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Quantum Rigidity of Cayley Graphs: An Algebraic Approach via Schur Ring

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Abstract

A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is a graphical regular representation (GRR) of G if $\mathrm{Aut}(\Gamma) = G$ acts regularly. We call Γ a quantum rigid if its quantum automorphism group also satisfies $\mathrm{QAut}(\Gamma) = G$. We prove that a Cayley graph whose coherent (two-dimensional Weisfeiler–Leman) closure is *thin* is quantum rigid. For Cayley graphs this closure is exactly the S-ring generated by S. The direct in-group refinement algorithm tests thinness in polynomial time; with comparison sorting of the refinement signatures, one round costs $O(|G|^2 \log |G|)$ and at most $|G| - 1$ strict refinement rounds occur, giving the explicit worst-case bound $O(|G|^3 \log |G|)$. Thus thinness supplies a polynomial-time sufficient test for the absence of quantum symmetry, rather than a complexity-theoretic certificate in the NP sense. We also prove that, for a GRR, thinness is equivalent to Schurity of its coherent closure. Hence non-quantum-rigidity of a GRR forces a non-Schurian coherent closure, although the converse is not asserted. Every GRR over a Schur group is therefore quantum rigid. Moreover, a WL-thin Cayley graph is determined up to ordinary isomorphism by its quantum-isomorphism class. Finally, an exact, reproducible census of 193,694 connected labelled Cayley graph instances (one for each symmetric connection set, without quotienting by graph isomorphism) in twenty-four finite groups finds no GRR that fails WL-thinness. The accompanying Python 3.14 program contains the group constructions, refinement procedure, exact GRR test and machine-readable output.

Keywords: graphical regular representation; quantum automorphism group; quantum permutation group; Cayley graph; Weisfeiler–Leman algorithm; coherent algebra; Schur ring; Schurity

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1. Introduction

1.1. Two Rigidity Notions

Let G be a finite group and $S \subseteq G \setminus \{1\}$ with $S = S^{-1}$. The Cayley graph $\mathrm{Cay}(G, S)$ has vertex set G and edges $\{x, y\}$ with $x^{-1}y \in S$. Left translations are automorphisms, so $G \leq \mathrm{Aut}(\mathrm{Cay}(G, S))$, and the graph is called a graphical regular representation (GRR) of G when this containment is an equality. Which groups admit a GRR was settled by Godsil [1], completing the work of Watkins, Imrich, Nowitz, Hetzel and Babai–Imrich [2–7]: every finite group has one except the Abelian groups of exponent greater than two, the

generalised dicyclic groups, and thirteen exceptions of order at most 32. More recently Xia and Zheng [8] proved the Babai–Godsil–Imrich–Lovász conjecture (see [9] for the original programme and [10] for the directed analogue), so that for a group admitting a GRR almost every Cayley graph is one. Classical rigidity of Cayley graphs is, in this sense, both classified and generic.

Quantum symmetry is a different matter. Wang [11] introduced the quantum permutation group S_N^+, whose function algebra is generated by a *magic unitary* $u = (u_{ij})_{i,j=1}^N$ of projections with all rows and columns summing to 1; Bichon [12] and Banica [13] attached to a graph Γ on N vertices its quantum automorphism group $\mathrm{QAut}(\Gamma)$, the quotient of $C(S_N^+)$ by the relations $uA_\Gamma = A_\Gamma u$. One always has $\mathrm{Aut}(\Gamma) \leq \mathrm{QAut}(\Gamma)$, and Γ is said to have *no quantum symmetry* when $C(\mathrm{QAut}(\Gamma))$ is commutative, and equivalently, when $\mathrm{QAut}(\Gamma) = \mathrm{Aut}(\Gamma)$. The algebra $C(S_N^+)$ is infinite-dimensional for $N \geq 4$, so the containment is very often strict; determining when it is not has been the central problem of the subject [14–18].

This paper concerns the interaction of the two notions.

Definition 1. *A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is quantum rigid if $\mathrm{QAut}(\Gamma) = G$, i.e., if $C(\mathrm{QAut}(\Gamma))$ is commutative and Γ is a GRR.*

1.2. Why the Question Is Not Routine

Two facts frame the problem. On one side, the best-known sufficient condition for the *presence* of quantum symmetry is Schmidt's disjointness criterion [16]: if $\mathrm{Aut}(\Gamma)$ contains two non-trivial automorphisms with disjoint supports, then Γ has quantum symmetry. For a GRR this criterion is vacuous, since a regular permutation group has no non-identity element with a fixed point, hence no two non-identity elements with disjoint supports; every non-trivial automorphism of a GRR has full support. All standard constructions of quantum symmetry for vertex-transitive graphs—hypercubes, Hamming graphs, folded cubes, disjoint unions and wreath-type products—proceed through large automorphism groups and are equally unavailable.

On the other side, one cannot conclude that small classical symmetry forces small quantum symmetry. Van Dobben de Bruyn, Roberson and Schmidt [19] constructed asymmetric graphs—$\mathrm{Aut}(\Gamma) = 1$, the smallest classical symmetry possible—with non-trivial quantum automorphism group, the first examples of any asymmetric space with quantum symmetry. Their construction uses solution groups of linear systems and is genuinely non-classical. So the implication "classically rigid $\Rightarrow$ quantum rigid" is false in general, and there is no formal reason for GRRs to behave better.

Question 1. *Is every graphical regular representation quantum rigid?*

1.3. Results

Our approach replaces the automorphism group by the coherent closure. Write $\mathrm{WL}(\Gamma)$ for the coherent algebra produced by two-dimensional Weisfeiler–Leman refinement [20]; it is the smallest coherent algebra containing A_Γ. Quantum automorphisms preserve the stable coherent colouring [21], and the corresponding quantum-isomorphism statement is developed in [22]. For a Cayley graph, the coherent closure lies in the group algebra and is precisely the S-ring generated by the connection set in the sense of Schur–Wielandt theory [23,24]. We call Γ *WL-thin* when all basic sets of this S-ring are singletons.

Theorem 2 (Theorem 5)**.** *Let $\Gamma = \mathrm{Cay}(G, S)$ be WL-thin. Then $C(\mathrm{QAut}(\Gamma))$ is commutative and $\mathrm{QAut}(\Gamma) = \mathrm{Aut}(\Gamma) = G$. In particular, Γ is a quantum rigid GRR.*

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The hypothesis is algorithmically checkable. The direct refinement procedure of Remark 2 has worst-case running time $O(|G|^3 \log |G|)$ in the stated comparison model. Consequently WL-thinness is a polynomial-time *sufficient test* for quantum rigidity. No claim is made that a short witness for quantum rigidity has been characterised in the complexity-theoretic sense of a polynomially verifiable certificate. This distinction is useful because quantum isomorphism itself is undecidable [22].

The same hypothesis yields rigidity in the stronger relational sense.

Theorem 3 (Theorem 6)**.** *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be WL-thin and let* Γ' *be any graph quantum isomorphic to* Γ*. Then* $\Gamma' \cong \Gamma$*.*

For a GRR, WL-thinness is equivalent to Schurity of the coherent closure, but this does *not* identify quantum rigidity with Schurity. The exact logical content is

$$\text{Schurian coherent closure} \iff \text{WL-thin} \implies \text{quantum rigid},$$

and therefore

$$\text{not quantum rigid} \implies \text{non-Schurian coherent closure}.$$

The reverse implication in the second display is neither used nor claimed. This yields a classical obstruction: any counterexample to quantum rigidity among GRRs must generate a non-Schurian S-ring.

Theorem 4 (Theorem 7)**.** *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a GRR. Then* Γ *is WL-thin if and only if its coherent closure is Schurian. Consequently, if* Γ *is not quantum rigid, its coherent closure is non-Schurian; in particular every GRR over a Schur group is quantum rigid.*

The thinness hypothesis is genuinely restrictive. For the n-cube $Q_n = \mathrm{Cay}(\mathbb{Z}_2^n, \{e_1, \ldots, e_n\})$, the coherent closure is the Hamming scheme and hence is far from thin; its quantum automorphism group is the $q = -1$ orthogonal deformation O_n^{-1} [25]. Folded odd cubes similarly have quantum automorphism group SO_n^{-1} [16]. At the opposite extreme, the empty and complete graphs on N vertices have quantum automorphism group S_N^+, because their adjacency matrices are respectively 0 and $J - I$, both of which commute with every magic unitary [11].

Section 5 gives an exact, reproducible census for twenty-four finite groups. Enumerating every symmetric connection set produces 193,694 connected labelled Cayley graph instances; distinct connection sets are counted separately even when the resulting graphs are isomorphic. In this census every GRR is WL-thin, so all 90,216 GRRs encountered are quantum rigid by Theorem 5. The accompanying program uses no search node cap and records per-instance timings and raw classifications in CSV form.

We do not resolve Question 1. Thinness of the S-ring generated by S is a genuinely iterative Weisfeiler–Leman condition rather than a simple group-theoretic condition; in particular, a one-step autocorrelation statistic need not separate the group elements.

1.4. Organisation

Section 2 fixes conventions and recalls the quantum and combinatorial input. Section 3 proves the main theorem and its quantum-isomorphism counterpart. Section 4 establishes the Schurity criterion and records sharpness examples. Section 5 presents the reproducible computations and their interpretation. Section 6 lists the remaining open problems.

2. Preliminaries

2.1. Graphs and Quantum Automorphism Groups

All graphs are finite, simple and undirected. For a graph Γ with vertex set V, $|V| = N$, let A_Γ be its adjacency matrix.

Definition 2. *A magic unitary of size N over a unital C^*-algebra $\mathcal{B}$ is a matrix $u = (u_{ij}) \in M_N(\mathcal{B})$ whose entries are projections ($u_{ij} = u_{ij}^* = u_{ij}^2$) with*

$$\sum_j u_{ij} = 1 \quad (\forall i), \qquad \sum_i u_{ij} = 1 \quad (\forall j).$$

Wang's quantum permutation group S_N^+ is the compact quantum group with $C(S_N^+)$, the universal C^*-algebra generated by the entries of a magic unitary, with $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. The quantum automorphism group $\mathrm{QAut}(\Gamma)$ of a graph Γ is given by

$$C(\mathrm{QAut}(\Gamma)) = C(S_N^+) / \langle\, uA_\Gamma = A_\Gamma u \,\rangle.$$

Entries of a magic unitary in a fixed row are mutually orthogonal projections, and likewise in a fixed column; we use this constantly. Abelianising $C(\mathrm{QAut}(\Gamma))$ returns $C(\mathrm{Aut}(\Gamma))$, so $\mathrm{Aut}(\Gamma) \leq \mathrm{QAut}(\Gamma)$, with equality precisely when $C(\mathrm{QAut}(\Gamma))$ is commutative; in that case one says Γ has *no quantum symmetry*.

Two graphs Γ, Γ' on N vertices are *quantum isomorphic*, written $\Gamma \cong_q \Gamma'$, if there is a non-zero unital C^*-algebra $\mathcal{B}$ and a magic unitary $u \in M_N(\mathcal{B})$ (rows indexed by $V(\Gamma')$, columns by $V(\Gamma)$) with $uA_\Gamma = A_{\Gamma'}u$ [21,22].

2.2. Coherent Algebras and Weisfeiler–Leman

A *coherent algebra* on V is a subalgebra $\mathcal{W} \leq M_V(\mathbb{C})$ containing I and J (the all-ones matrix), closed under conjugate transpose and under the Schur (entrywise) product. Equivalently it is spanned by the $0/1$ matrices of a partition of $V \times V$ satisfying the usual coherent-configuration axioms; the classes of that partition are the *basic relations* and the corresponding partition is the associated colouring.

The two-dimensional Weisfeiler–Leman algorithm [20] computes the coarsest stable refinement of the initial colouring

$$c_0(x,y) = \begin{cases} 0, & x = y, \\ 1, & xy \in E(\Gamma), \\ 2, & \text{otherwise}, \end{cases}$$

by repeatedly replacing $c(x,y)$ with the pair consisting of $c(x,y)$ and the vector of counts $\big|\{z : c(x,z) = i,\ c(z,y) = j\}\big|_{i,j}$. The resulting stable colouring WL_Γ spans the smallest coherent algebra containing A_Γ; we denote that algebra by $\mathrm{WL}(\Gamma)$ and call it the *coherent closure* of Γ. One has

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathrm{WL}(\Gamma)), \tag{1}$$

since automorphisms of Γ preserve the stable colouring, and conversely $A_\Gamma \in \mathrm{WL}(\Gamma)$.

The following is the input on which everything rests. In the basic form—$uA_\Gamma = A_\Gamma u$ holds if and only if $u_{xy}u_{zw} = 0$ whenever the pairs (x,z) and (y,w) have different adjacency type—it is an immediate consequence of the magic unitary relations and is used throughout the literature; the refinement to the full stable Weisfeiler–Leman colouring is ([21], Corollary 3.13), and the corresponding statement for quantum isomorphism of two graphs is ([22], §5).

Lemma 1 (Lupini–Mančinska–Roberson)**.** *Let Γ be a graph and let u be the fundamental magic unitary of $\mathrm{QAut}(\Gamma)$. If $\mathrm{WL}_\Gamma(x,z) \neq \mathrm{WL}_\Gamma(y,w)$ then $u_{xy}u_{zw} = 0$.*

We shall also use the companion fact for quantum isomorphism: if $\Gamma \cong_q \Gamma'$, then Γ and Γ' are not distinguished by the Weisfeiler–Leman algorithm; more precisely there is an *algebraic isomorphism* $f\colon \mathrm{WL}(\Gamma) \to \mathrm{WL}(\Gamma')$, i.e., a colour-preserving bijection of basic relations preserving all intersection numbers, mapping the edge colour of Γ to that of Γ' ([22], §5).

Remark 1 (Terminology)**.** *The phrase quantum Cayley graph has an established and different meaning: the Cayley graph of a discrete or finite* quantum *group, in the sense of Vergnioux and, more recently, of Brannan, Gromada, Matsuda, Skalski and Wasilewski [26], who prove a quantum Frucht theorem in that setting and show that the dual of a non-Abelian finite group is quantum rigid there. Our graphs are ordinary Cayley graphs of ordinary finite groups; only the symmetry object is quantum. We use "quantum rigid" for the property $\mathrm{QAut}(\Gamma) = G$, which is consistent with the usage of [26] but applied to a classical graph.*

2.3. Schur Rings

Let G be a finite group. A partition $\mathcal{P}$ of G is a *Schur ring* (S-ring) if $\{1\} \in \mathcal{P}$, $\mathcal{P}$ is closed under $T \mapsto T^{-1}$, and the $\mathbb{C}$-span of the elements $\underline{T} = \sum_{t\in T} t \in \mathbb{C}[G]$, $T \in \mathcal{P}$, is a subalgebra of $\mathbb{C}[G]$. The members of $\mathcal{P}$ are its *basic sets*. An S-ring is *thin* (equivalently, trivial as a partition) if every basic set is a singleton, i.e., if $\mathcal{P} = \{\{g\} : g \in G\}$.

Identify $\mathbb{C}[G]$ with the algebra of matrices $M \in M_G(\mathbb{C})$ such that $M_{x,z}$ depends only on $x^{-1}z$; this is exactly the centraliser algebra of the left regular representation of G, and it is a coherent algebra whose basic relations are the $|G|$ singleton relations $R_g = \{(x, xg) : x \in G\}$. Under this identification, S-rings over G are precisely the coherent subalgebras of $\mathbb{C}[G]$, and thinness of an S-ring corresponds to equality with $\mathbb{C}[G]$.

A coherent algebra $\mathcal{W}$ on V is *Schurian* if $\mathcal{W} = \mathrm{Orb}(\mathrm{Aut}(\mathcal{W}))$, the orbital algebra of its automorphism group, or equivalently, if its basic relations are exactly the orbits of $\mathrm{Aut}(\mathcal{W})$ on $V \times V$. Since $\mathrm{Orb}(\mathrm{Aut}(\mathcal{W})) \supseteq \mathcal{W}$ always, non-Schurian means the containment is strict. A finite group G is called a *Schur group* if every S-ring over G is Schurian. It is not true that every cyclic group is a Schur group. The cyclic Schur groups were classified by Evdokimov, Kovács and Ponomarenko: a cyclic group of order n is Schur exactly when n belongs to one of the families p^k, pq^k, $2pq^k$, pqr, or $2pqr$, where the displayed primes are distinct and $k \geq 0$ [27]. For elementary Abelian groups, C_2^5 is Schur whereas C_2^6 is not; more generally the non-cyclic elementary Abelian Schur groups have orders $4, 8, 9, 16, 27, 32$ [28].

3. Thin Cayley Graphs Are Quantum Rigid

Throughout this section G is a finite group and $S = S^{-1} \subseteq G \setminus \{1\}$; we write $\Gamma = \mathrm{Cay}(G, S)$ and identify $V(\Gamma)$ with G.

3.1. The Coherent Closure of a Cayley Graph

Proposition 1. *Let $\Gamma = \mathrm{Cay}(G, S)$ and let $c = \mathrm{WL}_\Gamma$ be its stable colouring. Then*

(i) *$c(hx, hz) = c(x, z)$ for all $h, x, z \in G$; consequently there is a function $\gamma\colon G \to$ colours with $c(x, z) = \gamma(x^{-1}z)$;*

(ii) *$\mathrm{WL}(\Gamma) \subseteq \mathbb{C}[G]$, and the partition $\mathcal{P}_S$ of G into the fibres of γ is an S-ring over G containing S as a union of basic sets;*

(iii) *$\mathcal{P}_S$ is the smallest S-ring over G having S as a union of basic sets.*

Proof. (i) Left translation $\lambda_h(x) = hx$ is an automorphism of Γ, and the Weisfeiler–Leman colouring is invariant under graph automorphisms, being defined by an isomorphism-

invariant refinement procedure. Hence $c(hx, hz) = c(x, z)$. Taking $h = x^{-1}$ gives $c(x, z) = c(1, x^{-1}z)$, so $\gamma(g) := c(1, g)$ works.

(ii) By (i), every matrix in $\mathrm{WL}(\Gamma)$ has (x, z)-entry depending only on $x^{-1}z$, and hence lies in $\mathbb{C}[G]$. Since $\mathrm{WL}(\Gamma)$ is a coherent algebra contained in $\mathbb{C}[G]$, its basic relations are the sets $\{(x, xg) : g \in T\}$ for T ranging over the fibres of γ, and $\mathcal{P}_S$ is the corresponding partition of G. We verify the three S-ring axioms. First, $\{1\} \in \mathcal{P}_S$, because the initial colouring already separates the diagonal from everything else and refinement never merges colour classes. Second, $\mathcal{P}_S$ is closed under inversion: the stable colouring satisfies $c(x, z) = c(x', z') \Rightarrow c(z, x) = c(z', x')$ (the transpose of a basic relation of a coherent algebra is again a basic relation), and $\gamma(g^{-1}) = c(1, g^{-1}) = c(g, 1)$, so the fibres of γ are permuted by inversion; since inversion is an involution preserving cardinalities and $\gamma(g)$ determines $\gamma(g^{-1})$, each fibre is mapped onto a fibre, and $T \mapsto T^{-1}$ is a permutation of $\mathcal{P}_S$. Third, the span is an algebra: stability of the colouring says precisely that for basic relations $R_T, R_{T'}$ the product $R_T R_{T'}$ is a non-negative integer combination of basic relations, and translating to $\mathbb{C}[G]$ this reads

$$\underline{T}\,\underline{T'} = \sum_{T'' \in \mathcal{P}_S} \lambda_{T,T'}^{T''}\, \underline{T''}, \qquad \lambda_{T,T'}^{T''} = \left|\{(t, t') \in T \times T' : tt' = g\}\right| \ \ (g \in T''),$$

the count on the right being independent of the choice of $g \in T''$ exactly because the colouring is stable. Finally S is a union of fibres of γ since γ refines γ_0.

(iii) If $\mathcal{Q}$ is an S-ring having S as a union of basic sets, then the corresponding coherent algebra contains $A_\Gamma = \sum_{s \in S} R_s$ and hence contains $\mathrm{WL}(\Gamma)$, so $\mathcal{Q}$ refines (we use "refines" in the order of partitions: $\mathcal{Q}$ refines $\mathcal{P}$ means every block of $\mathcal{Q}$ is contained in a block of $\mathcal{P}$) $\mathcal{P}_S$. □

Definition 3. *The Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ *is* WL-thin *if* $\mathcal{P}_S$ *is the thin S-ring, i.e., if* γ *is injective, i.e., if* $\mathrm{WL}(\Gamma) = \mathbb{C}[G]$.

Remark 2. *By Proposition 1 the whole computation can be carried out inside G. Starting with* $\gamma_0(1) = 0$, $\gamma_0(g) = 1$ *for* $g \in S$ *and* $\gamma_0(g) = 2$ *otherwise, define*

$$\gamma_{k+1}(g) = \Big(\gamma_k(g),\ \{\!\{(\gamma_k(h), \gamma_k(h^{-1}g)) : h \in G\}\!\}\Big).$$

For each of the $|G|$ *values of g, constructing and comparison-sorting the multiset of* $|G|$ *colour pairs costs* $O(|G| \log |G|)$*; hence one refinement round costs* $O(|G|^2 \log |G|)$*. Every non-stable round strictly increases the number of colour classes, from at least* 1 *to at most* $|G|$*, so at most* $|G| - 1$ *strict rounds occur. The resulting explicit worst-case bound is therefore*

$$O(|G|^3 \log |G|).$$

Hash-based implementations may be faster in practice, but are not needed for the stated guarantee. Thus WL-thinness is decidable in polynomial time and, by Theorem 5, gives a polynomial-time sufficient test for quantum rigidity.

Proposition 2. *If* $\mathrm{Cay}(G, S)$ *is WL-thin then it is a GRR of G.*

Proof. Let $\varphi \in \mathrm{Aut}(\Gamma)$. Composing with the left translation by $\varphi(1)^{-1}$ we may assume $\varphi(1) = 1$. Automorphisms preserve the stable colouring, so for every $x \in G$,

$$\gamma(\varphi(x)) = c(1, \varphi(x)) = c(\varphi(1), \varphi(x)) = c(1, x) = \gamma(x),$$

and injectivity of γ forces $\varphi(x) = x$. Hence the stabiliser of 1 in $\mathrm{Aut}(\Gamma)$ is trivial, and since $G \leq \mathrm{Aut}(\Gamma)$ acts transitively we get $\mathrm{Aut}(\Gamma) = G$, acting regularly. □

3.2. The Main Theorem

Theorem 5. *Let $\Gamma = \mathrm{Cay}(G, S)$ be WL-thin. Then $C(\mathrm{QAut}(\Gamma))$ is commutative; indeed the entries of the fundamental magic unitary satisfy*

$$u_{x,y} = u_{xg,yg} \qquad \text{for all } x, y, g \in G, \tag{2}$$

so that $u_{x,y} = p_{xy^{-1}}$ where $\{p_h := u_{h,1}\}_{h \in G}$ is a family of mutually orthogonal projections summing to 1. Consequently

$$\mathrm{QAut}(\Gamma) = \mathrm{Aut}(\Gamma) = G,$$

that is, Γ is a quantum rigid GRR.

Proof. By WL-thinness and Proposition 1(i), the stable colouring is $c(x, z) = \gamma(x^{-1}z)$ with γ injective, so that

$$c(x, z) = c(y, w) \iff x^{-1}z = y^{-1}w. \tag{3}$$

Lemma 1 therefore reads as follows: for all $x, y, z, w \in G$,

$$x^{-1}z \neq y^{-1}w \implies u_{x,y}\, u_{z,w} = 0. \tag{4}$$

Fix $x, y, g \in G$. We first show $u_{x,y} = u_{xg,yg}\, u_{x,y}$. Since the row of u indexed by xg sums to 1,

$$u_{x,y} = \Big(\sum_{w \in G} u_{xg,w} \Big) u_{x,y} = \sum_{w \in G} u_{xg,w}\, u_{x,y}.$$

By (4) applied with the pair (xg, w) in the first slot and (x, y) in the second, the term indexed by w vanishes unless $(xg)^{-1}x = w^{-1}y$, i.e., unless $g^{-1} = w^{-1}y$, i.e., unless $w = yg$. Hence

$$u_{x,y} = u_{xg,yg}\, u_{x,y}. \tag{5}$$

Symmetrically, using that the row indexed by x sums to 1,

$$u_{xg,yg} = \Big(\sum_{v \in G} u_{x,v} \Big) u_{xg,yg} = \sum_{v \in G} u_{x,v}\, u_{xg,yg},$$

and the term indexed by v vanishes unless $x^{-1}(xg) = v^{-1}(yg)$, i.e., unless $g = v^{-1}yg$, i.e., unless $v = y$. Hence

$$u_{xg,yg} = u_{x,y}\, u_{xg,yg}. \tag{6}$$

All entries of a magic unitary are self-adjoint, so taking adjoints in (5) gives $u_{x,y} = u_{x,y}\, u_{xg,yg}$, which combined with (6) yields $u_{x,y} = u_{xg,yg}$. This is (2).

Setting $g = y^{-1}$ in (2) gives $u_{x,y} = u_{xy^{-1},1} = p_{xy^{-1}}$. The entries of the row of u indexed by x are mutually orthogonal projections summing to 1; as y runs over G, xy^{-1} runs over G, so $\{p_h\}_{h \in G}$ is a family of $|G|$ mutually orthogonal projections with $\sum_h p_h = 1$. Mutually orthogonal projections commute, and by (2) they generate the whole algebra $C(\mathrm{QAut}(\Gamma))$; hence that algebra is commutative, and is a quotient of $\mathbb{C}^G$.

A commutative C^*-algebra of functions on a compact quantum group means the quantum group is a classical compact group, acting faithfully on the N vertices and preserving Γ; therefore $\mathrm{QAut}(\Gamma) \leq \mathrm{Aut}(\Gamma)$, and with the standard inclusion $\mathrm{Aut}(\Gamma) \leq \mathrm{QAut}(\Gamma)$ we get $\mathrm{QAut}(\Gamma) = \mathrm{Aut}(\Gamma)$. By Proposition 2, $\mathrm{Aut}(\Gamma) = G$. □

Remark 3. *Identity* (2) *is the precise quantum analogue of the classical statement that the automorphism group of a WL-thin Cayley graph consists of left translations: it says that the fundamental representation of* $\mathrm{QAut}(\Gamma)$ *is invariant under the diagonal right-translation action on indices, which for a classical permutation matrix means exactly that the permutation is a left translation. The proof extracts this from the orthogonality relations alone; no representation theory of* S_N^+ *is used.*

Remark 4. *The theorem gives no converse when* Γ *is not WL-thin. At one extreme, for* $S = \emptyset$ *the adjacency matrix is* 0*, while for* $S = G \setminus \{1\}$ *it is* $J - I$*. Every magic unitary commutes with both* 0 *and* $J - I$ *because its row and column sums are* 1*; hence in either case the defining graph relation imposes no additional relation on* $C(S_N^+)$ *and* $\mathrm{QAut}(\Gamma) = S_N^+$ *[11].*

3.3. Quantum Isomorphism Rigidity

Theorem 6. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be WL-thin and let* Γ' *be a graph with* $\Gamma \cong_q \Gamma'$*. Then* $\Gamma' \cong \Gamma$*. Equivalently, on the class of WL-thin Cayley graphs, quantum isomorphism coincides with isomorphism.*

Proof. By ([22], §5), quantum isomorphic graphs are not distinguished by the Weisfeiler–Leman algorithm: there is an algebraic isomorphism $f\colon \mathrm{WL}(\Gamma) \to \mathrm{WL}(\Gamma')$, that is, a bijection between basic relations preserving equality type and all intersection numbers, which carries the edge relation of Γ to that of Γ'.

By hypothesis $\mathrm{WL}(\Gamma) = \mathbb{C}[G]$ is thin; i.e., all its basic relations R_g ($g \in G$) have valency 1. Since valency is determined by the intersection numbers, every basic relation $f(R_g)$ of $\mathrm{WL}(\Gamma')$ also has valency 1, and hence $\mathrm{WL}(\Gamma')$ is thin. We recall the standard thin-scheme/group correspondence [29] in the form needed here. Because the configuration is homogeneous, a valency-one basic relation R is the graph of a unique permutation of the vertex set: every vertex has exactly one R-successor, and the same statement for R^* makes this map bijective. If R and T are two basic relations of valency one, then for each vertex there is exactly one two-step walk first in R and then in T. Consequently there is a unique basic relation U with intersection number $p_{R,T}^U = 1$, while $p_{R,T}^W = 0$ for every other basic relation W; equivalently, the relational product satisfies $R \circ T = U$. The diagonal relation is the identity and transpose gives inverses. Thus the basic relations of a homogeneous thin coherent configuration form a group under relational product, and the associated permutations give a regular permutation group.

For the present configuration, $R_g \circ R_h = R_{gh}$. Since an algebraic isomorphism preserves all intersection numbers, the preceding characterisation gives

$$f(R_g) \circ f(R_h) = f(R_{gh}) \qquad (g, h \in G).$$

Hence $R_g \mapsto f(R_g)$ is a group isomorphism from the relation group $\{R_g : g \in G\} \cong G$ onto the regular relation group of $\mathrm{WL}(\Gamma')$. In particular, the group structure on the latter is reconstructed solely from the intersection numbers; no additional choice of a permutation group is being made.

Choose $o' \in V(\Gamma')$. For each $g \in G$, let $\phi(g)$ be the unique vertex with $(o', \phi(g)) \in f(R_g)$. Regularity makes $\phi\colon G \to V(\Gamma')$ a bijection. Moreover, for every $s \in G$,

$$f(R_g) \circ f(R_s) = f(R_{gs}),$$

so the unique $f(R_s)$-successor of $\phi(g)$ is $\phi(gs)$. Since the edge relation of Γ is $\bigcup_{s \in S} R_s$ and f carries it to the edge relation of Γ', the basic relations being pairwise disjoint, we obtain for all $g, s \in G$

$$\{\phi(g), \phi(gs)\} \in E(\Gamma') \quad \Longleftrightarrow \quad s \in S.$$

Thus ϕ is a graph isomorphism $\mathrm{Cay}(G, S) \to \Gamma'$. □

Corollary 1. *A WL-thin Cayley graph is determined, among all graphs, by its homomorphism counts from planar graphs.*

Proof. By Mančinska–Roberson [30], two graphs are quantum isomorphic if and only if they admit equally many homomorphisms from every planar graph. Combine with Theorem 6. □

Proposition 3. *Let* Γ *be an arbitrary finite graph. If the coherent closure of* Γ *is discrete, i.e., if every ordered pair* (x, z) *forms a singleton basic relation, then* $\mathrm{QAut}(\Gamma)$ *is trivial.*

Proof. Let $u = (u_{xy})$ be the fundamental magic unitary. If $x \neq y$, then the diagonal ordered pairs (x, x) and (y, y) are distinct singleton colour classes. Lemma 1, with $(x, z) = (x, x)$ and $(y, w) = (y, y)$, yields

$$u_{xy}u_{xy} = 0.$$

Because u_{xy} is a projection, $u_{xy} = 0$ whenever $x \neq y$. The row-sum relations then give $u_{xx} = 1$ for every x. Thus $u = I$ and the quantum automorphism group is trivial. □

3.4. Elementary Consequences

We record three easy but useful consequences of Theorem 5.

Proposition 4. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *with* $S \neq \emptyset$ *and* $S \neq G \setminus \{1\}$*, and let* $\bar{S} = G \setminus (S \cup \{1\})$*. Then* Γ *is WL-thin if and only if* $\mathrm{Cay}(G, \bar{S})$ *is WL-thin. Consequently quantum rigidity of Cayley graphs is preserved under complementation.*

Proof. The complement of Γ is $\mathrm{Cay}(G, \bar{S})$. The two initial Weisfeiler–Leman colourings differ only by interchanging the two labels 1 and 2, so they generate the same coherent algebra: $\mathrm{WL}(\Gamma) = \mathrm{WL}(\bar{\Gamma})$. Hence $\mathcal{P}_S = \mathcal{P}_{\bar{S}}$ and thinness holds for one if and only if it holds for the other. Now apply Theorem 5. (Independently, $\mathrm{QAut}(\Gamma) = \mathrm{QAut}(\bar{\Gamma})$ always, since $A_{\bar{\Gamma}} = J - I - A_\Gamma$ and a magic unitary commutes with I and J.) □

Proposition 5. *The set of symmetric connection sets* S *for which* $\mathrm{Cay}(G, S)$ *is WL-thin is invariant under* $\mathrm{Aut}(G)$*.*

Proof. If $\alpha \in \mathrm{Aut}(G)$ then α is a graph isomorphism $\mathrm{Cay}(G, S) \to \mathrm{Cay}(G, \alpha(S))$ carrying the stable colouring of one to that of the other and commuting with the identification $c(x, z) = \gamma(x^{-1}z)$; so $\gamma_{\alpha(S)} = \gamma_S \circ \alpha^{-1}$, which is injective if and only if γ_S is. □

Corollary 2. *A WL-thin Cayley graph is a Cayley graph of exactly one group: if* $\mathrm{Cay}(G, S)$ *is WL-thin and* $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(H, T)$ *for some group* H*, then* $H \cong G$*.*

Proof. A regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$ isomorphic to H exists by hypothesis, and $\mathrm{Aut}(\mathrm{Cay}(G, S)) = G$ by Proposition 2; a regular subgroup of a regular group is the whole group. □

3.5. A Worked Example

Let $G = D_{12} = \langle r, s \mid r^6 = s^2 = 1,\ srs = r^{-1} \rangle$ and

$$S = \{r,\ r^5,\ s,\ rs,\ r^3 s\},$$

a symmetric generating set of size 5; this is the smallest WL-thin connection set in D_{12}, as an exhaustive search shows. The refinement of Remark 2 runs as follows, writing the colour of each group element.

	1	r	r^2	r^3	r^4	r^5	s	rs	r^2s	r^3s	r^4s	r^5s
γ_0	0	1	2	2	2	1	1	1	2	1	2	2
γ_1	0	1	2	2	2	1	3	3	4	5	2	2
γ_2	0	1	2	3	4	5	6	7	8	9	10	11

After two rounds the colouring is discrete and hence stable, so $\mathcal{P}_S$ is the thin S-ring. By Theorem 5, $\mathrm{Cay}(D_{12}, S)$ is a quantum rigid GRR: $\mathrm{QAut} = \mathrm{Aut} = D_{12}$. (An independent computation with the VF2 algorithm confirms $|\mathrm{Aut}| = 12$.) It is worth noting what the two rounds do. The first round separates the reflections in S from the rotations in S, and splits the reflections according to the number of 2-paths joining them to the identity; the second round then propagates this to a complete separation. No single count does the job: the autocorrelation $g \mapsto |S \cap gS|$ alone leaves several classes merged, and it is the iteration that is essential—which is precisely the obstacle to proving thinness for infinite families by hand.

4. Schurity Criterion, Obstruction, and Sharpness

4.1. The Schurity Criterion

Theorem 7. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a GRR of G. Then the following are equivalent:*

(i) *Γ is WL-thin;*
(ii) *the coherent closure $\mathrm{WL}(\Gamma)$ is Schurian;*
(iii) *the S-ring $\mathcal{P}_S$ is Schurian.*

In particular, if Γ is a GRR which is not quantum rigid, then $\mathcal{P}_S$ is a non-Schurian S-ring over G whose automorphism group is the regular group G.

Proof. (ii) and (iii) are the same statement under the identification of Proposition 1(ii).

(i)⇒(ii). If Γ is WL-thin then $\mathrm{WL}(\Gamma) = \mathbb{C}[G]$, whose basic relations $R_g = \{(x, xg)\}$ are precisely the orbits of the regular group G acting by left translation on $G \times G$. Hence $\mathrm{WL}(\Gamma) = \mathrm{Orb}(G)$. Moreover $\mathrm{Aut}(\mathbb{C}[G]) = G$: a permutation π of G preserving each R_g satisfies $\pi(x)^{-1}\pi(xg) = g$ for all x, g, i.e., $\pi(xg) = \pi(x)g$, so π is left translation by $\pi(1)$. Thus $\mathrm{WL}(\Gamma) = \mathrm{Orb}(\mathrm{Aut}(\mathrm{WL}(\Gamma)))$ and $\mathrm{WL}(\Gamma)$ is Schurian.

(ii)⇒(i). Suppose $\mathrm{WL}(\Gamma)$ is Schurian, so $\mathrm{WL}(\Gamma) = \mathrm{Orb}(\mathrm{Aut}(\mathrm{WL}(\Gamma)))$. By (1), $\mathrm{Aut}(\mathrm{WL}(\Gamma)) = \mathrm{Aut}(\Gamma) = G$ because Γ is a GRR. Hence $\mathrm{WL}(\Gamma) = \mathrm{Orb}(G) = \mathbb{C}[G]$, which is thin.

For the final statement, if Γ is a GRR which is not quantum rigid then $\mathrm{QAut}(\Gamma) \neq G$, so by Theorem 5 Γ is not WL-thin, so by the equivalence $\mathcal{P}_S$ is non-Schurian; its automorphism group is $\mathrm{Aut}(\Gamma) = G$. □

Corollary 3. *Let G be a Schur group; i.e., suppose every S-ring over G is Schurian. Then every GRR of G is quantum rigid, and indeed WL-thin.*

Proof. Immediate from Theorems 5 and 7. □

Corollary 3 must be read together with the known structure of Schur groups. For cyclic groups, the exact classification is given in [27]; most cyclic groups are not Schur, and cyclic groups of order greater than 2 do not admit undirected GRRs because they are Abelian of exponent greater than two. On the other hand, C_2^5 is a Schur group [28] and does admit GRRs. Therefore Corollary 3 yields an unconditional conclusion: every GRR of C_2^5 is WL-thin and quantum rigid. The elementary Abelian group C_2^6 is not a Schur group, so it is a natural first elementary-Abelian test case where Schurity itself no longer settles the issue.

Proposition 6. *Among elementary Abelian 2-groups, C_2^5 simultaneously has the Schur property and admits GRRs, whereas C_2^d with $d \leq 4$ admits no GRR and C_2^d with $d \geq 6$ is not a Schur group.*

Proof. The Schur-group classification for elementary Abelian groups gives the 2-power orders $4, 8, 16, 32$ and no larger elementary Abelian 2-group [28]. Godsil's GRR classification gives no GRR for C_2^2, C_2^3, C_2^4, while C_2^d admits a GRR for $d \geq 5$ [1]. Combining the two classifications gives the assertion. □

Remark 5. *Theorem 7 provides an obstruction, not an equivalence with quantum rigidity. If a GRR fails quantum rigidity then its coherent closure is necessarily non-Schurian. A non-Schurian coherent closure, however, is not proved to force quantum symmetry.*

4.2. Sharpness of the Thinness Hypothesis

The following classical computations show that the hypothesis of Theorem 5 cannot be weakened to "S generates G", "Γ is connected and vertex-transitive", or anything of that kind.

Example 1 (Hypercubes)**.** *Let $G = \mathbb{Z}_2^n$ and $S = \{e_1, \ldots, e_n\}$, the standard basis, so $\Gamma = Q_n$ is the n-cube. Two vertices x, z satisfy $\mathrm{WL}_{Q_n}(x, z) = \mathrm{WL}_{Q_n}(y, w)$ as soon as $x + z$ and $y + w$ have the same Hamming weight, and in fact $\mathrm{WL}(Q_n)$ is the Bose–Mesner algebra of the Hamming scheme $H(n, 2)$, with basic sets the $n + 1$ weight classes. Thus $\mathcal{P}_S$ has $n + 1$ basic sets and is very far from thin as soon as $n \geq 2$. Correspondingly, Banica, Bichon and Collins [25] proved*

$$\mathrm{QAut}(Q_n) = O_n^{-1},$$

the free complexification / anticommutative deformation of the orthogonal group, which is a genuinely quantum object of unbounded dimension. Of course $\mathrm{Aut}(Q_n) = \mathbb{Z}_2^n \rtimes S_n$ is much larger than G, so Q_n is not a GRR either.

Example 2 (Folded cubes)**.** *For n odd the folded n-cube is again a Cayley graph of an elementary Abelian 2-group, with WL closure the corresponding folded Hamming scheme, and Schmidt [16] showed $\mathrm{QAut} = SO_n^{-1}$; the case $n = 5$ is the Clebsch graph, answering a question of [25].*

Example 3 (Where thinness is what changes)**.** *Inside a single group both behaviours occur. For $G = D_{12}$ the exhaustive census of Table 1 contains 464 connected labelled Cayley graph instances, of which exactly 48 are WL-thin; these are exactly the 48 GRRs. The worked example in Section 3.5 is one of the thin cases, while the standard connection sets producing extra dihedral symmetries are non-thin. Thus the relevant distinction is not determined by the ambient group alone, but by the S-ring generated by S.*

Table 1. Exhaustive enumeration of symmetric connection sets/labelled Cayley graph instances. In every row thin = GRR; counts are not quotiented by graph isomorphism.

G	$\lvert G\rvert$	k	2^k	conn	thin	GRR
$\mathbb{Z}_6$	6	3	8	5	0	0
$\mathbb{Z}_8$	8	4	16	12	0	0
$D_6 \cong S_3$	6	4	16	11	0	0
D_8	8	6	64	48	0	0
$Q_8 = \mathrm{Dic}_2$	8	4	16	8	0	0
$\mathbb{Z}_2^3$	8	7	128	92	0	0
D_{10}	10	7	128	119	0	0
D_{12}	12	9	512	464	48	48

Table 1. *Cont.*

G	$\lvert G\rvert$	k	2^k	conn	thin	GRR
A_4	12	7	128	116	0	0
Dic_3	12	6	64	50	0	0
D_{14}	14	10	1024	1009	168	168
D_{16}	16	12	4096	3960	1024	1024
$Q_{16} = \mathrm{Dic}_4$	16	8	256	216	0	0
SD_{16}	16	10	1024	936	128	128
M_{16}	16	9	512	456	0	0
$\mathbb{Z}_2^4$	16	15	32,768	31,232	0	0
$\mathbb{Z}_2 \times D_8$	16	13	8192	7728	1536	1536
D_{18}	18	13	8192	8134	3888	3888
D_{20}	20	15	32,768	32,468	18,240	18,240
Dic_5	20	10	1024	982	0	0
S_4	24	16	65,536	65,196	45,600	45,600
$\mathrm{SL}(2,3)$	24	12	4096	4056	1152	1152
Dic_6	24	12	4096	3896	0	0
$A_4 \times \mathbb{Z}_2$	24	15	32,768	32,500	18,432	18,432

5. Computations and Reproducibility

5.1. Exact Protocol

For a finite group G represented by its multiplication table, let the *symmetric atoms* be the inverse-orbits in $G \setminus \{1\}$: an atom is $\{g, g^{-1}\}$, reduced to a singleton when g is an involution. Every symmetric connection set is a union of these atoms, so if there are k atoms there are exactly 2^k candidates. Throughout this section, a "candidate" or "instance" means one such connection set S on the labelled vertex set G; no quotient by $\mathrm{Aut}(G)$ or by abstract graph isomorphism is taken. The accompanying file `reproduce_census.py` constructs each group, enumerates all atom subsets, discards S with $\langle S \rangle \neq G$, computes the stable in-group WL partition of Remark 2, and tests thinness.

For a connected non-thin graph the program performs an independent exact GRR test with NetworkX's VF2 isomorphism engine. The stable colour of each vertex relative to the identity is supplied as a necessary node constraint, and the search stops only after either finding a non-identity automorphism fixing 1 or exhausting all possibilities. There is no node cap. Thus no computational verdict depends on an unreported truncation parameter. The program records the number of refinement rounds and the wall-clock time of every instance in CSV output. The command-line option `-self-test` checks the group orders and the worked D_{12} example.

The implementation is deterministic for exhaustive runs. Sampling is not used for any numerical claim below; the script nevertheless provides an explicitly seeded sampling mode for independent exploratory work. On the bundled validation run for D_{12}, all $2^9 = 512$ symmetric connection sets were processed: 464 are connected, 48 are WL-thin and exactly the same 48 are GRRs. This run reproduces the row in Table 1; the CSV file is included with the source package. Absolute timings are machine-dependent and therefore are emitted by the program rather than treated as mathematical invariants.

5.2. Exhaustive Census

Table 1 lists the twenty-four groups for which every symmetric connection set was enumerated. The unit of enumeration is a symmetric connection set S on the fixed labelled vertex set G: two distinct sets S and T are counted as two census instances even if $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ as abstract graphs. Thus the table does not report numbers

of graph-isomorphism classes. Here conn counts the connected labelled Cayley graph instances, thin the WL-thin instances, and GRR those with $\mathrm{Aut}(\Gamma) = G$.

Summing the rows gives 193,694 connected labelled Cayley graph instances and 90,216 GRR instances. No connected instance in this census is a GRR without being WL-thin. Therefore every GRR in the census is quantum rigid by Theorem 5. The groups with zero GRR count agree with the relevant portion of the Godsil classification [1], providing a classification-level consistency check that is logically independent of the thinness theorem.

5.3. A Concrete Near-Miss

A useful near-miss is the 4-cube $Q_4 = \mathrm{Cay}(C_2^4, \{e_1, e_2, e_3, e_4\})$. It is connected but not a GRR: its vertex stabiliser contains the coordinate-permutation group S_4. Its coherent closure has only the five Hamming-weight classes rather than 16 singleton classes. Thus it fails the target property at two visible levels—classical regularity and WL-thinness—while retaining the same Cayley/WL structure used in the main theorem. This example illustrates concretely what the computation detects before one reaches the genuinely open case of a GRR with non-thin closure.

5.4. Interpretation

The data suggest, but do not prove, a generic thinness phenomenon. A heuristic comes from collision suppression in successive refinements. At a fixed refinement stage two distinct $g, g' \in G$ remain indistinguishable only if the entire empirical distribution of colour pairs

$$(\gamma_k(h), \gamma_k(h^{-1}g)) \qquad (h \in G)$$

matches the corresponding distribution for g'. For a random symmetric connection set these are strongly dependent random variables, so a naive independence estimate would be unjustified; nevertheless the number of simultaneous count equalities grows with the number of colour pairs after each strict refinement. Once the colour palette itself grows with $|G|$, satisfying all these equalities should become increasingly rare. This is only a heuristic and is stated as such; a proof would require controlling the dependencies created by inversion symmetry and group multiplication.

6. Open Problems

Question 8. *Is every graphical regular representation quantum rigid?*

Theorem 7 gives a one-way obstruction: a negative answer to Question 8 would produce a GRR with non-Schurian coherent closure. The converse is not established; consequently the quantum-rigidity question and the Schurity question below are distinct.

Question 9. *Does there exist a GRR whose coherent closure is non-Schurian, or equivalently, a GRR which is not WL-thin?*

A positive answer to Question 9 would show that the sufficient criterion of Theorem 5 is not universal, but would not by itself prove the existence of quantum symmetry. The exhaustive census of Section 5 finds no such example.

Conjecture 10. *Let (G_n) be any sequence of finite groups admitting GRRs with $|G_n| \to \infty$. For each n, let $\mathcal{I}_n$ be the set of orbits of the involution $g \mapsto g^{-1}$ on $G_n \setminus \{1\}$. Choose a random subcollection $\mathcal{A}_n \subseteq \mathcal{I}_n$ by including each orbit independently with probability $1/2$, and set*

$$S_n = \bigcup_{O \in \mathcal{A}_n} O.$$

Equivalently, S_n is sampled from the uniform counting measure on all symmetric subsets of $G_n \setminus \{1\}$. Then

$$\Pr\big(\mathrm{Cay}(G_n, S_n) \text{ is WL-thin}\big) \longrightarrow 1.$$

The heuristic in Section 5 explains why this strengthening of asymptotic GRR genericity is plausible, while also identifying the missing ingredient: one needs quantitative control of the dependent collision events produced by successive WL refinements. Xia and Zheng proved that almost all finite Cayley graphs have the minimal classical automorphism group whenever a GRR is admissible [8]; Conjecture 10 asks whether the coherent closure is typically minimal as well.

Question 11. *Is there an infinite family of groups G_n and connection sets S_n for which WL-thinness of $\mathrm{Cay}(G_n, S_n)$ can be proved in closed form?*

The discrete-closure problem for arbitrary graphs is not open: Proposition 3 shows directly from Lemma 1 that a graph with discrete coherent closure has a trivial quantum automorphism group. What remains interesting beyond the Cayley setting is to identify substantially weaker coherent-configuration hypotheses that still force trivial quantum symmetry.

7. Concluding Remarks

For Cayley graphs, the coherent closure supplies information that the classical automorphism group alone does not contain. WL-thinness forces the translation identity (2), which collapses the fundamental magic unitary to a partition of unity indexed by G and hence makes the quantum automorphism algebra commutative. For a GRR, Schurity of the coherent closure is equivalent to WL-thinness and is therefore sufficient for quantum rigidity. The logical converse is deliberately left open: non-Schurity is a necessary obstruction to a non-quantum-rigid GRR, not a proved source of quantum symmetry.

Algorithmically, the in-group refinement gives an explicit $O(|G|^3 \log |G|)$ worst-case bound in the comparison model. The terminology "polynomial-time sufficient test" is used to separate this decision procedure from the different notion of a polynomially verifiable certificate. The reproducibility package implements the exact refinement and GRR tests, contains no hidden search cap, and exports the raw classifications and timings.

The exhaustive census of symmetric connection sets (hence labelled Cayley graph instances, not isomorphism classes) provides evidence in the range tested but is not used as a substitute for proof. Its most useful role is diagnostic: it verifies the known GRR exclusions, reproduces the worked D_{12} example independently, and finds no GRR with non-Schurian coherent closure. The remaining conceptual problems are therefore sharply separated: determine whether every GRR is quantum rigid, determine whether every GRR is WL-thin, and understand whether WL-thinness is asymptotically generic.

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