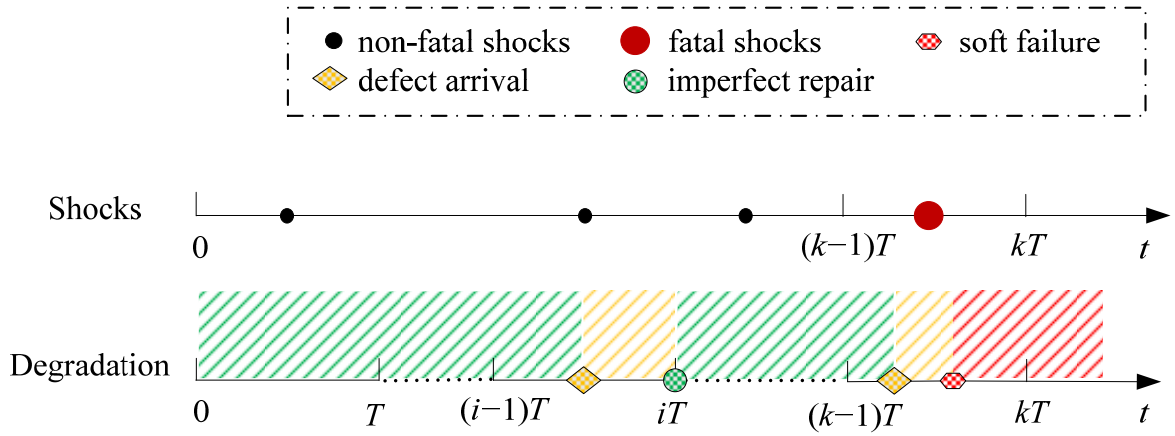



**Figure 7.** Evolution pattern of system state in Case 3.

Consequently, the probability of system failure due to the concurrent occurrence of degradation failure and a sudden shock is given by

$$P_{C3} = P_3 P_h. \tag{17}$$

Following the preceding analysis, the probability for a corrective replacement action is therefore given by the sum of the three cases, which is equal to

$$P_{cor}(L = kT) = P_{C1} + P_{C2} + P_{C3}. \tag{18}$$

### *4.3. Probability of Performing a Preventive Replacement*

If the system remains in a normal state by the $n$th inspection, a preventive replacement should be performed. This is because, as service time increases, the system can function normally for progressively shorter durations, and its ability to withstand external shocks is significantly diminished. Consequently, there is a high likelihood of failure occurring in the near term, which could lead to substantial economic losses. As shown in Figure 8, a preventive replacement requires the satisfaction of two conditions. First, no fatal shock must have reached the system before time $nT$. Second, the degradation process must remain in a normal state at time $nT$.

The probability of the event that no fatal shock arrives at the system before time $nT$ is expressed as follows.

$$\begin{aligned} P_{non} &= \textstyle\sum_{j=0}^{\infty} P(W_1 < D, W_2 < D, \ldots, W_j < D | N(nT) = j) P(N(nT) = j) \\ &= \textstyle\sum_{j=0}^{\infty} \Phi\left(\frac{D - \mu_W}{\sigma_W}\right)^j \frac{e^{-\lambda nT} (\lambda nT)^j}{j!}. \end{aligned} \tag{19}$$

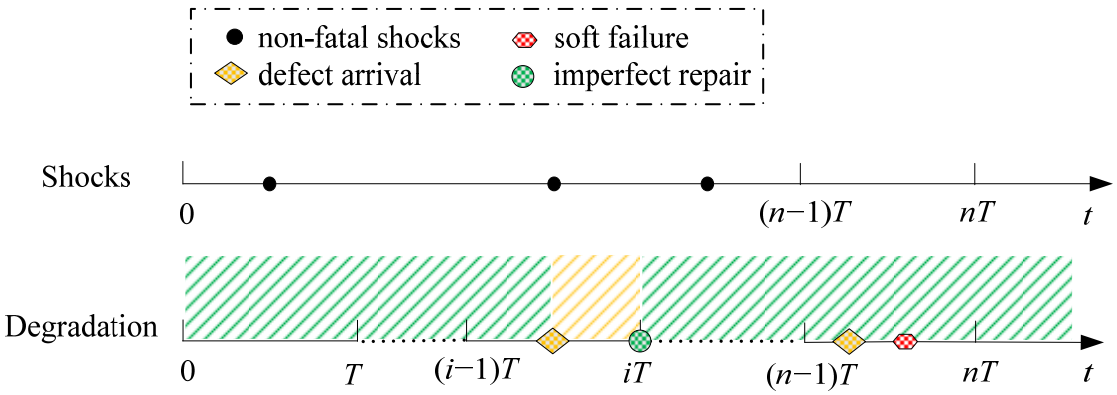



**Figure 8.** System state evolution under preventive replacement.

The probability of the evolutionary pattern of the degradation process can be calculated by

$$P_{sn} = \sum_{i=0}^{n-1} \frac{P(T,i)\int_{nT}^{\infty} f_X^i(u)du}{R_X((1-\omega)T_i)} = \sum_{i=0}^{n-1} \frac{P(T,i)R_X^i(nT)}{R_X((1-\omega)T_i)}, \tag{20}$$

where $R_X^i(nT) = R_X(nT - \omega T_i)$.

Therefore, the probability of preventive replacement is equal to

$$P_{pre}(L = nT) = P_{sn}P_{non}. \tag{21}$$

*4.4. Costs Associated with Operational Downtime*

The expected cost of downtime resulting from hidden failures is the product of the cost per unit downtime and the expected downtime duration. Based on the Mean Past Life (MPL) method [56], it is assumed that the system fails at time $T_f$, $(k-1)T < T_f \leq kT$. The resulting downtime length at the $k$th ($k$ = 1, 2, …, $n$) inspection is $kT - T_f$. Let $C_d$ represent the cost of downtime per unit. The total cost of downtime due to hidden failures is then given by

$$C_D = C_d \cdot E(kT - T_f \big| T_f \leq kT). \tag{22}$$

The expected system downtime is calculated using the following procedure.

**Step 1**. Manipulate the downtime length $kT - T_f$.

Let $kT - T_f = Z$, then $E(kT - T_f \big| T_f \leq kT) = E(0 < Z < kT) = E(Z|0 < Z < kT)$.

**Step 2.** Derive $E(Z|0 < Z < kT)$ based on the conditional expectation formula.

Let the cumulative distribution function and probability density function of $Z$ be $G(z)$ and $g(z)$, respectively. Employing the conditional expectation formula from Equation (23), $E(Z|0 < Z < kT)$ is expressed by Equation (24).

$$E_X(x|y) = \frac{\int xf(x,y)dx}{\int f(y)dy}, \tag{23}$$

$$E(Z|0 < Z < kT) = \frac{\int_0^{kT} zg(z)dz}{\int_0^{kT} g(z)dz}. \tag{24}$$

**Step 3.** Derive the numerator and denominator of the right-hand term in Equation (24) separately.

First, the molecular $\int_0^{kT} zg(z)dz$ is derived. Denote the cumulative distribution function and probability density function of the failure time as $F(x)$ and $f(x)$, respectively. Given that $kT - T_f = Z$, it follows that $G(z) = 1 - F(x)$ and $g(z) = -f(x)$. Therefore,

$$\begin{aligned} \int_0^{kT} zg(z)dz &= \int_0^{kT} (kT - x)(-f(x))d(kT - x) \\ &= \int_0^{kT} (kT - x)f(x)dx \\ &= kTF(x)|_0^{kT} - xF(x)|_0^{kT} + \int_0^{kT} F(x)dx \\ &= \int_0^{kT} F(x)dx, \end{aligned} \tag{25}$$

Then, we derive the denominator $\int_0^{kT} g(z)dz$.

$$\begin{aligned} \int_0^{kT} g(z)dz &= \int_0^{kT} (-f(x))d(kT - x) \\ &= \int_0^{kT} f(x)dx \\ &= F(kT) \end{aligned} \tag{26}$$

Substituting Equations (25) and (26) into Equation (24) yields the following result.

$$E(Z|0 < Z < kT) = \frac{\int_0^{kT} F(x)dx}{F(kT)}. \tag{27}$$

Consequently, the costs associated with downtime losses attributable to hidden failures are calculated as follows.

$$\begin{aligned} C_D &= C_d \cdot E(kT - T_f \Big| T_f \leq kT) = C_d \cdot E(Z|0 < Z < kT) \\ &= C_d \cdot \frac{\int_0^{kT} F(x)dx}{F(kT)}, \end{aligned} \tag{28}$$

where $F(x)$ is the failure distribution function of the multi-state system, which depends on the distributions of the random variables $X$, $Y$ and $W_j$.

## 5. Optimization of Inspection Scheduling to Minimize Expected Cost Rate

The primary objective of this paper is to determine the optimal inspection interval that minimizes the expected cost rate. This is achieved by developing and optimizing a maintenance model formulated as the expected cost rate, denoted ECR($T$), where $T$ represents the inspection interval and serves as the decision variable in the optimization framework.

Based on the probability of imperfect repair given in Equation (6), the probabilities of corrective and preventive replacements provided in Equations (18) and (21), and the cost contributions due to hidden failures derived in Equation (28), a comprehensive maintenance model of the multi-state system subject to competing failures and imperfect repair is constructed. The explicit expression of this model is presented in Equation (29).

$$\mathrm{ECR}(T) = \frac{\sum_{k=1}^{n} P_{cor}(L = kT)(kC_I + \sum_{i=0}^{k-1} C_R P(T,i) + C_C + C_D) + P_{pre}(L = nT)(nC_I + \sum_{i=0}^{n-1} C_R P(T,i) + C_P)}{\sum_{k=1}^{n} kTP_{cor}(L = kT) + nTP_{pre}(L = nT)}. \tag{29}$$

Although ECR($T$) is a univariate function of the decision variable $T$ and has an explicit analytical expression, the diversity and complexity of its constituent terms make it impractical to derive a closed-form solution for $T$ analytically. Therefore, this study employs a numerical approach to solve the maintenance model, thereby identifying the optimal inspection interval and its corresponding minimum expected cost rate. The simulation procedure is illustrated in Figure 9.

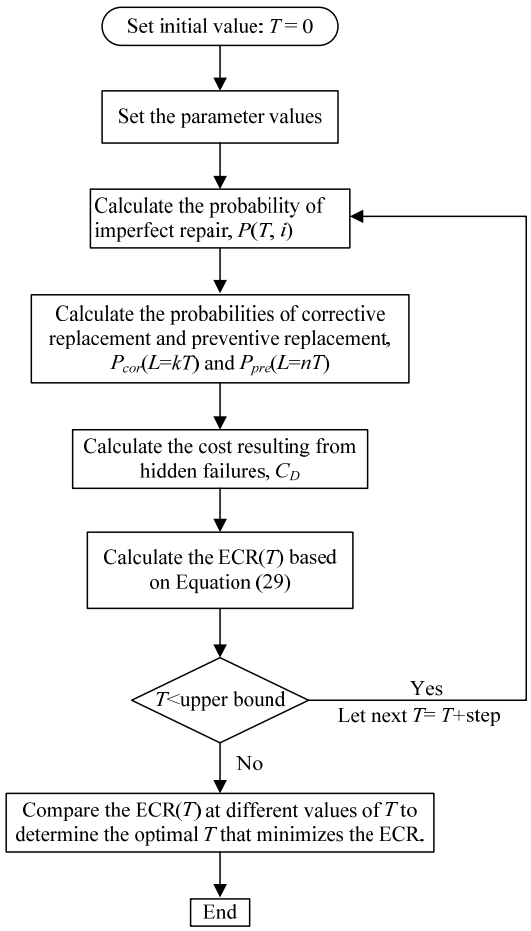



**Figure 9.** Simulation flow chart.

## 6. Numerical Example

Distribution system capacitor banks undergo continuous degradation from the start of their service life. In the initial stages, electrolyte consumption is minimal, and the system can be considered to be in a normal operating state. As the electrolyte depletes beyond a certain point, the capacitor bank enters a defective state. Once the electrolyte level falls below a critical threshold, the capacitor bank can no longer perform its intended function and eventually fails due to degradation.

In addition to inherent degradation, capacitor banks are also susceptible to external shocks, such as overvoltage or reverse voltage, which may cause sudden failure if the magnitude of any shock exceeds the sudden failure threshold. The condition of the capacitor bank can be assessed through periodic inspection, allowing for timely maintenance based on its identified state. This proactive approach helps ensure reliable performance and optimizes maintenance costs.

### *6.1. Maintenance Model Optimization and Analysis*

Regularly inspect the capacitor bank at intervals of length $T$. If defects are identified, repair the capacitor bank to restore it to normal working condition. However, the repairs are imperfect, meaning they cannot restore the capacitor bank to an as-new state. Assume that before the first imperfect repair, the normal lifespan $X$ and the delay time $Y$ of the system follow Weibull distributions with shape parameters $a_1$ and $a_2$, and scale parameters $b_1$ and $b_2$, respectively. The probability density function and survival function of $X$ are given by $f_X(t) = (b_1/a_1)(t/a_1)^{b_1-1}e^{-(\frac{t}{a_1})^{b_1}}$ and $R_X(t) = e^{-(\frac{t}{a_1})^{b_1}}$, respec-

tively. Similarly, the probability density function and survival function of $Y$ are given by $f_Y(t) = (b_2/a_2)(t/a_2)^{b_2-1}e^{-(\frac{t}{a_2})^{b_2}}$ and $R_Y(t) = e^{-(\frac{t}{a_2})^{b_2}}$, respectively.

Since imperfect repairs only affect the normal lifetime, it can be inferred from Equation (5) that after an imperfect repair is implemented at time $iT$, the probability density function of $X$ becomes

$$f_X^i(t) = (b_1/a_1)((t-\omega iT)/a_1)^{b_1-1}e^{-(\frac{t}{a_1})^{b_1}}.$$

Accordingly, the corresponding reliability function and lifetime distribution function are given by $R_X^i(t) = e^{-((t-\omega iT)/a_1)^{b_1}}$ and $F_X^i(t) = 1 - R_X^i(t) = 1 - e^{-((t-\omega iT)/a_1)^{b_1}}$, respectively. The assumed parameters of the two Weibull distributions are, respectively, $a_1 = 1$, $b_1 = 2$, $a_2 = 0.8$, $b_2 = 1$.

Assume that the shock arrival intensity parameter ($\lambda$) for the system is 1. The mean ($\mu_W$) and variance ($\sigma_W$) of the shock magnitudes are 5 and 4, respectively, and the sudden failure threshold ($D$) is 8. The minimum inspection interval is set to 1 month [57]. If defects are detected, the system undergoes imperfect repair with an improvement factor of $\omega = 0.8$. The downtime cost per unit time, denoted as $Cd$, is 100 (in units of \$100) [58]. Assumed values for the remaining cost parameters in the maintenance model are listed below.

- Cost of a single inspection, $C_I = 10$;
- Cost of an imperfect repair, $C_R = 40$;
- Cost of a preventive replacement, $C_P = 60$;
- Cost of a corrective replacement, $C_C = 800$.

Assume $n = 8$. The expected cost rates under different inspection intervals can be calculated using Equation (29). As shown in Figure 10, the horizontal axis represents the inspection interval $T$, and the vertical axis represents the corresponding expected cost rates. As shown in Figure 10, when the repair improvement factor is set to $\omega = 0.8$, the optimal inspection interval is 0.23 months (the unit used is hypothetical). At this interval, the maintenance cost of the system is minimized, with a minimal expected cost rate of \$384.5311.

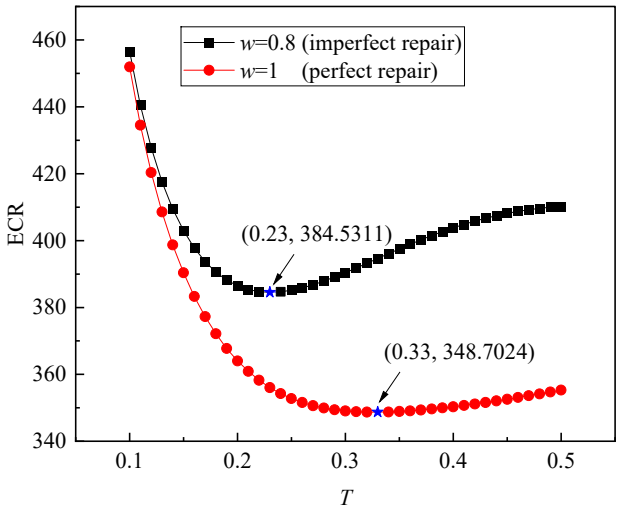



**Figure 10.** Expected cost rate at different inspection intervals. Note: The blue stars indicate the minimum values.

To better demonstrate the added value of our proposed model, we conducted a comparative analysis with the scenario that imperfect repair was not taken into account when constructing maintenance model. Figure 10 reveals that under perfect repair conditions,

the optimal inspection interval increases to 0.33 months, resulting in a further reduced minimized expected cost rate of \$348.7024. The comparison between the imperfect and perfect repair scenarios reveals that the system requires more frequent inspections under imperfect repair conditions. It indicates that when repairs are imperfect, a higher inspection frequency is necessary to mitigate the risks of defects and failures, thus helping to control the overall maintenance costs. Furthermore, the minimal expected cost rate is higher in the imperfect repair case (\$384.5311) than in the perfect repair case (\$348.7024), demonstrating that maintenance costs increase when repair effects are imperfect.

Indeed, accounting for imperfect repairs more closely aligns with real-world engineering contexts. Neglecting the imperfection of repairs during the design of inspection and maintenance strategies may lead to underestimation of the total maintenance cost, thereby hindering rational resource allocation decisions by maintenance engineers. Thus, it is essential to incorporate the effect of imperfect repair when formulating maintenance models for MSSs susceptible to competing failure modes.

### *6.2. Model Validation*

In this section, we introduce a simulation algorithm to test the correctness and accuracy of the constructed maintenance model considering competing failures and imperfect repairs for multi-state systems. Figure 11 illustrates the flowchart of a single simulation. The average cost rate can be obtained after the simulation in Figure 11 is run 100,000 times.

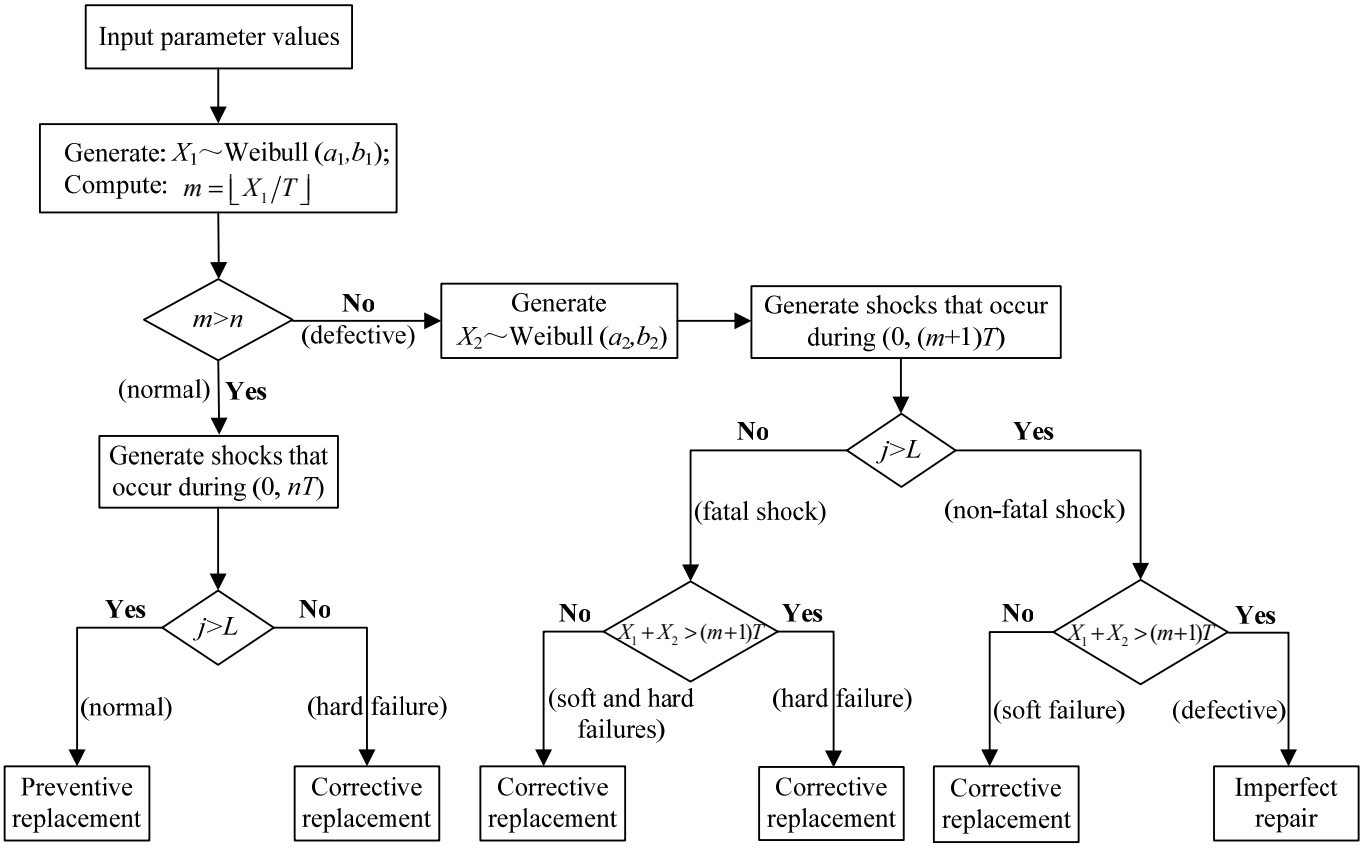



**Figure 11.** Simulation flowchart.

Based on the model parameter values provided in Section 6.1 and the simulation process illustrated in Figure 11, the expected maintenance cost rate of the system is calculated to be 383.7227. This result is nearly identical to the value of 384.5311 obtained through the optimization of the constructed maintenance model. Furthermore, multiple runs of

the simulation program yielded several expected cost rate values, all of which fluctuated within a narrow range around the theoretical solution. These findings demonstrate the correctness and robustness of the developed maintenance model.

### *6.3. Sensitivity Analysis of the Repair Improvement Factor*

This subsection analyzes the influence of the repair improvement factor $\omega$ on the maintenance model. The analysis is confined to the range of $0.6 \leq \omega \leq 1$, as it better reflects realistic engineering conditions. The optimal inspection interval $T^*$ and the corresponding minimum expected cost rate ECR($T^*$) for different values of $\omega$ are computed and presented in Figure 12.

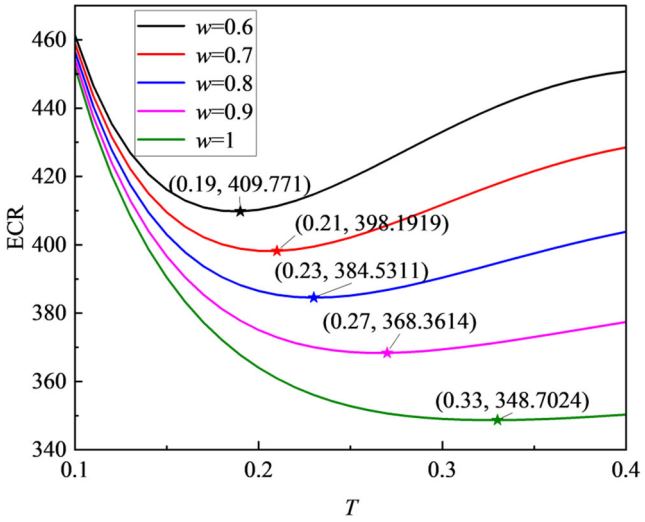



**Figure 12.** Sensitivity analysis of the improvement factor $\omega$. Note: Different colored stars represent the minimum expected cost rates corresponding to different $w$ values.

As shown in Figure 12, the ECR-T curve shifts downward as $\omega$ increases, indicating an inverse relationship between repair quality and maintenance costs. Specifically, when $\omega$ rises from 0.6 to 1, the optimal inspection interval $T^*$ increases from 0.19 to 0.23, while the minimum expected cost rate ECR($T^*$) decreases from 409.771 to 348.7024. These results suggest that improved repair effectiveness allows for less frequent inspections and effectively lowers overall maintenance expenses. The findings highlight that enhancing repair quality can significantly reduce both the inspection frequency and the total maintenance cost of capacitor banks. Therefore, in practical applications, maintenance engineers should focus on improving repair effectiveness to achieve substantial cost savings.

### *6.4. Sensitivity Analysis of Cost Parameters*

This section conducts a sensitivity analysis of the five cost parameters in the maintenance model.

#### 6.4.1. Sensitivity Analysis of a Single Cost Parameter

To perform sensitivity analysis for a given cost parameter, $\omega$ is first set to its default value of 0.8. The value of the target cost parameter is then varied across a defined range while keeping other parameters constant. These values are substituted into the maintenance model to determine the optimal inspection interval $T^*$ and the corresponding minimum expected cost rate ECR($T^*$) through model optimization. As summarized in Table 2 the sensitivities of five cost parameters, i.e., $C_I$, $C_R$, $C_P$, $C_C$, and $C_d$, to both $T^*$ and ECR($T^*$) are analyzed individually.

**Table 2.** Optimal solutions under different cost parameters.

| Cost Parameters | | | | | $T^*$ | ECR($T^*$) |
|---|---|---|---|---|---|---|
| $C_I$ | $C_R$ | $C_P$ | $C_C$ | $C_d$ | | |
| 6 | 40 | 60 | 800 | 100 | 0.21 | 366.3166 |
| 8 | 40 | 60 | 800 | 100 | 0.22 | 375.6450 |
| 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 12 | 40 | 60 | 800 | 100 | 0.24 | 393.0203 |
| 14 | 40 | 60 | 800 | 100 | 0.25 | 401.1472 |
| 10 | 20 | 60 | 800 | 100 | 0.22 | 338.0560 |
| 10 | 30 | 60 | 800 | 100 | 0.22 | 361.3959 |
| 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 10 | 50 | 60 | 800 | 100 | 0.24 | 407.5389 |
| 10 | 60 | 60 | 800 | 100 | 0.25 | 430.3200 |
| 10 | 40 | 40 | 800 | 100 | 0.22 | 376.6323 |
| 10 | 40 | 50 | 800 | 100 | 0.22 | 380.6841 |
| 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 10 | 40 | 70 | 800 | 100 | 0.24 | 388.2153 |
| 10 | 40 | 80 | 800 | 100 | 0.25 | 391.7374 |
| 10 | 40 | 60 | 600 | 100 | 0.26 | 334.0888 |
| 10 | 40 | 60 | 700 | 100 | 0.24 | 359.5363 |
| 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 10 | 40 | 60 | 900 | 100 | 0.22 | 409.1874 |
| 10 | 40 | 60 | 1000 | 100 | 0.21 | 433.6087 |
| 10 | 40 | 60 | 800 | 60 | 0.25 | 372.7835 |
| 10 | 40 | 60 | 800 | 80 | 0.24 | 378.8396 |
| 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 10 | 40 | 60 | 800 | 120 | 0.22 | 389.9357 |
| 10 | 40 | 60 | 800 | 140 | 0.22 | 395.1355 |

The analysis begins with $C_I$ by holding all other parameters fixed. $T^*$ and ECR($T^*$) are computed for varying values of $C_I$. The same procedure is repeated for the remaining four parameters, with all results compiled in Table 2 to facilitate visual interpretation of how each cost parameter influences $T^*$ and ECR($T^*$), the data from Table 2 are plotted in Figures 12 and 13. Figure 13 illustrates the sensitivity of ECR($T^*$) (in units of \$100) and $T^*$ to $C_I$, $C_R$, and $C_P$, while Figure 14 shows the sensitivity to $C_C$ and $C_d$.

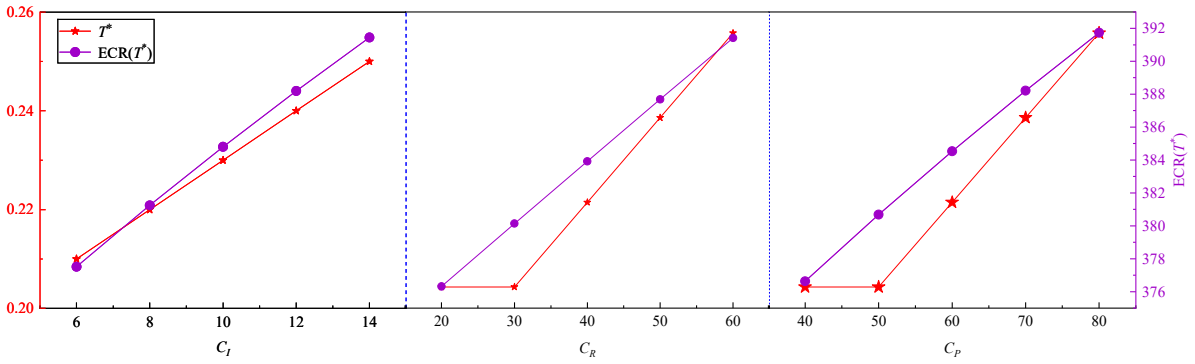



**Figure 13.** Sensitivities of ECR($T^*$) (×\$100) and $T^*$ to $C_I$, $C_R$, and $C_P$.

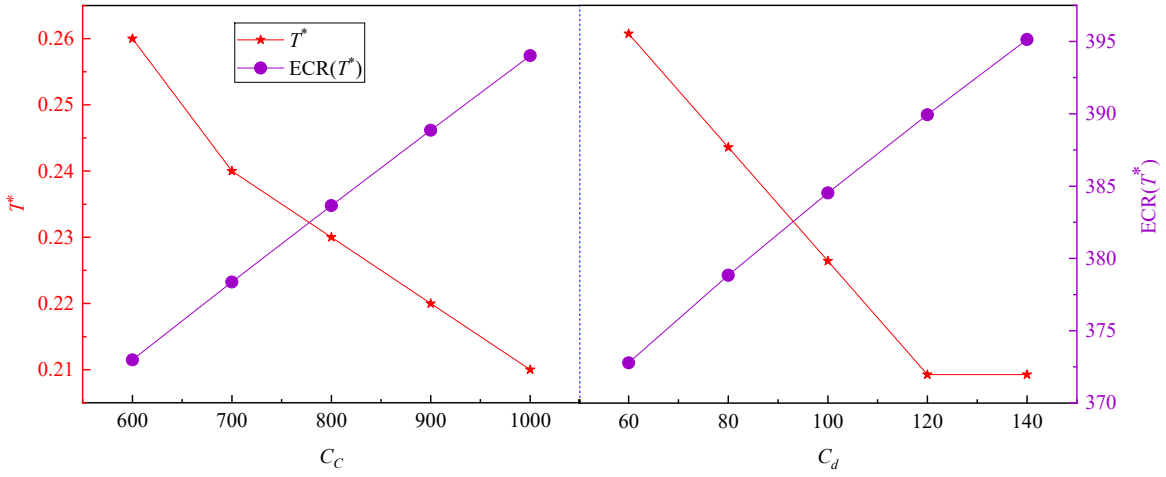



**Figure 14.** Sensitivities of ECR($T^*$) (×\$100) and $T^*$ to $C_C$ and $C_d$.

As shown in Figure 13, the optimal inspection interval $T^*$ and the minimum expected cost rate ECR($T^*$) exhibit consistent trends in response to changes in $C_I$, imperfect repair cost $C_R$, and preventive replacement cost $C_P$. Specifically, as these three parameters increase, both $T^*$ and ECR($T^*$) show an upward trend. In particular, ECR($T^*$) increases strictly monotonically with $C_I$, while $T$ also tends to lengthen as $C_I$ rises. This indicates that inspection cost significantly influences the total maintenance cost. To control expenses, maintenance managers should reduce inspection frequency when $C_I$ is high. Similarly, both $C_R$ and $C_P$ contribute to increased ECR($T^*$) and longer optimal inspection intervals, as also depicted in Figure 13. This result shows that $C_R$ and $C_P$ have a large impact on maintenance costs. It is therefore recommended that maintenance personnel implement effective measures to reduce these costs, thereby lowering the total cost of maintenance.

In addition, Figure 14 reveals that the $C_C$ and $C_d$ exhibit broadly similar influences on both ECR($T^*$) and $T^*$. Specifically, as $C_C$ increases, ECR($T^*$) rises from 334.0888 to 433.6087, while $T^*$ decreases from 0.26 to 0.21. This suggests that higher corrective replacement costs lead to an increase in the expected cost rate, and that more frequent inspections become necessary to mitigate overall maintenance expenses when $C_C$ is high.

#### 6.4.2. Sensitivity Analysis of Multiple Cost Parameters

Further, the five cost parameters were varied synchronously across percentage changes of −60%, −30%, 0%, +30% and +60% to observe their combined effect on the optimal inspection interval $T^*$ and the corresponding minimum expected cost rate ECR($T^*$). The adjusted parameter values for each variation level are presented in Table 3. Using a simulation algorithm, the values of $T^*$ and ECR($T^*$) under each percentages change were computed, and the results are shown in the last two columns of Table 3.

**Table 3.** Adjusted value of each cost parameter (in \$100 units) after a uniform change of ±%.

| Number | Percentage Change | $C_I$ | $C_R$ | $C_P$ | $C_C$ | $Cd$ | $T^*$ | ECR($T^*$) |
|---|---|---|---|---|---|---|---|---|
| 1 | −60% | 4 | 16 | 24 | 320 | 40 | 0.23 | 153.8124 |
| 2 | −30% | 7 | 28 | 42 | 560 | 70 | 0.23 | 269.1717 |
| 3 | 0% | 10 | 40 | 60 | 800 | 100 | 0.23 | 384.5311 |
| 4 | +30% | 13 | 52 | 78 | 1040 | 130 | 0.23 | 499.8904 |
| 5 | +60% | 16 | 64 | 96 | 1280 | 160 | 0.23 | 615.2497 |

The data from Table 3 are plotted in Figure 15 to facilitate further analysis. As shown in the figure, the optimal inspection interval $T^*$ for the capacitor bank remains constant at

0.23 months, forming a straight line, even as all five cost parameters vary synchronously by the same proportion. This indicates that the optimal inspection interval remains unchanged when the relative proportions among the cost parameters are maintained, regardless of absolute changes in their values.

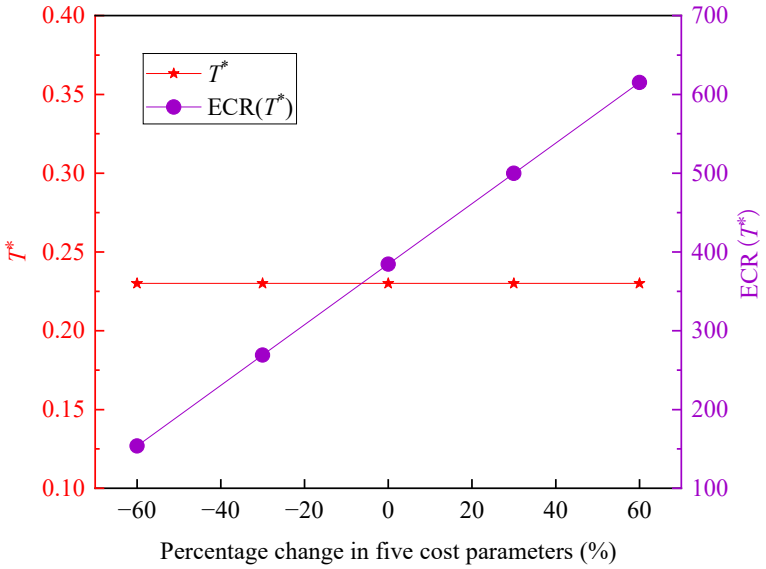



**Figure 15.** Sensitivity of ECR($T^*$) (×\$100) and $T^*$ to a uniform percentage change in cost parameters.

In contrast, the minimum expected cost rate ECR($T^*$) increases from 153.8124 to 615.2497 as the percentage change in cost parameters rises from −60% to 60%, demonstrating that ECR($T^*$) grows with increasing cost levels. Furthermore, the analysis reveals that for every 10% synchronized increase in the cost parameters, the maintenance cost increases by approximately 38.4531. The results indicate that the total maintenance cost of the capacitor bank is sensitive to the absolute values of the five cost parameters, whereas the optimal inspection interval is primarily determined by their relative ratios.

### *6.5. Sensitivity Analysis of Shock Arrival Rate and Degradation Distribution Parameters*

Finally, we analyze the influence of the shock arrival rate and degradation distribution parameters on the maintenance model. The optimal inspection interval $T^*$ and the corresponding minimal expected cost rate ECR($T^*$) for different values of shock arrival rate ($\lambda$) and degradation distribution parameters ($a_1$, $b_1$, $a_2$ and $b_2$) are obtained and presented in Table 4.

As shown in Table 4, as $\lambda$ increases, both $T^*$ and ECR($T^*$) exhibit an upward trend. More specifically, $T^*$ rises from 0.22 to 0.25, and ECR($T^*$) increases from 330.8078 to 434.8383. This indicates that more frequent shocks elevate the risk of system failure, necessitating more frequent and costly failure replacements, which in turn increases the overall maintenance cost. Interestingly, a higher shock arrival rate leads to a decrease in inspection frequency, rather than the expected increase.

We also observed that as $a_1$ and $a_2$ increase, $T^*$ gradually rises from 0.15 to 0.28 and from 0.17 to 0.24, respectively. Meanwhile, ECR($T^*$) progressively decreases from 515.7889 to 337.5672 and from 439.0810 to 370.5464, respectively. This indicates that an increase in the shape parameters of the degradation model leads to less frequent inspection strategies and results in lower maintenance costs. Furthermore, as $b_1$ increases, the trends of $T^*$ and ECR($T^*$) diverge. $T^*$ exhibits a decreasing trend, while ECR($T^*$) first increases and then decreases. This indicates that when $b_1$ becomes larger, the system should be inspected more frequently to minimize maintenance costs under different values of $b_1$. However, as

$b_2$ increases, $T^*$ first decreases and then increases, and ECR($T^*$) exhibits a decreasing trend. It implies that a larger scale parameter pertaining to the degradation model in the defective stage leads to a reduction in the minimal maintenance cost.

**Table 4.** Optimal solutions under different shock arrival rate and degradation distribution parameters.

| Parameters in Two Competing Failure Processes | | | | | $T^*$ | ECR($T^*$) |
|---|---|---|---|---|---|---|
| $\lambda$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ | | |
| 0.6 | 1 | 2 | 0.8 | 1 | 0.22 | 330.8078 |
| 0.8 | 1 | 2 | 0.8 | 1 | 0.22 | 358.1203 |
| 1 | 1 | 2 | 0.8 | 1 | 0.23 | 384.5311 |
| 1.2 | 1 | 2 | 0.8 | 1 | 0.24 | 410.1398 |
| 1.4 | 1 | 2 | 0.8 | 1 | 0.25 | 434.8383 |
| 1 | 0.6 | 2 | 0.8 | 1 | 0.15 | 515.7889 |
| 1 | 0.8 | 2 | 0.8 | 1 | 0.19 | 431.1158 |
| 1 | 1 | 2 | 0.8 | 1 | 0.23 | 384.5311 |
| 1 | 1.2 | 2 | 0.8 | 1 | 0.26 | 356.1640 |
| 1 | 1.4 | 2 | 0.8 | 1 | 0.28 | 337.5672 |
| 1 | 1 | 1 | 0.8 | 1 | 0.43 | 361.7211 |
| 1 | 1 | 1.5 | 0.8 | 1 | 0.30 | 384.3555 |
| 1 | 1 | 2 | 0.8 | 1 | 0.23 | 384.5311 |
| 1 | 1 | 2.5 | 0.8 | 1 | 0.22 | 378.8280 |
| 1 | 1 | 3 | 0.8 | 1 | 0.21 | 372.1564 |
| 1 | 1 | 2 | 0.4 | 1 | 0.17 | 439.0810 |
| 1 | 1 | 2 | 0.6 | 1 | 0.21 | 402.2313 |
| 1 | 1 | 2 | 0.8 | 1 | 0.23 | 384.5311 |
| 1 | 1 | 2 | 1.0 | 1 | 0.24 | 375.4384 |
| 1 | 1 | 2 | 1.2 | 1 | 0.24 | 370.5464 |
| 1 | 1 | 2 | 0.8 | 0.6 | 0.28 | 423.8319 |
| 1 | 1 | 2 | 0.8 | 0.8 | 0.24 | 401.5516 |
| 1 | 1 | 2 | 0.8 | 1 | 0.23 | 384.5311 |
| 1 | 1 | 2 | 0.8 | 1.2 | 0.23 | 372.4803 |
| 1 | 1 | 2 | 0.8 | 1.4 | 0.24 | 364.0272 |

These findings highlight that the shock arrival rate and degradation distribution parameters can significantly influence both the inspection frequency and the total maintenance cost. These analyses further validate the effectiveness and robustness of the constructed maintenance model.

## 7. Conclusions and Future Work

In this paper, we constructed a maintenance model for a multi-state system subject to two competing failure modes by considering imperfect repairs for system defects. A simulation algorithm was proposed to minimize the developed maintenance model of the ECR, thereby determining the optimal inspection interval and the minimal maintenance cost rate. The main findings are as follows:

(1) The constructed maintenance model for multi-state systems is valid and can effectively determine the optimal inspection interval and the minimal maintenance cost.

(2) Neglecting the imperfection of repairs during the design of inspection and maintenance strategies may lead to underestimation of the total maintenance cost. Thus, imperfect repair must be considered in maintenance models for MSSs.

(3) Enhancing repair quality can significantly reduce both the inspection frequency and the total maintenance cost. In practical applications, maintenance engineers should focus on improving repair effectiveness to achieve substantial cost savings.
(4) All cost parameters affect the optimal inspection interval and the minimal expected cost rate, among which the impact of inspection cost is the greatest. When the costs of corrective replacements and downtime are high, more frequent inspections become necessary to mitigate overall maintenance expenses.

These findings provide valuable insights to guide maintenance engineers in making informed inspection and maintenance decisions for MSSs under competing failures and imperfect repairs.

This study assumes perfect inspections, which is a simplification of reality. In practice, inspections can be imperfect due to human error, equipment sensitivity, or other environmental factors. Such imperfections might lead to an underestimation of the actual failure probability and maintenance cost. Consequently, the derived the optimal inspection interval may differ. Additionally, as both inspection and maintenance are time-consuming, the downtime cost incurred during system shutdowns should be incorporated into the maintenance model. Therefore, future work will explore model construction methods that account for imperfect inspection, inspection time and repair time to develop more practical maintenance strategies.

This research can be extended to more complex applications, such as modeling multi-state systems with more than three states, developing multi-component systems with economic and structural dependence, and incorporating opportunistic maintenance with component-specific shock thresholds.

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