

Article

A New Jounce Chaotic System: Dynamical Analysis, Adaptive Synchronization and Switching Parameter Modulation Applications

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Abstract

This paper presents and analyzes a new four-dimensional (4D) jounce hysteresis chaotic dynamical system and investigates its application to robust communication. An adaptive chaotic synchronization scheme is developed based on the proposed 4D jounce system, in which an adaptive controller is designed at the receiver to achieve robust synchronization between the master and slave systems in the presence of uncertain system parameters. Lyapunov stability theory is employed to derive the adaptive control laws and parameter-update rules required for online estimation of the uncertain receiver-side parameters. For chaos-based communication, multiple information signals are encoded by modulating selected parameters of the chaotic transmitter system. At the receiver, the adaptive parameter estimator tracks the transmitted time-varying parameter, enabling reliable information recovery through a midpoint-thresholding decision scheme after the initial transient period. The performance and robustness of the proposed communication framework are evaluated through comprehensive numerical simulations under additive white Gaussian noise (AWGN) at different noise levels. The results demonstrate accurate chaotic synchronization, reliable information recovery, and robustness against channel noise, indicating the potential of the proposed 4D jounce chaotic system for robust and noise-tolerant communication applications.

Keywords: jounce chaotic system; synchronization; adaptive control; chaos-based communication; switching parameter modulation

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1. Introduction

Over the past several decades, chaotic systems have attracted increasing attention in both theoretical research and practical circuit implementation because of their distinctive nonlinear dynamics and broad range of applications. They have been investigated extensively in various fields, including engineering [1–3], biology and ecological models [4,5],

data encryption [6], physics [7], navigation systems, image processing [8,9], and military and wireless communications [10,11]. Chaotic systems exhibit several distinctive properties, including high sensitivity to small variations in initial conditions, complex and difficult-to-predict trajectories, and broadband noise-like behavior [12]. These characteristics make chaotic systems particularly attractive for secure communication and information-encryption applications, where signal complexity, synchronization, and resistance to unauthorized interception are important considerations.

In general, only a limited number of coupled structures can achieve autonomous synchronization. However, many other systems require external control to achieve synchronization. In nonlinear systems, chaos synchronization remains a crucial research topic and an important approach. The first scheme for chaos synchronization was proposed by Pecora and Carroll and termed the Pecora–Carroll (PC) synchronization method, which is widely regarded as a pioneering contribution in the corresponding subject of studies.

The PC method deals with two identical chaotic systems referred to as the master and slave systems, respectively, with different initial conditions [13]. Many studies have since been proposed to achieve synchronization using time-delay feedback, the OGY method, and adaptive synchronization. Other approaches include active control [14,15], nonlinear control, linear feedback control [16,17], sliding mode control [18], lag synchronization [19] and =projective synchronization [12]. Studies have shown that adaptive control can be utilized in some chaotic systems (e.g., the Lorenz and Rössler systems) [20–22]. The strategies of adaptive control are utilized when some or all parameters of chaotic systems are unknown. The important features of the adaptive control method include good transient performance, fast response, easy implementation in practical applications and robustness against uncertainties [22]. Existing studies have demonstrated chaotic parameter modulation (CPM) for secure communication using various low-dimensional and hyperchaotic systems, with adaptive synchronization enabling recovery of the transmitted information [23].

In this paper, the nonlinear analysis of a new jounce chaotic model is considered. Bifurcation diagrams, the Lyapunov exponent spectrum, power spectral density, bicoherence, and spectral entropy have been used to explore the proposed system dynamics. The new model exhibits several features that are beneficial for robust noise-tolerant communication applications. Such as a wide range of system parameters exhibits chaotic behavior and hysteretic dynamical behavior. Hysteresis modeling is important in structural and mechanical systems for representing nonlinear, history-dependent behavior. Various physical and phenomenological models have been developed, including the well-known Bouc–Wen model, which describes hysteresis based on both current and past input values. Chaotic behavior has also been investigated in systems incorporating such hysteretic dynamics [24–27].

Chaos synchronization of the proposed model is studied in this work, where two jounce systems are arranged in a master–slave configuration and synchronized with each other. The slave chaotic system is considered uncertain. The synchronization process is achieved based on Lyapunov stability theory and an adaptive synchronization method. An adaptive control law and a parameter estimation update rule for unknown slave system parameters are given for self-synchronization of the jounce chaotic systems. The complete system structure is employed in a robust communication application. A chaotic switching scheme has been used for modulation of the transmitted information signal into one of the parameters of the transmitter (master) system. It is important to study robust communication in the presence of unknown parameters and noise, which can destroy or even disrupt synchronization [28].

The main contributions of this study can be summarized as follows: The majority of earlier synchronization techniques assumed that the parameters of chaotic systems are known. Although the values of these parameters are not exactly known a priori in many applications, they are inevitably disrupted by external influences such as voltage variation and

temperature. As a result, these issues might degrade synchronization performance or even cause synchronization failure. Therefore, the study of synchronizing chaotic systems with uncertain parameters should be considered with high interest. Moreover, comparatively limited attention has been given to CPM-based communication using four-dimensional jounce hysteresis chaotic systems with uncertain parameters, particularly for binary image transmission over noisy communication channels. The proposed work addresses this gap by integrating a novel 4D jounce hysteresis chaotic system with adaptive synchronization, dynamic parameter estimation, and binary CPM for reliable information and image recovery under different channel-noise conditions. Due to the uniquely complicated structure of the jounce hysteretic chaotic system, it is challenging to build an appropriate Lyapunov function to demonstrate the stability of the error dynamics, since such a function depends on how to estimate the uncertain key parameters of the slave system. Also, for the secure communication application, to increase the security level, many cryptosystems leverage nonlinear dynamical systems with complex dynamics such as chaotic oscillators with strong nonlinear and hysteretic behavior.

The structure of this paper is summarized as follows: Section 2 presents the mathematical model of a new 4D jounce chaotic hysteresis system, along with the dynamical analysis of the system using modern nonlinear analysis tools. Section 3 explains the details of the synchronization theory for the new jounce master and slave chaotic systems and the design of the adaptive control strategy. The simulation results of the suggested synchronization algorithm are illustrated in Section 4. Section 5 presents robust noise-tolerant communication applications using switching modulation or chaotic parameter modulation, where the information signals (a binary pseudorandom (PN) code and a binary image) are sent via noiseless and noisy channels. Finally, Section 6 discusses the conclusions of the work reported in this manuscript.

2. Mathematical Modeling and Dynamical Analysis of the New Jounce Hysteresis Chaotic System

In this section, the dynamics and behavior of the new 4D chaotic jounce system are presented and discussed. The model is derived from the RC-OTA hysteretic oscillator presented in [25]. The proposed system introduces a cross-coupling state feedback term $(-\alpha(z + u))$ directly into the first state equation. In contrast, in Ref. [25], $\dot{x} = y$ acts as a pure kinematic integrator. Then, the new proposed chaotic model can be represented by the following system of equations:

$$\begin{cases} \dot{x} = y - \alpha(z + u) \\ \dot{y} = z + u + \beta y - x \\ \dot{z} = \gamma F_1(x, z) \\ \dot{u} = \gamma F_2(x, u) \end{cases} \quad (1)$$

where $x(t)$, $y(t)$, $z(t)$ and $u(t) \in \mathbb{R}$ are the state variables; $F_n(\bullet) : \mathbb{R} \rightarrow \mathbb{R}$ with $n = 1, 2$ are nonlinear functions; and α , β and γ are system parameters. The β parameter acts as the active gain and negative damping coefficient, physically representing the active OTA transconductance gain that supplies power to sustain chaotic trajectories; γ represents the gain controlling the nonlinearity effect and serves as the nonlinear transition rate, establishing a fast-slow timescale separation under singular perturbation that mathematically models sharp relay switching while physically capturing the finite switching bandwidth, slew rate, and component delays. The α parameter functions as the feedforward velocity coupling factor, mathematically skewing the derivative field ($\dot{x} = y - \alpha(z + u)$) to adjust attractor geometry while physically accounting for direct feedforward transconductance

or parasitic capacitive feedthrough between the non-linear hysteresis generator and the primary RC-integrating node.

To generate a three-scroll chaotic attractor, the nonlinearities can be written as follows:

$$\begin{cases} F_1(x, z) = (-z + \operatorname{sgn}[x - 1 + \operatorname{sgn}(z)] + 0.5) \\ F_2(x, u) = (-u + \operatorname{sgn}[x + 1 + \operatorname{sgn}(u)] - 0.5) \end{cases} \quad (2)$$

where the standard symmetric signum convention has been adopted:

$$\operatorname{sgn}(\theta) = \begin{cases} +1, & \theta > 0 \\ 0, & \theta = 0 \\ -1, & \theta < 0 \end{cases} \quad (3)$$

The nonlinear functions in (2) particularly exhibit hysteresis dynamics [25]. By introducing $-\alpha(z + u)$, the fast hysteretic switching variables ($z(t)$ and $u(t)$) are fed back directly into the rate of change of state $x(t)$, fundamentally altering the differential structure of the vector field. Moreover, the state feedback parameter (α) acts as an adjustable spatial controller that dynamically expands the distance between scroll centers in the phase plane. Using (1) and (2), one can find equilibrium points of the proposed jounce chaotic system as follows:

$$\begin{cases} y - \alpha(z + u) = 0 \\ z + u + \beta y - x = 0 \\ \gamma F_1(x, z) = 0 \\ \gamma F_2(x, u) = 0 \end{cases} \quad (4)$$

The analytical solution of (4) gives three equilibrium points that lie within smooth sub-regions of the vector field located at $E_1 = (0, 0, -0.5, 0.5)$, $E_2 = (2 + 2\alpha\beta, 2\alpha, 1.5, 0.5)$, and $E_3 = (-2 - 2\alpha\beta, -2\alpha, -0.5, -1.5)$, with two of the equilibrium points depending on bifurcation parameters.

Within the interior of all sub-regions surrounding E_1 , E_2 , and E_3 , the partial derivatives of F_1 and F_2 with respect to x are zero almost everywhere. Consequently, the Jacobian matrix (J) is uniform across all three equilibria:

$$J = \begin{bmatrix} 0 & 1 & -\alpha & -\alpha \\ -1 & \beta & 1 & 1 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{bmatrix} \quad (5)$$

After solving the characteristic equation, i.e., $\det(\lambda I - J) = 0$, the resulting analytical eigenvalues are given by

$$\lambda_{1,2} = -\gamma, \quad \text{and} \quad \lambda_{3,4} = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}. \quad (6)$$

Nominal parameter values of $\alpha = 0.7$, $\beta = 0.05$, and $\gamma = 15$ result in $\lambda_{1,2} = -15$ and $\lambda_{3,4} = 0.025 \pm i0.9997$. Thus, E_1 , E_2 , and E_3 are saddle foci of index 2. The combination of stable real contraction and unstable planar spiral expansion drives the trajectory between hysteresis thresholds, generating three-scroll chaos. In contrast, one more equilibrium point lies at the intersection of the two switching manifolds located at $E_4 = (0, 0, 0.5, -0.5)$. Consequently, the conventional Jacobian criterion is not directly applicable to E_4 . Using

the Filippov interpretation, the one-sided vector fields at E_4 are $f^+(E_4) = [0, 0, 15, 15]^T$ and $f^-(E_4) = [0, 0, -15, -15]^T$, giving $0 \in \text{co}\{f^-, f^+\}$. Hence, E_4 is a Filippov equilibrium.

The bifurcation diagrams of system (1) are shown in Figure 1. Bifurcation diagrams are a useful tool for understanding the behavior of complex systems and can provide insights into the underlying dynamics of these systems. They can be used to study the qualitative behavior of a system and help to identify transitions between different behaviors or regimes. Such a diagram can be used in a wide range of applications, including in the study of biological systems, financial systems, and engineering systems [29]. The α and β parameters serve as control parameters for investigation of the system's dynamical behavior. Figure 1a shows that system (1) displays robust chaos characterized by linearly expanding phase-space amplitude before an abrupt transition to periodic behavior. For $\alpha \in [0, 0.71]$, it displays a continuous chaotic regime. At $\alpha \approx 1.35$, the broad chaotic attractor collapses suddenly, where for $\alpha \in [1.35, 1.6]$, the dynamics signify transition to a stable periodic limit cycle, while Figure 1b indicates that the system has a narrowband periodic window for $\beta \in [0, 0.018)$ and chaotic dynamic across $\beta \in [0.018, 0.3]$ range.

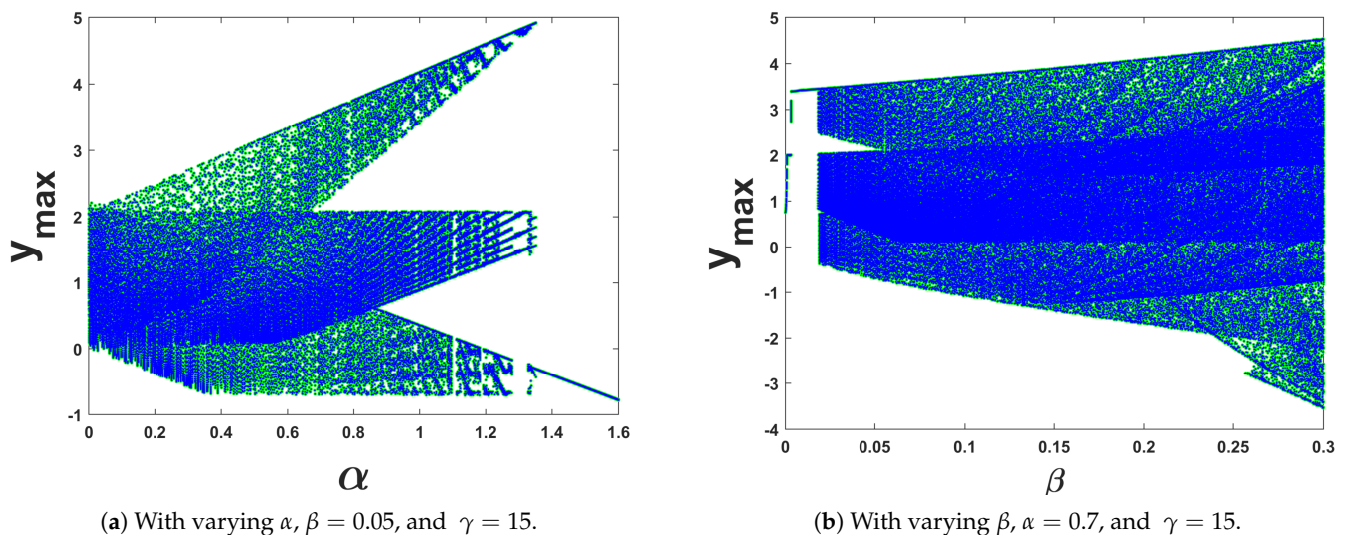


Figure 1. Bifurcation diagrams of jounce chaotic system (1) with nonlinearity (2).

Another contemporary nonlinear analysis tool used to measure the sensitivity of a system to initial conditions is called the Lyapunov exponent spectrum, which is commonly used in the study of chaotic systems, named after Russian mathematician Aleksandr Lyapunov, who developed it in the late 19th century. Lyapunov exponents are estimated by adopting the classical algorithm proposed by Wolf and co-workers [30,31]. Figure 2 shows the Lyapunov exponent spectrum for system (1) with α and β as control parameters. The plot indicates one positive Lyapunov exponent in some range of system parameters, which is a clear signature of chaotic behavior, as the sensitivity to initial conditions leads to unpredictable behavior over time.

All bifurcation diagrams and Lyapunov exponent spectra were computed using a fixed-step fourth-order Runge–Kutta integration scheme with a step size of $\Delta t = 0.001$ s over a total simulation duration of $T_{\text{total}} = 500$ s (500,000 iterations). To eliminate transient dynamic effects, the initial 30% of simulation data was discarded prior to sampling local state maxima and accumulating Lyapunov exponents. The full Lyapunov spectrum $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ was evaluated using the Benettin Gram–Schmidt reorthonormalization algorithm applied at every time step.

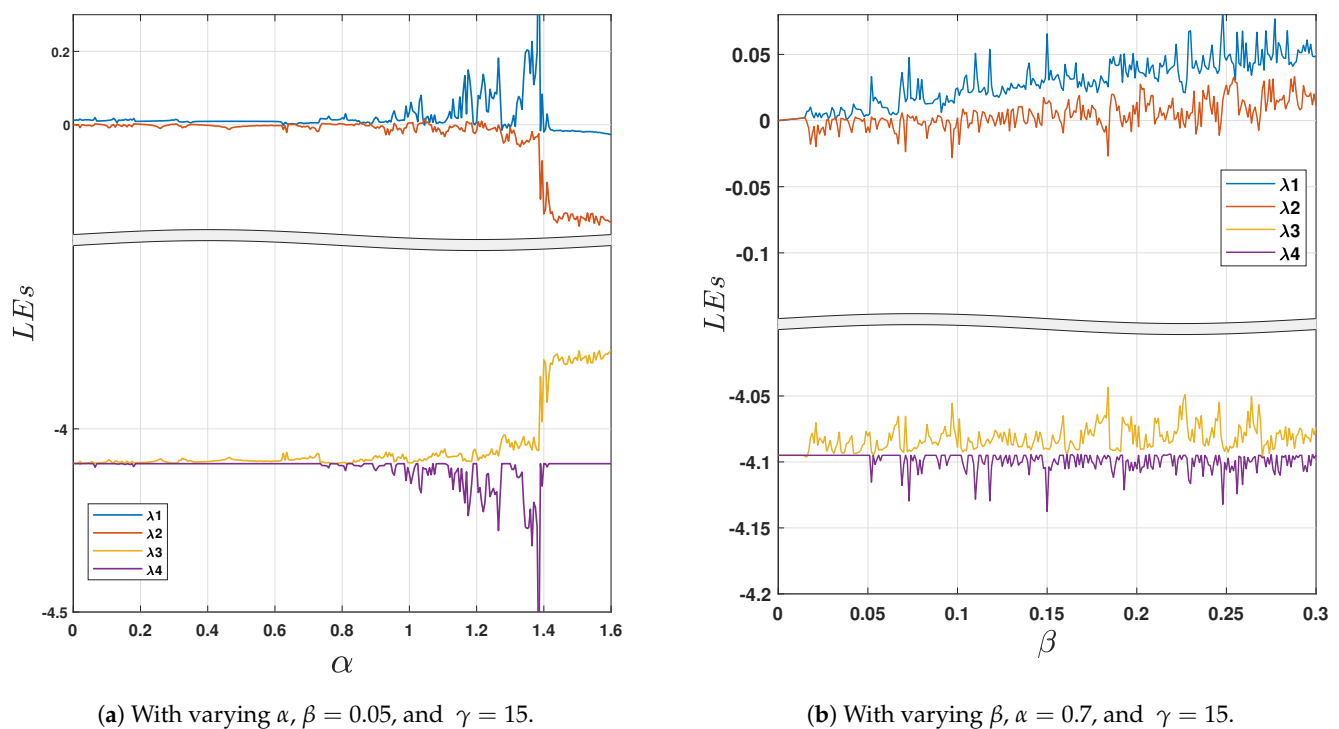


Figure 2. Lyapunov exponents spectrum of jounce chaotic system (1) with nonlinearity (2).

The bicoherence diagram of the system is shown in Figure 3. Bicoherence is a measure of nonlinearity in a time series or signal. It is based on the concept of bivariate coherence, which measures the degree of linear relationship between two signals. Bicoherence extends this concept to higher-order relationships between three or more signals. Bicoherence is often used in the analysis of complex systems, such as those found in physics, biology, and engineering. It can provide insights into the interactions and dynamics of the system and can be used to identify patterns and correlations that may not be apparent using other techniques. One of the main advantages of bicoherence is that it can detect nonlinear relationships that may not be apparent using other techniques, such as linear regression or cross-correlation. This makes it particularly useful for the analysis of complex systems, where nonlinear interactions may be important. In the bicoherence plot, clusters of high bicoherence indicate strong nonlinear phase coupling and can serve as an indicator of potential chaotic behavior in the system [32]. Moreover, Figure 3 shows the power spectral density (PSD), which measures energy distribution across frequencies to evaluate non-periodic dynamics, and underlying spectral trends. The observed continuous, broadband spectral profile reflects elevated signal complexity and non-periodic behavior, representing a characteristic hallmark consistent with the system's underlying chaotic dynamics.

To quantitatively evaluate the structural complexity and pseudo-randomness of the proposed 4D jounce chaotic model, a two-dimensional Spectral Entropy (SE) map is computed across the parameter plane of $\alpha \in [0, 1.6]$ and $\beta \in [0, 0.3]$. As depicted in the 2D SE map given in Figure 4, the system exhibits elevated spectral entropy values ranging over the vast majority of the parametric space. This broad distribution of high spectral entropy reflects sustained signal complexity and broad frequency spread over extended parameter regions, demonstrating that high spectral complexity is not restricted to fragile, and isolated operating points. These features support the presence of rich nonlinear interactions and serve as characteristic signatures consistent with the system's underlying chaotic dynamics. From an engineering perspective, this wide-range entropy distribution

provides significant advantages for secure communication and binary image encryption schemes [33]. The initial conditions of system (1) are selected to be in the basin of attraction of the strange chaotic attractor except the equilibrium points. The considered initial conditions are $(x_0, y_0, z_0, u_0) = (1, -0.5, 1.5, 0.5)$. According to bifurcation diagrams, one can select the system parameters as $\alpha = 0.7$, $\beta = 0.05$, and $\gamma = 15$. Along with the defined nonlinearities given in (2), the simulation results of the hysteretic model show the existence of a strange chaotic attractor, as depicted in Figure 5.

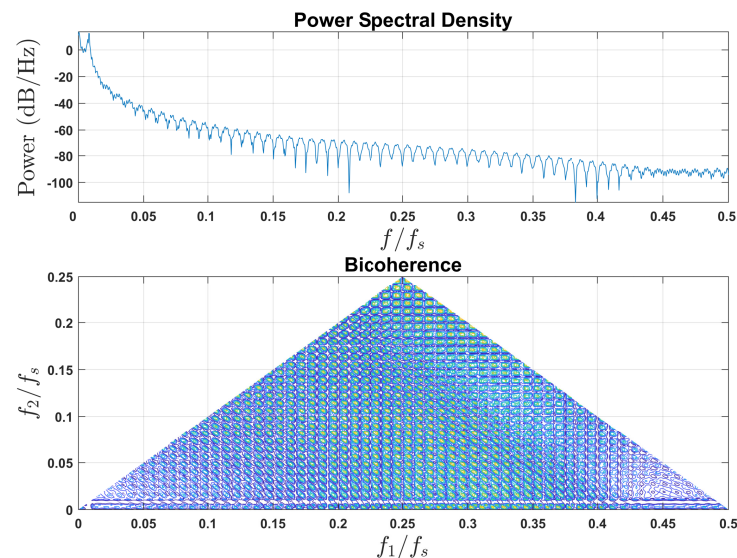


Figure 3. Bicoherence and PSD of the jounce chaotic model (1) for $\alpha = 0.7$, $\beta = 0.05$, and $\gamma = 15$.

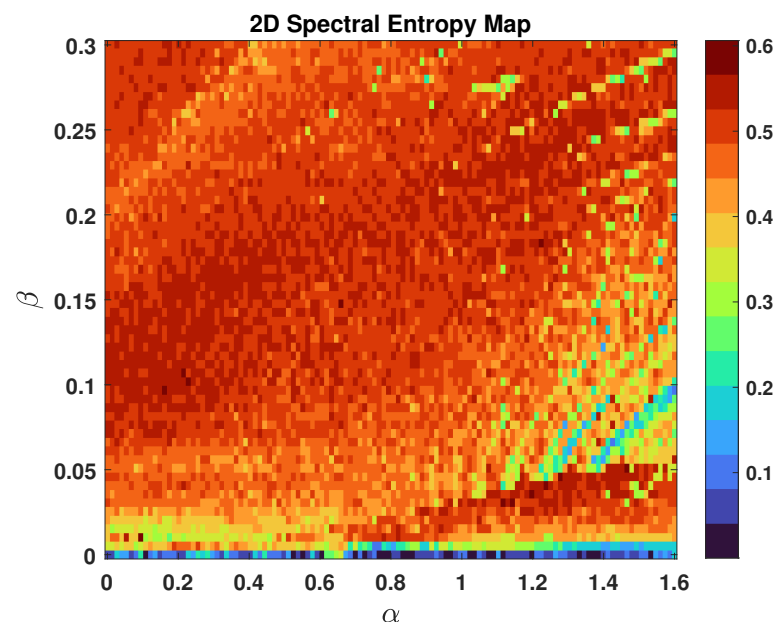


Figure 4. Spectral entropy diagram of the jounce chaotic model (1) for $\gamma = 15$.

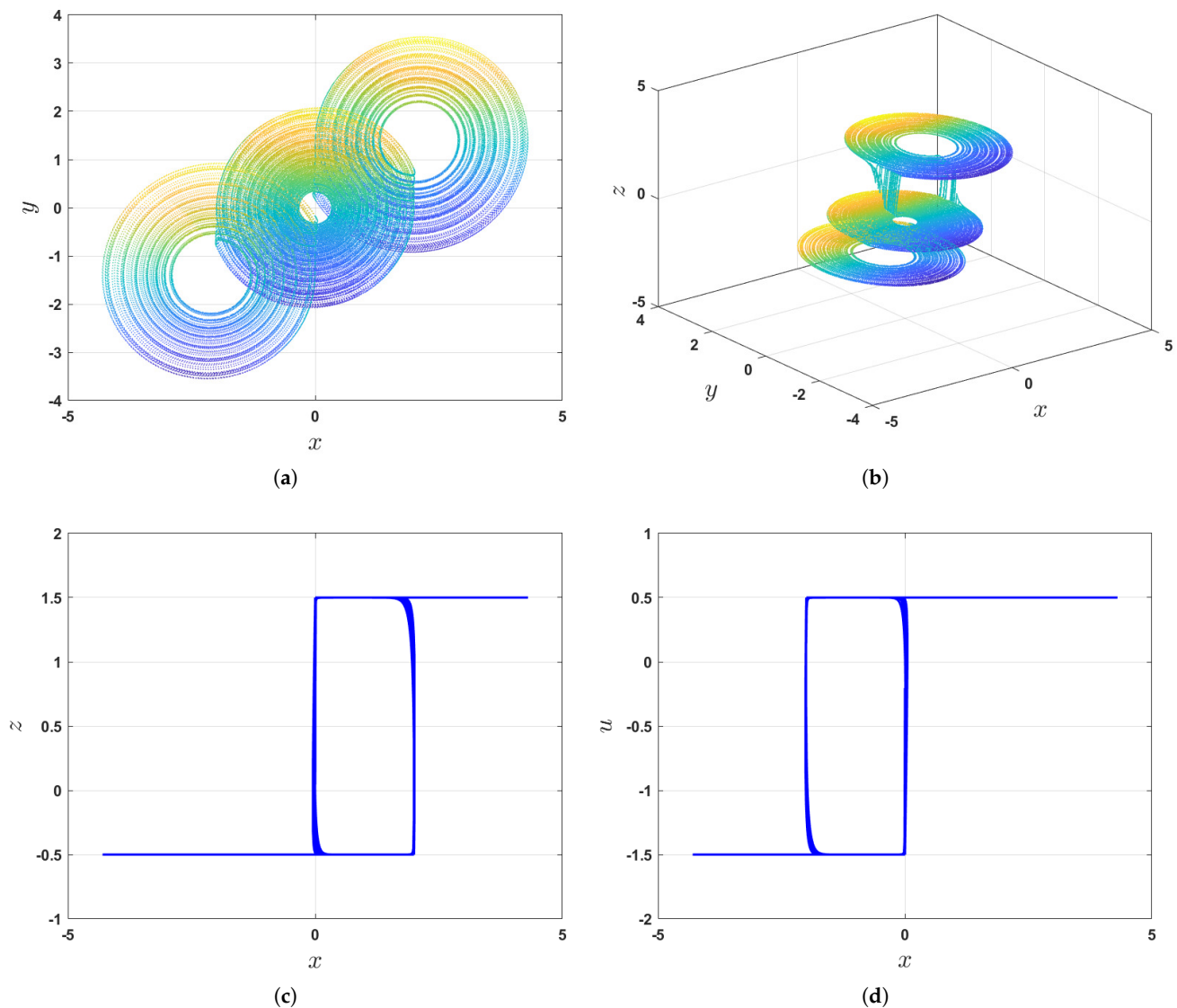


Figure 5. (a) Chaotic attractor of x versus y . (b) 3D view of the chaotic attractor in xyz -space. (c) The nonlinear $F_1(x, z)$ function. (d) The nonlinear $F_2(x, u)$ functions. Where $(x_0, y_0, z_0, u_0) = (1, -0.5, 1.5, 0.5)$, $\alpha = 0.7$, $\beta = 0.05$, and $\gamma = 15$.

3. Synchronization via Adaptive Control

The synchronization process of the new suggested jounce hysteretic model is a challenging task due to the fact that the system exhibits complex dynamics due to rapid changes in its dynamical behavior. Moreover, the uncertainty of the system parameters of the slave system further increases the synchronization difficulty and hinders direct coupling and the applicability of the classical PC synchronization method. In addition, it is demonstrated that the synchronization between systems depends on initial conditions [13]. The adaptive control algorithm is designed to achieve the synchronization of two identical 4D chaotic systems. In this design, the slave system (8) is employed as a copy of the master system (7) with an appropriate controller to be designed such that the slave is able to estimate the parameters of the master system and achieve synchronization error convergence, despite the presence of parameter uncertainties in the slave system.

The master system (1) with nonlinearity (2) is expressed as follows:

$$\begin{cases} \dot{x}_m = y_m - a_1(z_m + u_m) \\ \dot{y}_m = z_m + u_m + a_2 y_m - x_m \\ \dot{z}_m = a_3(-z_m + \operatorname{sgn}[x_m - 1 + \operatorname{sgn}(z_m)] + 0.5) \\ \dot{u}_m = a_4(-u_m + \operatorname{sgn}[x_m + 1 + \operatorname{sgn}(u_m)] - 0.5) \end{cases} \quad (7)$$

Suppose a slave system (1) with nonlinearity (2) in the following form:

$$\begin{cases} \dot{x}_s = y_s - b_1(z_s + u_s) + N_1 \\ \dot{y}_s = z_s + u_s + b_2 y_s - x_s + N_2 \\ \dot{z}_s = b_3(-z_s + \operatorname{sgn}[x_s - 1 + \operatorname{sgn}(z_s)] + 0.5) + N_3 \\ \dot{u}_s = b_4(-u_s + \operatorname{sgn}[x_s + 1 + \operatorname{sgn}(u_s)] - 0.5) + N_4 \end{cases} \quad (8)$$

where subscript m and s refer to the master and slave systems, respectively; a_i ($i = 1, 2, 3, 4$) refers to the master parameters; b_i ($i = 1, 2, 3, 4$) refers to the slave parameters; and N_i ($i = 1, 2, 3, 4$) are the controller system equations. Consequently, according to the proposed methodology, the synchronization error is defined as follows:

$$\begin{cases} e_1 = x_s - x_m \\ e_2 = y_s - y_m \\ e_3 = z_s - z_m \\ e_4 = u_s - u_m \end{cases} \quad (9)$$

The error dynamics can be written as follows:

$$\begin{cases} \dot{e}_1 = e_2 - b_1 e_3 - b_1 e_4 - (b_1 - a_1)z_m - (b_1 - a_1)u_m + N_1 \\ \dot{e}_2 = -e_1 + b_2 e_2 + e_3 + e_4 + (b_2 - a_2)y_m + N_2 \\ \dot{e}_3 = -b_3 e_3 + (b_3 - a_3) \operatorname{sgn}[x_m - 1 + \operatorname{sgn}(z_m)] - (b_3 - a_3)z_m + 0.5(b_3 - a_3) \\ \quad + b_3 \Delta \operatorname{sgn}_3 + N_3 \\ \dot{e}_4 = -b_4 e_4 + (b_4 - a_4) \operatorname{sgn}[x_m + 1 + \operatorname{sgn}(u_m)] - (b_4 - a_4)u_m - 0.5(b_4 - a_4) \\ \quad + b_4 \Delta \operatorname{sgn}_4 + N_4 \end{cases} \quad (10)$$

where $\Delta \operatorname{sgn}_3 = \operatorname{sgn}[x_s - 1 + \operatorname{sgn}(z_s)] - \operatorname{sgn}[x_m - 1 + \operatorname{sgn}(z_m)]$ represents the first hysteresis switching difference and the second hysteresis switching difference is expressed as $\Delta \operatorname{sgn}_4 = \operatorname{sgn}[x_s + 1 + \operatorname{sgn}(u_s)] - \operatorname{sgn}[x_m + 1 + \operatorname{sgn}(u_m)]$. Then, the transient switching mismatch can be defined as $\Delta \operatorname{sgn}_i \in \{-2, 0, 2\}$, where $i = 3, 4$. The uncompensated dynamic term ($d_i(t) = b_i \Delta \operatorname{sgn}_i(t)$) constitutes a uniformly bounded perturbation ($\|d_i(t)\| \leq 2|b_i|$).

Noting that the error should stabilize to the origin or $\lim_{t \rightarrow \infty} \|\mathbf{e}\| = 0$, the error vector is defined as follows: $\mathbf{e} = [e_1, e_2, e_3, e_4]^T$. To this end, the nonlinear feedback control laws can be designed as follows:

$$\begin{cases} N_1(t) = -e_1 - e_2 + b_1 e_3 + b_1 e_4 \\ N_2(t) = e_1 - e_2 - e_3 - e_4 - b_2 e_2 \\ N_3(t) = -e_3 + b_3 e_3 - b_3 \Delta \operatorname{sgn}_3 \\ N_4(t) = -e_4 + b_4 e_4 - b_4 \Delta \operatorname{sgn}_4 \end{cases} \quad (11)$$

The parameter updating rules of the slave system are given by

$$\begin{cases} \dot{b}_1 &= z_m e_1 + u_m e_1 \\ \dot{b}_2 &= -y_m e_2 \\ \dot{b}_3 &= z_m e_3 - 0.5e_3 - e_3 \operatorname{sgn}[x_m - 1 + \operatorname{sgn}(z_m)] \\ \dot{b}_4 &= u_m e_4 + 0.5e_4 - e_4 \operatorname{sgn}[x_m + 1 + \operatorname{sgn}(u_m)] \end{cases} \quad (12)$$

Theorem 1. Consider a master system (7) and slave system (8) under control signals (11) and parameter update rules (12). Then, provided the master-system trajectories are bounded and the resulting error trajectory satisfies the regularity conditions required by Barbalat's lemma, the state synchronization error asymptotically converges to zero: $\lim_{t \rightarrow \infty} \|e(t)\| = 0$.

Proof. Choose the positive-definite candidate Lyapunov function:

$$V(t) = \frac{1}{2} (e_1^2 + e_2^2 + e_3^2 + e_4^2 + \mathbf{e}_d^T \mathbf{e}_d) = \frac{1}{2} (\mathbf{e}^T \mathbf{e} + \mathbf{e}_d^T \mathbf{e}_d) \quad (13)$$

where

$$\mathbf{e}_d = \mathbf{b}(t) - \mathbf{a} \quad (14)$$

and $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ and $\mathbf{b} = [b_1, b_2, b_3, b_4]^T$ denote the master and slave parameter vectors, respectively.

The time derivative of $V(t)$ along the trajectory of the error dynamic system (10) is given by

$$\dot{V}(t) = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 + (b_1 - a_1) \dot{b}_1 + (b_2 - a_2) \dot{b}_2 + (b_3 - a_3) \dot{b}_3 + (b_4 - a_4) \dot{b}_4. \quad (15)$$

Substituting the explicit error dynamics from (10) yields

$$\begin{cases} \dot{V}(t) &= (e_2 - b_1 e_3 - b_1 e_4 - (b_1 - a_1) z_m - (b_1 - a_1) u_m + N_1) e_1 \\ &+ (-e_1 + b_2 e_2 + e_3 + e_4 + (b_2 - a_2) y_m + N_2) e_2 \\ &+ (-b_3 e_3 - (b_3 - a_3) z_m + 0.5(b_3 - a_3) + (b_3 - a_3) \operatorname{sgn}[x_m - 1 + \operatorname{sgn}(z_m)] \\ &+ b_3 \Delta \operatorname{sgn}_3 + N_3) e_3 \\ &+ (-b_4 e_4 - (b_4 - a_4) u_m - 0.5(b_4 - a_4) + (b_4 - a_4) \operatorname{sgn}[x_m + 1 + \operatorname{sgn}(u_m)] \\ &+ b_4 \Delta \operatorname{sgn}_4 + N_4) e_4 \\ &+ (b_1(t) - a_1) \dot{b}_1(t) + (b_2(t) - a_2) \dot{b}_2(t) + (b_3(t) - a_3) \dot{b}_3(t) + (b_4(t) - a_4) \dot{b}_4(t). \end{cases} \quad (16)$$

Substituting the controllers ($N_i(t)$) from (11) and parameter adaptation rules ($\dot{b}_i(t)$) from (12) into the above equation causes all parameter mismatch terms ($b_i - a_i$) to cancel identically. The Lyapunov derivative reduces strictly to

$$\dot{V}(t) = -e_1^2 - e_2^2 - e_3^2 - e_4^2 = -\|\mathbf{e}(t)\|^2 \leq 0 \quad (17)$$

Since $\dot{V}(t) \leq 0$, $V(t)$ is non-increasing and bounded from below by zero ($0 \leq V(t) \leq V(0)$), which establishes that $V(t) \in \mathcal{L}_\infty$. This implies that the synchronization errors ($\mathbf{e}(t)$) and estimation errors ($\mathbf{e}_d(t)$) (and, hence, parameter estimates ($b_i(t)$)) are globally uniformly bounded for all $t \geq 0$.

It is clear that $\dot{V}(t)$ is negative semi-definite on the state space $(\mathbf{e}, \mathbf{e}_d)$, and Barbalat's Lemma is applied to prove asymptotic convergence:

1. Integrability:

Integrating $\dot{V}(t) = -\|\mathbf{e}(t)\|^2$ from $t = 0$ to $t \rightarrow \infty$:

$$\int_0^\infty \|\mathbf{e}(\tau)\|^2 d\tau = V(0) - V(\infty) < \infty \quad (18)$$

Thus, $\|\mathbf{e}(t)\|^2 \in \mathcal{L}_1$.

2. Uniform Continuity:

Since $\mathbf{e}(t)$, $\mathbf{e}_d(t)$, and the master states are all bounded ($\mathcal{L}_\infty$), the error derivatives ($\dot{\mathbf{e}}(t)$) defined in (10) are bounded. Differentiating $\|\mathbf{e}(t)\|^2$ gives

$$\frac{d}{dt} \|\mathbf{e}(t)\|^2 = 2\mathbf{e}^T(t)\dot{\mathbf{e}}(t). \quad (19)$$

Since $\mathbf{e}(t)$ and $\dot{\mathbf{e}}(t)$ are bounded, $\frac{d}{dt} \|\mathbf{e}(t)\|^2 \in \mathcal{L}_\infty$, confirming that $\|\mathbf{e}(t)\|^2$ is uniformly continuous.

According to Barbalat's Lemma, an integrable and uniformly continuous function vanishes asymptotically:

$$\lim_{t \rightarrow \infty} \|\mathbf{e}(t)\|^2 = 0 \implies \lim_{t \rightarrow \infty} \mathbf{e}(t) = 0 \quad (20)$$

Hence, the slave system asymptotically synchronizes with the master system. This completes the proof. $\square$

Proposition 1. Consider the decoupled subsystem governing the parameter estimation error ($e_{d,2}(t) = b_2(t) - a_2$) associated with a constant unknown parameter (a_2):

$$\begin{cases} \dot{e}_2(t) &= -e_2(t) + y_m(t)e_{d,2}(t) \\ \dot{e}_{d,2}(t) &= -y_m(t)e_2(t) \end{cases} \quad (21)$$

If the scalar regressor ($\phi_2(t) = y_m(t)$) satisfies the Persistence of Excitation (PE) condition, i.e.,

$$\int_t^{t+T_2} y_m^2(\tau) d\tau \geq \mu_2 > 0, \quad \forall t \geq 0, \quad (22)$$

for finite constants ($T_2 > 0$ and $\mu_2 > 0$), then under the standard regularity assumptions of adaptive parameter estimation, the parameter estimation error associated with a_2 converges asymptotically to zero:

$$\lim_{t \rightarrow \infty} (b_2(t) - a_2) = 0$$

The PE condition in Proposition 1 can be examined numerically for the chaotic trajectory considered in this study. Since the relevant scalar regressor is $\phi_2(t) = y_m(t)$, the corresponding PE measure is defined as the integral of $y_m^2(\tau)$ over a sliding time window of length T_2 . For a window length of $T_2 = 20$ s, the resulting scalar PE test is shown in Figure 6a. The sliding-window integral remains strictly positive throughout the considered simulation interval, indicating that the regressor ($y_m(t)$) provides a nonzero amount of excitation within every 20 s window.

To further examine the dependence of the excitation on the selected window length, the minimum value of the sliding-window integral is evaluated for different values of T_2 . The results are presented in Figure 6b. The lower bound remains positive over the investigated range of window lengths, indicating that the excitation energy associated with $\phi_2(t) = y_m(t)$ does not vanish within the examined sliding windows. These results there-

fore numerically support the scalar PE assumption used in Proposition 1 for the considered chaotic trajectory.

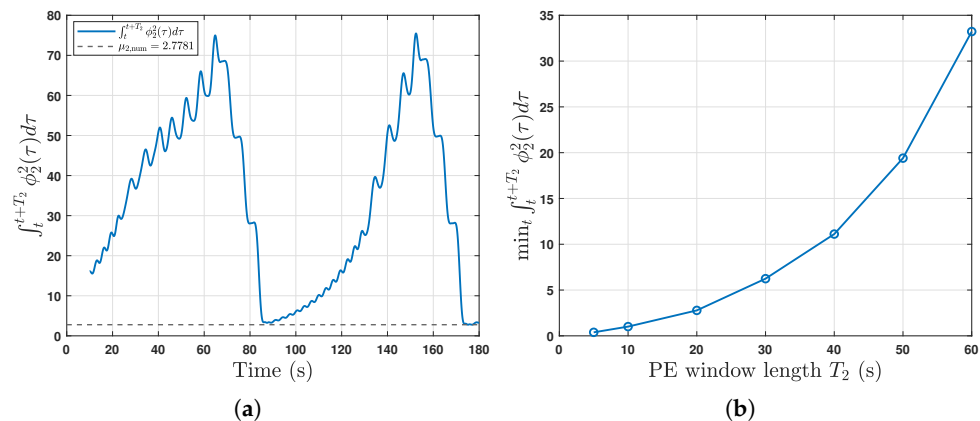


Figure 6. Persistence of excitation tests of the jounce chaotic model (1) for $\alpha = 0.7$, $\beta = 0.05$, and $\gamma = 15$. (a) Scalar PE test for $\phi_2(t) = y_m(t)$ and $T_2 = 20$ s. (b) Scalar PE lower bound versus window length.

4. Simulation Results

The adaptive-control synchronization methodology proposed in Section 3 was simulated using two different initial conditions, the initial conditions for the master system selected as $(x_0, y_0, z_0, u_0) = (1, -0.5, 1.5, 0.5)$ and those for slave system selected as $(x_0, y_0, z_0, u_0) = (1.3, -0.8, 0.8, -0.2)$. Simulation results are shown in Figures 7–9. Figure 7 depicts the estimation of system parameters after the controller system is implemented. It can be observed from Figure 7 that the slave parameters follow and converge to the master parameters. Moreover, the state-variable time responses of both master and slave systems are shown in Figure 8. It is clear that the slave state variables rapidly converge to the corresponding master states with good accuracy. Figure 9 illustrates the asymptotic convergence of the synchronization errors to the origin.

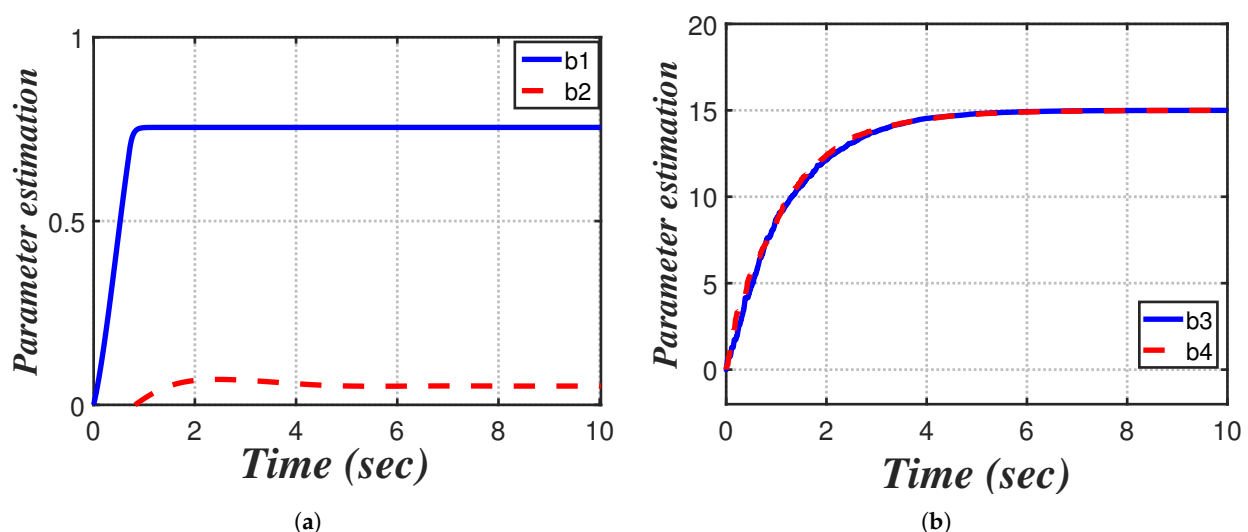


Figure 7. System parameter estimation time series. (a) Identification of system parameters for both $a_1 = 0.7$ and $a_2 = 0.05$. (b) Identification of system parameters for both $a_3 = 15$ and $a_4 = 15$.

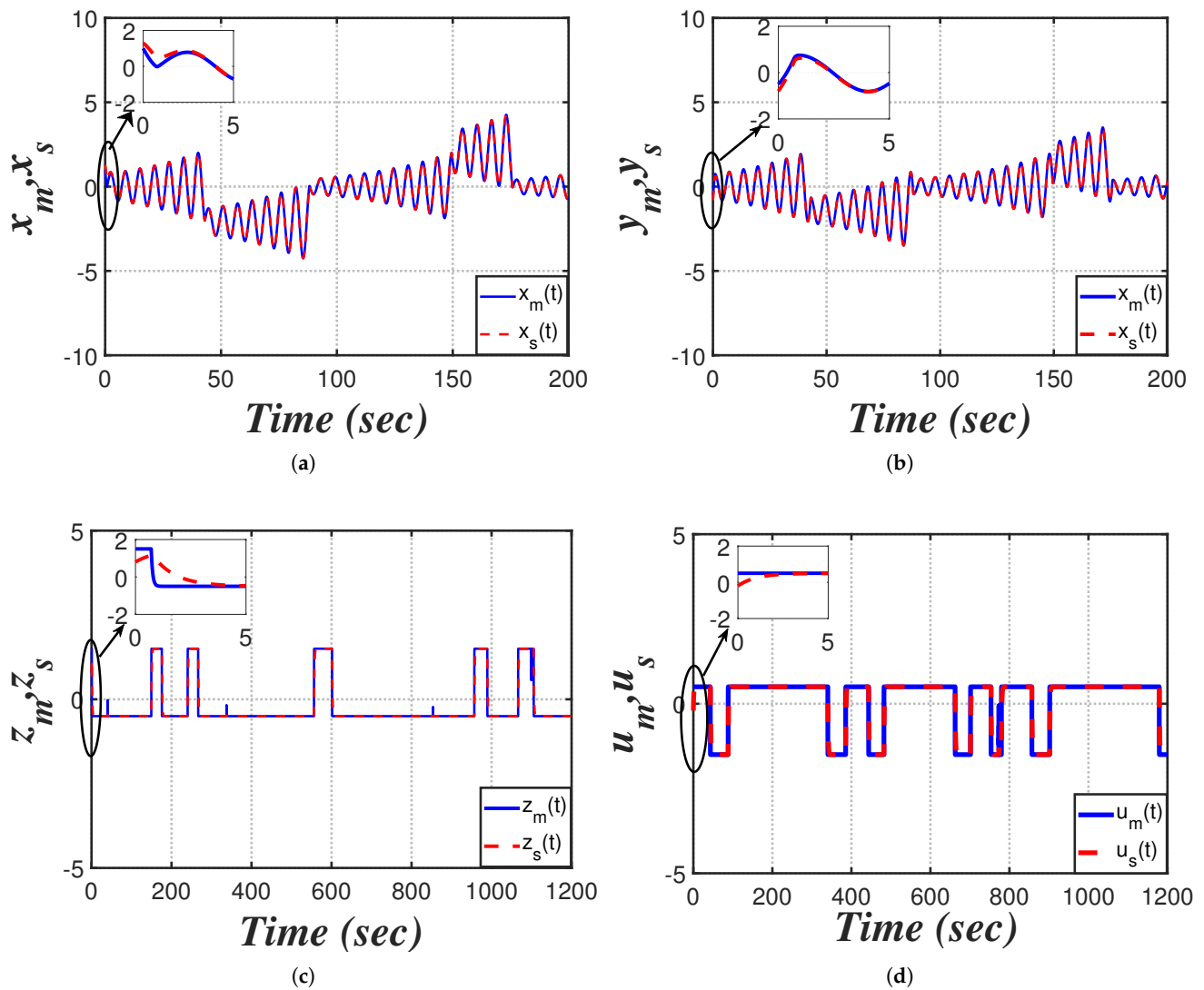


Figure 8. Time series of master and slave signals: (a) $x_m(t)$ versus $x_s(t)$, (b) $y_m(t)$ versus $y_s(t)$, (c) $z_m(t)$ versus $z_s(t)$, and (d) $u_m(t)$ versus $u_s(t)$.

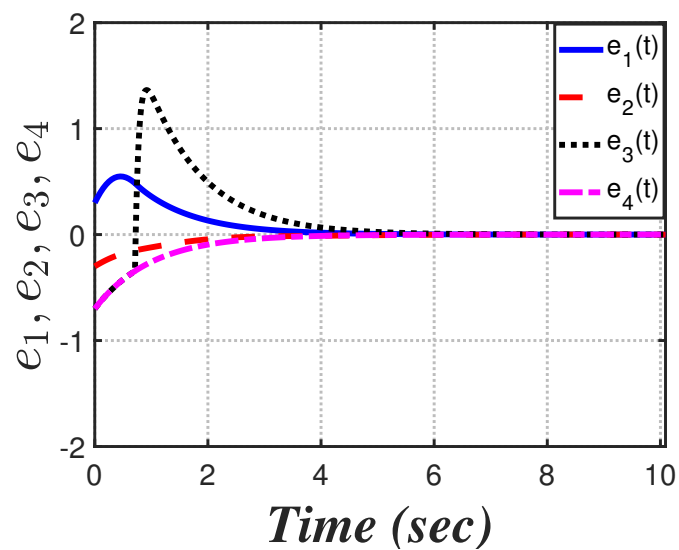


Figure 9. The synchronization errors: $e_1(t)$, $e_2(t)$, $e_3(t)$, and $e_4(t)$.

5. Chaos-Based Communication Applications

Chaotic signals have many features, such as a broadband spectrum, high bicoherence, unpredictability of the chaotic signal, and the sensitivity of the chaotic system to any initial condition variation. These inherent characteristics are beneficial for secure communication applications [12,34]. The aim of this section is to build a robust, noise-tolerant and chaos-based communication system for sending information signals through public channels. The strategy of a secure communication system using adaptive control synchronization is proposed as shown in Figure 10. The information message is hidden in one of the transmitter (master) parameters, then transmitted through the public channel. After receiving the transmitted signal at the receiver (slave) side, the original message is decoded using the controller. The proposed methodology is based on chaotic switching modulation (CSM) or chaotic parameter modulation (CPM). The chaotic transmitter (master) system is expressed as follows:

$$\begin{cases} \dot{x}_m = y_m - a_1(z_m + u_m) \\ \dot{y}_m = z_m + u_m + \beta(t) y_m - x_m \\ \dot{z}_m = a_3(-z_m + \text{sgn}[x_m - 1 + \text{sgn}(z_m)] + 0.5) \\ \dot{u}_m = a_4(-u_m + \text{sgn}[x_m + 1 + \text{sgn}(u_m)] - 0.5) \end{cases} \quad (23)$$

In this work, the $\beta(t)$ parameter is used to modulate the information signal. The information message is inserted into the master's second system parameters (a_2). The considered parameter value is switched between 0.02 and 0.05 when the transmitted bit is zero and one, respectively. These values are selected based on the bifurcation diagram of the control parameter β shown in Figure 1b such that the system remains chaotic in these two cases [35].

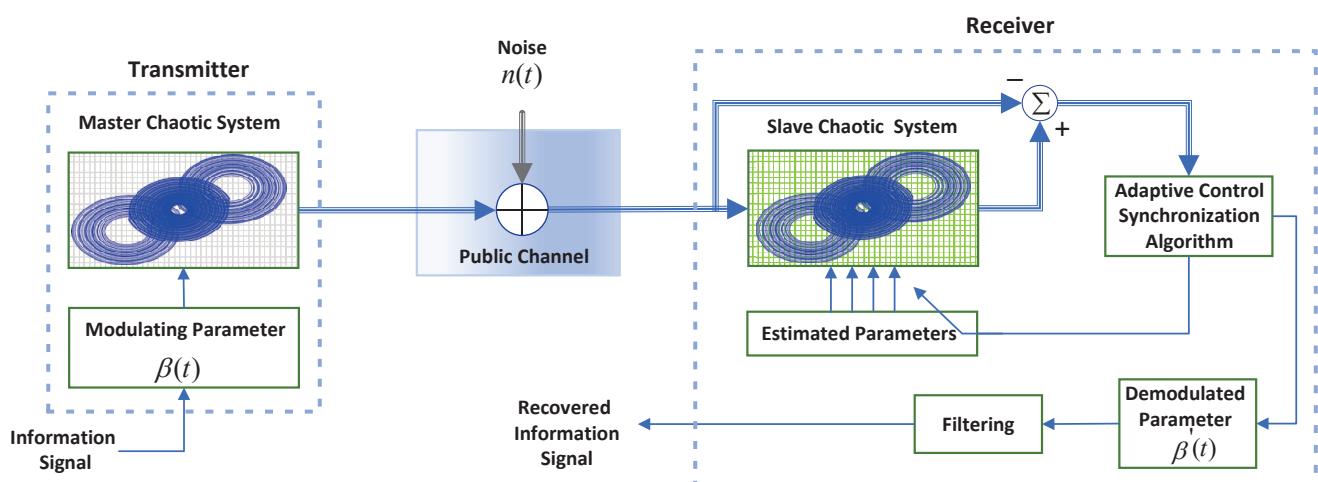


Figure 10. Diagram of the chaotic switching parameter modulation block.

The performance of the designed communication platform is evaluated by sending a binary information message. The sequence is generated using a pseudorandom (PN) code sequence as $2^7 - 1$. Then, adaptive synchronization is implemented between two systems (master–slave), and the b_2 parameter is estimated on the receiver side; therefore, the originally sent information message can be accurately demodulated. It is worth noting that the sample time of the digital source should be selected according to the response time of the receiver (slave) system. Numerical simulations are conducted with noiseless and noisy public channels, as shown in Figures 11–15. It is clear that there is a significant confirmation of the effectiveness of the proposed method at different signal-to-noise ratios (SNRs). For the different simulation cases, the system parameters were selected as follows: $a_1 = 0.7$ and $a_2 = 0.02$ when the transmitted bit is zero and $a_2 = 0.05$ when the bit is one,

and $a_3 = a_4 = 15$. The initial conditions are set to $(x_{m0}, y_{m0}, z_{m0}, u_{m0}) = (1, -0.5, 1.5, 0.5)$ and $(x_{s0}, y_{s0}, z_{s0}, u_{s0}) = (1.3, -0.8, 0.8, -0.2)$. The results for the ideal case are presented in Figure 11. Rapid synchronization between the drive and response (master–slave) systems is achieved, as illustrated in Figure 11a,b. Figure 11c shows the hidden transmitted message embedded in the master-system parameter ($\beta(t)$). The recovered message ($\beta'(t)$) is shown in Figure 11d. On the other hand, in the real channel, a specific value of AWGN (a model at the wired communication) with zero mean channel noise is added to the time series of the transmitted signals (x_m, y_m, z_m, u_m) . The transmitted signal with a noise channel is represented as follows:

$$\begin{cases} x_r(t) = x_m(t) + n_1(t), \\ y_r(t) = y_m(t) + n_2(t), \\ z_r(t) = z_m(t) + n_3(t), \\ u_r(t) = u_m(t) + n_4(t). \end{cases} \quad (24)$$

where n_i ($i = 1, 2, 3, 4$) refers to white Gaussian noise. The Signal-to-Noise Ratio (SNR), in decibels, is calculated based on the average power of the transmitted chaotic carrier relative to the variance of the channel noise:

$$\text{SNR}_{\text{dB}} = 10 \log_{10} \left(\frac{P_{\text{signal}}}{P_{\text{noise}}} \right) = 10 \log_{10} \left(\frac{\sigma_{\text{signal}}^2}{\sigma_{\text{noise}}^2} \right) \quad (25)$$

where $P_{\text{signal}} = \sigma_{\text{signal}}^2 = \frac{1}{T} \int_0^T x_m^2(t) dt$ represents the signal power of the transmitted state variable over simulation period T and $P_{\text{noise}} = \sigma_{\text{noise}}^2$ is the channel noise variance.

The obtained results are depicted in Figures 12–15. Figures 12–15 demonstrate the performance of the proposed robust communication scheme under different AWGN channel conditions ($\text{SNR} = 40, 35, 30$, and 25 dB). At 40 – 30 dB, the master and slave systems remain well synchronized, and the recovered information precisely matches the transmitted signal, indicating reliable communication. At 25 dB, increased noise causes noticeable distortion in the synchronization trajectory and significant degradation of the recovered signal. Overall, the results indicate that the proposed scheme maintains reliable synchronization and information recovery at relatively high SNR levels, while its performance deteriorates as channel noise increases.

Image Transmission over a Noisy Channel

As a further application of the proposed chaotic communication system, robust transmission of a binary image from the master system to the receiver is considered. The image-transmission procedure consists of converting the input image into a binary format suitable for chaotic encoding and transmission. First, the input image (I_{raw}) is resized to a standardized dimension of $M_1 \times M_2$ pixels. A global intensity threshold is then applied to convert the resized image into a binary representation, producing the binary spatial matrix (I_{orig}), which is subsequently encoded and transmitted through the chaotic communication channel:

$$I_{\text{orig}} \in \{0, 1\}^{M_1 \times M_2} \quad (26)$$

which produces a total sequence length of $M = M_1 \cdot M_2$ bits. The 2D binary matrix is subsequently vectorized line-by-line into a 1D binary bit-stream:

$$B = [b_1, b_2, \dots, b_M], \quad b_k \in \{0, 1\} \quad (27)$$

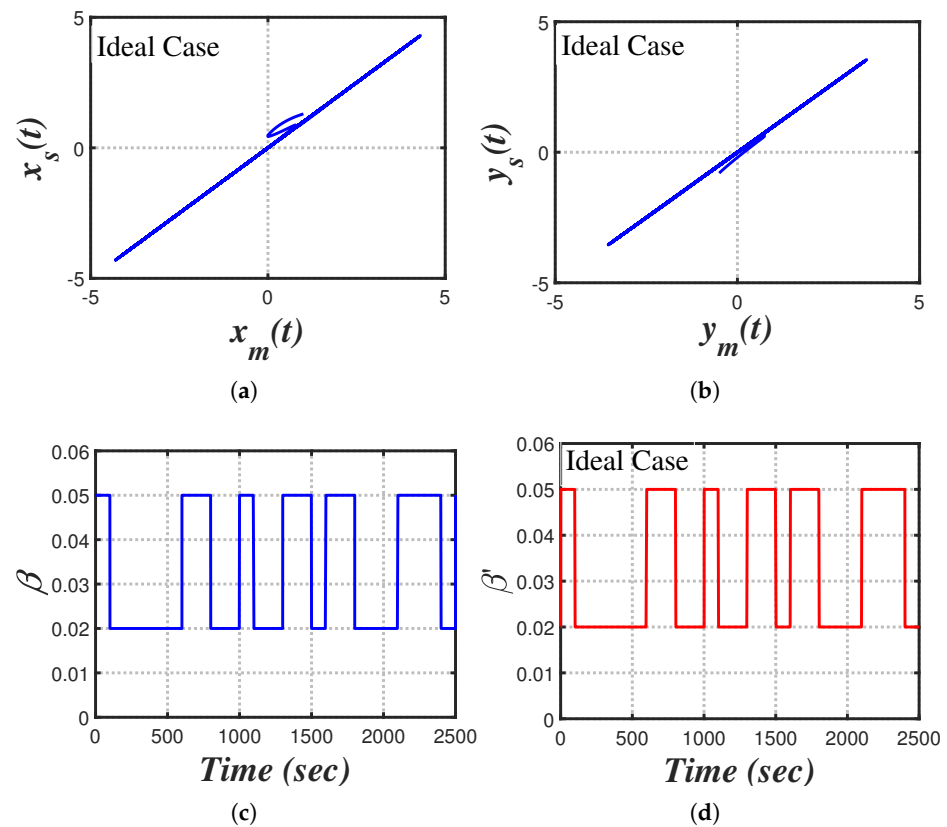


Figure 11. Communication profile via noiseless channel: (a) the synchronization between $x_m(t)$ and $x_s(t)$; (b) the synchronization between $y_m(t)$ and $y_s(t)$; (c) the transmitted information; (d) the recovered information.

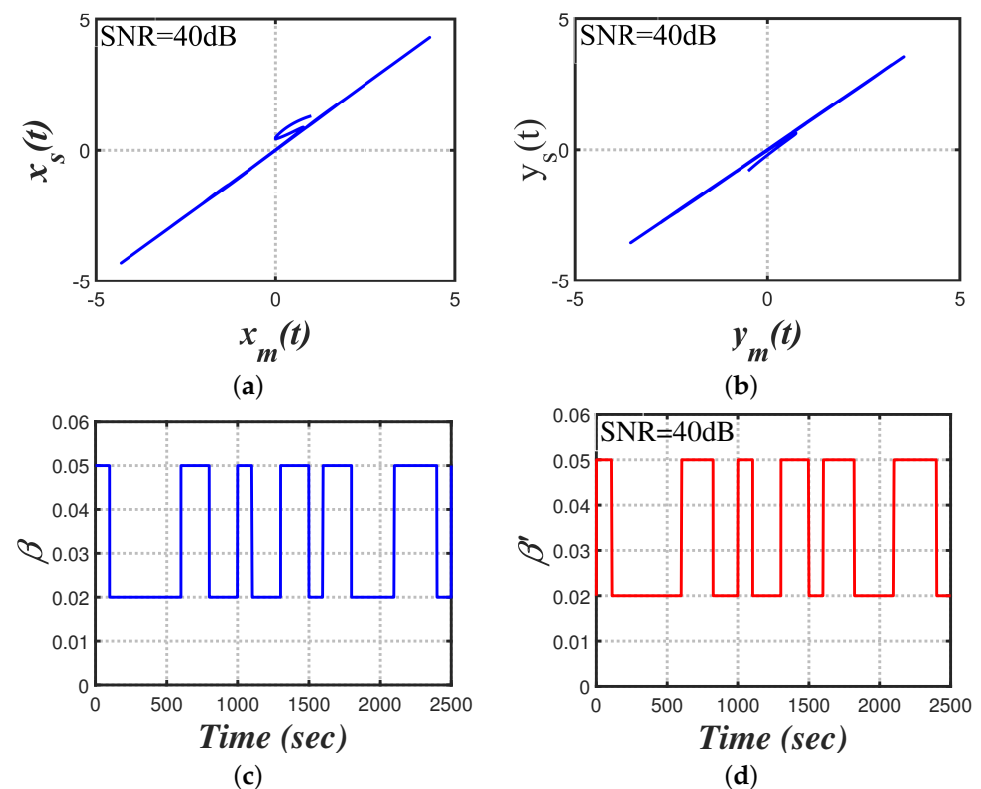


Figure 12. Communication profile via noisy channel at $SNR = 40$ dB: (a) the synchronization between $x_m(t)$ and $x_s(t)$; (b) the synchronization between $y_m(t)$ and $y_s(t)$; (c) the transmitted information; (d) the recovered information.

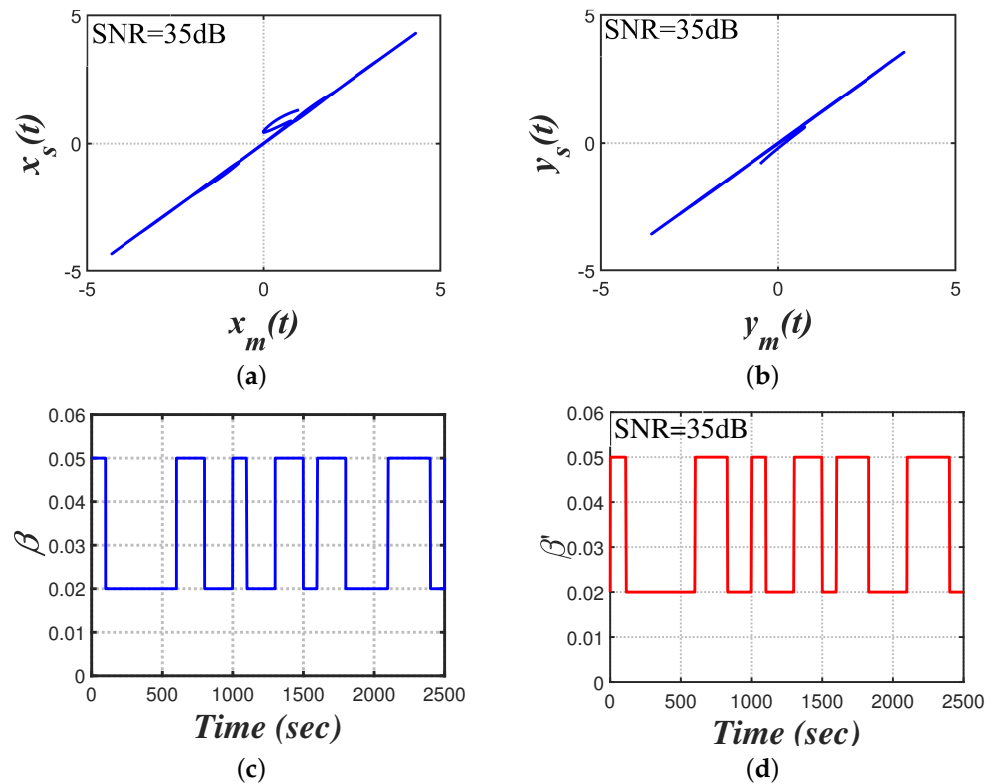


Figure 13. Communication profile via noisy channel at SNR = 35 dB: (a) the synchronization between $x_m(t)$ and $x_s(t)$; (b) the synchronization between $y_m(t)$ and $y_s(t)$; (c) the transmitted information; (d) the recovered information.

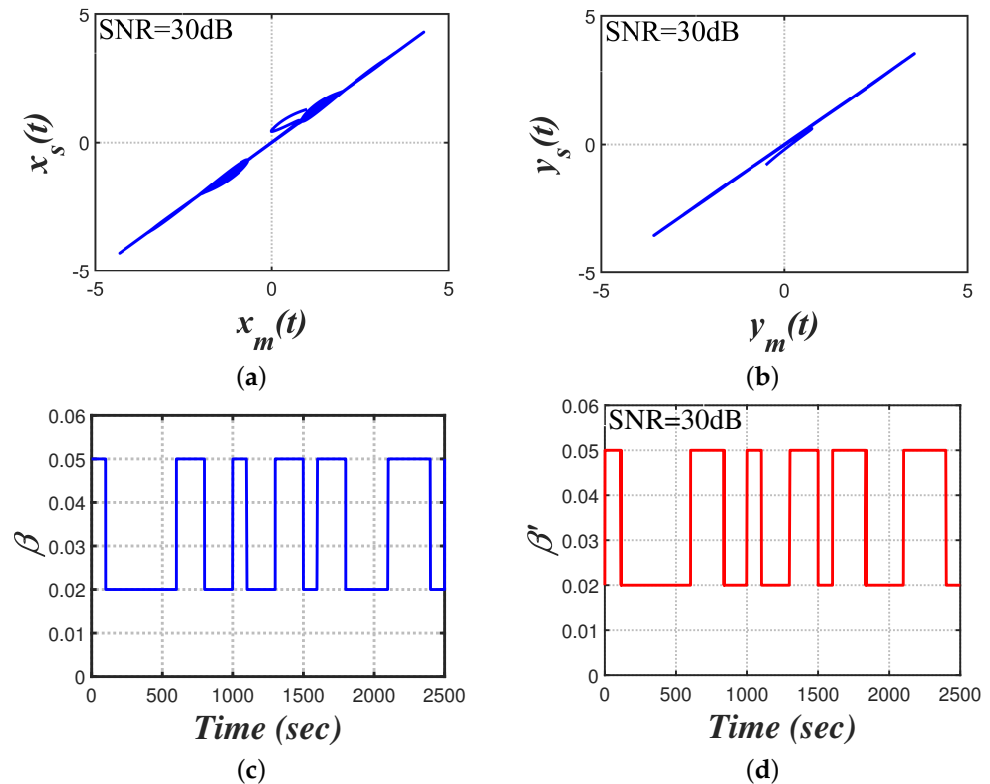


Figure 14. Communication profile via noisy channel at SNR = 30 dB: (a) the synchronization between $x_m(t)$ and $x_s(t)$; (b) the synchronization between $y_m(t)$ and $y_s(t)$; (c) the transmitted information; (d) the recovered information.

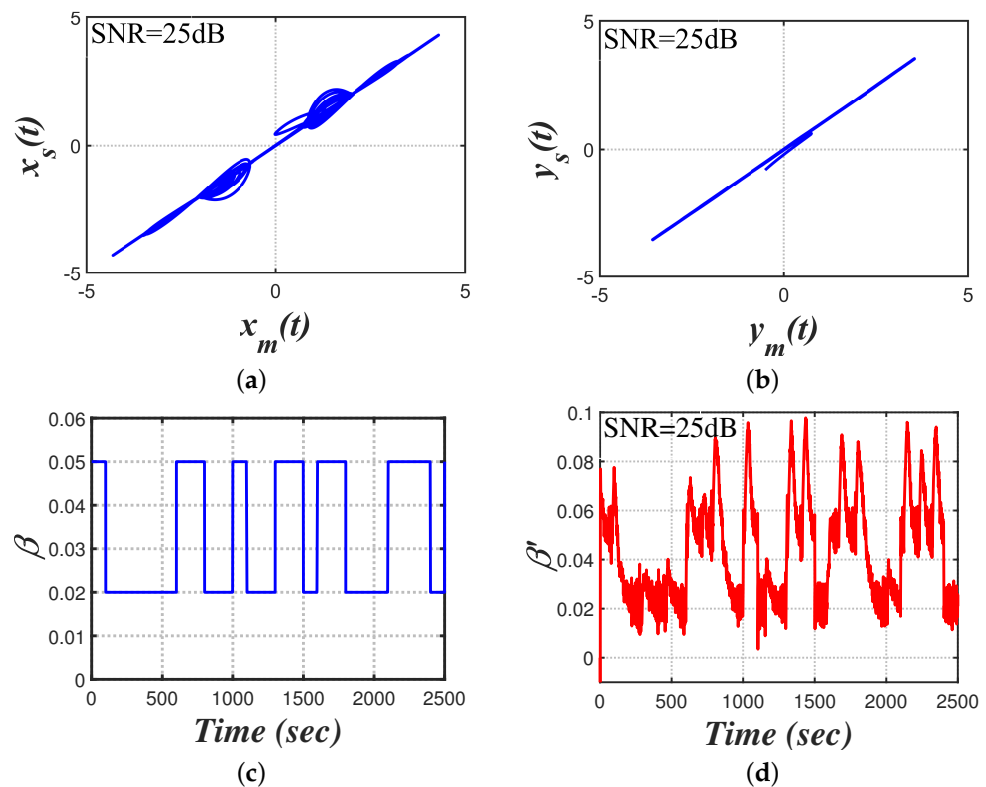


Figure 15. Communication profile via noisy channel at $SNR = 25$ dB: (a) the synchronization between $x_m(t)$ and $x_s(t)$; (b) the synchronization between $y_m(t)$ and $y_s(t)$; (c) the transmitted information; (d) the recovered information.

In the proposed Chaotic Parameter Modulation (CPM) scheme, each bit (b_k) directly modulates the control parameter ($\beta(t)$) of the 4D jounce master system over a fixed bit interval (T_{bit}). Specifically, the parameter mapping is defined as

$$\beta(t) = \begin{cases} \beta_0 = 0.02, & \text{if } b_k = 0, \\ \beta_1 = 0.05, & \text{if } b_k = 1, \end{cases} \quad \text{for } (k-1)T_{\text{bit}} \leq t < kT_{\text{bit}}, \quad (28)$$

thereby translating the discrete digital information stream into a continuous piecewise parameter signal ($\beta(t)$). The encoding process relies on the nonlinear dynamics of a continuous-time 4D jounce master system. As the control parameter ($\beta(t)$) varies dynamically according to the bit sequence (B), the system generates complex, non-periodic chaotic trajectories across its state variables (x_m, y_m, z_m, u_m). The transmitted information is naturally embedded within the primary chaotic carrier state ($x_m(t)$). To visualize the encoding image prior to receiver demodulation, the continuous state ($x_m(t)$) is sampled and averaged over each bit duration (T_{bit}), normalized to the dynamic range $[0, 1]$, and reshaped into an $M_1 \times M_2$ matrix. As depicted in Figure 16b, the resulting encoded image exhibits a fully pseudo-random, noise-like visual texture. The high structural masking ensures that the encoded signal shares virtually zero visual or statistical correlation with the original binary plain image shown in Figure 16a, effectively concealing the underlying visual information from adversarial interception.

Recovery of the transmitted information is achieved at the slave receiver through adaptive chaotic synchronization and dynamic parameter estimation. The receiver employs continuous feedback controllers (N_1, N_2, N_3, N_4), together with four adaptive update laws ($\dot{b}_1, \dot{b}_2, \dot{b}_3, \dot{b}_4$). In particular, the second parameter estimator ($b_2(t)$) is designed to track the unknown transmitter control parameter ($\beta(t)$), enabling the encoded information bits to be recovered through an appropriate thresholding decision.

Figure 16e illustrates the real-time tracking response, where the green curve represents the continuous adaptive parameter estimation ($b_2(t)$) following the black step waveform of the transmitted parameter ($\beta(t)$).

To recover the binary sequence, $b_2(t)$ is sampled exclusively during the second half of each bit interval to allow the initial synchronization transient to settle, then averaged over that interval, denoted as $\bar{b}_2$. The recovered bit ($\hat{b}_k$) is determined by evaluating the averaged estimate against a midpoint decision threshold (β_{th}):

$$\beta_{th} = \frac{\beta_0 + \beta_1}{2} = 0.035. \quad (29)$$

The decision logic rule is specified by:

$$\hat{b}_k = \begin{cases} 1, & \text{if } \bar{b}_2 > \beta_{th}, \\ 0, & \text{if } \bar{b}_2 \leq \beta_{th}. \end{cases} \quad (30)$$

The estimated sequence ($\hat{B} = [\hat{b}_1, \hat{b}_2, \dots, \hat{b}_M]$) is reordered back into a $M_1 \times M_2$ binary grid for reconstruction of the recovered spatial image ($I_{recovered}$).

Remark 1. (Dwell-Time Requirement for Piecewise Parameter Modulation)

The modulated parameter ($\beta(t)$) is not a continuously time-varying non-stationary function; rather, it is a piecewise-constant square-wave signal defined on discrete time intervals:

$$\beta(t) = \beta_k \in \{\beta_0, \beta_1\} = \{0.02, 0.05\}, \quad \forall t \in [(k-1)T_{bit}, kT_{bit}) \quad (31)$$

Within every open bit interval ($I_k = ((k-1)T_{bit}, kT_{bit})$), the derivative of the master parameter is strictly zero:

$$\dot{\beta}(t) = 0, \quad \forall t \in I_k \quad (32)$$

Consequently, the $-\dot{\beta}(t)$ term vanishes almost everywhere in time. Within each bit interval (I_k), the autonomous error dynamics analyzed in Theorem 1 hold identically, ensuring that $\dot{V}(t) \leq 0$ and the tracking error ($e_{d2}(t) = b_2(t) - \beta(t)$) converges asymptotically toward zero. At the exact switching boundary ($t_k = kT_{bit}$), $\beta(t)$ undergoes a step discontinuity of magnitude $\Delta\beta = |\beta_1 - \beta_0| = 0.03$. This step change creates an instantaneous reset in the parameter error state:

$$e_{d2}(t_k^+) = e_{d2}(t_k^-) \mp \Delta\beta \quad (33)$$

This jump re-initializes the Lyapunov function ($V(t_k^+)$) to a bounded non-zero initial value. To guarantee that the system re-synchronizes before the next bit switch, the bit period (T_{bit}) (dwell time) is chosen to satisfy the strict dwell-time condition:

$$T_{bit} > \tau_{sync} \quad (34)$$

where τ_{sync} is the convergence settling time of the error system, providing ample time for the Lyapunov function to decay exponentially back to zero long before the arrival of the next switching instant. Moreover, to ensure that the switching transient resulting from $\Delta\beta$ does not impair message demodulation, the decision logic in (30) evaluates the temporal average ($\bar{b}_2$) exclusively over the second half of each bit interval:

$$t \in \left[\left(k - \frac{1}{2} \right) T_{bit}, kT_{bit} \right) \quad (35)$$

This evaluation window completely masks out the initial transient settling period ($(t \in [t_k, t_k + \tau_{sync}])$), ensuring that thresholding decisions are made solely on steady-state parameter estimates.

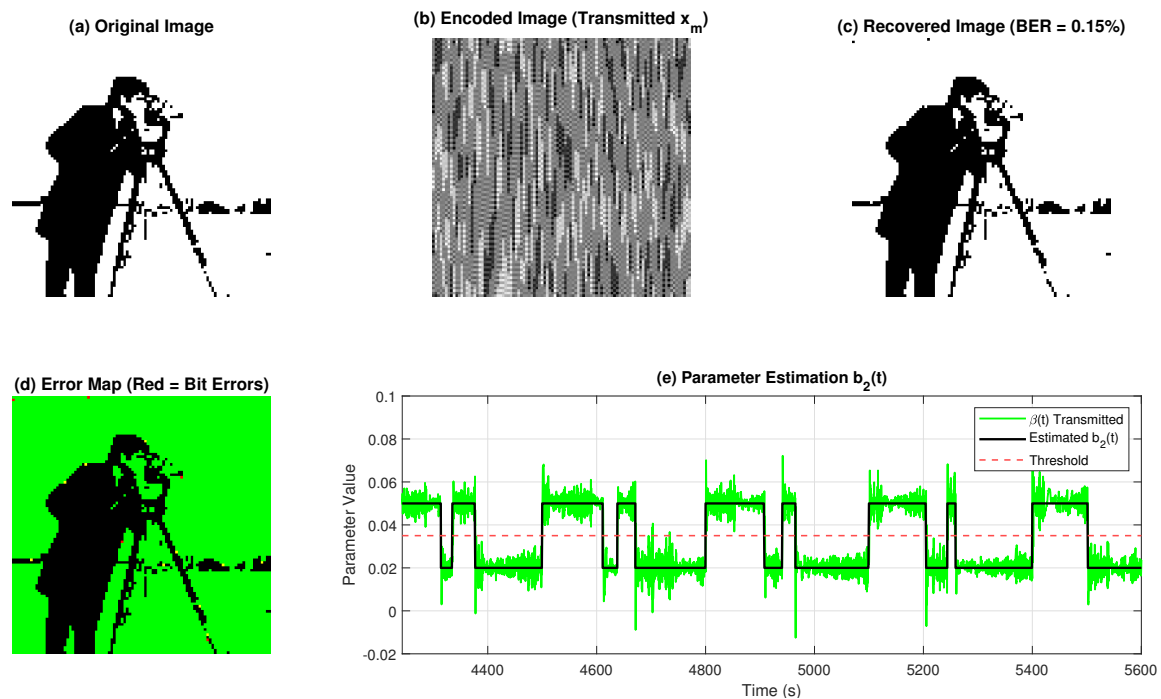


Figure 16. Visual compilation of the image transmission process, parameter tracking, and image recovery under noise with $SNR = 40$ dB. (a) The plain image. (b) Encoded image. (c) Recovered image. (d) Error map. (e) Transmitted signal and the estimated information signal.

To evaluate the robustness of the proposed communication system under practical channel degradation, AWGN noise components (n_1, n_2, n_3, n_4) were added to the four transmitted state signals. The system performance was evaluated at SNR levels of 40 dB, 35 dB, 30 dB, 25 dB, and 20 dB, as illustrated in Figures 16–20, respectively.

The numerical results summarized in Table 1 demonstrate that the proposed adaptive synchronization controller with switching parameter modulation maintains high information-recovery fidelity across SNR levels of 40, 35, and 30 dB and moderate and high distortion at 25 dB and 20 dB SNR levels. These results confirm the robustness of the proposed 4D jounce chaotic parameter modulation framework under noisy-channel conditions. The modulated transmission signal effectively obscures the structural characteristics and spatial correlations of the original image, thereby reducing the risk of direct visual interpretation during transmission.

Table 1. Switching parameter modulation performance metrics under AWGN channel degradation with a binary image as the information signal.

Channel SNR (dB)	Bit Error Rate (BER)	Reconstruction Fidelity
40 dB	0.15%	Near-perfect reconstruction
35 dB	0.24%	High visual clarity
30 dB	1.39%	Highly recognizable image, sparse visual errors
25 dB	6.79%	Moderate distortion due to transient noise
20 dB	15.61%	High distortion due to large channel noise

The encoded image successfully reduces the structural redundancy and spatial correlation present in the input image. Concurrently, the red error masks demonstrate that the bit errors introduced under noisy channel conditions (30 dB $\rightarrow$ 40 dB) are predominantly manifested as localized pixel-level anomalies rather than causing catastrophic desynchronization during the decoding and recovery process. The close agreement between the estimated parameter ($b_2(t)$) and the decision threshold (β_{th}) proves that the adaptive controller pro-

vides adequate noise suppression during the steady-state evaluation window, ensuring resilient binary image transmission over noisy continuous-time communication channels.

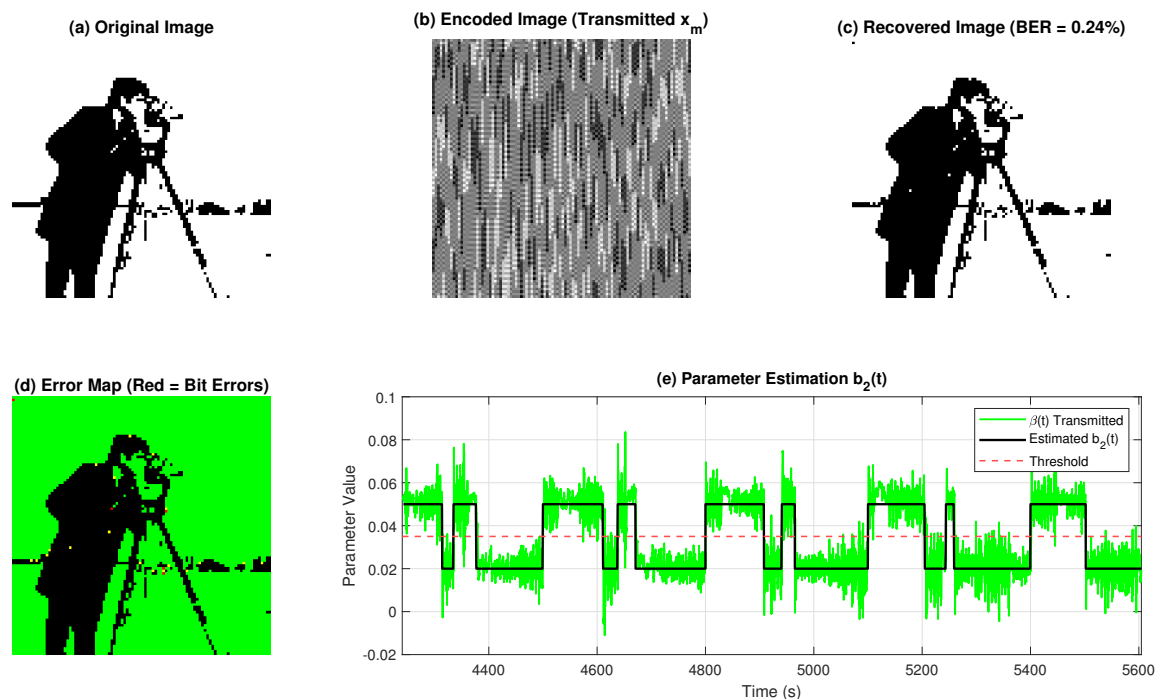


Figure 17. Visual compilation of the image transmission process, parameter tracking, and image recovery under noise with $SNR = 35$ dB. (a) The plain image. (b) Encoded image. (c) Recovered image. (d) Error map. (e) Transmitted signal and the estimated information signal.

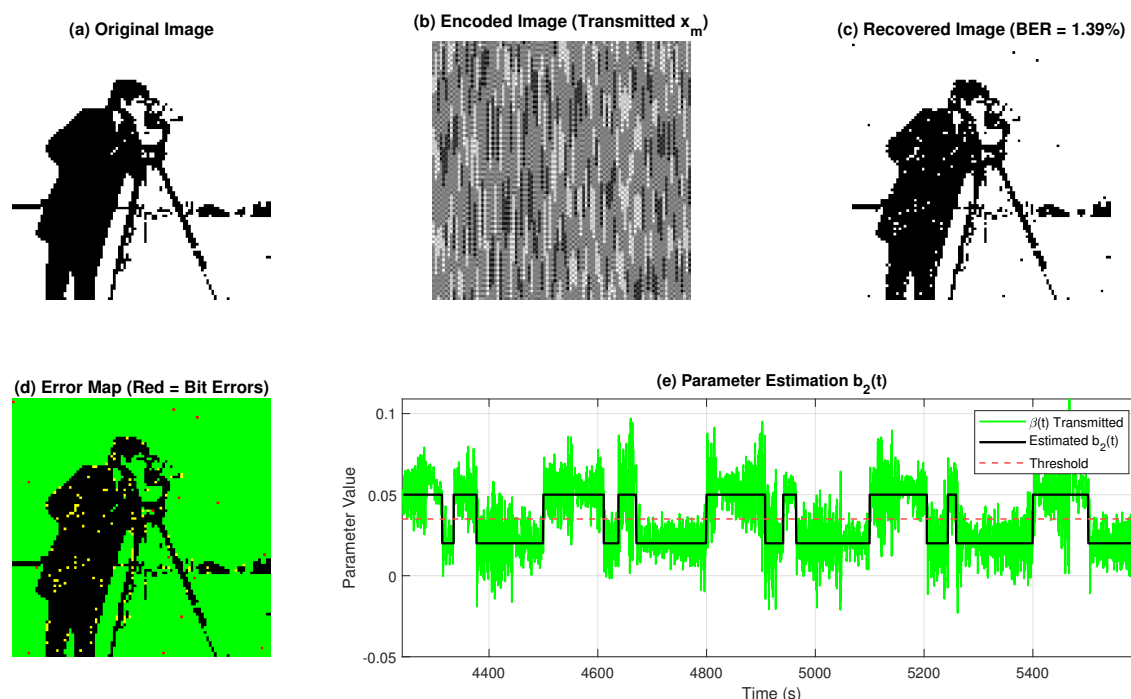


Figure 18. Visual compilation of the image transmission process, parameter tracking, and image recovery under noise with $SNR = 30$ dB. (a) The plain image. (b) Encoded image. (c) Recovered image. (d) Error map. (e) Transmitted signal and the estimated information signal.

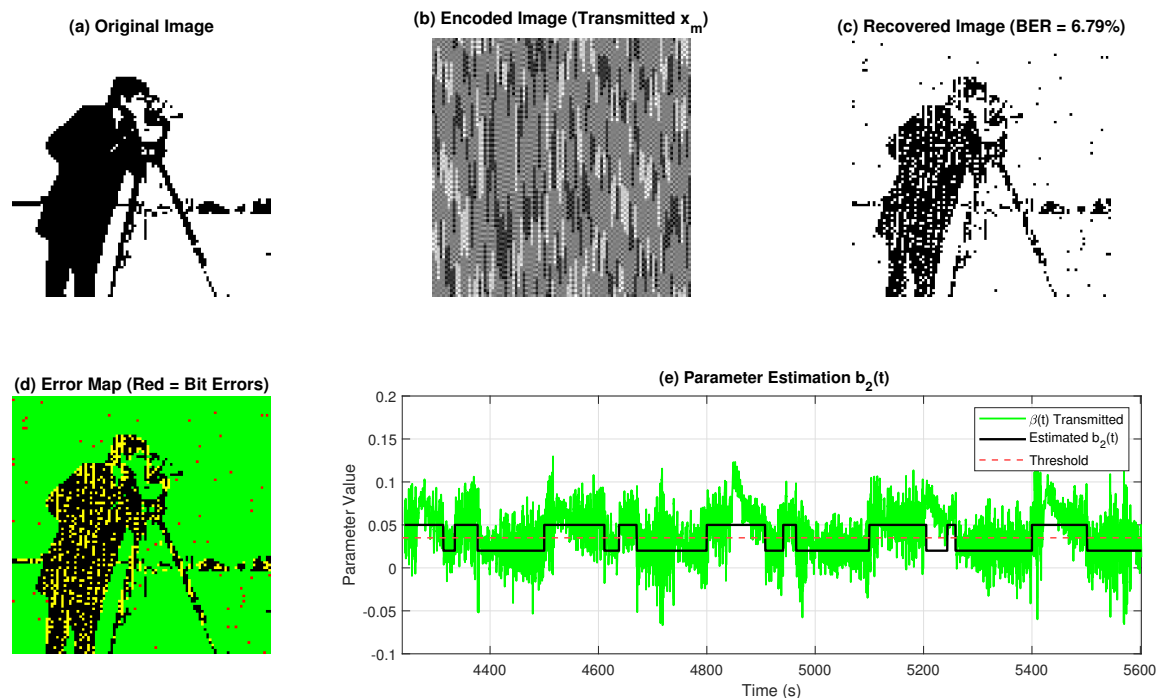


Figure 19. Visual compilation of the image transmission process, parameter tracking, and image recovery under noise with $SNR = 25$ dB. (a) The plain image. (b) Encoded image. (c) Recovered image. (d) Error map. (e) Transmitted signal and the estimated information signal.

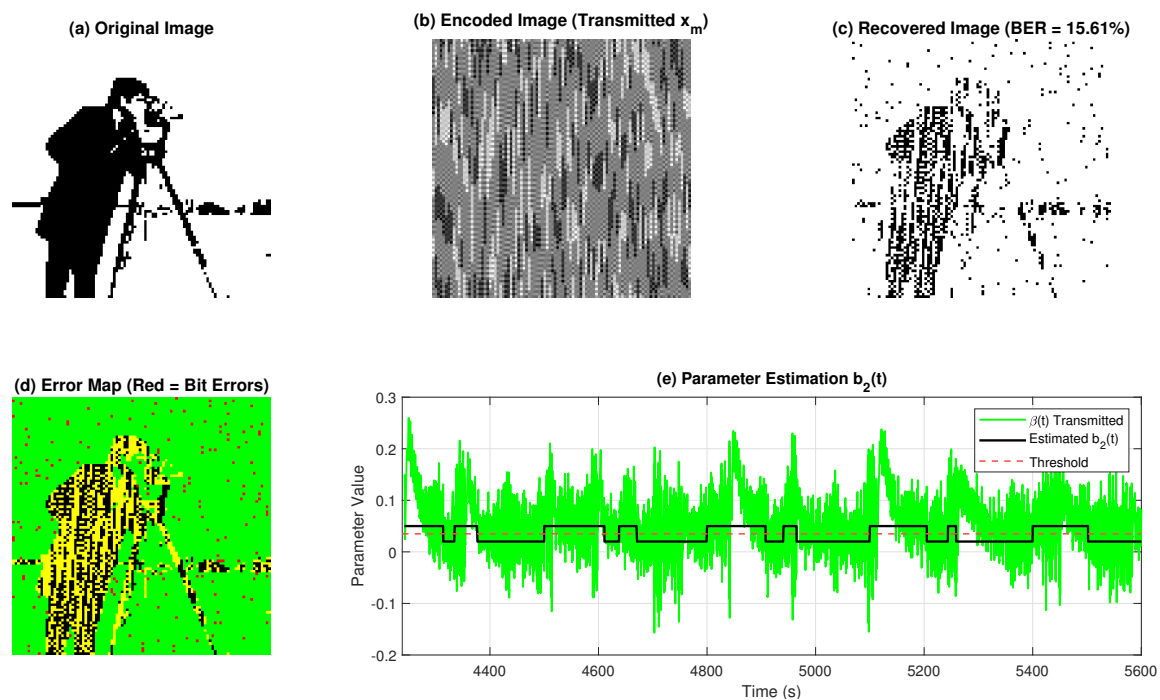


Figure 20. Visual compilation of the image transmission process, parameter tracking, and image recovery under noise with $SNR = 20$ dB. (a) The plain image. (b) Encoded image. (c) Recovered image. (d) Error map. (e) Transmitted signal and the estimated information signal.

6. Conclusions

This paper introduced a new four-dimensional jounce chaotic system exhibiting rich nonlinear dynamics and complex multi-scroll chaotic attractors. The dynamical behavior of the proposed system was thoroughly investigated using nonlinear analysis tools, such as bifurcation diagrams, Lyapunov exponent spectra, power spectral density (PSD),

and bicoherence, which collectively confirm the presence of complex chaotic oscillations over a wide range of system parameters. These analyses demonstrate that the proposed system possesses the complexity and unpredictability required for applications in information processing. An effective scheme of adaptive control-based synchronization was also implemented. By constructing suitable adaptive control laws and parameter update rules based on Lyapunov stability theory, the convergence of the synchronization errors was achieved while simultaneously estimating the uncertain system parameters. Numerical simulations verified the effectiveness of the proposed adaptive controller, demonstrating rapid synchronization and accurate parameter estimation, thereby supporting the theoretical stability analysis. The synchronization framework was subsequently applied to develop a robust chaotic communication system based on the uncertain Jounce chaotic model on both the transmitter (master) and receiver (slave) sides. In the proposed Chaotic Parameter Modulation (CPM) scheme, binary information is encoded by modulating the control parameter of the master system. The simulation results demonstrate reliable recovery of the transmitted information, including binary sequences and binary images, under both noiseless and noisy public-channel conditions. The combination of complex chaotic dynamics and adaptive synchronization provides a robust mechanism for reliable information transmission and increases the difficulty of unauthorized signal interpretation. The proposed Jounce chaotic system represents a promising candidate for nonlinear control and robust noise-tolerant communication applications due to its simple mathematical structure, rich dynamic characteristics, and effective synchronization capability. Future work may focus on hardware implementation of the proposed system using FPGA, DSP, or embedded platforms; further evaluation under channel noise and parameter perturbations; and extension of the proposed model to fractional-order and hyperchaotic systems. In addition, more advanced chaotic communication schemes could be investigated by employing multiple modulation parameters or hybrid chaotic synchronization techniques to enhance transmission reliability and security.

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